NRL Report 6002

Elements of Normal Mode Theory

G. J. O'HARA AND P. F. CUNNIFF

Structures Branch
Mechanics Division

November 15, 1963

Reprinted
August 12, 1964

U.S. NAVAL RESEARCH LABORATORY
Washington, D.C.

UNCLASSIFIED

CONTENTS

SYMBOLS

A dot over a symbol indicates differentiation with respect to time.

A_n	n^{th} Fourier cosine coefficient
$A(t)$	Indicial admittance
B_n	n^{th} Fourier sine coefficient
C_a	proportionality factor for characteristic load
$D_a(t)$	Duhamel integral for base motion
$F_i(t)$	applied force acting on m_i
F_{ia}	inertia force plus applied force at m_i in the a^{th} mode
$\bar{F}_{ia}$	characteristic load acting on m_i in the a^{th} mode
$G(t)$	impulsive response
I_i	impulse applied to m_i
K	spring stiffness for single-degree-of-freedom system
K_{ij}	stiffness coefficient
M	total mass of a structure
M_a	apparent mass in mode a
$N_a(t)$	Duhamel integral for an applied force
$\bar{P}_a$	participation factor in mode a
Q_{ia}	inertia force acting on m_i in the a^{th} mode
T	time
T_0	period of a function
V_0	velocity step
W	work
$\underline{X}_i$	absolute displacement of m_i
X_i	relative displacement between m_i and the base
X_{ia}	relative displacement between m_i and the base in the a^{th} mode
$\bar{X}_{ia}$	normal mode shape for mode a
Z	base motion
g	acceleration due to gravity
m_i	i^{th} mass
$q_a(t)$	time function for displacement
r_{jk}	relative displacement between m_j and m_k
t	time
w_i	weight of i^{th} point
y_i	perpendicular distance between m_i and the base
β_k	weighting function
γ_{cj}	stress coefficient
δ_{ji}	influence coefficient
$\theta(t)$	rotational motion of a base
σ_c	stress at point c
$\bar{\sigma}_{ca}$	characteristic modal stress at point c due to characteristic loads from applied force
$\bar{\bar{\sigma}}_{ca}$	characteristic modal stress at point c due to characteristic loads from base motion
ϕ_{ia}	orthogonal function
ω	$\sqrt{K/m}$, the natural frequency for an undamped single-degree-of-freedom system
ω_a	natural frequency of mode a for an undamped multi-degree-of-freedom system
Ω	frequency of an applied vibratory force of constant amplitude

UNCLASSIFIED

Elements of Normal Mode Theory

G. J. O'HARA AND P. F. CUNNIFF

Structures Branch
Mechanics Division

Elementary normal mode theory is derived and used in defining the dynamic response of linear elastic structures. The theory is derived from the definition of a linear elastic structure by using D'Alembert's principle and only those mathematical methods which are familiar to most engineers and are no more complex than necessary. The cases of free vibrations and response to applied forces and base motion are examined in detail. Each normal mode is shown to respond to dynamic loads as a single-degree-of-freedom system with specific characteristics. Equations are developed for stress and deflection. It is shown that these can be converted to the form where the stresses or deflections are considered to be composed of two parts: one which ignores inertial effects (the static solution as a function of time) and one which represents a dynamic correction. Generalized Fourier expansions and characteristic load theorems are derived. The general problem of stresses and deflections is presented for arbitrary applied forces and base motions as well as for steady-state conditions.

INTRODUCTION

The response of linear elastic structures to dynamic loadings has been the subject of several earlier works, including NRL reports (1, 2). Recently, normal mode theory has become more widely used and accepted as a tool for structural design and analysis. It was felt that a new and clearer presentation of the background theory and derivation of the equations was necessary to help those who use this technique in their design work.

This report is deliberately limited in its use of mathematical methods to those which are no more complex than necessary. No background knowledge of Fourier transforms, Laplace transforms, Hamilton's equations, or Lagrange's equations is assumed. These general methods are so powerful for this type of problem that solutions are produced with deceptive ease. The feeling of really understanding the problem is lost while following the operational rules. This report is a self-contained reference text which includes many steps not often published. However, no claim is made to originality.

There are two basic approaches for the analytical representation of a linear elastic structure responding to dynamic forces. One method breaks the structure into a finite number of concentrated masses which are restrained by a weightless structure which has the same strength properties as the real structure. Such systems are called lumped parameter systems and have their governing equations of motion in the form of ordinary differential equations. The second method treats the structure as a continuous elastic body (an infinite number of masses) in which (at least segmentally) the material is assumed to be homogenous, isotropic, and to follow Hooke's law. These systems are called distributed parameter systems and have partial differential equations for their equations of motion. Most engineering structures are too complex to be solved by this second method.

The primary concern of this report is to find the motions and stresses of undamped linear elastic structures which are idealized as lumped parameter systems. It should be noted that the derivations can be converted to those for distributed systems by replacing the influence coefficients with Green's functions, and replacing the summations over all the masses by integrations with respect to the mass. A thorough understanding of the lumped parameter derivations will place the reader in a very advantageous position when dealing with problems of structures idealized as distributed parameter systems.

The usual assumptions concerning linear elasticity are made in this report. In addition, it is assumed that all applied forces and deflections are parallel. Only structures which rest on a base are considered.

NRL Problem F0 2-05; Projects RR 009-03-45-5752 and SF 013-10-01, 1793; 2760, 2962. This is an interim report on one phase of the problem; work is continuing on this and other phases. Manuscript submitted July 26, 1963.

THE SINGLE-DEGREE-OF-FREEDOM SYSTEM

Differential Equations

Since the undamped single-degree-of-freedom system is the simplest possible vibratory lumped parameter system, some of its properties are reviewed. Consider a structure which is idealized as a concentrated mass supported by a linear spring. The structure has been modeled as an undamped single-degree-of-freedom oscillator (Fig. 1). Motion and an applied force are indicated by Z and F, respectively, while $\underline{X}$ indicates the *absolute* response of the mass. The equation of motion is

$$\ddot{\underline{X}} + \omega^2 (\underline{X} - Z) = F/m \tag{1}$$

where $\omega^2 = K/m$. The relative motion of the mass (displacement with respect to the base) is designated by X, where $X = \underline{X} - Z$. If there is no base motion, the relative and absolute motions of the mass become equal. For this case Eq. (1) becomes

$$\ddot{X} + \omega^2 X = F/m. \tag{2}$$

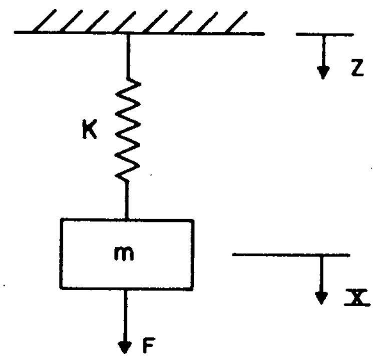

Fig. 1 — Undamped oscillator

The solution of Eq. (2) is composed of two parts, namely, the particular solution and the complementary solution. The particular solution for such an equation is developed in the next section. The complementary solution of Eq. (2) is

$$X_c = C_1 \cos \omega t + C_2 \sin \omega t.$$

Therefore,

$$X = X_c + X_p$$

$$X = C_1 \cos \omega t + C_2 \sin \omega t + X_p$$

where X_p is the particular solution. The constants C_1 and C_2 are found by using the known initial conditions on displacement and velocity, that is, $X(0)$ and $\dot{X}(0)$.

If there is no applied force, Eq. (2) becomes

$$\ddot{X} + \omega^2 X = 0. \tag{3}$$

This is the well-known equation of free vibration (3), which describes the motion possible in the absence of applied forces or base motions. The solution of Eq. (3) is

$$X = X(0) \cos \omega t + \frac{\dot{X}(0)}{\omega} \sin \omega t \tag{4}$$

where $C_1 = X(0)$ and $C_2 = \dot{X}(0)/\omega$ from the initial conditions. Let Eq. (1) be written as

$$\ddot{\underline{X}} + \omega^2 \underline{X} = \frac{F}{m} + \omega^2 Z \tag{5}$$

or

$$\ddot{X} + \omega^2 X = \frac{F}{m} - \ddot{Z}. \tag{6}$$

The form of Eqs. (5) and (6) is the same. However, one is for absolute motion and one is for relative motion. *This distinction should be clearly understood.* To find the general solution of Eqs. (5) and (6), the particular solution must be added to the complementary solution. The particular solution is of the form of a superposition integral called a Duhamel integral. This integral is derived in the following subsection.

Duhamel Integrals

It is characteristic of linear differential equations that solutions can be superposed. Thus, if λ_1, λ_2, and λ_3 are individual solutions of such an equation, then the complete solution is

$$X = \sum_{i=1}^{3} \lambda_i.$$

The indicial admittance and the impulsive response are two quantities which are used to represent superposition integral solutions for a simple oscillator. The indicial admittance A is the response of the oscillator to a unit step of the disturbing force $1(t)$, and the impulsive response

G is the response of the oscillator to a unit impulse. Since a unit step is the integral of a unit impulse (Figs. 2a-2d),

$$A(t) = \int_0^t G(T)\, dT \qquad (7)$$

because superposition holds.

Assume that a unit step of force as shown in Fig. 2a is applied to the oscillator and there is no base motion. Equation (6) becomes

$$\ddot{X} + \omega^2 X = \frac{1}{m}.$$

The method of undetermined coefficients can be used to find a particular solution. The assumption that $X = C_0$ yields $\omega^2 C_0 = 1/m$; after rearrangement this yields $C_0 = 1/K$. Thus the solution is

$$X = \frac{1}{K} + C_1 \sin \omega t + C_2 \cos \omega t$$

where the C's are determined from the condition that at $t = 0$, $X = 0$ and $\dot{X} = 0$. Then

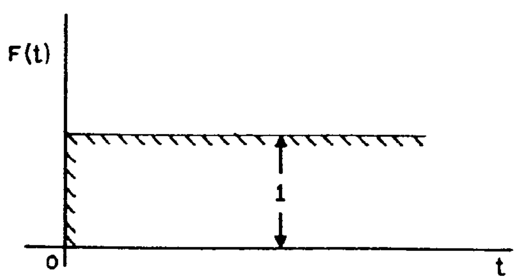

Fig. 2a – Unit step of force

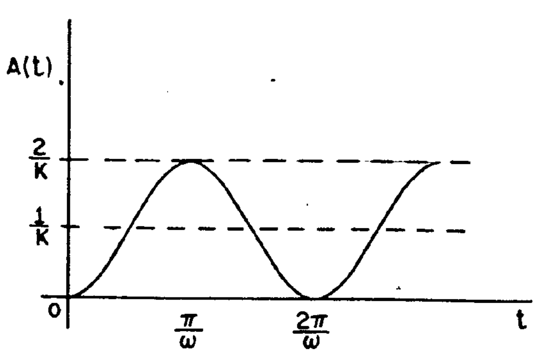

Fig. 2b – Indicial admittance: the response to a unit step of force

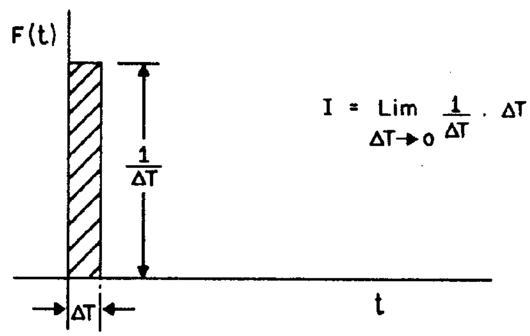

Fig. 2c – Unit impulse

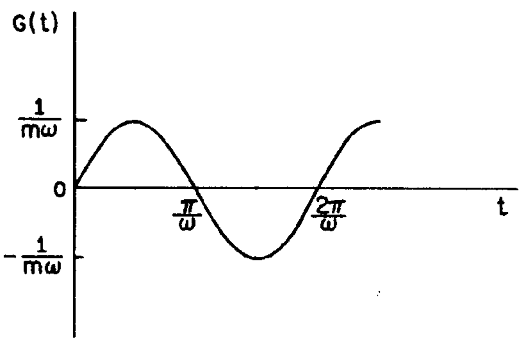

Fig. 2d – Impulsive response: the response to a unit impulse

$$X = A(t) = \frac{1}{K}(1 - \cos \omega t). \qquad (8)$$

This response is shown in Fig. 2b. Using Eq. (7)

$$G(t) = \frac{1}{m\omega} \sin \omega t. \qquad (9)$$

This response is shown in Fig. 2d.

Consider the problem of finding the response to a general transient force as shown in Fig. 3. The principle of superposition can be used to find the solution in terms of the indicial admittance or the impulsive response. The method consists of breaking up the forcing function into a number of steps at equal time intervals and summing the response to these steps. The response at any time t is a function of the elapsed time, $t - T$, from the application of the step ΔF. Writing $\Delta F = (\Delta F/\Delta T)/\Delta T$, we obtain

$$X(t) = F(0)A(t) + \sum_{T=\Delta T}^{t} \frac{\Delta F}{\Delta T} A(t-T)\, \Delta T. \qquad (10)$$

From the fundamental theorem of integral

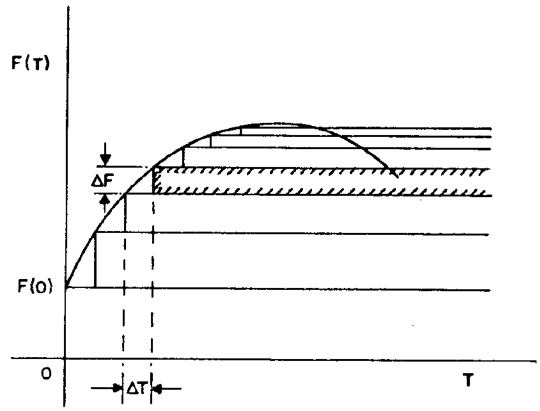

Fig. 3 – General forcing function partitioned into a number of steps of equal time intervals

calculus the limit of a sum of the form of Eq. (10) as $\Delta T \to 0$ is

$$X = F(0)A(t) + \int_0^t \frac{dF}{dT} A(t-T)\, dT. \quad (11)$$

Integration by parts gives

$$X = F(t)A(0) + \int_0^t F(T)G(t-T)\, dT. \quad (12)$$

For the undamped linear oscillator, Eq. (11) becomes

$$X = \frac{F(t)-F(0)}{K} + \frac{F(0)}{K}(1-\cos \omega t)$$

$$-\frac{1}{K}\int_0^t \dot{F}(T) \cos \omega(t-T)\, dT \quad (13)$$

and Eq. (12) becomes

$$X = \frac{1}{m\omega}\int_0^t F(T) \sin \omega(t-T)\, dT. \quad (14)$$

These are two of the possible forms for Duhamel integrals. Inspection of Eq. (6) immediately leads to a Duhamel integral for base motion alone:

$$X = -\frac{1}{\omega}\int_0^t \ddot{Z}(T) \sin \omega(t-T)\, dT. \quad (15)$$

Appendix A discusses differentiation of integrals in the form of Eq. (15).

INFLUENCE AND STIFFNESS COEFFICIENTS

If a static force F_i is applied to a linear elastic structure which is fixed to an immovable base, the deflection due to distortion of any point on the structure is proportional to the force, or

$$\overline{X}_j = X_j = \delta_{ji} F_i. \quad (16)$$

This is simply the definition of a linear elastic structure, and the proportionality factor δ_{ji} is called an influence coefficient. It reads as the deflection at j due to a unit force applied at i. In the introduction it was assumed that applied forces and deflections were parallel to each other. If more than one force is applied to the structure, then the principle of superposition is used to find the deflection at any point or set of points. Thus:

$$X_1 = \delta_{11}F_1 + \delta_{12}F_2 + \ldots + \delta_{1n}F_n$$

$$X_2 = \delta_{21}F_1 + \delta_{22}F_2 + \ldots + \delta_{2n}F_n$$

$$\vdots$$

$$X_n = \delta_{n1}F_1 + \delta_{n2}F_2 + \ldots + \delta_{nn}F_n$$

which may be written as

$$X_j = \sum_{i=1}^{n} \delta_{ji} F_i, \qquad j = 1, 2, \ldots, n. \quad (17)$$

Unless otherwise indicated, from here onward, all summations are taken from $i = 1$ to $i = n$. For example,

$$\sum_i = \sum_{i=1}^{n} \text{ and } \sum_a = \sum_{a=1}^{n}.$$

There is a relationship between the influence coefficients of the form δ_{ik} and δ_{ki}, known as Maxwell's law of reciprocal deflections. To show this, first apply a force F_j and then a force F_i. Calculate the internal work. Then reverse the procedure of loading and calculate the work. Since for a linear elastic structure the energy depends only upon the applied loads and final deflections, the work done in both cases is the same.

In the first case, when load F_j is applied the work done is

$$W = \frac{1}{2} F_j X_j = \frac{1}{2} \delta_{jj} F_j^2$$

and when load F_i is applied the work done is $W = \frac{1}{2} F_i X_i +$ (work done when F_j moves due to F_i)

$$= \frac{1}{2} \delta_{ii} F_i^2 + F_j \delta_{ji} F_i.$$

The total work is

$$\frac{1}{2} \delta_{jj} F_j^2 + \frac{1}{2} \delta_{ii} F_i^2 + F_j \delta_{ji} F_i.$$

In the second case, application of the loads in reverse order gives the total work as

$$\frac{1}{2} \delta_{ii} F_i^2 + \frac{1}{2} \delta_{jj} F_j^2 + F_i \delta_{ij} F_j.$$

To satisfy the equality of energy, $\delta_{ij} = \delta_{ji}$. Therefore, the array of influence coefficients

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2n} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \delta_{n1} & \delta_{n2} & \dots & \delta_{nn} \end{bmatrix}$$

is symmetric about the principal diagonal.

A stiffness coefficient K_{ij} is the force required at i when the structure is loaded in such manner that all points are restrained from moving except j, which moves a unit distance in the negative direction. If Eq. (17) is solved for the forces, there results

$$F_i = \sum_j K_{ij} X_j.$$

Note that $K_{ij} \neq 1/\delta_{ij}$ (see Appendix B). In an analogous manner it can be shown that $K_{ji} = K_{ij}$, so that the array of stiffness coefficients is symmetric about the principal diagonal. For some structures it may be more convenient to use stiffness coefficients than influence coefficients in defining mode shapes and natural frequencies.

GENERALIZED FOURIER EXPANSIONS

This section reviews some important properties of generalized Fourier expansions which are used in the remainder of this report. As an introduction, it is noted that the decisive property of the set of functions $\{\cos(2n\pi t/T_0)$ and $\sin(2n\pi t/T_0)\}$ which allows the arbitrary function $f(t)$ to be expanded in the form

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi t}{T_0} + \sum_{n=1}^{\infty} B_n \sin \frac{2n\pi t}{T_0}$$

between the limits 0 and T_0 is that the integral of the product of any two of these functions which are distinct is zero. Sines and cosines are by no means the only functions with this property. In fact, they are perhaps only the simplest example of the infinity of such possible functions.

Orthogonality

If a sequence of n real functions $\{\phi_{ka}\}$ has the property that over some interval (finite or infinite)

$$\sum_k \beta_k \phi_{ka} \phi_{kb} = 0$$

and

$$\sum_k \beta_k \phi_{ka}^2 \neq 0$$

then the ϕ's are said to be orthogonal with respect to a weighting function β_k.

Completeness

If there exists no function F_i, except the identically zero function, with the property that

$$\sum_i \beta_i \phi_{ia} F_i = 0$$

for all members of an orthogonal set $\{\phi_{ia}\}$, then the set $\{\phi_{ia}\}$ is complete. If one of the members of the set $\{\phi_{ib}\}$ were omitted, the resulting set is not complete since

$$\sum_i \beta_i \phi_{ia} \phi_{ib} = 0.$$

Expansions

An arbitrary function γ_i ($i = 1, \dots, n$) has a formal expansion which is analogous to its Fourier expansion. Let

$$\gamma_i = b_1 \phi_{i1} + b_2 \phi_{i2} + \dots + b_n \phi_{in}$$

where the ϕ's form a complete set. Multiply both sides by $\beta_i\phi_{ia}$ and sum on i:

$$\sum_i \beta_i\phi_{ia}\gamma_i = b_1\sum_i \beta_i\phi_{ia}\phi_{i1} + \ldots$$

$$+ b_a\sum_i \beta_i\phi_{ia}^2 + \ldots$$

$$+ b_n\sum_i \beta_i\phi_{ia}\phi_{in}.$$

Now from the orthogonality conditions all the sums on the right are zero, except the one which contains the ϕ_{ia}^2. Therefore,

$$b_a = \frac{\sum_i\beta_i\phi_{ia}\gamma_i}{\sum_i\beta_i\phi_{ia}^2}.$$

This leads to the expansion

$$\gamma_i = \sum_a \frac{\phi_{ia}\sum_j\beta_j\phi_{ja}\gamma_j}{\sum_j\beta_j\phi_{ja}^2}. \quad (18)$$

Orthonormal Functions

Some authors set

$$\sum_i\beta_i\phi_{ia}^2 = H$$

as a "normalizing" condition, while others set

$$\sum_i\beta_i\phi_{ia}^2 = 1.$$

This last technique is a favorite of mathematicians because it creates an orthonormal set. An orthonormal set has the property that

$$\sum_j \psi_{ja}\psi_{jb} = 0$$

and

$$\sum_j \psi_{ja}^2 = 1.$$

Any set of orthogonal functions can be converted into an orthonormal set. In fact, let

$$\eta_{ia} = \phi_{ia}\Big/\sqrt{\sum_i\beta_i\phi_{ia}^2}\,.$$

Now let

$$\psi_{ia} = \sqrt{\beta_i}\;\eta_{ia}.$$

Then

$$\sum_i \psi_{ia}\psi_{ib} = \sum_i \beta_i\eta_{ia}\eta_{ib} = 0$$

and

$$\sum_i \psi_{ia}^2 = \sum_i \beta_i\eta_{ia}^2 = \frac{\sum_i\beta_i\phi_{ia}^2}{\sum_i\beta_i\phi_{ia}^2} = 1$$

since the original ϕ's are orthogonal. Therefore, it is no specialization to assume that an orthogonal set is also orthonormal. This fact is *not* used in this report, because it is desired to present normal mode theory as simply as possible and to present end results that are directly useful for calculation purposes.

FREE VIBRATIONS

Normal Modes

Assume that a weightless structure attached to a fixed base is carrying a set of n concentrated masses m_i which are attached at the n points i. Consider its free vibrations, that is, the possible motions in the absence of external forces. This is done by introducing D'Alembert's principle, which states that a system in motion can be considered to be in equilibrium at any instant if appropriate inertia forces $-m_i\ddot{\underline{X}}_i$ are applied to the system. For the case of the freely vibrating structure, simply apply these inertia forces to view the structure as being in a state of equilibrium. The set of forces on the structure is now treated as a static problem. Recall that for an elastically distorted structure in equilibrium

$$X_i = \sum_j \delta_{ij}F_j. \quad (17)$$

For free vibrations the only forces on the structure are the inertia forces, so

$$X_i = -\sum_j \delta_{ij}m_j\ddot{\underline{X}}_j. \quad (19)$$

This is a set of n differential equations with constant coefficients expressing the X_i's in terms of the $\ddot{\underline{X}}_j$'s. Since there is no base motion, $\underline{X}_j = X_j$ and $\ddot{\underline{X}}_j = \ddot{X}_j$. Equation (19) is rewritten

$$X_i = -\sum_j \delta_{ij} m_j \ddot{X}_j. \qquad (19')$$

To obtain a solution try $X_i = \overline{X}_i \sin(\omega t + \beta)$, which is usually done in the single-degree-of-freedom system. Then

$$\overline{X}_i \sin(\omega t + \beta) = \omega^2 \sin(\omega t + \beta) \sum_j \delta_{ij} m_j \overline{X}_j$$

or

$$\overline{X}_i = \omega^2 \sum_j \delta_{ij} m_j \overline{X}_j \qquad (i = 1, 2, \ldots, n). \quad (20)$$

There may be a solution of the problem if these n algebraic equations can be solved for the displacement ratios and the ω's. When written out, this set is

$$(\omega^2 m_1 \delta_{11} - 1)\overline{X}_1 + \omega^2 m_2 \delta_{12} \overline{X}_2 + \ldots + \omega^2 m_n \delta_{1n} \overline{X}_n = 0$$

$$\omega^2 m_1 \delta_{21} \overline{X}_1 + (\omega^2 m_2 \delta_{22} - 1)\overline{X}_2 + \ldots + \omega^2 m_n \delta_{2n} \overline{X}_n = 0$$

$$\vdots$$

$$\omega^2 m_1 \delta_{n1} \overline{X}_1 + \omega^2 m_2 \delta_{n2} \overline{X}_2 + \ldots + (\omega^2 m_n \delta_{nn} - 1)\overline{X}_n = 0.$$

Inspection of this set shows that it is a set of linear algebraic equations all of which are equal to zero. If a solution is to exist other than the trivial one where all the $\overline{X}_j$'s equal zero (static equilibrium case), it occurs only for those values of ω which make the determinant of the coefficients of the $\overline{X}_j$'s equal to zero (4-6). This leads to an algebraic equation of degree n in ω^2 usually called the frequency equation. Since undamped structures are considered, these roots are real and positive (4). These frequencies are called the fixed base natural frequencies of the system oscillating in the absence of external forces. Except for a few special cases they will be distinct. Those systems which have a pair or more of equal roots are called degenerate systems. Other techniques for solving such a set of equations treat them as an eigenvalue–eigenvector problem, which is a characteristic value problem with latent roots (5).

For the systems where the roots of ω^2 are all distinct, the ratios of amplitudes of the masses can be found by the back substitution solution of the set of equations, which is defined by

$$\overline{X}_{ia} = \omega_a^2 \sum_j \delta_{ij} m_j \overline{X}_{ja}. \qquad (21)$$

Note the following:

1. A subscript a has been added to the $\overline{X}_j$'s to identify those which correspond with ω_a.
2. The ω_a's are called the fixed base natural frequencies of the system.
3. The sets of the $\overline{X}_{ja}$'s are called the normal mode shapes, and are defined by Eq. (21) for each mode a.
4. Equation (21) is still satisfied if all the $\overline{X}_{ja}$'s are multiplied by any factor C. This means that the *ratios* of the displacements have been found for each mode and not the absolute values. This is not too surprising because the fixed base natural frequencies of a linear elastic structure are not amplitude dependent.
5. The ratios of the numerical values of the $\overline{X}_{ia}$'s can be arbitrarily fixed in any convenient fashion. One technique sets the amplitude of one of them equal to unity. The remaining amplitudes then become some multiple of unity.
6. For the degenerate systems, back substitution in Eq. (21) does not produce the set of mode shapes. Other techniques such as matrix deflation or special forms of adjoint matrices can be used (5). It is assumed that these mode shapes can be found in order to proceed.

Orthogonality of the Normal Modes

There is some additional information which can be obtained about the normal mode shapes. They are orthogonal to each other. To establish this, multiply both sides of Eq. (21) by $m_i \overline{X}_{ib}$ and sum on i. This gives

$$\sum_i m_i \overline{X}_{ib} \overline{X}_{ia} = \omega_a^2 \sum_i m_i \overline{X}_{ib} \sum_j \delta_{ij} m_j \overline{X}_{ja}$$

which can be written as

$$\sum_i m_i \overline{X}_{ib} \overline{X}_{ia} = \omega_a^2 \sum_j m_j \overline{X}_{ja} \sum_i \delta_{ji} m_i \overline{X}_{ib}$$

since $\delta_{ij} = \delta_{ji}$. Changing the order of subscripts in Eq. (21) gives

$$\frac{\overline{X}_{jb}}{\omega_b^2} = \sum_i \delta_{ji} m_i \overline{X}_{ib}. \qquad (21')$$

The right side of this equation appears in the previous one, so

$$\sum_i m_i \bar{X}_{ib} \bar{X}_{ia} = \frac{\omega_a^2}{\omega_b^2} \sum_j m_j \bar{X}_{ja} \bar{X}_{jb}.$$

Now, since

$$\sum_i m_i \bar{X}_{ib} \bar{X}_{ia} = \sum_j m_j \bar{X}_{ja} \bar{X}_{jb}$$

the previous equation becomes

$$\left(1 - \frac{\omega_a^2}{\omega_b^2}\right) \sum_j m_j \bar{X}_{jb} \bar{X}_{ja} = 0.$$

There are two possible cases; $b = a$, or $b \neq a$. When $b = a$, the term in the brackets becomes zero and the summation becomes

$$\sum_j m_j \bar{X}_{ja}^2.$$

This is a series of positive terms which cannot be zero. When $b \neq a$, the term in the brackets is not zero, so that the summation term must be zero. This yields the orthogonality conditions

$$\sum_j m_j \bar{X}_{ja}^2 \neq 0 \tag{22}$$

and

$$\sum_j m_j \bar{X}_{ja} \bar{X}_{jb} = 0. \tag{23}$$

Equation (23) shows that the normal modes are independent of each other.

Type of Normal Mode Solution

The distortion of the structure is completely described if the set of X_i's is found. However, when defining the normal shapes, they were found to be relative ratios which are orthogonal to each other. Let mode response at point i be $X_{ia}(t)$. There is no bar over the letter because it is actual response that is desired. If $X_{ia}(t)$ is found, the total response X_i can be found by superposition, that is,

$$X_i = \sum_a X_{ia}.$$

The problem resolves itself into finding the X_{ia}'s. At each i in each mode a there is a relative amplitude of $\bar{X}_{ia}$. There must be a function which converts the $\bar{X}_{ia}$ to X_{ia}. The technique used is very similar to the concept of the separation of variables in the solution of certain partial differential equations. That is, a solution will be sought in the form

$$X_{ia} = \bar{X}_{ia} q_a$$

so that

$$X_i = \sum_a \bar{X}_{ia} q_a \tag{24}$$

and

$$\ddot{X}_i = \sum_a \bar{X}_{ia} \ddot{q}_a. \tag{25}$$

Now if q_a is found, the free vibration problem is solved. Substitution of Eqs. (24) and (25) into the original set of differential equations (Eq. 19′) yields

$$\sum_a \bar{X}_{ia} q_a = -\sum_j \delta_{ij} m_j \sum_a \bar{X}_{ja} \ddot{q}_a. \tag{26}$$

By transposition

$$\sum_a \left(\ddot{q}_a \sum_j \delta_{ij} m_j \bar{X}_{ja} + \bar{X}_{ia} q_a\right) = 0$$

and by using Eq. (21) this becomes

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a\right) \bar{X}_{ia} = 0. \tag{27}$$

The orthogonality relationship can now be used. Multiply both sides of Eq. (27) by $m_i \bar{X}_{ib}$ and sum on i:

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a\right) \sum_i m_i \bar{X}_{ib} \bar{X}_{ia} = 0.$$

There is only one case when the summation over i is not equal to zero: when $a = b$. The summation over all the modes a is reduced to

$$\frac{\ddot{q}_b}{\omega_b^2} + q_b = 0$$

or

$$\ddot{q}_a + \omega_a^2 q_a = 0 \tag{28}$$

where the subscript b is arbitrarily changed to a. Equation (28) is in the form of Eq. (3) and has the free vibration solution

$$q_a = q_a(0) \cos \omega_a t + \frac{\dot{q}_a(0)}{\omega_a} \sin \omega_a t. \quad (29)$$

Returning to Eq. (24),

$$X_i = \sum_a \bar{X}_{ia} q_a(0) \cos \omega_a t + \sum_a \bar{X}_{ia} \frac{\dot{q}_a(0)}{\omega_a} \sin \omega_a t. \quad (30)$$

Initial Conditions

Assume that the general initial conditions at $t=0$ are $X_i = X_i(0)$, $\dot{X}_i = \dot{X}_i(0)$. Equation (30) yields

$$X_i(0) = \sum_a \bar{X}_{ia} q_a(0). \quad (31)$$

Upon differentiating and introducing the initial condition on velocity, Eq. (30) yields

$$\dot{X}_i(0) = \sum_a \bar{X}_{ia} \dot{q}_a(0). \quad (32)$$

Again the orthogonality relationship can be used by multiplying both sides of Eqs. (31) and (32) by $m_i \bar{X}_{ib}$ and summing on i:

$$\sum_i m_i \bar{X}_{ib} X_i(0) = \sum_a q_a(0) \sum_i m_i \bar{X}_{ib} \bar{X}_{ia}$$

and

$$\sum_i m_i \bar{X}_{ib} \dot{X}_i(0) = \sum_a \dot{q}_a(0) \sum_i m_i \bar{X}_{ib} \bar{X}_{ia}.$$

Therefore,

$$q_a(0) = \frac{\sum_i m_i \bar{X}_{ia} X_i(0)}{\sum_i m_i \bar{X}_{ia}^2} \quad (33)$$

and

$$\dot{q}_a(0) = \frac{\sum_i m_i \bar{X}_{ia} \dot{X}_i(0)}{\sum_i m_i \bar{X}_{ia}^2}. \quad (34)$$

Substitution of Eqs. (33) and (34) into Eqs. (31) and (32) respectively produces

$$X_i(0) = \sum_a \frac{\bar{X}_{ia} \sum_j m_j \bar{X}_{ja} X_j(0)}{\sum_j m_j \bar{X}_{ja}^2}$$

and

$$\dot{X}_i(0) = \sum_a \frac{\bar{X}_{ia} \sum_j m_j \bar{X}_{ja} \dot{X}_j(0)}{\sum_j m_j \bar{X}_{ja}^2}.$$

This expansion of an arbitrary function into a series of modal functions is called a generalized Fourier expansion. Substitution of Eqs. (33) and (34) into Eq. (30) gives the complete normal mode solution for free vibrations, that is,

$$X_i = \sum_a \frac{\bar{X}_{ia} \sum_j m_j \bar{X}_{ja} X_j(0)}{\sum_j m_j \bar{X}_{ja}^2} \cos \omega_a t + \sum_a \frac{\bar{X}_{ia} \sum_j m_j \bar{X}_{ja} \dot{X}_j(0)}{\omega_a \sum_j m_j \bar{X}_{ja}^2} \sin \omega_a t. \quad (35)$$

Several important points related to free vibrations of undamped linear elastic structures are now summarized.

1. The system is described by ordinary differential equations which are linear and have constant coefficients. This allows superposition.
2. There are as many modes and natural frequencies as there are independent masses, although some frequencies may be redundant.
3. Each normal mode is periodic, of frequency ω_a, and the relative amplitudes of each of the vibrating masses is fixed for this mode.
4. The normal modes are orthogonal to each other.
5. A solution of the form of a linear combination of the normal modes is possible, that is,

$$X_i = \sum_a X_{ia} = \sum_a \bar{X}_{ia} q_a$$

for any possible deflected position X_i.

6. An arbitrary set of values, such as $X_i(0)$ and $\dot{X}_i(0)$, which are assigned to each mass point can be expanded into a series involving the mode shape functions, and each resulting coefficient of the series assigned to a different mode (generalized Fourier expansion).
7. Equation (35) shows that while each normal mode vibrates freely in a periodic fashion, the resulting motion need not be periodic for any mass

point. It is, in fact, aperiodic except for the case when the roots of the frequency equation are commensurable, like a Fourier series.

8. The term $\overline{X}_{ia}$ may be read as the a^{th} mode shape number at the mass point i. Unlike influence and stiffness coefficients, the array of the $\overline{X}_{ia}$ as defined by

$$X_i = \sum_a \overline{X}_{ia} q_a$$

is not symmetrical, so that $\overline{X}_{cd}$ is not of necessity equal to $\overline{X}_{dc}$.

RESPONSE TO AN APPLIED FORCE

Consider a structure which rests on an immovable base, and suppose a force F_k, applied to m_k, is time dependent but independent of structural reaction. Using D'Alembert's principle and influence coefficients, the distortion of the structure is described by the n equations

$$X_i = -\sum_j \delta_{ij} m_j \ddot{X}_j + \delta_{ik} F_k. \tag{36}$$

A solution of the form

$$X_i = \sum_a \overline{X}_{ia} q_a \tag{24}$$

is sought. Substitution of Eqs. (24) and (25) into Eq. (36) yields

$$\sum_a \overline{X}_{ia} q_a = -\sum_a \ddot{q}_a \sum_j \delta_{ij} m_j \overline{X}_{ja} + \delta_{ik} F_k.$$

Transposing,

$$\sum_a \left(q_a \overline{X}_{ia} + \ddot{q}_a \sum_j \delta_{ij} m_j \overline{X}_{ja} \right) = \delta_{ik} F_k.$$

Using Eq. (21) this may be written as

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a \right) \overline{X}_{ia} = \delta_{ik} F_k. \tag{37}$$

The left side is the same as in free vibrations. The influence coefficient δ_{ik} times the force F_k must be brought into the parentheses. Therefore, this expression is expanded into a series of the mode shapes. Let

$$\delta_{ik} F_k = \sum_a \overline{X}_{ia} \Delta_{ka}. \tag{38}$$

Multiply both sides by $m_i \overline{X}_{ib}$ and sum on i:

$$F_k \sum_i m_i \overline{X}_{ib} \delta_{ik} = \sum_a \Delta_{ka} \sum_i m_i \overline{X}_{ib} \overline{X}_{ia}.$$

The left side is $F_k \overline{X}_{kb}/\omega_b^2$ by Eq. (21) and the right side reduces from a series in a to a single term by virtue of orthogonality. So,

$$\frac{F_k \overline{X}_{kb}}{\omega_b^2} = \Delta_{kb} \sum_i m_i \overline{X}_{ib}^2.$$

Therefore, upon changing subscripts,

$$\Delta_{ka} = \frac{F_k \overline{X}_{ka}}{\omega_a^2 \sum_i m_i \overline{X}_{ia}^2}.$$

Equation (38) becomes

$$\delta_{ik} F_k = F_k \sum_a \frac{\overline{X}_{ia} \overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2}. \tag{39}$$

The influence coefficient is given by a normal mode expansion of the form

$$\delta_{ik} = \sum_a \frac{\overline{X}_{ia} \overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2}. \tag{39'}$$

For direct influence coefficients this expression is a series of positive terms only, that is,

$$\delta_{kk} = \sum_a \frac{\overline{X}_{ka}^2}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2}. \tag{39''}$$

Substitute Eq. (39) into Eq. (37) to obtain

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a \right) \overline{X}_{ia} = \sum_a \frac{\overline{X}_{ia} \overline{X}_{ka} F_k}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2}.$$

Transposing,

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a - \frac{\overline{X}_{ka} F_k}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \right) \overline{X}_{ia} = 0.$$

Now the orthogonality relationship is applied. Thus,

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a - \frac{\overline{X}_{ka} F_k}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \right) \sum_i m_i \overline{X}_{ib} \overline{X}_{ia} = 0.$$

Therefore,

$$\ddot{q}_a + \omega_a^2 q_a = \frac{\overline{X}_{ka} F_k}{\sum_j m_j \overline{X}_{ja}^2}. \qquad (40)$$

Equation (40) is in the same form as Eq. (6) for a simple oscillator when Z is zero. The simple oscillator was shown to have Eq. (14) as a particular solution. This same expression is applicable provided ω_a replaces ω, q_a replaces X, and $\overline{X}_{ka}F_k(T)/\sum_j m_j \overline{X}_{ja}^2$ replaces $F(T)/m$. Hence,

$$q_a = \frac{\overline{X}_{ka}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t F_k(T) \sin \omega_a (t - T)\, dT.$$

The solution for X_i is

$$X_i = \sum_a \frac{\overline{X}_{ia} \overline{X}_{ka}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t F_k(T) \sin \omega_a (t - T)\, dT. \qquad (41)$$

This is the response equation due to an applied force $F_k(T)$ for a structure which is initially at rest. If the structure is not at rest, then Eq. (35) should be added to Eq. (41). It is noted that the ratio of the response between points i and j in mode a is

$$\frac{X_{ia}}{X_{ja}} = \frac{\overline{X}_{ia}}{\overline{X}_{ja}}.$$

If more than one force is applied at the same time, superposition is used to solve the problem. Since the derivation assumed the force to be applied at m_k, sum the d applied forces. In this case

$$X_i = \sum_a \frac{\overline{X}_{ia}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t \left[\sum_{k=1}^{d} \overline{X}_{ka} F_k(T)\right] \sin \omega_a (t - T)\, dT \qquad (42)$$

for the particular solution.

RESPONSE TO BASE MOTION

Suppose a structure initially at rest is attached to some base. Assume that this base undergoes a motion $Z(T)$ which is a known time dependent function.

Consider the equations of an elastically distorted structure:

$$X_i = \sum_j \delta_{ij} F_j. \qquad (17)$$

Using D'Alembert's principle, this becomes

$$X_i = -\sum_j \delta_{ij} m_j \ddot{\underline{X}}_j \qquad (19)$$

where $\ddot{\underline{X}}_i$ equals the absolute acceleration of m_i. Since $X_i = \underline{X}_i - Z$, Eq. (19) is written

$$X_i = -\sum_j \delta_{ij} m_j (\ddot{X}_j + \ddot{Z}). \qquad (43)$$

This expresses the relative displacement X_i in terms of the relative acceleration of m_j and the base acceleration. The usual means of solution is used again. Let

$$X_i = \sum_a \overline{X}_{ia} q_a$$

and substitute into Eq. (43):

$$\sum_a \overline{X}_{ia} q_a = -\sum_a \ddot{q}_a \sum_j \delta_{ij} m_j \overline{X}_{ja} - \ddot{Z} \sum_j \delta_{ij} m_j.$$

Making use of an expansion in terms of the modes for $\sum_j \delta_{ij} m_j$ leads to

$$\sum_a \left(\frac{\ddot{q}_a}{\omega_a^2} + q_a + \frac{\ddot{Z} \sum_j m_j \overline{X}_{ja}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \right) \overline{X}_{ia} = 0.$$

The orthogonality conditions give

$$\ddot{q}_a + \omega_a^2 q_a = -\frac{\sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}_{ja}^2} \ddot{Z}.$$

This equation is in the form of the equation of relative motion for a simple oscillator if there is a base motion and no applied force, namely Eq. (6) without the F/m term. A particular solution is given by Eq. (15). This same expression is applicable provided ω_a replaces ω, q_a replaces X, and $\ddot{Z} \sum_j m_j \overline{X}_{ja} / \sum_j m_j \overline{X}_{ja}^2$ replaces $\ddot{Z}$. Hence,

$$q_a = -\frac{\sum_j m_j \overline{X}_{ja}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t \ddot{Z}(T) \sin \omega_a (t - T)\, dT$$

and

$$X_i = -\sum_a \frac{\overline{X}_{ia} \sum_j m_j \overline{X}_{ja}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t \ddot{Z}(T) \sin \omega_a (t - T)\, dT. \qquad (44)$$

This is a general equation for the relative response of a linear elastic structure when the base motion is a known function of time. The absolute motion of m_i is $\underline{X}_i = X_i + Z$, so

$$\underline{X}_i = Z(t)$$

$$-\sum_a \frac{\overline{X}_{ia}\sum_j m_j \overline{X}_{ja}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t \ddot{Z}(T) \sin \omega_a (t-T)\, dT.$$

Before proceeding to some special topics it might be well to consider some important points which have arisen in this discussion of applied forces and foundation motions.

1. A solution in the form of a linear combination of normal modes was obtained.
2. Each normal mode acts as a single-degree-of-freedom system with specific characteristics when responding to applied forces or base motions.
3. Since the equations of motion are linear, the "initial" conditions for a structure can be accounted for by adding their equation of motion to the Duhamel integral solutions.
4. A generalized Fourier expansion for the influence coefficients was obtained in terms of the mode shapes and natural frequencies.
5. The ratios of the deflections X_{ia} were found to be the same as the ratios of $\overline{X}_{ia}$'s.

SPECIAL TOPICS

Response to a Step Function of Force

Equation (8) is the response of an oscillator to a unit step in force. For a step in force equal to F, the response is

$$X = \frac{F}{m\omega^2} - \frac{F}{m\omega^2} \cos \omega t.$$

If the corresponding equivalent terms of the multi-degree-of-freedom model are again substituted for the oscillator response, the normal mode solution is

$$X_i = F_k \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2}$$

$$-F_k \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \cos \omega_a t.$$

Using Eq. (39′), it is written as

$$X_i = \delta_{ik}F_k - F_k \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \cos \omega_a t. \quad (45)$$

This produces the interesting result that the deflection is the static deflection, $\delta_{ik}\ F_k$, plus a dynamic correction.

A similiar equation for X_k is

$$X_k = \delta_{kk}F_k - F_k \sum_a \frac{\overline{X}_{ka}^2}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \cos \omega_a t. \quad (46)$$

This equation shows that each normal mode term $\overline{X}_{ka}^2/\omega_a^2 \sum_j m_j \overline{X}_{ja}^2$ is positive. Therefore, the maximum possible deflection of X_k would occur if all the cos $\omega_a t$ values were simultaneously -1. This leads to the result that

$$|X_k|_{max,\ max} = 2\delta_{kk}F_k.$$

Inspection of Eq. (45) shows that the term $\overline{X}_{ia}\overline{X}_{ka}/\omega_a^2 \sum_j m_j \overline{X}_{ja}^2$ is negative as well as positive. Therefore, for a step force applied at k,

$$|X_i|_{max,\ max} \geqslant 2\delta_{ik}F_k.$$

If there are many such applied forces F_k, superposition is used to write

$$X_i = \sum_{k=1}^{d} \delta_{ik}F_k - \sum_a \frac{\overline{X}_{ia} \sum_{k=1}^{d} F_k \overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}_{ja}^2} \cos \omega_a t \quad (47)$$

which is again the static deflection plus a dynamic correction.

Impulse

Using the oscillator response to impulse, the response to impulse applied at mass k is

$$X_i = I_k \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \sin \omega_a t \quad (48)$$

where I_k is I times the unit impulse. So,

$$\dot{X}_i = I_k \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\sum_j m_j \overline{X}_{ja}^2} \cos \omega_a t.$$

Since the structure rests on a base and the masses were assumed to be capable of independent movement, then at $t = 0$, the velocity of m_i must be zero, so that

$$\sum_a \frac{\bar{X}_{ia}\bar{X}_{ka}}{\sum_j m_j \bar{X}^2_{ja}} = 0. \tag{49}$$

Similarly, the velocity of the mass which is struck by the impulse is I_k/m_k at $t = 0$. Therefore,

$$\sum_a \frac{\bar{X}^2_{ka}}{\sum_j m_j \bar{X}^2_{ja}} = \frac{1}{m_k}. \tag{50}$$

If there are many applied impulses all occurring at the same time, the solution by superposition is

$$X_i = \sum_a \frac{\bar{X}_{ia}\sum_k I_k \bar{X}_{ka}}{\omega_a \sum_j m_j \bar{X}^2_{ja}} \sin \omega_a t. \tag{51}$$

Sudden Motion of the Base

Consider the response of a structure initially at rest to a step change in the velocity of the base. The single-degree-of-freedom solution is $X = (-V_0/\omega) \sin \omega t$, and $\dot{X} = -V_0 \cos \omega t$, where V_0 is the velocity step. Therefore, the normal mode solution is

$$\dot{X}_i = -V_0 \sum_a \frac{\bar{X}_{ia}\sum_j m_j \bar{X}_{ja}}{\sum_j m_j \bar{X}^2_{ja}} \cos \omega_a t. \tag{52}$$

At $t = 0$, the absolute velocity of each mass must be zero, so that its velocity *relative* to the base must be $-V_0$. When applied to the above equation, this means that

$$\sum_a \frac{\bar{X}_{ia}\sum m_j \bar{X}_{ja}}{\sum_j m_j \bar{X}^2_{ja}} = 1. \tag{53}$$

General Disturbing Force

Starting with the previous solution of Eq. (41), integration by parts yields

$$X_i = F_k(t) \sum_a \frac{\bar{X}_{ia}\bar{X}_{ka}}{\omega_a^2 \sum_j m_j \bar{X}^2_{ja}}$$

$$- F_k(0) \sum_a \frac{\bar{X}_{ia}\bar{X}_{ka}}{\omega_a^2 \sum_j m_j \bar{X}^2_{ja}} \cos \omega_a t$$

$$- \sum_a \frac{\bar{X}_{ia}\bar{X}_{ka}}{\omega_a^2 \sum_j m_j \bar{X}^2_{ja}} \int_0^t \dot{F}_k(T) \cos \omega_a (t - T)\, dT.$$

If $F_k(0) = 0$, this reduces to

$$X_i = \delta_{ik} F_k(t) - \sum_a \frac{\bar{X}_{ia}\bar{X}_{ka}}{\omega_a^2 \sum_j m_j \bar{X}^2_{ja}} \int_0^t \dot{F}_k(T) \cos \omega_a (t - T)\, dT.$$

This equation indicates that the deflection can be considered to be composed of two parts: the response, ignoring inertia effects (the statical component as a function of time), and a series of terms which represent the dynamic correction factor. This equation has the advantage that if $F(t)$ and $\dot{F}(t)$ contain no discontinuities, all of the statical component is accounted for when the mode series is cut off at some mode. For many applied forces, follow the same procedure which was used in deriving Eq. (42) from Eq. (41).

Equivalent Forces for Base Motion

As another special case let $F_k(t) = -m_k C(t)$, that is, the force on a mass is proportional to that mass. Assume that such forces are applied to each mass and $C(t)$ is not a function of k. Then Eq. (42) becomes

$$X_i = -\sum_a \frac{\bar{X}_{ia}\sum_k m_k \bar{X}_{ka}}{\omega_a \sum_j m_j \bar{X}^2_{ja}} \int_0^t C(T) \sin \omega_a (t - T)\, dT.$$

However, this is precisely Eq. (44), if $C(T) = \ddot{Z}(T)$. Therefore, the displacement response for many applied forces can be converted to the relative displacement response due to base motion by the substitution $F_k(T) = -m_k \ddot{Z}(T)$ and summing over all k. This is not too surprising since it is a principle of mechanics that acceleration of the frame of reference is indistinguishable from a change in the gravity field. This is precisely the meaning of "let $F_k(T)$ equal $-m_k Z(T)$ and sum over all k."

Reciprocity

As in the static case, there is a reciprocity theorem for the dynamic response of a linear elastic structure. Consider any forcing function $F(t)$. First apply F at k and measure the displacement at i,

$$X_i = \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t F_k(T) \sin \omega_a (t-T)\, dT.$$

Now apply the force at i and measure the displacement at k:

$$X_k = \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t F_i(T) \sin \omega_a (t-T)\, dT.$$

If $F_i(T) = F_k(T)$, then $X_i = X_k$.

This reciprocity theorem has many uses and is especially advantageous in impedance applications.

Relative Motion Between Masses

Sometimes it is necessary to know the relative motion between two mass points j and k. Let r_{jk} be this motion. For many applied forces

$$r_{jk} = \sum_a \frac{(\overline{X}_{ja} - \overline{X}_{ka})}{\omega_a \sum_i m_i \overline{X}_{ia}^2} \int_0^t \left[\sum_{i=1}^{d} \overline{X}_{ia} F_i(T)\right] \sin \omega_a (t-T)\, dT \tag{54}$$

and for foundation motion

$$r_{jk} = -\sum_a \frac{(\overline{X}_{ja} - \overline{X}_{ka})\sum_i m_i \overline{X}_{ia}}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t \ddot{Z}(T) \sin \omega_a (t-T)\, dT. \tag{55}$$

Steady-State Vibrations

In this section the response to disturbing functions of the periodic type is derived. Consider the steady-state response of a system to a forcing function of the type $F_k = F_k \sin \Omega t$. The equations of motion $X_i = \sum_a \overline{X}_{ia} q_a$ and $\ddot{q}_a + \omega_a^2 q_a = (\overline{X}_{ka} F_k \sin \Omega t)/\sum_j m_j \overline{X}_{ja}^2$ are now solved. First assume $\Omega \neq \omega_a$. Then,

$$X_i = F_k \sin \Omega t \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka}}{\omega_a^2\left(1 - \frac{\Omega^2}{\omega_a^2}\right)\sum_j m_j \overline{X}_{ja}^2} \tag{56}$$

plus the solution which involves the modal responses at their own frequencies. For a structure initially at rest this is

$$-F_k \Omega \sum_a \frac{\overline{X}_{ia}\overline{X}_{ka} \sin \omega_a t}{\omega_a^3\left(1 - \frac{\Omega^2}{\omega_a^2}\right)\sum_j m_j \overline{X}_{ja}^2}.$$

This second part of the solution is usually ignored, and only the first part is considered. The result is called the "steady-state response." If the function $F_k = F_k \cos \Omega t$ is applied, it is only necessary to exchange $\cos \Omega t$ for $\sin \Omega t$ in the steady-state solution to find the response. That portion which is usually ignored has the set of $\sin \omega_a t$'s replaced by $\cos \omega_a t$'s.

As in the single-degree-of-freedom case, for $\Omega << \omega_1$ (the fundamental natural frequency), the steady-state solution becomes

$$X_i \approx \delta_{ik} F_k \sin \Omega t.$$

The steady-state response for many applied forces all of the same phase becomes

$$X_i = \sin \Omega t \sum_a \frac{\overline{X}_{ia} \sum_{k=1}^{d} F_k \overline{X}_{ka}}{\omega_a^2\left(1 - \frac{\Omega^2}{\omega_a^2}\right)\sum_j m_j \overline{X}_{ja}^2}. \tag{57}$$

Letting $F_k = -m_k \ddot{Z}$ and summing on k, the steady-state response to a periodic base motion is

$$X_i = -\ddot{Z} \sin \Omega t \sum_a \frac{\overline{X}_{ia} \sum_k m_k \overline{X}_{ka}}{\omega_a^2\left(1 - \frac{\Omega^2}{\omega_a^2}\right)\sum_j m_j \overline{X}_{ja}^2}. \tag{58}$$

It may occur that Ω coincides with one of the ω_a's, say ω_b. Then for this mode the solution changes. Consider

$$\ddot{q}_b + \omega_b^2 q_b = \frac{\overline{X}_{kb} F_k \sin \omega_b t}{\sum_j m_j \overline{X}_{jb}^2}.$$

This has the particular solution

$$q_b = -\frac{tF_k\overline{X}_{kb}\cos\omega_b t}{2\omega_b\sum_j m_j\overline{X}^2_{jb}}$$

So the steady-state solution is

$$X_i = \left(\sum_{a=1}^{b-1} + \sum_{a=b+1}^{n}\right)\left[\frac{\overline{X}_{ia}\overline{X}_{ka}F_k\sin\omega_a t}{\omega_a^2\left(1-\frac{\omega_h^2}{\omega_a^2}\right)\sum_j m_j\overline{X}^2_{ja}}\right]$$

$$-\frac{tF_k\overline{X}_{kb}\overline{X}_{ib}}{2\omega_b\sum_j m_j\overline{X}^2_{jb}}\cos\omega_b t. \qquad (59)$$

Mode b grows with time because of the presence of the t term. This is an example of resonant buildup.

Consider now the problem of a periodic disturbing force $F_k(t)$ of period T_0 which is not a simple sine or cosine function; the steady-state response is desired. A general method of solution is to let

$$F_k(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi t}{T_0}$$

or

$$F_k(t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi t}{T_0}$$

where T_0 is the period. These are the Fourier cosine and sine half range expansions (6, 7) of the functions. The coefficients are given by Refs. 6 and 7:

$$A_0 = \frac{2}{T_0}\int_0^{T_0} F_k(t)\,dt$$

$$A_n = \frac{2}{T_0}\int_0^{T_0} F_k(t)\cos\frac{n\pi t}{T_0}\,dt$$

$$B_n = \frac{2}{T_0}\int_0^{T_0} F_k(t)\sin\frac{n\pi t}{T_0}\,dt.$$

Reference 7 gives a practical way of calculating these coefficients.

Since the response to sine and cosine functions is known, replace $F_k\sin\Omega t$ in Eq. (56) by

$$\sum_{n=1}^{\infty} B_n\sin\frac{n\pi t}{T_0},$$

to obtain

$$X_i = \sum_a\sum_{n=1}^{\infty}\frac{\overline{X}_{ia}\overline{X}_{ka}B_n\sin n\pi t/T_0}{\omega_a^2\left(1-\frac{n^2\pi^2}{T_0^2\omega_a^2}\right)\sum_j m_j\overline{X}^2_{ja}}. \qquad (60)$$

CHARACTERISTIC LOAD THEOREMS

A concept known as characteristic shape is now discussed. A knowledge of these theorems will be of considerable help when considering stress.

Definition

A load distributed over a structure in such a manner that the load intensity is proportional to the product of mass and mode shape for a particular frequency is called a characteristic load. That is,

$$\overline{F}_{ia} = E_a m_i\overline{X}_{ia}. \qquad (61)$$

There is no loss of generality if the proportionality constant E_a is replaced by the product of ω_a^2 and a new proportionality constant C_a. Equation (61) becomes

$$\overline{F}_{ia} = \omega_a^2 C_a m_i\overline{X}_{ia}. \qquad (61')$$

Statical Theorem

"If a structure is loaded statically with the a^{th} characteristic load, then the deflection curve is proportional to the a^{th} normal mode shape."

Proof of this theorem is as follows: Consider the static deflection of the j^{th} point on a structure. This deflection is

$$X_j = \sum_i \delta_{ji}F_i. \qquad (17)$$

If the set of F_i's form a characteristic load, then Eq. (17) may be written as $X_j = \omega_a^2 C_a\sum_i\delta_{ji}m_i\overline{X}_{ia}$. Equation (21) states that

$$\overline{X}_{ja} = \omega_a^2\sum_i\delta_{ji}m_i\overline{X}_{ia}. \qquad (21)$$

Hence

$$X_j = C_a\overline{X}_{ja}.$$

The deflection of any point j is proportional to the normal mode displacement, thus proving the theorem.

Virtual Work Theorem

"If a structure loaded with a characteristic load of the a^{th} mode is subjected to a virtual displacement corresponding to the b^{th} mode, then the characteristic load does no virtual work."

Proof of this theorem is as follows: The work done at point j under these conditions is

$$\frac{1}{2}\bar{F}_{ja}\bar{X}_{jb} = \frac{C_a}{2}\omega_a^2 m_j \bar{X}_{ja}\bar{X}_{jb}.$$

The total work is the summation over all points j, or

$$W = \frac{C_a}{2}\omega_a^2\sum_j m_j \bar{X}_{ja}\bar{X}_{jb}.$$

From orthogonality conditions

$$\sum_j m_j \bar{X}_{ja}\bar{X}_{jb} = 0.$$

Hence the work is 0.

Transient and Steady-State Response

"When a structure is loaded with a characteristic load which varies as some function of time, it responds as a single-degree-of-freedom system with the natural frequency and mode shape corresponding to the characteristic load."

Proof of this theorem is as follows: Consider the response to a transient force system,

$$X_i = \sum_a \frac{\bar{X}_{ia}}{\omega_a\sum_j m_j \bar{X}_{ja}^2}\int_0^t\left[\sum_k \bar{X}_{ka}F_k(T)\right]\sin\omega_a(t-T)\,dT.$$

Let the $F_k(T)$'s form a characteristic load for mode b. Then if $f(t)$ is a function of time,

$$X_i = \sum_a \frac{\bar{X}_{ia}}{\omega_a\sum_j m_j \bar{X}_{ja}^2}\int_0^t\left[\sum_k \omega_b^2 C_b f(T) m_k \bar{X}_{kb}\bar{X}_{ka}\right]\sin\omega_a(t-T)\,dT.$$

The Duhamel integral has a value only when $a = b$, because of the orthogonality conditions, so that

$$X_i = \bar{X}_{ib}\omega_b C_b\int_0^t f(T)\sin\omega_b(t-T)\,dT$$

thus proving the theorem.

As a special case consider a structure subjected to a set of steady-state driving forces $F_k(t) = F_k \sin\lambda t$. If the F's form a normal characteristic load for mode b,

$$X_i = \sum_a \frac{\bar{X}_{ia}C_b\omega_b^2\sum_k m_k \bar{X}_{kb}\bar{X}_{ka}\sin\lambda t}{\omega_a^2\left(1-\frac{\lambda^2}{\omega_a^2}\right)\sum_j m_j \bar{X}_{ja}^2}.$$

from the orthogonality relationship,

$$X_i = \frac{C_b\bar{X}_{ib}\sin\lambda t}{(1-\lambda^2/\omega_b^2)}.$$

This equation indicates motion is only possible for the mode for which the characteristic load is applied, because the sum has been reduced to one term.

Characteristic Shape Coefficients

In this section on characteristic shape theorems, much use is made of a proportionality factor C_a, but nothing has been said about how to calculate it. Since normal mode expansions are intimately connected with the theory of generalized Fourier expansions, it is possible to compute the coefficients C_a in the normal manner by means of the orthogonality relationships.

Consider a structure of n masses subjected to the action of a set of n forces which remain proportional to each other in time (some of these forces may be zero). The i^{th} force may be expressed as

$$F_i = F_i f(t).$$

If the set of F_i's form a characteristic load which is summed over all modes, then

$$F_i f(t) = f(t)\sum_a C_a m_i \bar{X}_{ia}\omega_a^2.$$

To compute the coefficient C_a, multiply both sides of this equation by $\overline{X}_{ib}$ and sum over i to get

$$\sum_i F_i \overline{X}_{ib} = \sum_a C_a \omega_a^2 \sum_i m_i \overline{X}_{ia} \overline{X}_{ib}.$$

This expression only has a value when $a = b$, so

$$\sum_i F_i \overline{X}_{ib} = C_b \omega_b^2 \sum_i m_i \overline{X}_{ib}^2.$$

The generalized Fourier coefficient is then

$$C_a = \frac{\sum_i F_i \overline{X}_{ia}}{\omega_a^2 \sum_i m_i \overline{X}_{ia}^2}.$$

It is now possible to write out the response of mass k as

$$X_k = \sum_a C_a \overline{X}_{ka} \omega_a \int_0^t f(T) \sin \omega_a (t - T)\, dT.$$

Note that for a single applied force F_i this immediately reduces to

$$X_k = \sum_a \frac{\overline{X}_{ia} \overline{X}_{ka} F_i}{\omega_a \sum_j m_j \overline{X}_{ja}^2} \int_0^t f(T) \sin \omega_a (t - T)\, dT.$$

FORCES AND STRESSES

Single Applied Force

Before proceeding with a discussion of stress, it is necessary to determine the inertia loadings that the masses apply to the structure. It has been shown that each normal mode acts as a single-degree-of-freedom system with certain characteristics. If the absolute acceleration of each mass point m_i is found, the inertia forces can be added to the structure as a loading by D'Alembert's principle.

Consider the case of an applied force at m_k with no base motion. The q_a equation is

$$\ddot{q}_a + \omega_a^2 q_a = \frac{\overline{X}_{ka} F_k(t)}{\sum_j m_j \overline{X}_{ja}^2}.$$

Solving for $\ddot{q}_a$,

$$\ddot{q}_a = \frac{\overline{X}_{ka} F_k(t)}{\sum_j m_j \overline{X}_{ja}^2} - \omega_a^2 q_a.$$

Since

$$\ddot{X}_i = \sum_a \overline{X}_{ia} \ddot{q}_a$$

then

$$\ddot{X}_i = \sum_a \frac{\overline{X}_{ia} \overline{X}_{ka} F_k(t)}{\sum_j m_j \overline{X}_{ja}^2} - \sum_a \omega_a^2 \overline{X}_{ia} q_a.$$

It has been shown that

$$\sum_a \frac{\overline{X}_{ia} \overline{X}_{ka}}{\sum_j m_j \overline{X}_{ja}^2} = 0 \tag{49}$$

and

$$\sum_a \frac{\overline{X}_{ka}^2}{\sum_j m_j \overline{X}_{ja}^2} = \frac{1}{m_k} \tag{50}$$

so that for any mass *but* the k^{th} (where the force is applied)

$$\underline{\ddot{X}}_i = \ddot{X}_i = -\sum_a \omega_a^2 \overline{X}_{ia} q_a$$

and for the k^{th} mass

$$\underline{\ddot{X}}_k = \ddot{X}_k = \frac{F_k(t)}{m_k} - \sum_a \omega_a^2 \overline{X}_{ka} q_a.$$

The inertia loadings are

$$Q_i = \sum_a \omega_a^2 m_i \overline{X}_{ia} q_a \qquad (i \neq k)$$

and

$$Q_k = -F_k(t) + \sum_a \omega_a^2 m_k \overline{X}_{ka} q_a.$$

These equations describe the inertial loadings for each mass point. At m_k, there is an external force F_k. The sum of the forces on m_k is the net applied force:

$$Q_K + F_K(t) = \sum_a \omega_a^2 m_k \overline{X}_{Ka} q_a.$$

The structure is therefore loaded in mode a by a force system of the form

$$F_{ia} = \omega_a^2 m_i \overline{X}_{ia} q_a \qquad \text{(for all } i\text{)}. \tag{62}$$

These forces acting on each mass in mode a are characteristic loads if it is recalled that

$$q_a = \frac{\overline{X}_{ka}}{\omega_a \sum_j m_j \overline{X}^2_{ja}} \int_0^t F_k(T) \sin \omega_a(t-T)\, dT.$$

Equation (62) may be rewritten

$$F_{ia} = \omega_a^2 m_i \overline{X}_{ia} C'_a N_a(t) \qquad (63)$$

where

$$C'_a = \frac{\overline{X}_{ka}}{\omega_a^2 \sum_j m_j \overline{X}^2_{ja}}$$

$$N_a(t) = \omega_a \int_0^t F_k(T) \sin \omega_a(t-T)\, dT.$$

Therefore,

$$F_{ia} = \overline{F}_{ia} N_a(t)$$

where

$$\overline{F}_{ia} = \omega_a^2 m_i \overline{X}_{ia} C'_a. \qquad (63')$$

Many Applied Forces

Consider the case where there are many applied forces acting on a structure which vary as different functions of time. The $\ddot{q}_a$ equation is

$$\ddot{q}_a = \frac{\sum_{k=1}^{d} \overline{X}_{ka} F_k}{\sum_j m_j \overline{X}^2_{ja}} - \omega_a^2 q_a$$

so that

$$\ddot{\underline{X}}_{ia} = \ddot{X}_{ia} = \frac{\overline{X}_{ia} \sum_{k=1}^{d} \overline{X}_{ka} F_k}{\sum_j m_j \overline{X}^2_{ja}} - \omega_a^2 \overline{X}_{ia} q_a$$

$$= \frac{\overline{X}_{ia}}{\sum_j m_j \overline{X}^2_{ja}} (\overline{X}_{1a} F_1 + \overline{X}_{2a} F_2$$

$$+ \ldots + \overline{X}_{ia} F_i + \ldots + \overline{X}_{da} F_d)$$

$$- \omega_a^2 \overline{X}_{ia} q_a.$$

Therefore,

$$\ddot{X}_i = \sum_a \ddot{X}_{ia} = \sum_a \frac{\overline{X}^2_{ia} F_i}{\sum_j m_j \overline{X}^2_{ja}} - \sum_a \omega_a^2 \overline{X}_{ia} q_a$$

$$= F_i/m_i - \sum_a \omega_a^2 \overline{X}_{ia} q_a$$

using Eqs. (49) and (50). The inertia loadings are

$$Q_i = -m_i \ddot{\underline{X}}_i = -F_i + \sum_a \omega_a^2 m_i \overline{X}_{ia} q_a$$

The net force acting on each mass is

$$Q_i + F_i = \sum_a \omega_a^2 m_i \overline{X}_{ia} q_a \qquad \text{(for all } i\text{)}.$$

These loads are characteristic loads of the form

$$F_{ia} = \omega_a^2 m_i \overline{X}_{ia} C''_a N'_a(t) \qquad (64)$$

where

$$C''_a = \frac{1}{\omega_a^2 \sum_j m_j \overline{X}^2_{ja}}$$

$$N'_a(t) = \omega_a \int_0^t \left[\sum_{k=1}^{d} \overline{X}_{ka} F_k(T) \right] \sin \omega_a(t-T) dT$$

Alternative Form

In deriving the characteristic loads for a single applied force and many applied forces, it has been assumed that the summation over all of the modes is performed. This is required in order to use Eqs. (49) and (50). The following presentation derives the equations one may use for the case where the summation does not include all the modes, say the first "u" modes of the "n" modes present.

Single Applied Force

The absolute acceleration of m_i is

$$\ddot{X}_i = \sum_{a=1}^{u} \frac{\overline{X}_{ia} \overline{X}_{Ka} F_K(t)}{\sum_j m_j \overline{X}^2_{ja}} - \sum_{a=1}^{u} \omega_a^2 \overline{X}_{ia} q_a.$$

The inertia load at each point is

$$Q_i = \sum_{a=1}^{u} \frac{m_i \overline{X}_{ia} \overline{X}_{Ka} F_K(t)}{\sum_j m_j \overline{X}^2_{ja}} + \sum_{a=1}^{u} \omega_a^2 m_i \overline{X}_{ia} q_a.$$

The net force on m_K is

$$F_K(t) + Q_K = F_K(t) - \sum_{a=1}^{u} \frac{m_K \overline{X}_{Ka}^2 F_K(t)}{\sum_j m_j \overline{X}_{ja}^2} + \sum_{a=1}^{u} \omega_a^2 m_i \overline{X}_{ia} q_a.$$

Many Applied Forces

The absolute acceleration in mode "a" is

$$\ddot{X}_{ia} = \frac{\overline{X}_{ia} \sum_{K=1}^{d} \overline{X}_{Ka} F_K}{\sum_j m_j \overline{X}_{ja}^2} - \omega_a^2 \overline{X}_{ia} q_a.$$

Therefore,

$$\ddot{X}_i = \sum_{a=1}^{u} \ddot{X}_{ia} = \sum_{a=1}^{u} \frac{\overline{X}_{ia} \sum_{K=1}^{d} \overline{X}_{Ka} F_K}{\sum_j m_j \overline{X}_{ja}^2} - \sum_{a=1}^{u} \omega_a^2 \overline{X}_{ia} q_a.$$

The inertia loadings are

$$Q_i = - m_i \ddot{X}_i = - \sum_{a=1}^{u} \frac{m_i \overline{X}_{ia} \sum_{K=1}^{d} \overline{X}_{Ka} F_K}{\sum_j m_j \overline{X}_{ja}^2} + \sum_{a=1}^{u} \omega_a^2 m_i \overline{X}_{ia} q_a.$$

The net force acting on each mass is

$$F_i + Q_i = F_i - \sum_{a=1}^{u} \frac{m_i \overline{X}_{ia} \sum_{K=1}^{d} \overline{X}_{Ka} F_K}{\sum_j m_j \overline{X}_{ja}^2} + \sum_{a=1}^{u} \omega_a^2 m_i \overline{X}_{ia} q_a.$$

Base Motion

Consider the case of base motion alone. Equation (6) becomes $\ddot{X} + \omega^2 X = - \ddot{Z}$ or $\ddot{\underline{X}} = - \omega^2 X$. The a^{th} mode equation for a multi-degree-of-freedom system can be written by replacing $\ddot{\underline{X}}$ by $\ddot{\underline{X}}_{ia}$, ω by ω_a, and X by $\overline{X}_{ia} q_a$. Therefore,

$$\ddot{\underline{X}}_{ia} = - \omega_a^2 \overline{X}_{ia} q_a$$

so that the inertia force acting on each mass is

$$Q_{ia} = - m_i \ddot{\underline{X}}_{ia} = \omega_a^2 m_i \overline{X}_{ia} q_a \text{ (for all } i). \quad (65)$$

Once again, these inertia forces are characteristic loads acting on each mass in mode a. If the expression for q_a is introduced into Eq. (65), it becomes

$$Q_{ia} = - m_i \overline{X}_{ia} \overline{P}_a D_a(t)$$

where

$$\overline{P}a = \frac{\sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}_{ja}^2}$$

$$D_a(t) = \omega_a \int_0^t \ddot{Z}(T) \sin \omega_a (t - T)\, dT.$$

The term $\overline{P}_a$ is sometimes called the participation factor.

Stresses

The concept of a characteristic load is used to show a practical procedure for stress calculation. It is true that the stress at some point c is proportional to a load, say F_k. This is a direct consequence of the assumption of linear elasticity. For example, if the load is doubled, the stress is doubled. Therefore,

$$\sigma_c = \gamma_{ck} F_k$$

where γ_{ck} is the stress at c due to a unit load at k, and can be considered to be a stress coefficient. If there are many applied forces

$$\sigma_c = \sum_{k=1}^{d} \gamma_{ck} F_k. \quad (66)$$

Suppose there are n applied forces and two stresses of interest. Then

$$\sigma_c = \gamma_{c1} F_1 + \gamma_{c2} F_2 + \ldots + \gamma_{cn} F_n$$

$$\sigma_g = \gamma_{g1} F_1 + \gamma_{g2} F_2 + \ldots + \gamma_{gn} F_n.$$

Note that the array of the γ's need not be square, and there is no reciprocal relationship.

Suppose the set of F_k's applied to the structure is a characteristic load for mode a. A "characteristic modal stress" at point c could then be determined as

$$\bar{\sigma}_{ca} = \sum_j \gamma_{cj} \bar{F}_{ja}. \tag{67}$$

It has been shown that each of the characteristic loads vary in time proportional to the solution of the Duhamel integral

$$N_a(t) = \omega_a \int_0^t F_k(T) \sin \omega_a(t-T)\, dT.$$

Therefore,

$$\sigma_{ca} = \bar{\sigma}_{ca} N_a(t)$$

so that

$$\sigma_c = \sum_a \sigma_{ca} = \sum_a \bar{\sigma}_{ca} N_a(t). \tag{68}$$

The procedure is to find the modal characteristic stress in mode a due to the characteristic loads on each mass, multiply them by the proper Duhamel integrals, and sum over the modes.

In order to show that the stress can be considered to be composed of two parts as in the case of response, first expand γ_{cj} by the generalized Fourier expansion theorem and then integrate Eq. (68) by parts. By proper substitution the desired result will be found.

The generalized Fourier expansion for γ_{cj} is obtained directly from Eq. (18) by replacing ϕ_{ia} and β_i by $\bar{X}_{ia}$ and m_i, respectively. Therefore,

$$\gamma_{cj} = \sum_a \frac{\bar{X}_{ja} \sum_i m_i \bar{X}_{ia} \gamma_{ci}}{\sum_i m_i \bar{X}_{ia}^2}. \tag{69}$$

Now,

$$\bar{\sigma}_{ca} = \sum_j \gamma_{cj} \bar{F}_{ja}$$

where the $\bar{F}_{ja}$ are characteristic forces, defined by Eq. (63′), so

$$\bar{\sigma}_{ca} = \sum_j \frac{\gamma_{cj} m_j \bar{X}_{ja} \bar{X}_{ka}}{\sum_i m_i \bar{X}_{ia}^2} = \frac{\bar{X}_{ka} \sum_j m_j \bar{X}_{ja} \gamma_{cj}}{\sum_i m_i \bar{X}_{ia}^2}.$$

Summing on all a,

$$\sum_a \bar{\sigma}_{ca} = \sum_a \bar{X}_{ka} \frac{\sum_i m_i \bar{X}_{ia} \gamma_{ci}}{\sum_j m_j \bar{X}_{ja}^2} = \gamma_{ck} \tag{70}$$

by Eq. (69).

Now integrate Eq. (68) by parts:

$$\sigma_c = \sum_a \bar{\sigma}_{ca} F_k(t) - F_k(0) \sum_a \bar{\sigma}_{ca} \cos \omega_a t$$
$$- \sum_a \bar{\sigma}_{ca} \int_0^t \dot{F}_k(T) \cos \omega_a (t-T)\, dT.$$

Using Eq. (70),

$$\sigma_c = \gamma_{ck} F_k(t) - F_k(0) \sum_a \bar{\sigma}_{ca} \cos \omega_a t$$
$$- \sum_a \bar{\sigma}_{ca} \int_0^t \dot{F}_k(T) \cos \omega_a (t-T)\, dT.$$

This states that the stress at c can be considered to be composed of two parts: the stress, ignoring inertia effects, and a series of terms which represent a dynamic correction factor.

To find a similar expression for base motion let

$$F_k(t) = -m_k \ddot{Z}(t)$$

and sum on all k. Hence,

$$\sigma_c = -\ddot{Z}(t) \sum_k m_k \gamma_{ck} + \ddot{Z}(0) \sum_a \bar{\bar{\sigma}}_{ca} \cos \omega_a t$$
$$+ \sum_a \bar{\bar{\sigma}}_{ca} \int_0^t \dddot{Z}(T) \cos \omega_a (t-T)\, dT$$

where $\bar{\bar{\sigma}}_{ca}$ is the characteristic modal stress at point c due to the motion of the base. Multiply the numerator and denominator of the first term by g (the acceleration due to gravity):

$$-\frac{\ddot{Z}(t)}{g} \sum_k w_k \gamma_{ck}.$$

Now

$$\sum_k w_k \gamma_{ck}$$

is the static stress due to the structure's own weight; so,

$$\sigma_c = -\frac{\ddot{Z}(t)}{g}(\sigma_{st}) + \ddot{Z}(0)\sum_a \bar{\bar{\sigma}}_{ca}\cos\omega_a t$$
$$+ \sum_a \bar{\bar{\sigma}}_{ca}\int_0^t \dddot{Z}(T)\cos\omega_a(t-T)\,dT$$

which is again in the form of a statical component and a dynamic correction.

Stresses Using Deflections

Some engineers prefer to use deflections as a means of finding stresses. It has been demonstrated that each mode responds in a characteristic mode shape. Then for unit deflection of any mass j in each of the modes (provided this modal deflection is not zero, $\bar{X}_{ja} \neq 0$) there is a characteristic stress at c due to deflection of the mode. Call this $\bar{\sigma}^j_{ca}$. Then

$$\sigma_c = \sum_a \frac{\bar{\sigma}^j_{ca}\bar{X}_{ja}\bar{X}_{ka}}{\omega_a\sum_j m_j\bar{X}^2_{ja}}\int_0^t F_k(T)\sin\omega_a(t-T)\,dT$$

and

$$\sigma_c = -\sum_a \frac{\bar{\sigma}^j_{ca}\bar{X}_{ja}\sum_i m_i\bar{X}_{ia}}{\omega_a\sum_j m_j\bar{X}^2_{ja}}\int_0^t$$
$$\ddot{Z}(T)\sin\omega_a(t-T)\,dT.$$

Stress and Deflection Checking

If the purpose of an analysis is to compute stresses or deflections, normal-mode theory indicated that the proper equation must be solved as a function of time and the separate modal responses added together timewise to produce the desired results. For engineering purposes, however, it is often quite satisfactory to use shock spectra for calculating these structural effects. A shock spectrum is a plot of the maximum absolute values of the relative displacement, times scaling factors if desired, of a set of either damped or undamped single-degree-of-freedom oscillators with negligible mass which have been subjected to the shock motion. As a second definition, instead of using the maximum absolute values of the relative displacement, a shock spectrum may plot the maximum positive and maximum negative values of the relative displacement.

It is noted that the time to the peak values is ignored in the definition of a shock spectrum. In the case of the second definition of a shock spectrum, this technique gives two stresses or deflections for each point, a maximum of maximum positive values, and a maximum of maximum negative values.

For example, consider a structure of n-degrees-of-freedom, subjected to a base motion, where stress is the object of consideration. The stress contribution of mode a at point c is

$$\sigma_{ca} = \bar{\bar{\sigma}}_{ca}D_a(t).$$

This can be thought of as having two parts, $\bar{\bar{\sigma}}_{ca}$ and $D_a(t)$. When $\bar{\bar{\sigma}}_{ca}$ is computed, it is either positive or negative. If only the maximum values of $D_a(t)$ are used, then there are two stresses of interest: the maximum positive and maximum negative values. For each mode there are then two products $\bar{\bar{\sigma}}_{ca}D^+_a$ (product positive) and $\bar{\bar{\sigma}}_{ca}D^-_a$ (product negative). If the two sums

$$\sum_a \bar{\bar{\sigma}}_{ca}D^+_a \text{ and } \sum_a \bar{\bar{\sigma}}_{ca}D^-_a$$

are formed, then a conservative value for both the positive and negative maximum stresses will be found. This method will give a better result than simply using maximum absolute values and is, of course, still conservative because of the neglect of the time to peak stress in each mode.

Sometimes only a design shock spectrum (8-11) is provided. A design shock spectrum is a plot of values for use of the analyst in predicting the stresses, etc., in a contemplated structure for which no measured shock spectrum exists. It is noted that a design shock spectrum for a particular structure is composed of carefully selected information and is not just a combination or envelope of data points taken from existing shock spectrum curves for similar types of structures. Since this is a set of maximum of maximum values there is no way of knowing whether D_a is positive or negative. A possible procedure is to argue that the largest stress (or deflection) at c occurs, and because of phasing, etc., a "statistical expected value" of the rest is added to it. This results (for stress) in a formula like

$$|\sigma_c| = |\sigma_{cb}| + \sqrt{\left[\sum_a(\sigma_{ca})^2\right] - (\sigma_{cb})^2} \quad (71)$$

where $|\sigma_{cb}|$ is the largest stress at c caused by a mode. Such a formula should never be used for an intermediate step, but only for a final result.

EFFECTIVE MASS WITH BASE MOTION

The question often arises as to the dynamic reaction of a structure on its base. The problem is to replace the actual structure by a set of simple oscillators, such that the force transmitted across the base is precisely the same for the simple oscillators as for the structure. These oscillators must have the same fixed base frequencies as the structure, so their frequencies must coincide with normal mode frequencies in order that the time variation of the forces be correct. The question then arises as to how much mass each oscillator should be assigned. Consider the case of a complex structure subjected to a unidirectional translation shock motion $\ddot{Z}(t)$, applied at the base with no rotation. The absolute acceleration of the point i in mode a is

$$\ddot{\underline{X}}_{ia} = \frac{\overline{X}_{ia}\sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}^2_{ja}} D_a(t).$$

From Newton's laws the force exerted by this mass i is

$$Q_{ia} = -\frac{m_i \overline{X}_{ia}\sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}^2_{ja}} D_a(t).$$

The force present in mode a is then the sum of the individual forces, or

$$Q_a = -\frac{(\sum_i m_i \overline{X}_{ia})^2}{\sum_j m_j \overline{X}^2_{ja}} D_a(t).$$

Now the force exerted upon the foundation by a simple oscillator is

$$F = -MD(t)$$

where M is the total mass. Therefore the effective mass acting in mode a must be

$$M_a = \frac{(\sum_i m_i \overline{X}_{ia})^2}{\sum_j m_j \overline{X}^2_{ja}}. \qquad (72)$$

It is of interest to compare the sum of the modal masses with the total mass of the structure. The sum of the modal masses is

$$M' = \sum_a \frac{(\sum_i m_i \overline{X}_{ia})^2}{\sum_j m_j \overline{X}^2_{ja}}.$$

This may be written as

$$M' = \sum_i m_i \sum_a \frac{\overline{X}_{ia}\sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}^2_{ja}}.$$

It was shown that

$$\sum_a \frac{\overline{X}_{ia}\sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}^2_{ja}} = 1. \qquad (53)$$

Therefore,

$$M' = \sum_i m_i = M. \qquad (73)$$

Equation (73) then indicates that the sum of all the effective masses M_a for the total number of modes is equal to the total mass of the actual structure. Since the effective mass is always a positive quantity, this enables one to estimate the amount remaining in the other modes after a few modes have been calculated.

The location of these simple oscillators is prescribed by the fact that the moment of each must equal that of its corresponding mode. Choose a point on the base to take moments about. Then if the perpendicular distance to mass i is y_i, the moment of the a^{th} mode is

$$\text{torque}_{(a)} = -\frac{\sum_i m_i \overline{X}_{ia} y_i \sum_j m_j \overline{X}_{ja}}{\sum_j m_j \overline{X}^2_{ja}} D_a(t). \qquad (74)$$

The distance to the oscillator for the a^{th} mode is the torque divided by the force or

$$\bar{y}_a = \frac{\sum_i m_i \overline{X}_{ia} y_i}{\sum_i m_i \overline{X}_{ia}}. \qquad (75)$$

SUDDEN CHANGE IN THE GRAVITY FIELD

As an example of theoretical interest showing how normal mode theory may be used in the abstract sense, consider a structure vibrating freely after a sudden change in the gravity field, say from zero to gravity g. The maximum deflection of the point i in mode a from the new equilibrium position becomes

$$X_{ia} = \frac{g\bar{X}_{ia}\sum_k m_k \bar{X}_{ka}}{\omega_a^2 \sum_j m_j \bar{X}_{ja}^2}.$$

The maximum potential energy of mode a becomes

$$\frac{g}{2}\sum_i m_i X_{ia} = \frac{g^2(\sum_i m_i \bar{X}_{ia})^2}{2\omega_a^2 \sum_j m_j \bar{X}_{ja}^2}.$$

In mode a the maximum kinetic energy of vibration is

$$\frac{1}{2}\sum_i m_i X_{ia}^2 \omega_a^2$$

where X_{ia} is the *actual* modal displacement at j. During a free vibration the potential energy must equal the kinetic energy, so

$$\frac{g^2(\sum_i m_i \bar{X}_{ia})^2}{2\omega_a^2 \sum_j m_j \bar{X}_{ja}^2} = \frac{1}{2}\omega_a^2 \sum_i m_i X_{ia}^2.$$

Now from the section on effective mass with base motion

$$M_a = \frac{(\sum_i m_i \bar{X}_{ia})^2}{\sum_j m_j \bar{X}_{ja}^2} \qquad (72)$$

is always positive and less than M, the total mass. Therefore

$$\frac{g^2 M_a}{\omega_a^4 M} = \frac{\sum_j m_j X_{ja}^2}{M}.$$

Taking the square root of both sides

$$X_a(\mathrm{rms}) = \frac{g}{\omega_a^2}\sqrt{\frac{M_a}{M}}$$

where X_a(rms) is the root-mean-square value of the actual amplitude of the mode shape. This equation shows that X_a(rms) tends to vanish in the higher modes because

$$\sqrt{\frac{M_a}{M}} < 1$$

and

$$\frac{g}{\omega_a^2} \to 0 \text{ as } \omega_a \to \infty.$$

A similar result may be obtained for strain energy in the following manner:

$$\int_v \frac{\bar{\sigma}^2(v)\,dv}{2E} = \frac{1}{2}\sum_a \omega_a^2 \sum_i m_i \bar{X}_{ia}^2$$

is divided by the total mass, giving

$$\frac{1}{M}\int_v \frac{\bar{\sigma}^2(v)}{2E}\,dv = \frac{1}{2}\sum_a \omega_a^2 X_a^2\ (ms).$$

However, since

$$X_a^2\ (ms) = \frac{g^2 M_a}{\omega_a^4 M}$$

then

$$\int_r \frac{\bar{\sigma}^2(v)}{E}\,dv = M\sum_a \frac{g^2 M_a}{\omega_a^2 M}.$$

The strain energy in a mode tends to go down as mode number goes up since

$$\frac{1}{\omega_a^2} \to 0 \text{ as } \omega_a \to \infty.$$

Although these results do not seem to have immediate practical significance they were included, as stated before, to show an example of the abstract use of normal mode theory.

RESPONSE TO BASE ROCKING

Sometimes the base of a structure undergoes a rocking motion. If the time history of this motion is known, then the concept of many applied forces and changes in the gravity field can be used to solve this problem. Since this report deals only with those structures which deflect parallel to the applied loads, the results presented here will be for the relative motion of a mass parallel to the base and perpendicular to a line drawn through the center of rotation. That is, only components of inertial forces parallel to the base are considered. This is a good approximation if the motions are small.

If the base rotational acceleration is $\ddot{\theta}(t)$, then each mass has a force $-m_j y_i \ddot{\theta}(t)$ applied to it. Summing these forces gives:

$$X_i = -\sum_a \frac{\overline{X}_{ia}\sum_j m_j \overline{X}_{ja} y_j}{\omega_a \sum_j m_j \overline{X}^2_{ja}} \int_0^t \ddot{\theta}(T) \sin \omega_a (t-T)\, dT. \qquad (76)$$

The absolute motion of m_i is

$$\underline{X}_i = X_i + y_i \theta(t). \qquad (77)$$

The remaining equations for acceleration, stress, etc., can be derived in a similar manner as for those equations presented in this report.

Appendix C shows that a structure under unitranslation and small rotation may be treated as a lumped parameter system of n masses for unidirectional translation under the following restrictions:

1. The sum of the n masses equals the total mass.

2. The n masses are so placed on the model that the moment of inertia about the center of gravity equals the moment of inertia of the original structure about the center of gravity.

SUMMARY

Some important points brought out in this discussion of normal mode theory are the following:

1. Undamped linear elastic structures have ordinary linear differential equations with constant coefficients as their equations of motion. This allows the principle of superposition to be used.

2. The normal mode shapes may be defined by using either stiffness or influence coefficients. Maxwell's law of reciprocal deflections holds for both cases.

3. There are as many normal modes and natural frequencies as there are independent masses, although some frequencies may be redundant.

4. In a free vibration each normal mode is periodic with frequency ω_a and the relative amplitudes of each of the vibrating masses is fixed in this mode.

5. The normal modes are orthogonal to each other.

6. In all cases investigated a solution was found in the form of a linear combination of the normal mode solutions. That is,

$$X_i = \sum_a X_{ia} = \sum_a \overline{X}_{ia} q_a.$$

7. The theory of generalized Fourier expansions was presented, which allowed an arbitrary velocity and displacement set of initial conditions to be used in the solution of the free vibration problem.

8. For free vibrations each normal mode vibrates in a periodic fashion, but the resulting motion need not be periodic.

9. The array of the normal mode coefficients is not symmetrical.

10. Each normal mode acts as a single-degree-of-freedom system with specific characteristics when responding to the applied forces or base motions.

11. The solutions were presented for structures initially at rest. Since superposition holds, the initial conditions may be accounted for simply by adding their equation of motion to the solutions which were derived.

12. It was shown that the stress or deflection at a point could be considered to be a component which ignored inertial effects, plus a dynamic correction.

13. For the single-degree-of-freedom system it is often stated that the maximum stress or deflection due to a step change in force is double the static case. This was shown to be true for deflection of the mass where the force was applied but not true for the other masses.

14. It was demonstrated that the results for a structure responding to many applied forces can be converted to the relative response due to base motion by the principle of mechanics which states that the acceleration of a frame of reference is indistinguishable from a change in the gravity field.

15. As in the static case a reciprocity theorem for the dynamic response of a linear elastic structure was shown.

16. The special cases of response to impulses, sudden base motions, steps of forces, and steady-state vibrations were discussed.

17. The characteristic load theorems were proven.

18. The distribution of forces applied to the structure by applied forces or base motions were shown to be characteristic loads for each of the modes.

19. Stress coefficients and their use were introduced.

20. The use of characteristic loads and stress coefficients was shown to be an advantageous way of solving the stress problem.

21. Three approaches were shown for the problem of stress and deflection checking: (a) use of Duhamel integrals as time functions, (b) use of both the maximum positive and negative values of these integrals, and (c) use of the maximum absolute value of the integrals.

22. The effective mass and its location for a structure subjected to a base motion were derived.

23. The effect upon modal stress and deflection of a sudden change in the gravity field was discussed.

24. The response of a structure to base rocking was discussed.

REFERENCES

1. Blake, R.E. and Swick, E.S., "Dynamics of Linear Elastic Structures," NRL Report 4420, Oct. 1954

2. O'Hara, G.J., "Notes on Dynamics of Linear Elastic Structures," NRL Report 5387, Oct. 1959.

3. Timoshenko, S., and Young, D.H., "Vibration Problems in Engineering," 3rd edition, Princeton:Van Nostrand, 1955

4. Goldstein, H., "Classical Mechanics," Cambridge:Addison-Wesley, p. 322, 1950

5. Frazer, R.A., Duncan, W.J., and Collar, A.R., "Elementary Matrices," New York:MacMillan, 1947

6. Wylie, C.R., Jr., "Advanced Engineering Mathematics," New York:McGraw-Hill, 1951

7. O'Hara, G.J., "A Numerical Procedure for Shock and Fourier Analysis," NRL Report 5772, June 1962

8. O'Hara, G.J., "Effect Upon Shock Spectra of the Dynamic Reaction of Structures," NRL Report 5236, Dec. 1958

9. O'Hara, G.J., "Shock Spectra and Design Shock Spectra," NRL Report 5386, Nov. 1959

10. O'Hara, G.J., "Impedance and Shock Spectra," *J. Acoustical Soc. Amer.* 31:1300 (Oct. 1959)

11. Belsheim, R.O., and O'Hara, G.J., "Shock Design of Shipboard Equipment," Part I, "Dynamic Design Analysis Method," NavShips 250-423-30, May 1961

APPENDIX A

DIFFERENTIATION OF AN INTEGRAL

The Duhamel integrals given in the single-degree-of-freedom section were for relative displacement. Sometimes the velocity or acceleration might be desired, and one may wish to differentiate one of these integrals. The following is known as Leibnitz's rule and its derivation may be found in almost any advanced calculus or advanced engineering mathematics textbook.* Given

$$\phi(t) = \int_{a(t)}^{b(t)} f(T, t)\, dT$$

then

$$\frac{d\phi}{dt} = \int_{a(t)}^{b(t)} \frac{df}{dt}\, dT + f[b(t), t]\frac{db}{dt} - f[a(t), t]\frac{da}{dt}.$$

For example:

$$X = -\frac{1}{\omega}\int_0^t \ddot{Z}(T) \sin \omega(t-T)\, dT$$

$$\dot{X} = -\int_0^t \ddot{Z}(T) \cos \omega(t-T)\, dT - \left[-\frac{\ddot{Z}(t)}{\omega} \sin \omega(0)\right](1) - \left[-\frac{\ddot{Z}(0)}{\omega} \sin \omega t\right](0).$$

Therefore,

$$\dot{X} = -\int_0^t \ddot{Z}(T) \cos \omega(t-T)\, dT.$$

Differentiation again yields

$$\ddot{X} = \omega \int_0^t \ddot{Z}(T) \sin \omega(t-T)\, dT - \ddot{Z}(t).$$

Note that

$$\underline{\ddot{X}} = \ddot{X} + \ddot{Z} = -\omega^2 X = \omega \int_0^t \ddot{Z}(T) \sin \omega(t-T)\, dT.$$

APPENDIX B

USE OF STIFFNESS COEFFICIENTS

In the section on influence and stiffness coefficients it was stated: "A stiffness coefficient, denoted by the symbol K_{ij}, is the force required at i when the structure is loaded in such a manner that all points are restrained from moving except j, which moves a unit distance in the negative direction." If Eq. (17) is solved for the forces, there results

$$F_i = \sum_j K_{ij} X_j.$$

For free vibrations apply D'Alembert's principle:

$$-m_i \ddot{X}_i = \sum_j K_{ij} X_j. \qquad \text{(B1)}$$

Assuming a solution in the form $X_i = \overline{X}_i \sin(\omega t + \beta)$ yields

$$m_i \overline{X}_i \omega^2 \sin(\omega t + \beta) = \sin(\omega t + \beta) \sum_j K_{ij} \overline{X}_j$$

or

$$m_i \overline{X}_i \omega^2 = \sum_j K_{ij} \overline{X}_j.$$

This is a set of equations of the form

$$(m_1\omega^2 - K_{11})\overline{X}_1 - K_{12}\overline{X}_2 + \ldots + -K_{1n}\overline{X}_n = 0$$

$$-K_{21}\overline{X}_1 + (m_2\omega^2 - K_{22})\overline{X}_2 + \ldots + -K_{2n}\overline{X}_n = 0$$

$$\vdots$$

$$-K_{n1}\overline{X}_1 - K_{n2}\overline{X}_2 + \ldots + (m_n\omega^2 - K_{nn})\overline{X}_n = 0.$$

*For example, C.R. Wylie, Jr., "Advanced Engineering Mathematics," New York: McGraw-Hill, 1951.

Inspection shows that it is a set of linear algebraic equations equal to zero. The determinant of the coefficient of the $\overline{X}_j$'s is the frequency equation, and

$$\overline{X}_{ia} = \frac{1}{m_i\omega_a^2} \sum_j K_{ij}\overline{X}_{ja}. \qquad \text{(B2)}$$

The normal modes are the same ones which were found before, so they are orthogonal.

Multiply both sides by $m_i\overline{X}_{ib}$ and sum on i:

$$\sum_i m_i\overline{X}_{ib}\overline{X}_{ia} = \frac{1}{\omega_a^2} \sum_i \overline{X}_{ib}\sum_j K_{ij}\overline{X}_{ja}$$

$$= \frac{1}{\omega_a^2} \sum_j \overline{X}_{ja}\sum_i K_{ji}\overline{X}_{ib}$$

since $K_{ij} = K_{ji}$. Now

$$\sum_i K_{ji}\overline{X}_{ib} = m_j\overline{X}_{jb}\omega_b^2$$

so

$$\sum_i m_i\overline{X}_{ib}\overline{X}_{ia} = \frac{\omega_b^2}{\omega_a^2}\sum_j m_j\overline{X}_{jb}\overline{X}_{ja}.$$

This reduces at once to

$$\left(1 - \frac{\omega_b^2}{\omega_a^2}\right) \sum_i m_i\overline{X}_{ib}\overline{X}_{ia} = 0$$

which is the same expression as was found previously for the orthogonality conditions.

Letting

$$X_i = \sum_a \overline{X}_{ia}q_a$$

we have from Eq. (B1)

$$m_i\sum_a \overline{X}_{ia}\ddot{q}_a + \sum_j K_{ij}\sum_a \overline{X}_{ja}q_a = 0$$

which may be written as

$$\sum_a\left(m_i\overline{X}_{ia}\ddot{q}_a + q_a\sum_j K_{ij}\overline{X}_{ja}\right) = 0$$

or

$$\sum_a (\ddot{q}_a + \omega_a^2 q_a)m_i\overline{X}_{ia} = 0$$

since

$$m_i\overline{X}_{ia}\omega_a^2 = \sum_j K_{ij}\overline{X}_{ja}.$$

Multiply by $\overline{X}_{ib}$ and sum on i. Then

$$\ddot{q}_a + \omega_a^2 q_a = 0$$

as before.

This is as far as the use of stiffness coefficients is demonstrated, because the results for deflections, stress, etc., are the same as for influence coefficients. Note that

$$K_{ij} \neq \frac{1}{\delta_{ij}}.$$

APPENDIX C

REPLACEMENT OF ROTARY INERTIAS BY MASSES

The equations of normal mode theory for unidirectional motion may be used for the case of a structure which has unitranslation and rotation in a plane provided certain modifications are made to the lumped-parameter model representing the structure. The only restriction in what follows is that the rotation, θ, must be small.

As an example consider the body shown in Fig. C1. The mass of the body is M, and r is the radius of gyration of the body about the center

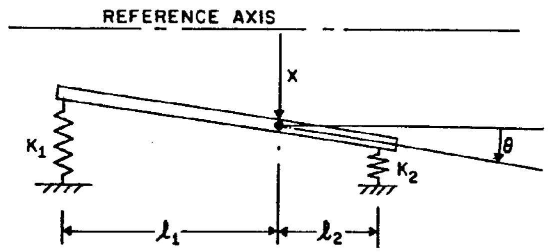

Fig. C1 – Example of translation and rotation of a beam of gravity. Let x and θ be the translational and rotational displacements about the center of gravity respectively. The equations of motion for the body in free vibrations are

$$M\ddot{x} + K_1(x - \ell_1\theta) + K_2(x + \ell_2\theta) = 0 \quad \text{(C1)}$$

$$Mr^2\ddot{\theta} - K_1\ell_1(x - \ell_1\theta) + K_2\ell_2(x + \ell_2\theta) = 0. \quad \text{(C2)}$$

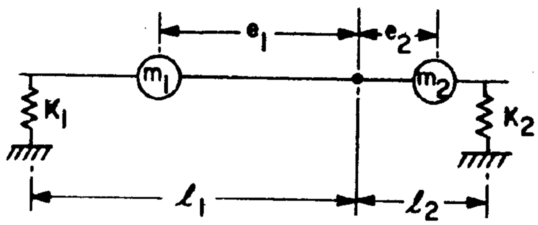

Fig. C2 – Replacement of the beam by two masses

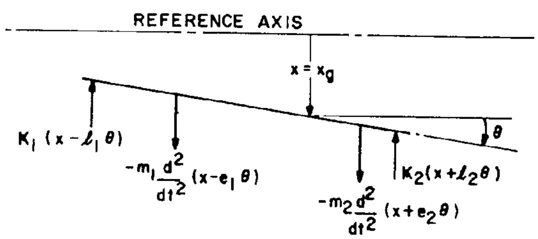

Fig. C3 – Free-body diagram showing the forces acting on the beam

Suppose this structure is replaced by two masses such that $m_1 + m_2 = M$, and the masses are placed at distances e_1 and e_2, respectively, from the center of gravity such that $m_1e_1 = m_2e_2$. Find e_1 and e_2 so that the new structure is equivalent to the original structure. Figure C2 shows the masses on the model of the structure. Figure C3 shows this model at some instant after motion of the structure occurs.

The equation of motion in the x-direction is

$$m_1 \frac{d^2}{dt^2}(x - e_1\theta) + m_2 \frac{d^2}{dt^2}(x + e_2\theta)$$

$$+ K_1(x - \ell_1\theta) + K_2(x + \ell_2\theta) = 0$$

or

$$(m_1 + m_2)\ddot{x} + (-m_1e_1 + m_2e_2)\ddot{\theta}$$

$$+ K_1(x - \ell_1\theta) + K_2(x + \ell_2\theta) = 0.$$

Now,

$$-m_1e_1 + m_2e_2 = 0$$

and

$$m_1 + m_2 = M.$$

Therefore,

$$M\ddot{x} + K_1(x - \ell_1\theta) + K_2(x + \ell_2\theta) = 0. \quad \text{(C3)}$$

The equation of motion in the θ-direction is

$$(m_1e_1^2 + m_2e_2^2)\ddot{\theta} - K_1\ell_1(x - \ell_1\theta)$$

$$+ K_2\ell_2(x + \ell_2\theta) = 0. \quad \text{(C4)}$$

Comparing Eqs. (C1) and (C2) with (C3) and (C4), respectively, if

$$Mr^2 = m_1e_1^2 + m_2e_2^2$$

the two sets of equations are the same. Therefore, a structure which translates and rotates may be idealized by a lumped parameter model of two masses such that the influence coefficients are found only for unidirectional motion, provided the following restrictions are met: (a) the sum of the two masses equals the total mass, and (b) the two masses are so placed on the model that the moment of inertia about the center of gravity equals the moment of inertia of the original structure about the center of gravity.

Therefore, all equations developed in this report for unidirectional motion are applicable.

It is noted that a logical selection of the two masses is $m_1 = m_2$. It then follows that $e_1 = e_2 = r$.