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Field Reversals of "Paleomagnetic" Type in Coupled Disk Dynamos

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CONTENTS

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Rikitake's adaptation of Bullard's simplified disk-dynamo model for the dynamo theory of the geomagnetic dipole was investigated further by means of an analog computer. A modified model having generalized interaction, which was suggested by an axial configuration, was also studied. A striking similarity is observed between the computer solutions and the geomagnetic polarity reversals inferred from paleomagnetic studies. Thus the magnetofluiddynamic origin of the earth's field becomes more plausible.

INTRODUCTION

Many theories have been advanced to explain the origin of the earth's main dipole field. Some magnetofluiddynamic process operating as a homogeneous dynamo in the earth's core is now generally accepted, with fair certainty, as the basic mechanism for a valid theory. The formidable nature of the mathematics of the problem has made development of a theory most difficult. As a result, much of the earlier effort had been on steady-state considerations of the theoretical homogeneous dynamo. Some important results are concerned with the fundamental question of the ability of the steady-state dynamo to maintain itself. Backus and Chandrasekhar (1) have established cases for rigorous kinematic impossibility; for different cases, Herzenberg (2) has established possibility. Also, axial symmetry of the revolving fluid has been prohibited by the Cowling theorem (3), with extension by Backus and Chandrasekhar.

The results of paleomagnetic studies have strongly suggested that the present polarity of the earth's main dipole field was just the opposite at an earlier time in geological history. In fact, polarity reversals are thought to have occurred at various times during the existence of the earth. This evidence of polarity reversal from remanent rock magnetism is now generally conceded to require recognition in any theoretical model of the earth's main dipole field. Bullard and Gellman (Ref. 4, p. 258) and Runcorn (5) have indicated that the dipole may be highly sensitive to the details of convection. Thus, polarity reversal might conceivably be due to shifts from one steady-state dynamo to another, caused by extra-magnetofluiddynamic changes in the earth. Despite much progress, the steady-state results are far from complete.

Though the theory of nonsteady dynamos is even more difficult, many geophysicists believe that study may shed some light on dipole variation and polarity reversal. A clever analogy due to Bullard (6) started an interesting series of discoveries relating to polarity reversal capabilities of the unsimplified, nonsteady system. Bullard introduced a disk-dynamo model in place of the vastly more complicated homogeneous dynamo thought to be functioning within the core of the earth. The working principles of the disk dynamo were known, and the interaction of the current and the rotational velocity in this model could be precisely formulated. The disk-dynamo model remains theoretical, nevertheless, because the large dimensions of the earth process do not lend themselves to reproduction on the small scale attainable in the laboratory. The partial differential equations of the magnetofluiddynamic prototype became the ordinary differential equations of lumped circuit theory and rigid rotation. The essential nonlinearity ($\vec{v} \times \vec{B}$, $\vec{J} \times \vec{B}$) of closed-system, magnetomechanical interaction was retained. Admittedly, the disk-dynamo model has to be interpreted with great caution since it is far removed from the details of the prototype. With regard to field reversals, Bullard's single-disk, self-exciting, dynamo model proved somewhat disappointing, for he found that current oscillations observed in the solutions of the equations were fundamentally incapable of exhibiting opposite polarity. Rikitake (7) worked out the theoretical model for a system of two, mutually exciting, disk dynamos. He was able to carry out solutions of the equations of this system far enough to establish that the oscillating dynamo-currents could indeed change sign. Allen (8) published the solution of a single current for the same system, corroborating Rikitake's results. Allan's solution was of sufficient length to reveal another feature. A series of small oscillations, or ripples, which remained almost entirely of one polarity was followed by another series of opposite polarity. This, feature seemed to suggest that the ripples were superimposed first on one level of equilibrium, and then on a second level of opposite polarity. Thus, in principle, a theoretical disk-dynamo system model had been shown capable of accounting for the field reversals indicated by the results of paleomagnetic studies. Paleomagnetic evidence, however, seems to indicate a rather special type of variation: intervals of more or less random dura-

NRL Problem E01-01; Projects RR 010-04-41-5950 and BuWeps RAE 50R-035/652-1/F012-11-001. This is a final report on one phase of the problem; work on other phases of the problem continues.
NOTE: One of the authors (J.H.M.) is formerly of NRL.

tion and of fairly constant intensity in one polarity are separated by relatively rapid polarity reversals between.

We have used an analog computer to obtain solutions to the equations of the Rikitake two-disk system, with and without modification. The forte of this computer is that continuous solutions of systems of ordinary nonlinear differential equations are possible. Solutions for significantly long durations have been achieved with relative ease.

The purpose of this report is to show that a remarkable approximation to the special type of variation is possible. For a more extensive background of the problem in general, the reader may refer to two recent review articles (9,10).

MODELS

The basic Bullard-type single-disk dynamo model is shown schematically in Fig. 1. Voltage is generated radially in the disk as it rotates in the coil field created by the current. For high angular velocities the smallest initial field or current is sufficient to self-excite the system. Connections between coil and disk may be visualized as some equivalent form of sliding contact. For the coil connections and direction of rotation shown, this dynamo is regenerative. If either is changed the dynamo is degenerative. If both are changed the dynamo is again regenerative.

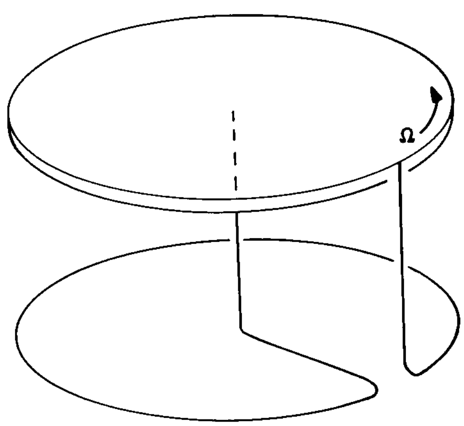

Fig. 1 - Bullard-type, self-excited, disk-dynamo model

Figure 2 illustrates Rikitake's two-disk system. Disk I and Coil I are coupled electrically, as are Disk II and Coil II. Magnetic coupling occurs between Coil II and Disk I and also between Coil I and Disk II. The two halves, left and right, are considered to be separated sufficiently to exclude the presence of any other coupling. Rikitake's dynamo is regenerative for the coil connections and directions of rotation shown.

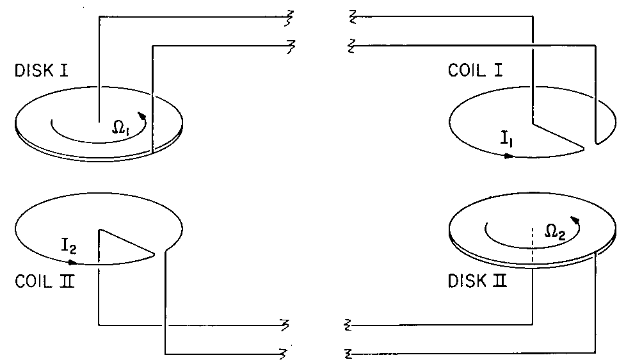

Fig. 2 - Rikitake-type, two-disk, mutually excited dynamo model. The break in the drawing serves to indicate that the individual sections, left and right, are sufficiently separated to exclude interaction except the mutual excitation.

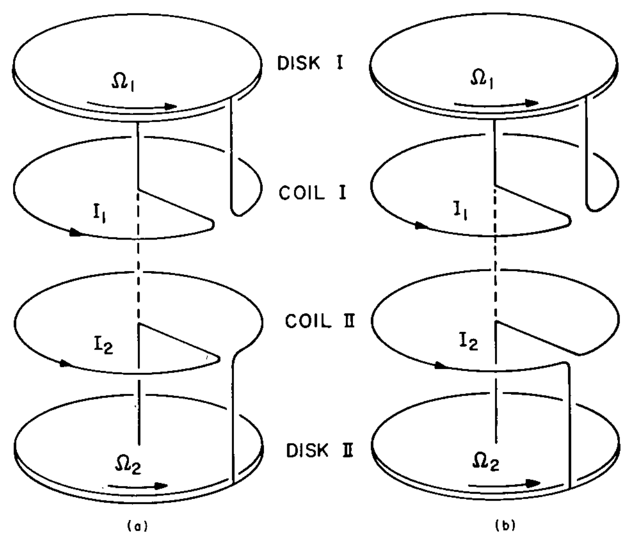

Fig. 3 - Axially stacked two-disk system. The dynamo in (b) differs from the dynamo in (a) in that the connections of Coil II have been reversed.

Figure 3(a) shows the two-disk system of Fig. 2 rearranged in an axial configuration, still regenerative. For ease of illustration the coils have been drawn closer to the disks which are electrically coupled. Figure 3(b) differs from 3(a) in that connections of the lower coil have been changed to make the system degenerative. The axial configuration of these figures requires the consideration of additional magnetic coupling, namely, the mutual inductance between Coil I and Coil II, and the mutual inductances between Coil I and Disk I and between Coil II and Disk II.

EQUATIONS

Rikitake System

The Rikitake system equations are

$$L\dot{I}_1^* - \alpha I_2^* \Omega_1^* + RI_1^* = 0 \tag{1}$$

$$L\dot{I}_2^* \mp \alpha I_1^* \Omega_2^* + RI_2^* = 0 \tag{2}$$

$$C\Omega_1^* + \alpha I_1^* I_2^* - G = 0 \quad (3)$$

$$C\dot{\Omega}_2^* \pm \alpha I_1^* I_2^* - G = 0, \quad (4)$$

where

$$\dot{I}_1^* \equiv \frac{dI_1^*}{dt^*},$$

etc., are time derivatives. The asterisks denote that the variables are real, dimensional quantities, the subscripts 1 and 2 refer to the two disk-and-coil circuits, I is current, and Ω is angular velocity. The parameters of the system are symmetrical. L and R are the self-inductance and resistance, respectively, of the circuits; C is the moment of inertia of the disks; and G is the common, external, driving torque on the disks. The symbol α, called M by Rikitake and Bullard, is the factor which remains explicit from the term $2\pi\alpha$, which represents the mutual inductance between the coil and the periphery of the disk. For the two-disk system, this mutual inductance is between Coil II and Disk I and between Coil I and Disk II. We prefer to reserve the symbol M for later use. Although somewhat trivial, the lower signs have been added to include the degenerate type of disk rotation and coil connection.

Modified Rikitake System

Generalizing the interaction of the axial arrangement of Fig. 3 requires the addition of two new parameters. We shall now let M represent the mutual inductance between the coils. The factor β remains explicit from the term $2\pi\beta$, which represents the mutual inductance between Coil I and the periphery of Disk I, and between Coil II and the periphery of Disk II. The equations become, in dimensional form,

$$L\dot{I}_1^* + M\dot{I}_2^* - (\alpha I_2^* + \beta I_1^*)\,\Omega_1^* + RI_1^* = 0 \quad (5)$$

$$L\dot{I}_2^* + M\dot{I}_1^* \mp (\alpha I_1^* + \beta I_2^*)\,\Omega_2^* + RI_2^* = 0 \quad (6)$$

$$C\dot{\Omega}_1^* + (\alpha I_2^* + \beta I_1^*)\,I_1^* - G = 0 \quad (7)$$

$$C\dot{\Omega}_2^* \pm (\alpha I_1^* + \beta I_2^*)\,I_2^* - G = 0. \quad (8)$$

The upper and lower signs correspond to the models in Figs. 3(a) and 3(b) respectively.

Dimensionless Forms

Solution of the equations by computer was facilitated by first making the variables dimensionless in order to provide for easy adjust~~ment~~ of parameters. This was done by means of the following Rikitake-type relations, which we refer to as the K-form and the P-form. The K-form relation is:

$$I_i^* \equiv \sqrt{\frac{G}{\alpha}}\quad I_i \quad (9)$$

$$\Omega_i^* \equiv \sqrt{\frac{GL}{C\alpha}}\quad \Omega_i \quad (10)$$

$$t^* \equiv \sqrt{\frac{CL}{G\alpha}}\quad t. \quad (11)$$

The P-form relation is:

$$I_i^* \equiv \frac{R}{\alpha}\sqrt{\frac{C}{L}}\,I_i \quad (12)$$

$$\Omega_i^* \equiv \frac{R}{\alpha}\,\Omega_i \quad (13)$$

$$t^* \equiv \frac{L}{R}\,t\,. \quad (14)$$

For both forms, $i = 1,2$.

The K-form of the resulting Rikitake equations having dimensionless variables is:

$$\dot{I}_1 - I_2\,\Omega_1 + KI_1 = 0 \quad (15)$$

$$\dot{I}_2 \mp I_1\,\Omega_2 + KI_2 = 0 \quad (16)$$

$$\dot{\Omega}_1 + I_1\,I_2 - 1 = 0 \quad (17)$$

$$\dot{\Omega}_2 \pm I_1\,I_2 - 1 = 0\,, \quad (18)$$

where

$$K \equiv R\sqrt{\frac{C}{L\alpha G}}\,.$$

The P-form of the resulting Rikitake equations with dimensionless variables is:

$$\dot{I}_1 - I_2\,\Omega_1 + I_1 = 0 \quad (19)$$

$$\dot{I}_2 \mp I_1\,\Omega_2 + I_2 = 0 \quad (20)$$

$$\dot{\Omega}_1 + I_1\,I_2 - P = 0 \quad (21)$$

$$\dot{\Omega}_2 \pm I_1\,I_2 - P = 0\,, \quad (22)$$

where $P \equiv 1/K^2$.

The K-form of the modified Rikitake equations for the axial system (Fig. 3), generalized in interaction, is:

$$\dot{I}_1 + Q\dot{I}_2 - (I_2 + FI_1)\,\Omega_1 + KI_1 = 0 \tag{23}$$

$$\dot{I}_2 + Q\dot{I}_1 \mp (I_1 + FI_2)\,\Omega_2 + KI_2 = 0 \tag{24}$$

$$\dot{\Omega}_1 + (I_2 + FI_1)\,I_1 \; - 1 \; = 0 \tag{25}$$

$$\dot{\Omega}_2 \pm (I_1 + FI_2)\,I_2 \; - 1 \; = 0\,, \tag{26}$$

where $Q \equiv M/L$ and $F \equiv \beta/\alpha$.

The P-form for the modified system is:

$$\dot{I}_1 + Q\dot{I}_2 - (I_2 + FI_1)\,\Omega_1 + I_1 = 0 \tag{27}$$

$$\dot{I}_2 + Q\dot{I}_1 \mp (I_1 + FI_2)\,\Omega_2 + I_2 = 0 \tag{28}$$

$$\dot{\Omega}_1 + (I_2 + FI_1)\,I_1 \; - P = 0 \tag{29}$$

$$\dot{\Omega}_2 \pm (I_1 + FI_2)\,I_2 \; - P = 0\,. \tag{30}$$

RESULTS

The study performed on the analog computer for a variety of parameters and initial conditions resulted in many solutions of paleomagnetic interest. In the search for interesting behavior, solutions were observed after each change of a fairly systematic variation of parameters and initial conditions. Occasionally, rapid changes of style were observed within narrow limits of parameter and initial condition changes. Although good coverage is thought to have been obtained, the search was not automated and cannot be claimed to have been completely exhaustive. The recorded results were obtained in terms of current versus time (both dimensionless) where current represents the field.

Solutions to the Rikitake Equations (P-Form)

The first group of solutions are from the dimensionless P-form equations of the Rikitake system, namely, Eqs. (19)-(22), using the upper signs (regenerative case) and the value $P = 1.000$. Figure 4(a) shows the individual currents in a typical reversing type of behavior. As can be seen, both curves consist of series of small oscillations in one polarity followed by another series usually in the opposite polarity. Thus polarity reversals have occurred between the series of small oscillations. The upper curve (I_2) resembles strongly the one given by Allan (8). However, for this curve, portions of the

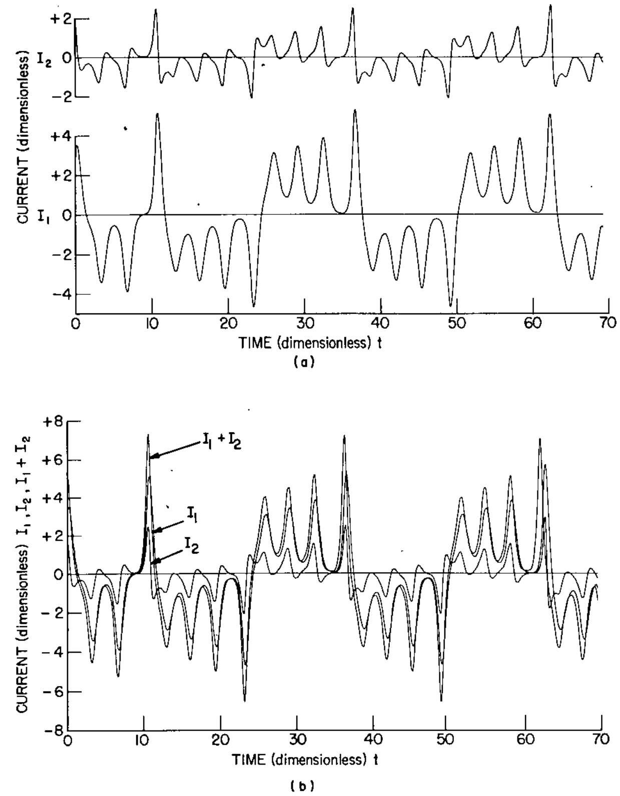

Fig. 4 - (a) Two individual currents I_1 and I_2 for $P = 1.000$. Initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, $\Omega_2 = 0.037$. (b) Repeat runs of the same curves, and their sum $I_1 + I_2$.

small oscillations in a series of one polarity frequently extend across the zero-current line into the opposite polarity. Significantly, for the current in the lower curve (I_1), the series of small oscillations change polarity in their entirety. The small oscillations, or ripples, are distinctly separated from the zero line until reversal action takes place, i.e., the small oscillations do not cross the zero line. We consider this distinctly separated type of solution to be far more appropriate for representing the rather special type of earth's field reversal behavior suggested by paleomagnetic data. Figure 4(b) shows three solutions. Two of these are repeat runs of the individual currents for comparison with the third, which is the sum of the two. Several attempts were found necessary to sequentially record this series to obtain results which would be equivalent to a simultaneous recording of the three curves. Even so, a slight scatter is apparent beyond the value of 60 on the dimensionless time scale.

The highly sensitive nature of the equations in regions of zero current is illustrated in Fig. 5(a) for the parameter value $P = 1$. The second run departs drastically from the first at $t = 48$. The departure is seen to consist of opposite-polarity reversals which approximate a mirror image of the first run. The small

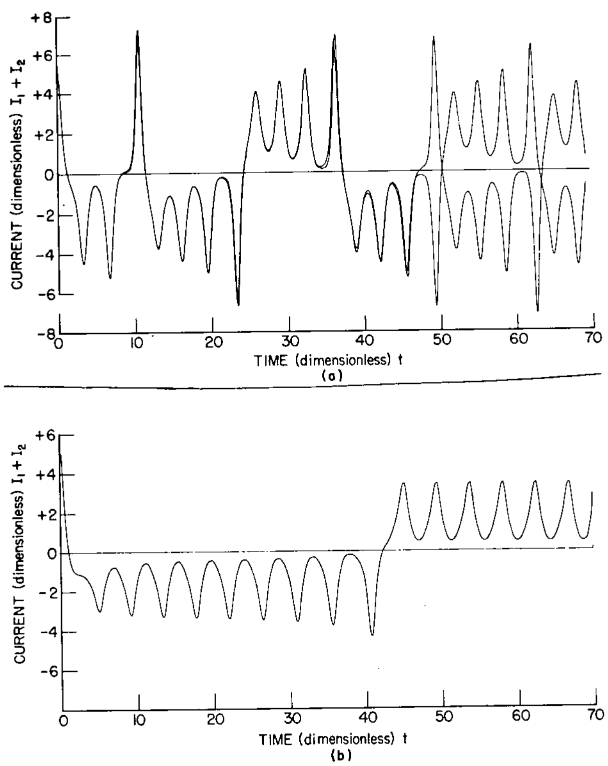

Fig. 5 - Two runs of the current sum $I_1 + I_2$ for (a) $P = 1.000$ and (b) $P = 0.550$. Initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, and $\Omega_2 = 0.037$.

difference which is observable at $t = 35$ may have been the cause of the drastic departure at $t = 48$. The highly sensitive regions at $t = 9$ and 22 were successfully repeated on the second computer run. The single excursion above the zero line at $t = 9$ is considered to be an initial reversal to positive polarity followed by an immediate return to negative polarity.

Although repeatability varied somewhat with parameters and initial conditions, these curves are rather typical. That is, approximately three or four high-sensitivity regions is the limit that can usually be successfully retraced. Certainly the solutions are subject to reversal whenever they enter the highly sensitive region about zero current. Somewhat longer repeatability is illustrated in Fig. 5(b) for a lower value of the parameter P. Here, however, just one highly sensitive region has been encountered, exclusive of the polarity change at the very start.

Although several individual-current solution recordings are presented, the sum of the currents was recorded for many of the solutions in this first group. One reason was that in the early phases of the work we attached significance to the sum for axially stacked disk and coil models. Later we realized that the sum in an axial variation of the Rikitake system would not be very realistic without inclusion of the resulting interaction. There was, however, a practical advantage. The sum of the currents in the Rikitake system, for any conditions observed, always displayed characteristics similar to those of the individual current which is distinctly separated from zero. This is due to the small contribution from the other individual current. A distinctly separated curve appeared on I_1 for some runs and on I_2 for others, depending on the particular parameters and initial conditions chosen. Thus the sum provided a quick monitor of the most interesting, distinctly separated, individual current when utilizing a single recorder.

The effects of P-parameter variation for one set of initial conditions may be seen from comparison of Figs. 6(a), (b), and (c). The relatively low P-value of Fig. 6(a) results in there being many more ripples during the intervals between reversals. Also, the current amplitude is smaller. The nature of the ripples of Fig. 6(a) may be seen more clearly by referring to Fig. 5(b), both for a P-value of 0.550. Although the solution of Fig. 7 was based upon a slightly different initial condition ($\Omega_2 = 0.0$), it serves to show typical detail of the ripple content of Fig. 6(b). Both solutions are for a P-value of 1.000. P-values lower than about 0.550 resulted in solutions (not shown) which no longer reversed.

Figure 8(a) illustrates an interesting type of behavior. The individual current shows preference for extended intervals in one polarity separated by brief multiple, or even single, excursions into the opposite polarity. With some changes in the initial conditions, the individual current of Fig. 8(b) exhibits, initially, similar behavior to Fig. 8(a), but in the opposite polarity. It then reverses polarity and settles to a very long oscillatory mode. We have sometimes visually monitored solutions of this type, long after the recorder came to the end of its sheet of paper, by using digital and electronic voltmeters. Nothing was observed which might indicate that the oscillatory mode would ever be disturbed. Eventually, drift and other signs of deterioration of computer performance became evident, as could be expected from the outrageously long computing time demand.

Figure 9(a) shows a solution of special paleomagnetic interest. Although the time scale may not be that thought precisely desirable from rock magnetism studies, the run shows that quite ideal representations are indeed possible with a disk-dynamo system model. Figures 9(b) and (c), obtained on a later date for the same conditions used in Fig. 9(a), illustrate another point of the repeatability difficulties. When repeating the longer runs, the sequence of reversals changed in somewhat random fashion, Nevertheless, the overall behavior patterns associated with the conditions used can usually be observed in the repeated solution.

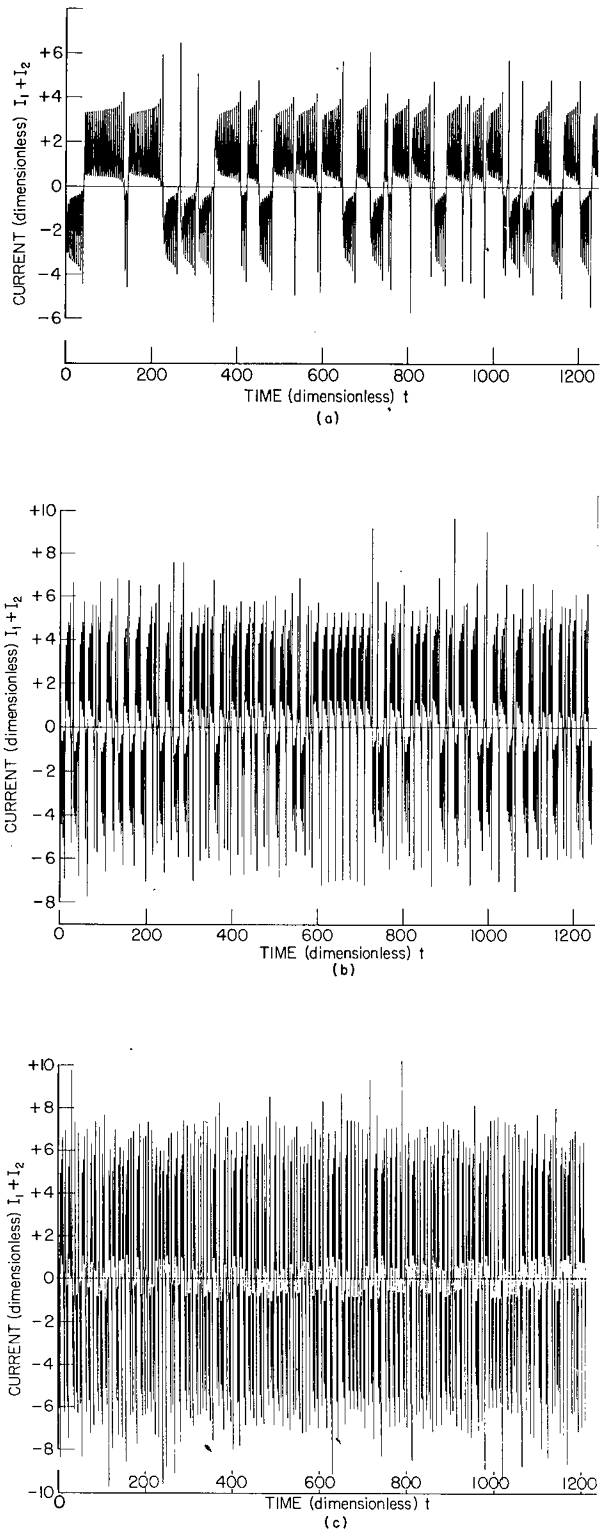

Fig. 6 - Current sum $I_1 + I_2$ for (a) $P = 0.550$, (b) $P = 1.000$, and (c) $P = 1.400$. Initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, and $\Omega_2 = 0.037$.

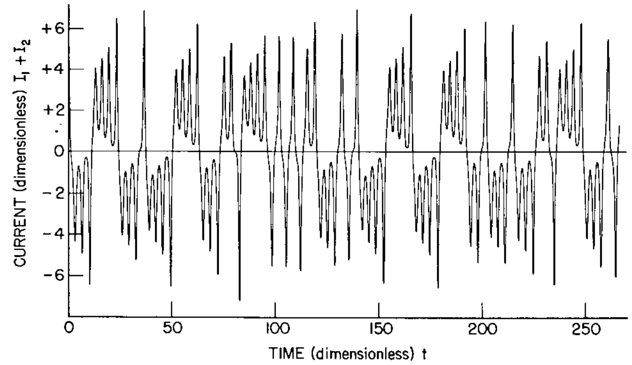

Fig. 7 - Expanded scale of current sum for $P = 1.000$ (see Fig. 6(b)) to illustrate typical ripple content. Initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, and $\Omega_2 = 0.000$ (slightly different value than that of Fig. 6(b)).

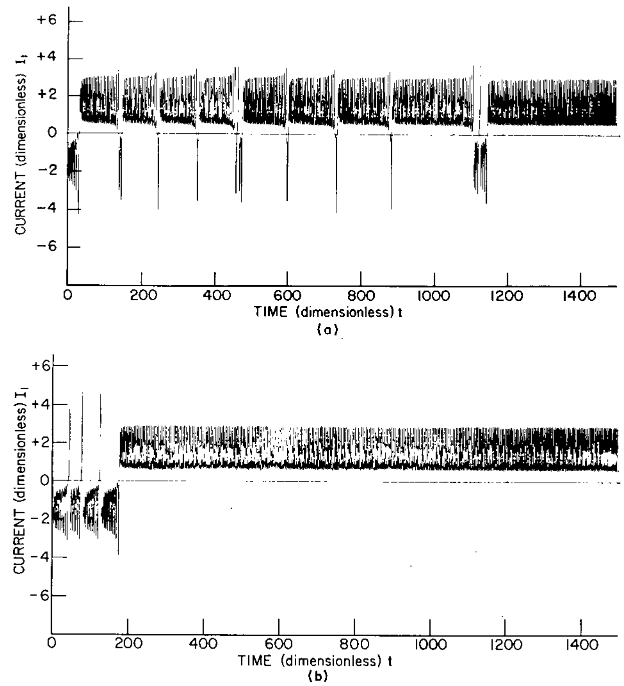

Fig. 8 - Individual current for $P = 0.580$. (a) Initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, and $\Omega_2 = 0.037$; (b) Initial conditions: $I_1 = 3.000$, $I_2 = 1.915$, $\Omega_1 = 3.802$, and $\Omega_2 = 0.0$.

The pattern of the more frequent reversals toward the right-hand side of these curves seemed quite dependent upon the exact manner in which the first reversal to positive polarity occurred. Probably we were somewhat fortunate to capture the long stay above the line in Fig. 9(a). On further attempts (not shown) we were able to reproduce Fig. 9(a) as far as $t = 675$. However, small refinements of the earlier settings proved necessary to accomplish this.

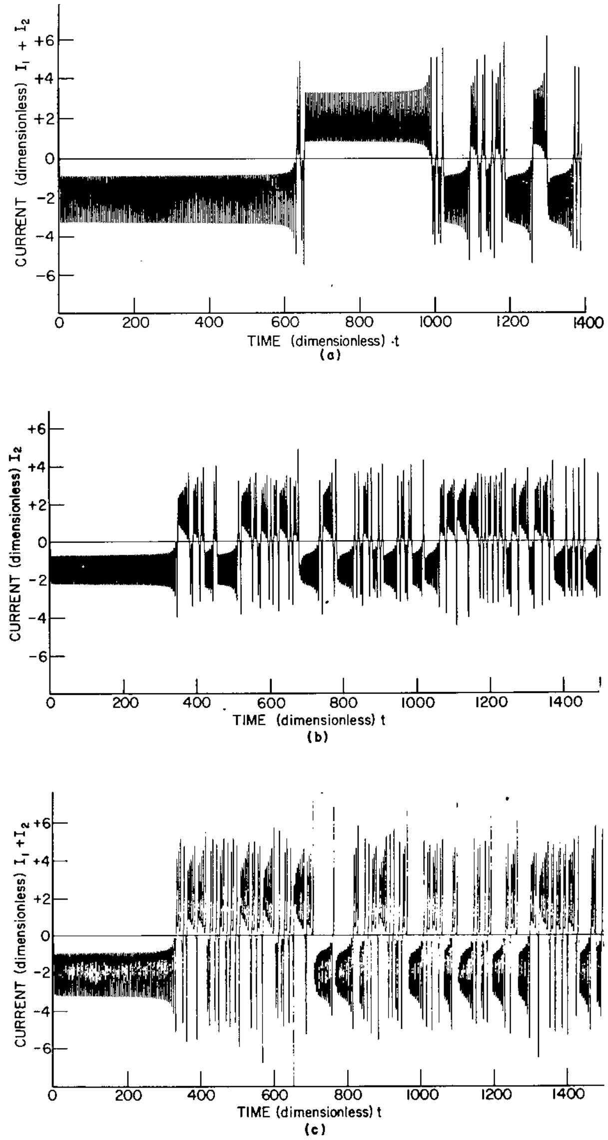

Fig. 9 - Three separate runs for $P = 0.715$ with initial conditions: $I_1 = 3.300$, $I_2 = 0.0$, $\Omega_1 = 0.0$, and $\Omega_2 = 2.400$. (a) illustrates a long duration in the upper polarity which occurred after three rapid-order reversals; (b) and (c) illustrate the general similarity of style for the more random reversals of the longer runs. (a) and (c) are for the current sum $I_1 + I_2$, and (b) is for the individual current I_2.

Figures 10(a) and 10(b) present the only solutions appearing in this report which were recorded simultaneously during a single run, using two recorders. A comparison of these figures illustrates our observations concerning the relation between the individual currents and their sums for the Rikitake system, i.e., the sum of the currents was found representative of one of the individual currents for all of the parameters and initial conditions attempted. The shorter

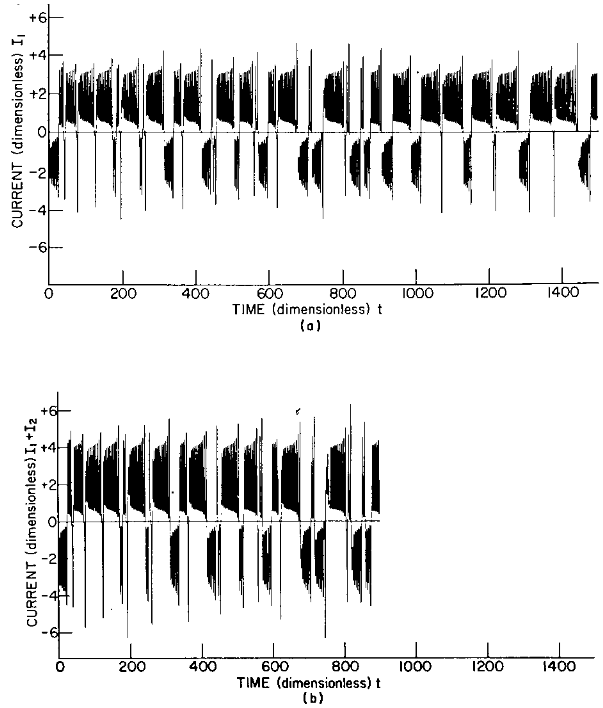

Fig. 10 - Simultaneous recording of (a) the individual current I_1 and (b) the current sum $I_1 + I_2$, both for $P = 0.600$ and the initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, and $\Omega_2 = 0.037$.

length of run recorded for the current sum in Fig. 10(b) is a result of the smaller size of the second recorder.

Solutions to the Modified Equations (*K*-Form)

The next group of curves presented in this report are recorded solutions of Eqs. (23)-(26) using the upper signs. These are the dimensionless *K*-form of the equations which contain the generalized interaction terms suggested by the axial system shown in Fig. 3(a). Only the sum of the currents was recorded for this second group of solutions. Here, however, the sum is not necessarily an indicator of the behavior of either of the individual currents. With the addition of mutual coupling, the current sum or the resultant field of all the currents in the system seems to be the only consistent representation of the total field. Figures 11(a) and (b) illustrate just one of the many interesting types of behavior encountered in the second group of solutions. The *K*-value of zero for Fig. 11(a) is probably unrealistic. A changing frequency of the small oscillations may be observed in both of these solutions. Figures 12(a)-(f) illustrate a variety of styles of behavior found in these solutions. It is apparent that the particular set of initial conditions and parameters does not forever determine the reversal pattern.

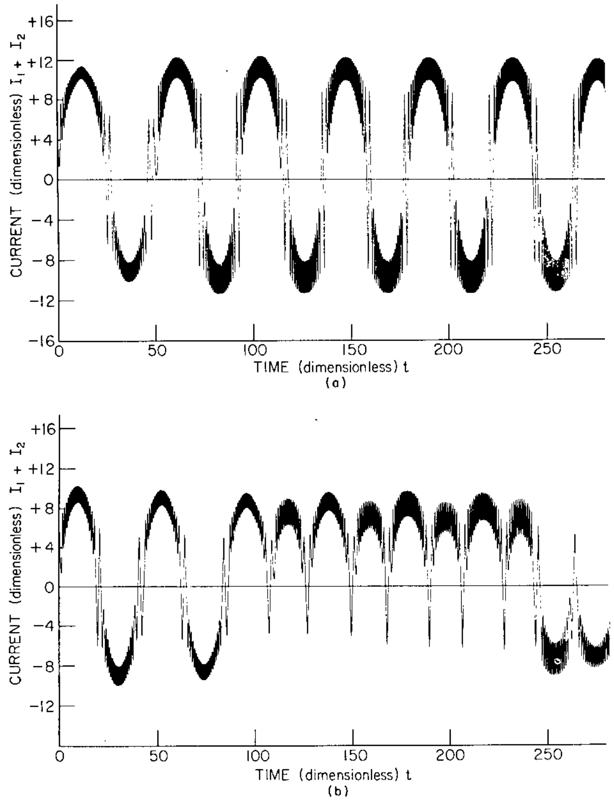

Fig. 11 - The current sum $I_1 + I_2$ for (a) $K = 0.0$, $F = 0.0075$, and $Q = 0.0850$, and (b) $K = 0.0025$, $F = 0.0075$, and $Q = 0.0850$. Initial conditions: $I_1 = 3.000$, $I_2 = 1.914$, $\Omega_1 = 4.000$, and $\Omega_2 = 0.0$.

Much larger values of the F- and Q-parameters resulted in solutions (not shown) which did not reverse, except for an occasional polarity change at the very start. The ripples usually settled to constant, nonzero, steady-state values.

Some solutions (not shown) were obtained for the model of Fig. 3(b) using Eqs. (27)-(30) with the lower signs (degenerative case). In general, these solutions settled rapidly to zero from the initial conditions. One band of initial conditions and parameters was found which resulted, at first, in a reversal with almost every oscillation, but then soon settled to zero.

DISCUSSION

Attempts to predict in advance the time of occurrence of the next reversal in the solutions proved unsuccessful. While increasing ripple amplitude of the currents is frequently evident prior to reversal, no reliable criteria could be established. A linearization technique was tried with the hope that differences between linear and nonlinear solutions might foretell the occurrence of the next reversal. However, no advanced indication was obtained from this technique.

Figure 13 is an idealized pictorial representation of some of the important characteristics of the earth's field reversals as indicated from paleomagnetic studies. Originally, it seemed conceivable in the present study that parameter and initial condition values might be found which would give solutions roughly constant in one polarity, between the reversals. The solid line of the sketch of Fig. 13 illustrates this idea. However, no recurring-reversal solutions without ripple were found, even though an extensive parameter and initial condition search was made. Actually, the presence of ripples during the interval in one polarity, illustrated by the dashed line of the sketch, does not clash with paleomagnetic evidence. Allan (8) cites several references containing evidence of field intensity variation. A lack of additional evidence may be due to inability of paleomagnetic techniques to resolve the ripple content.

The interval of time during reversal, from paleomagnetic evidence, should be of the order of 10,000 years. The field should remain in one polarity for intervals of 100,000 years to 2 or 3 million years, with 500,000 years considered typical (10,11). For comparison, the time scales of the solutions presented may be changed to dimensional units (seconds) using Eq. (14) for the P-form and Eq. (11) for the K-form. It is difficult to estimate reliable values of the quantities needed for these time-scale conversions, particularly of L and G. If Eq. (11) is multiplied by $1/K$, there results the optional form

$$t^* = K\left(\frac{L}{R}\right)t \ . \tag{31}$$

Thus the time-scale conversion for the K-form solutions differs from the P-form conversion only by the factor K. The alternate comparison may be obtained by multiplying Eq. (14) by $1/\sqrt{P}$, giving the optional form

$$t^* = \sqrt{P}\ \sqrt{\frac{CL}{G\alpha}}\ t \ . \tag{32}$$

It is thought that the task of estimating reliable values of L and R, or of C, L, G, and α, for representing the earth's core is best left to geophysicists specializing in these matters. Hence, no direct attempt will be made to convert the solutions to dimensional time. Nevertheless, it is worth pointing out that the ripple frequencies of the first group of solutions (Figs. 4-11) are within a factor of 2 or 3 of those of Rikitake's and Allan's. Thus it may be inferred that the dimensional times of these solutions to the Rikitake equations would be considered insufficient by a factor of about 500 for representing paleomagnetic times.

The second group of solutions correctly represents the behavior of a two-disk dynamo system with mutual interaction in axial arrangement, and they may be of

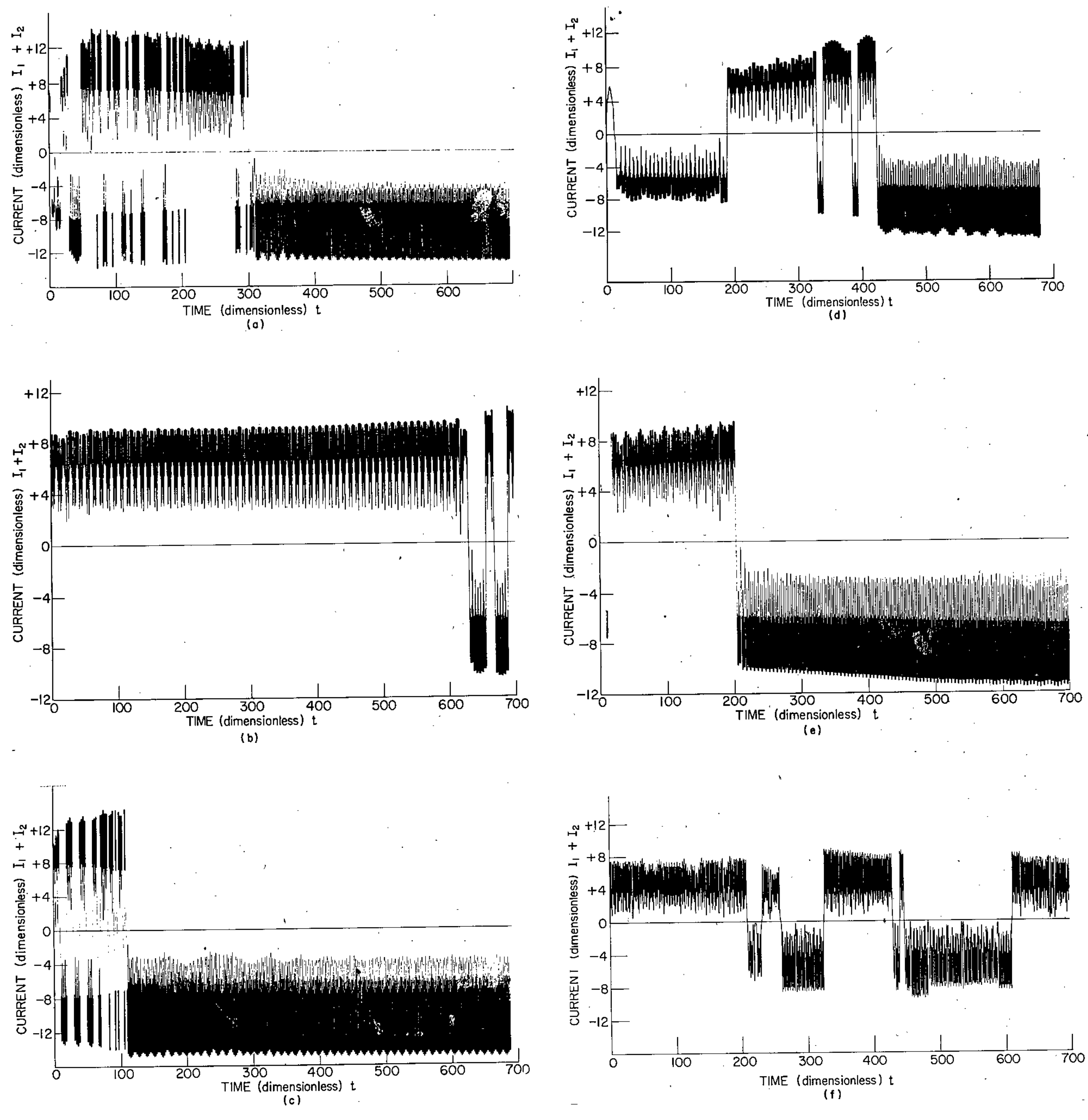

Fig. 12 - The current sum $I_1 + I_2$ for (a) $K = 0.0002$, $F = 0.0400$, and $Q = 0.4000$, with initial conditions: $I_1 = 6.6000$, $I_2 = 0.0040$, $\Omega_1 = 0.8000$, and $\Omega_2 = 0.8000$; (b) $K = 1.5 \times 10^{-5}$, $F = 0.0210$, and $Q = 0.4000$, with initial conditions: $I_1 = 6.1800$, $I_2 = 2.3800$, $\Omega_1 = 0.5000$, and $\Omega_2 = 1.9400$; (c) $K = 0.0001$, $F = 0.0400$, and $Q = 0.4000$, with initial conditions: $I_1 = 5.6000$, $I_2 = 4.4000$, $\Omega_1 = 3.4400$, and $\Omega_2 = 0.0$; (d) $K = 0.0001$, $F = 0.0200$, and $Q = 0.4000$, with initial conditions: $I_1 = 1.4000$, $I_2 = 0.7660$, $\Omega_1 = 1.8000$, and $\Omega_2 = 0.0$; (e) $K = 1 \times 10^{-5}$, $F = 0.0285$, and $Q = 0.5000$, with initial conditions: $I_1 = 0.7640$, $I_2 = 1.2000$, $\Omega_1 = 0.0040$, and $\Omega_2 = 1.6000$; and (f) $K = 1 \times 10^{-5}$, $F = 0.1050$, and $Q = 0.4000$, with initial conditions: $I_1 = 2.7880$, $I_2 = 4.2380$, $\Omega_1 = 1.8080$, and $\Omega_2 = 1.0440$.

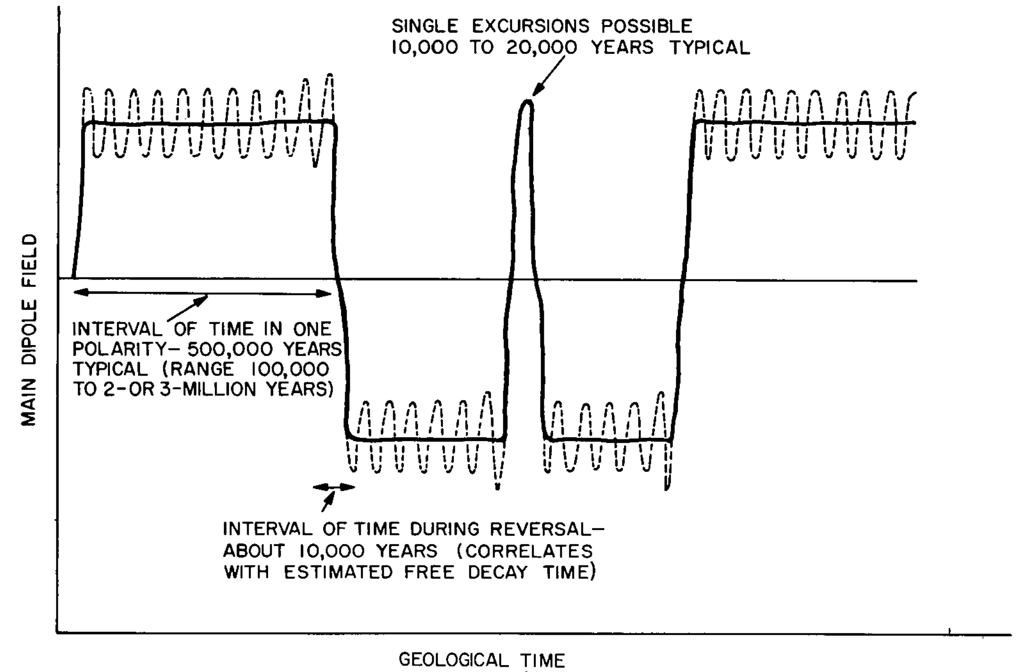

Fig. 13 - Idealized pictorial representation of important characteristics of earth's field reversals indicated from paleomagnetic studies

some interest *per se*. However, Backus (12) has found that even nonsteady axisymmetric dynamos are incapable of long-term support of external dipole moments. Obviously, the axial arrangement of the dynamo systems of Fig. 3 must not be taken too literally. The interactions must be construed in a highly schematic way, not confined to the configuration which suggested them.

Paleomagnetically interesting behavior of the modified equations was obtained from the K-form using relatively low values of K. Those solutions with very low values of K are probably the most realistic for representing the earth's core (see Ref. (7), p. 104). However, the dimensionless ripple frequencies of the second group of solutions are not greatly different from those obtained for the unmodified equations. If Eq. (31) were used, these very low K-values would cause the dimensional times of the modified solutions to be unreasonably small for interpretation of paleomagnetic behavior.

Probably further efforts should be made to reevaluate the parameters L, R, α, C, and G. Should significant revision result, a more careful search for interesting solution behavior for the new values would seem to be in order.

Despite the difficulties inherent in determining proper dimensional time scales, many of the solutions presented show remarkable similarity to the reversals of the earth's main dipole field. Clearly there is a choice of an extremely wide range of intervals in one polarity regardless of any final adjustment to the time scales. Acceptable ratios (10/1 to 100/1) of the interval in one polarity to the interval during reversal are found. Paleomagnetic evidence further suggests that single reversals or excursions of short duration into opposite polarity may have occurred. This feature is present in most of these solutions.

The use of an analog computer has resulted in a significant extension of disk-dynamo model solutions. In repeat runs using the same parameter values, exact reproducibility was not obtained beyond several ripples of the solutions. Presumably this is due to repeated approaches of the solutions to a mathematical singularity. Increased accuracy and reproducibility, resulting in a postponement of scatter, could be expected from a digital computer utilizing a large number of digits. But the analog method insures that the exact mathematical formulation does not mislead one into expecting unnatural physical exactness.

The simplification from the homogeneous dynamo, thought to exist in the earth's core, to disk-dynamo models is a drastic one, so an interpretation of the results of the simplified models must be cautious. However, the solutions presented in this report emphasize the capability of disk-dynamo system models to perform reversals satisfying paleomagnetic criteria. Thus the magnetofluiddynamic origin of reversals of the earth's field becomes more plausible.

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REFERENCES

1. Backus, G. E., and Chandrasekhar, S., "On Cowling's Theorem on the Impossibility of Self-Maintained Axisymmetric Homogeneous Dynamos," Proc. Natl. Acad. Sci. U. S. **42**:105-109 (1956)
2. Herzenberg, A., "Geomagnetic Dynamos," Phil. Trans. Roy. Soc. London **250**(A):543-583 (1958)
3. Cowling, T. G., "The Magnetic Field of Sunspots," Monthly Notices Roy. Astron. Soc. **94**:39-48 (1933)
4. Bullard, E. C., and Gellman, H., "Homogeneous Dynamos and Terrestrial Magnetism," Phil. Trans. Roy. Soc. London **247**(A):213-278 (1954)
5. Runcorn, S. K., "Core Motions and Reversals of the Geomagnetic Field," Ann. géophys. **11**:73-79 (1955)
6. Bullard, E. C., "The Stability of a Homopolar Dynamo," Proc. Cambridge Phil. Soc. **51**:744-760 (1955)
7. Rikitake, T., "Oscillations of a System of Disk Dynamos," Proc. Cambridge Phil. Soc. **54**:89-105 (1958)
8. Allan, D. W., "Reversals of the Earth's Magnetic Field," Nature **182**:469-470 (1958)
9. Hide, R., and Roberts, P. H., "The Origin of the Main Geomagnetic Field," Chapter 2 of "Physics and Chemistry of the Earth," **4**:27-98, New York:Pergamon Press, 1961
10. Doell, R. R., and Cox, A., "Paleomagnetism," in "Advances in Geophysics," **8**:221-313, New York:Academic Press, 1961
11. Cox, A., and Doell, R. R., "Review of Paleomagnetism," Bull. Geol. Soc. Am. **71**(No. 6):645-768 (1960)
12. Backus, G., "The Axisymmetric Self-Excited Fluid Dynamo," Astrophys. J. **125**:500-524 (1957)

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