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Analytic Formulas for the Acoustic Pressure Distribution Between a Spherical Transducer and a Concentric Spherical Baffle

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A sphere having arbitrary distribution of normal velocity over its surface and surrounded (partially) by a spherical baffle board radiates sound into infinite space. In the region between the radiating sphere and the baffle board the resultant sound field consists of a set of incoming and outgoing waves described by an appropriate set of spherical wave functions. These functions are adjusted to yield a normal component of particle velocity distribution which matches the velocity distribution at the surface of the sphere. In the region outside the baffle the sound field corresponds to outgoing waves only. At the junction of the inner and outer regions over those portions of arc where no baffle exists the condition of continuity requires that pressures and particle velocities agree as the boundary is crossed. This requirement of continuity equates the infinite series of spherical wave functions in the outer region to an infinite series of different spherical wave functions in the inner region. The result is an infinite set of simultaneous equations in an infinite set of unknown expansion constants. Further treatment of these constants varies with the acoustic properties of the baffle. Baffles are considered which have surfaces of either zero pressure or zero normal components of particle velocity and the corresponding sets of equations are derived. A brief discussion is also made of the case where the baffle has an acoustic impedance which is neither zero nor infinite.

INTRODUCTION

When practical generators of sound are installed on ships, some portion of their surfaces look directly into the adjacent ship hull. The proximity of the hull alters the conditions of radiation in complex ways. In this report the transducer-hull system is idealized to the state of a radiating sphere inside a concentric spherical baffle having a shape that may be described by tesseral harmonics (1). The problem is thus reduced to a simple boundary value problem in three dimensions, and the solution sought for is the distribution of pressure in the region between the transducer and the baffle, and in the region outside the entire system. A similar problem of finding the acoustic pressure distribution between a radiating cylinder and a concentric baffle has been studied in Ref. 2.

BOUNDARY VALUE PROBLEM AND ITS SOLUTION

In the region outside the entire system (called region II) the pressure distribution due to the generation of sound by the transducer may be written in the form of a series of spherical wave functions which satisfy the condition for outgoing waves. If the time variable is expressed in complex form by $e^{-i\omega t}$, the amplitude and phase of acoustic pressure is given in spherical coordinates (r, θ, ϕ) by the real part of

$$p_{II}(r,\theta,\varphi,t) = \sum_{m,n}^{\infty} [A_{mn} h_n^{(1)}(kr) Y_{C\,mn}(\theta,\phi) + B_{mn} h_n^{(1)}(kr) Y_{S\,mn}(\theta,\phi)] e^{-i\omega t} \qquad (1)$$

in which $h_n^{(1),(2)}(kr) = j_n(kr) \pm i n_n(kr)$ = the spherical Hankel function of the first and second kind,

$Y_{Cmn}(\theta, \phi) = \cos m\phi P_n^m(\cos\theta)$,
$Y_{Smn}(\theta, \phi) = \sin m\phi P_n^m(\cos\theta)$,
A_{mn}, B_{mn} are arbitrary constants to be determined by conditions at the inner boundary of region II, *i.e.*, at $r = b$.

At $r = b_+$ (*i.e.*, at r slightly greater than b) the amplitude of acoustic pressure is made discontinuous by the presence of the baffle. Initially the hull baffle will be considered acoustically soft,

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acoustically hard, or alternately soft and hard as in layers. Baffles having mechanical (complex) impedances with phase angles that are neither 0 nor 90 degrees will not be studied in the first part of this report. At $r = b_+$ then we suppose that

$$p_{II}(b_+, \theta, \phi) = 0, \quad \theta_1 < \theta < \theta_2, \text{etc.},$$
$$\phi_1 < \phi < \phi_2, \text{etc.},$$
$$\neq 0 \text{ elsewhere on "surface } r = b_+\text{."} \quad (2)$$

This is the condition of a pressure release baffle over portions of the surface $r = b_+$. We now desire to expand the *function* expressed by Eq. (2) into a Fourier-type spherical harmonic series, continuous everywhere, given by

$$p_{II}(b_+, \theta, \phi) = \sum_{m,n=0}^{\infty} L_{mn}(kb_+) Y_{cmn}(\theta, \phi) + \sum_{m,n=0}^{\infty} M_{mn}(kb_+) Y_{smn}(\theta, \phi). \quad (3a)$$

Multiplying the amplitudes of both sides of Eq. (3a) by $Y_{cpq} \sin\theta$ and integrating over θ from 0 to π and over ϕ from 0 to 2π we find that all terms L_{mn} vanish unless $p = m$ and $q = n$. For particular mn, therefore, we have

$$L_{mn} = \frac{1}{N_{cn}{}^m} \left\{ \sum_{\mu,\nu}^{\infty} A_{\mu\nu} h_\nu{}^{(1)}(kb_+) T_{cc}\begin{bmatrix}\mu, & m\\ \nu, & n\end{bmatrix} + \sum_{\sigma,\tau}^{\infty} B_{\sigma\tau} h_\tau{}^{(1)}(kb_+) T_{sc}\begin{bmatrix}\sigma, & m\\ \tau, & n\end{bmatrix}\right\} \quad (3b)$$

with

$$N_{\alpha n}{}^m = \int_0^{2\pi}\int_0^{\pi} (Y_{\alpha mn})^2 \sin u \, du dv, \quad \alpha = c \text{ or } s \quad (3c)$$

and

$$T_{\alpha\beta}\begin{bmatrix}\gamma, & \epsilon\\ \delta, & \eta\end{bmatrix} \equiv \int_v \int_u H[1] Y_{\alpha\gamma\delta} Y_{\beta\epsilon\eta} \sin u \, du dv \quad (3d)$$

where H [1] is the unit function on the surface $r = r_0$, *i.e.*,

$$H[1] = 0, \quad u, < u < u_2, \text{etc.}, v_1 < v < v_2, \text{etc.},$$
$$= 1, \text{ elsewhere on the surface } r = r_0.$$

Note that u, v correspond to θ, ϕ respectively. By application of similar methods to the odd spherical harmonics we find

$$M_{mn} = \frac{1}{N_{sn}{}^m} \left\{ \sum_{\mu,\nu}^{\infty} A_{\mu\nu} h_\nu{}^{(1)}(kb_+) T_{sc}\begin{bmatrix}\mu, & m\\ \nu, & n\end{bmatrix} + \sum_{\sigma,\tau}^{\infty} B_{\sigma\tau} h_\tau{}^{(1)}(kb_+) T_{ss}\begin{bmatrix}\sigma, & m\\ \tau, & \nu\end{bmatrix}\right\}. \quad (3e)$$

The pressure in region II which meets a pressure discontinuity requirement at $r = b_+$ thus becomes

$$p_{II}(b_+, \theta, \phi, t) = \sum_{m,n}^{\infty} e^{-i\omega t} \left\{ \frac{Y_{cmn}(\theta, \phi)}{N_{cn}{}^m} \left[\sum_{\mu,\nu}^{\infty} A_{\mu\nu} h_\nu{}^{(1)}(kb_+) T_{cc}\begin{bmatrix}\mu, & m\\ \nu, & n\end{bmatrix} + \sum_{\sigma,\tau}^{\infty} B_{\sigma\tau} h_\tau{}^{(1)}(kb_+) T_{sc}\begin{bmatrix}\sigma, & m\\ \tau, & n\end{bmatrix}\right] + \frac{Y_{smn}(\theta, \phi)}{N_{sn}{}^m} \left[\sum_{\mu,\nu}^{\infty} B_{\mu\nu} h_\nu{}^{(1)}(kb_+) T_{ss}\begin{bmatrix}\mu, & m\\ \nu, & n\end{bmatrix} + \sum_{\sigma,\tau}^{\infty} A_{\sigma,\tau} h_\tau{}^{(1)}(kb_+) T_{sc}\begin{bmatrix}\sigma, & m\\ \tau, & n\end{bmatrix}\right]\right\}. \quad (4)$$

This formula is valid for all θ *and all* ϕ *at* $r = b_+$.

The normal component of particle velocity in region *II* ($= V_{II}$) is deduced directly to be

$$V_{II}(r, \theta, \phi) = \frac{1}{ik\rho c} \frac{\partial p_{II}}{\partial r} \quad (5a)$$

$$= \frac{1}{ik\rho c} \left[\sum_{m,n} A_{mn} h_n{}^{(1)\prime}(kr) Y_{cmn} + B_{mn} h_n{}^{(1)\prime}(kr) Y_{smn} \right] \quad (5b)$$

where the prime mark indicates the ordinary derivative with respect to r.

In region I between the baffle and the transducer the acoustic pressure satisfies the conditions of standing waves. We can write

$$p_I(r, \theta, \phi, t) = e^{-i\omega t} \sum_{m,n} [C_{mn} h_n^{(1)}(kr) + D_{mn} h_n^{(2)}(kr)] Y_{cmn}(\theta, \phi) + e^{-i\omega t} \sum_{m,n} [E_{mn} h_n^{(1)}(kr) + F_{mn} h_n^{(2)}(kr)] Y_{smn}(\theta, \phi). \tag{6a}$$

The normal component of particle velocity in region I thus becomes

$$V_I(v, \theta, \phi, t) = \frac{e^{-i\omega t}}{ik\rho c} \sum_{m,n} [C_{mn} h_n^{(1)\prime}(kr) + D_{mn} h_n^{(2)\prime}(kr)] Y_{cmn}(\theta, \phi) + \frac{e^{-i\omega t}}{ik\rho c} \sum_{m,n} [E_{mn} h_n^{(1)\prime}(kr) + F_{mn} h_n^{(2)\prime}(kr)] Y_{smn}(\theta, \phi). \tag{6b}$$

At the surface of the transducer (*i.e.*, at $r = a$) we specify a complex velocity such that

$$V_1(a, \theta, \phi, t) = U(\theta, \phi) e^{-i\omega t}. \tag{7a}$$

The velocity $U(\theta, \phi)$ can be discontinuous. Using the orthogonality conditions of Y_{smn} and Y_{cmn} once more we see that

$$D_{mn} h_n^{(2)\prime}(ka) = -C_{mn} h_n^{(1)\prime}(ka) + \frac{U_{cn}{}^m}{N_{cn}{}^m} ik\rho c \tag{7b}$$

where

$$U_{pn}{}^m = \int_0^{2\pi} \int_0^{\pi} U(u, v) Y_{pmn}(u, v) \sin u \, du dv, \quad p = \cos s. \tag{7c}$$

The pressure and particle velocity in region I therefore reduce to the following formulas:

$$p_I(r, \theta, \phi, t) = \sum_{m,n}^{\infty} \left\{ C_{mn} h_n^{(1)}(kr) + \left[\frac{\frac{U_{cn}{}^m}{N_{cn}{}^m} ik\rho c - C_{mn} h_n^{(1)\prime}(ka)}{h_n^{(2)\prime}(ka)} h_n^{(2)}(kr) \right] \right\} \times Y_{cmn}(\theta, \phi) e^{-i\omega t} + \sum_{m,n}^{\infty} \left\{ E_{mn} h_n^{(1)}(kr) + \left[\frac{\frac{U_{sn}{}^m}{N_{sn}{}^m} ik\rho c - E_{mn} h_n^{(1)\prime}(ka)}{h_n^{(2)\prime}(ka)} \right] h_n^{(2)}(kr) \right\} \times Y_{smn}(\theta, \phi) e^{-i\omega t} \tag{8a}$$

and

$$V_1(v, \theta, \phi) = \frac{1}{ik\rho c} \frac{\partial p_I}{\partial r}. \tag{8b}$$

When $r = b_-$ (*i.e.*, at r slightly less than b) we require that the velocity $V_1(b_-, \theta, \phi)$ have a discontinuity such that

$$V_1 \ (b_-, \theta, \phi) = 0, \theta_1 < \theta < \theta_2, \text{etc.}, \phi_1 < \phi < \phi_2, \text{etc.}, \neq 0, \text{elsewhere on the surface } r = b_-. \tag{9a}$$

Note that the pressure discontinuity at $r = b_+$ and the velocity discontinuity at $r = b_-$ coincide in angular distribution.

As in the earlier case of pressure $p_{II}(b_+, \theta, \phi)$ we desire to expand the *function* expressed by Eq. (9a) into a Fourier-type spherical harmonic series which is continuous everywhere. We write

$$V_1(b_-, \theta, \phi) = \sum_{m,n}^{\infty} R_{mn}(kb_-) Y_{cmn}(\theta, \phi) + \sum_{m,n}^{\infty} S_{mn}(kb_-) Y_{smn}(\theta, \phi).$$

Application of the procedures utilized above finally lead up to the equation

$$V_I(b_-,\theta,\phi)=\frac{1}{ik\rho_C}\sum_{m,n}\frac{Y_{Cmn}}{N_{Cn}{}^m}\Bigg(\Bigg\{\sum_{\mu,\nu}C_{\mu\nu}h_\nu{}^{(1)\prime}(kb_-)$$

$$+\left[\frac{\dfrac{U_{C\nu}{}^\mu}{N_{C\nu}{}^\mu}ik\rho\;-C_{\mu\nu}h_\nu{}^{(1)\prime}(ka)}{h_\nu{}^{(2)\prime}(ka)}\right]h_\nu{}^{(2)\prime}(kb_-)\Bigg\}T_{CC}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}$$

$$+\Bigg\{\sum_{\sigma,\tau}E_{\sigma\tau}h_\tau{}^{(1)\prime}(kb_-)$$

$$+\left[\frac{\dfrac{U_{S\tau}{}^\sigma}{N_{S\tau}{}^\sigma}k\rho_C-E_{\sigma\tau}h_\tau{}^{(1)\prime}(ka)}{h_\tau{}^{(2)\prime}(ka)}\right]h_\tau{}^{(2)\prime}(kb_-)\Bigg\}$$

$$\times T_{SC}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}\Bigg\}\Bigg)$$

$$+\frac{1}{ik\rho_C}\sum_{m,n}\frac{Y_{Smn}}{N_{Sn}{}^m}\Bigg(\Bigg\{\sum_{\mu,\nu}C_{\mu\nu}h_\nu{}^{(1)\prime}(kb_-)$$

$$+\left[\frac{\dfrac{U_{C\nu}{}^\mu ik\rho_C}{N_{C\nu}{}^\mu}-C_{\mu\nu}h_\nu{}^{(1)\prime}(ka)}{h_\nu{}^{(2)\prime}(ka)}\right]h_\nu{}^{(2)\prime}(kb_-)\Bigg\}$$

$$\times T_{SC}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}$$

$$+\Bigg\{\sum_{\sigma,\tau}E_{\sigma\tau}h_\tau{}^{(1)\prime}(kb_-)$$

$$+\left[\frac{\dfrac{U_{S\tau}{}^\sigma ik\rho_C}{N_{S\tau}{}^\sigma}-E_{\sigma\tau}h_\tau{}^{(1)\prime}(ka)}{h_\tau{}^{(2)\prime}(ka)}\right]h_\tau{}^{(2)\prime}(kb_-)\Bigg\}$$

$$\times T_{SS}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}\Bigg\}\Bigg). \tag{9b}$$

We now let $b_+ \to b_- \to b$; *i.e.*, the baffle is considered a mathematical surface. At the boundary $r = b$ between regions I and II there must be continuity of acoustic pressure and particle velocity in the unbaffled region. We require therefore that the following boundary conditions be satisfied:

$$\left.\begin{aligned} p_I(b,\theta,\phi)&=p_{II}(b,\theta,\phi)\\ V_I(b,\theta,\phi)&=V_{II}(b,\theta,\phi)\end{aligned}\right\}\quad\text{in the unbaffled region.}\qquad \begin{aligned}&(10a)\\&(10b)\end{aligned}$$

It is convenient at this point to define the following entities:

$$L_\lambda(kr)=h_\lambda{}^{(1)}(kr)-\frac{h_\lambda{}^{(1)\prime}(ka)}{h_\lambda{}^{(2)\prime}(ka)}h_\lambda{}^{(2)}(kr)$$

$$L_\lambda{}'(kr)=\frac{dL_\lambda(kr)}{dr}$$

$$\mathsf{C}_{\kappa\lambda}(kr)=C_{\kappa\lambda}L_\lambda(kr)$$

$$\mathsf{C}_{\kappa\lambda}{}'(kr)=C_{\kappa\lambda}L_\lambda{}'(kr)$$

$$\mathsf{E}_{\kappa\lambda}(kr)=E_{\kappa\lambda}L_\lambda(kr)$$

$$\mathsf{E}_{\kappa\lambda}{}'(kr)=E_{\kappa\lambda}L_\lambda{}'(kr)$$

$$\mathsf{U}_{\xi\lambda}{}^\kappa(kr)=\frac{U_{\xi\lambda}{}^\kappa}{N_{\xi\lambda}{}^\kappa}(ik\rho_C)\frac{h_\lambda{}^{(2)}(kr)}{h_\lambda{}^{(2)\prime}(ka)}$$

$$\mathsf{U}_{\xi\lambda}{}^\kappa(kr)=\frac{d\mathsf{U}_{\xi\lambda}{}^\kappa(kr)}{dr}$$

$$\mathsf{A}_{\kappa\lambda}(kr)=A_{\kappa\lambda}h_\lambda{}^{(1)}(kr),\ \mathsf{B}_{\kappa\lambda}(kr)=B_{\kappa\lambda}h_\lambda{}^{(1)}(kr)$$

$$\mathsf{A}_{\kappa\lambda}{}'(kr)=\frac{d\mathsf{A}_{\kappa\lambda}}{dr},\ \mathsf{B}_{\kappa\lambda}{}'(kr)=\frac{d\mathsf{B}_{\kappa\lambda}}{dr}.$$

The pressure requirement may be fulfilled by extracting from Eqs. (4) and (8a) all entities corresponding to particular Y_{Cmn} and Y_{Smn}. We thus obtain for the coefficients of Y_{Cmn}

$$\mathsf{C}_{mn}(kb)=\frac{1}{N_{Cn}{}^m}\Bigg\{\sum_{\mu,\nu}\mathsf{A}_{\mu\nu}T_{CC}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}$$

$$+\sum_{\sigma,\tau}\mathsf{B}_{\sigma\tau}T_{SC}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}\Bigg\}-\mathsf{U}_{Cn}{}^m(kb). \tag{11a}$$

For the coefficients of Y_{smn} we obtain

$$\mathsf{E}_{mn}(kb)=\frac{1}{N_{sn}{}^{m}}\left\{\sum_{\mu,\nu}\mathsf{B}_{\mu\nu}T_{ss}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}\right.$$

$$\left.+\sum_{\sigma,\tau}\mathsf{A}_{\sigma\tau}T_{sc}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}\right\}-\mathsf{U}_{sn}{}^{m}(kb). \qquad (11b)$$

The particle velocity requirement may be fulfilled by extracting from Eqs. (5b) and (9b) all similar corresponding entities. We thus obtain for the coefficients of Y_{cmn},

$$\mathsf{A}_{mn}'(kb)=\sum_{\mu,\nu}\left[\mathsf{C}_{\mu\nu}'(kb)+\mathsf{U}_{c\nu}{}^{\mu\prime}(kb)\right]\frac{T_{cc}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}}{N_{cn}{}^{m}}$$

$$+\sum_{\sigma,\tau}\left[\mathsf{E}_{\sigma\tau}'(kb)+\mathsf{U}_{s\tau}{}^{\sigma}(kb)\right]\frac{T_{sc}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}}{N_{cn}{}^{m}}. \qquad (12a)$$

For the coefficients of Y_{smn} we obtain

$$\mathsf{B}_{mn}'(kb)=\sum_{\mu,\nu}\left(\mathsf{C}_{\mu\nu}'(kb)+\mathsf{U}_{c\nu}{}^{\mu\prime}(kb)\right)\frac{T_{sc}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}}{N_{sn}{}^{m}}$$

$$+\sum_{\sigma,\tau}\left(\mathsf{E}_{\sigma\tau}'(kb)+\mathsf{U}_{s\tau}{}'^{\sigma}(kb)\right)\frac{T_{ss}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}}{N_{sn}{}^{m}}. \qquad (12b)$$

The set of four equations (11a), (11b), (12a), and (12b) are simultaneous in the four unknowns A_{mn}, B_{mn}, C_{mn}, and E_{mn}. In any physical application this set has a unique solution. Hence the pressure distribution in regions *I* and *II* can be obtained.

PARTICULAR CASES

A general solution of Eqs. (11a), (11b), (12a), and (12b) seems impractical. The first simplification that may be made in these equations is to set $m=0$. The baffle will then be rotationally symmetrical. Figures 1a and 1b shows sketches of types of baffles in which $m\neq 0$ and $m=0$ respectively. We will consider in this report only those cases for which $m=0$. A first consequence of this selection is that $B_{mn}=E_{mn}=T_{sc}=T_{ss}=Y_{smn}=0$. Hence the set of equations that require solution reduces to

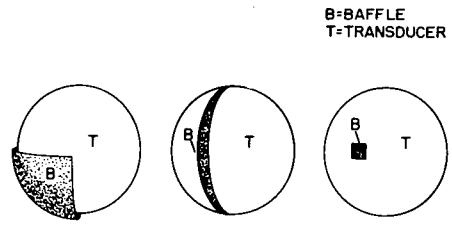

Fig. 1a—Baffles for which $m\neq 0$

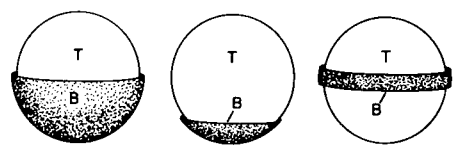

Fig. 1b—Baffles for which $m=0$

$$\mathsf{C}_{0n}(kb)=\sum_{\nu}A_{0\nu}\frac{T_{cc}}{N_{cn}{}^{0}}\begin{bmatrix}0, 0\\ \nu, n\end{bmatrix}-\mathsf{U}_{cn}{}^{0}(kb) \qquad (13a)$$

and

$$\mathsf{A}_{0n}'(kb)=\sum_{\nu}\left[\mathsf{C}_{0\nu}'(kb)+\mathsf{U}_{c\nu}{}'^{0}(kb)\right]\frac{T_{cc}\begin{bmatrix}0, 0\\ \nu, n\end{bmatrix}}{N_{cn}{}^{0}}. \qquad (13b)$$

Substituting Eq. (13b) into (13a) we obtain

$$C_{0n}(kb)=\left\{\sum_{\nu}\frac{h_{\nu}{}^{(1)}(kb)}{h_{\nu}{}^{(1)\prime}(kb)}\sum_{q}\left[C_{0q}L_{q}'(kb)\right.\right.$$

$$\left.+\mathsf{U}_{cq}{}'^{0}(kb)\right]\frac{T_{cc}\begin{bmatrix}0, 0\\ q, \nu\end{bmatrix}T_{cc}\begin{bmatrix}0, 0\\ \nu, n\end{bmatrix}}{N_{c\nu}{}^{0}N_{cn}{}^{0}}$$

$$\left.-\mathsf{U}_{cn}{}^{0}(kb)\right\}\Big/L_{n}(kb). \qquad (14)$$

Further manipulation yields

$$C_{0n} - \sum_{q=0}^{\infty} C_{0q}\,\alpha_{qn} = \beta_n, \qquad n = 0, 1, 2, \ldots, \infty \tag{15a}$$

in which

$$\alpha_{qn} = \frac{\left[h_q^{(1)\prime}(kb) - \dfrac{h_q^{(1)\prime}(ka)}{h_q^{(2)\prime}(ka)}\,h_q^{(2)\prime}(kb)\right]\displaystyle\sum_{\nu=0}^{\infty}\frac{h_\nu^{(1)}(kb)}{h_\nu^{(1)\prime}(kb)}\,\frac{T_{cc}\begin{bmatrix}0,0\\ \nu,n\end{bmatrix}T_{cc}\begin{bmatrix}0,0\\ q,\nu\end{bmatrix}}{N_{c^n}{}^0 N_{c^\nu}{}^0}}{\left[h_n^{(1)}(kb) - \dfrac{h_n^{(1)\prime}(ka)}{h_n^{(2)\prime}(ka)}\,h_n^{(2)}(kb)\right]} \tag{15b}$$

$$\beta_n = \frac{\displaystyle\sum_{r=0}^{\infty}\frac{U_{c^r}{}^0 h_r^{(2)\prime}(kb)\,ik\rho_c}{N_{c^r}{}^0 h_r^{(2)\prime}(ka)}\sum_{\nu=0}^{\infty}\frac{h_\nu^{(1)}(kb)}{h_\nu^{(1)\prime}(kb)}\,\frac{T_{cc}\begin{bmatrix}0,0\\ \nu,n\end{bmatrix}T_{cc}\begin{bmatrix}0,0\\ r,\nu\end{bmatrix}}{N_{c^n}{}^0 N_{c^\nu}{}^0} - \dfrac{U_{c^n}{}^0 h_n^{(2)}(kb)\,ik\rho_c}{N_{c^n}{}^0 h_n^{(2)\prime}(ka)}}{\left[h_n^{(1)}(kb) - \dfrac{h_n^{(1)\prime}(ka)}{h_n^{(2)\prime}(ka)}\,h_n^{(2)}(kb)\right]}. \tag{15c}$$

Equations (15) in the unknowns C_{0n} are infinite in number and complex. A practical procedure for evaluating these unknowns is to consider finite number of equations ranging from subscript $n = a$ to subscript $n = b$, these limits depending in turn upon the values of q which make the largest contributions to the coefficients α_{qn}. To obtain the first n coefficients ($= C_{0n}$) it is clear that some number of coefficients N greater than n must be solved for. By increasing N in steps one can determine the successive values of the first n coefficients and test these for convergence.

The real and imaginary parts of the first n coefficients (C_{0n}) of Eq. (15a) are solved separately. From these are obtained the complex coefficients A_{0n} by means of Eq. (13b). When these coefficients are obtained they are substituted into the infinite series given by Eqs. (5) and (8a) to yield the pressure distribution in regions II and I respectively. For either regions I or II let Re_n be the real part of the nth term in the series, and Im_n be the imaginary part. Then the magnitude of pressure $|p|$ is

$$|p| = \left[\left(\sum_n Re_n Y_{c^{0n}}\right)^2 + \left(\sum_n Im_n Y_{c^{0n}}\right)^2\right]^{1/2}. \tag{16a}$$

The phase Ω of the pressure field will be given by

$$\tan\Omega = \frac{\sum_n Im_n Y_{c^{0n}}}{\sum_n Re_n Y_{c^{0n}}}. \tag{16b}$$

The complex pressure P at any point will then be

$$P = |p|\,e^{i(\Omega - \omega t)}.$$

In region II when the field point is in the far field we may write

$$A_{0n} = |A_{0n}|e^{i\alpha_{0n}}$$

$$h_n^{(1)}(kr) \to e^{i[kr - \pi(n+1)/2]}$$

$$p_{II} \to \sum_n \frac{|A_{0n}|Y_{c^{0n}}e^{i\lambda_n}e^{-i\omega t}}{kr} \tag{17a}$$

$$\lambda_n = kr - \pi(n+1)/2 + \alpha_{0n}. \tag{17b}$$

The far field pressure distribution $P(\phi, \theta)$ thus reduces to

$$P(\phi,\theta) = \sqrt{\left(\sum_n \frac{A_{0n}Y_{c^{0n}}\cos\lambda_n}{kr}\right)^2 + \left(\sum_n \frac{A_{0n}Y_{c^{0n}}\sin\lambda_n}{kr}\right)^2}. \tag{17c}$$

ALTERNATIVE BOUNDARY CONDITIONS

For the case of a spherical baffle which is acoustically hard on the outer boundary the boundary requirements that the acoustic pressure and particle velocity be continuous across that portion of the boundary where there is no baffle lead to the following four simultaneous equations:

$$\mathsf{A}_{mn}(kb)=\sum_{\mu,\nu}[\mathsf{C}_{\mu\nu}(kb)+\mathsf{U}_{C\nu}{}^{\mu}]\frac{T_{cc}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}}{N_{Cn}{}^{m}}+\sum_{\sigma,\tau}[\mathsf{E}_{\sigma\tau}+\mathsf{U}_{S\tau}{}^{\sigma}]\frac{T_{sc}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}}{N_{Cn}{}^{m}} \tag{18a}$$

$$\mathsf{B}_{mn}(kb)=\sum_{\mu,\nu}[\mathsf{C}_{\mu\nu}(kb)+\mathsf{U}_{C\nu}{}^{\mu}]\frac{T_{sc}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}}{N_{Sn}{}^{m}}+\sum_{\sigma,\tau}[\mathsf{E}_{\sigma\tau}(kb)+\mathsf{U}_{S\tau}{}^{\sigma}(kb)]\frac{T_{ss}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}}{N_{Sn}{}^{m}} \tag{18b}$$

$$\mathsf{C}_{mn}'(kb)=\sum_{\mu,\nu}\mathsf{A}_{\mu\nu}'\frac{T_{cc}}{N_{Cn}{}^{m}}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}+\sum_{\sigma,\tau}\mathsf{B}_{\sigma\tau}'\frac{T_{sc}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}}{N_{Cn}{}^{m}}-\mathsf{U}_{Cn}{}^{m\prime}(kb) \tag{18c}$$

$$\mathsf{E}_{mn}'(kb)=\sum_{\sigma,\tau}\mathsf{A}_{\sigma\tau}'\frac{T_{cs}\begin{bmatrix}\sigma, m\\ \tau, n\end{bmatrix}}{N_{Sn}{}^{m}}+\sum_{\mu,\nu}\mathsf{B}_{\mu\nu}'(kb)\frac{T_{ss}\begin{bmatrix}\mu, m\\ \nu, n\end{bmatrix}}{N_{Sn}{}^{m}}-\mathsf{U}_{Sn}{}^{m\prime}(kb). \tag{18d}$$

For pressure fields which have no variation of magnitude or phase in the coordinate ϕ (*i.e.*, rotationally symmetric fields) we can set $m = 0$, and reduce the above set of four equations to the following two equations:

$$\mathsf{A}_{0n}(kb)=\sum_{\nu}\left[\mathsf{C}_{0m}(kb)+\mathsf{U}_{C\nu}{}^{0}(kb)\right]\frac{T_{cc}\begin{bmatrix}0, 0\\ \nu, n\end{bmatrix}}{N_{Cn}{}^{0}} \tag{19a}$$

$$\mathsf{C}_{0n}'(kb)=\sum_{\nu}A_{0\nu}'\frac{T_{cc}}{N_{Cn}{}^{0}}\begin{bmatrix}0, 0\\ \nu, n\end{bmatrix}-\mathsf{U}_{Cn}{}^{0\prime}(kb). \tag{19b}$$

Upon substituting Eq. (19a) in Eq. (19b) we obtain a formula for the constants C_{0n}:

$$C_{0n}(kb)=\left\{\sum_{\nu}\frac{h_{\nu}{}^{(1)\prime}(kb)}{h_{\nu}{}^{(1)}(kb)}\sum_{q}\left[C_{0q}L_q(kb)+\mathsf{U}_{Cq}{}^{0}(kb)\right]\frac{T_{cc}\begin{bmatrix}0, 0\\ q, \nu\end{bmatrix}T_{cc}\begin{bmatrix}0, 0\\ \nu, n\end{bmatrix}}{N_{C\nu}{}^{0}N_{Cn}{}^{0}}-\mathsf{U}_{Cn}{}^{0\prime}(kb)\right\}\Big/L_n'(kb). \tag{20}$$

This formula should be compared to the solution given by Eq. (14) in which the roles of pressure and particle velocity are interchanged.

BAFFLES WITH FINITE ACOUSTIC IMPEDANCE

Baffles with finite acoustic impedances were excluded from the first part of this report. Boundary value problems involving these baffles are computable, in principle, by the same methods which were applied in detail above (2). For example we may suppose that a specific pressure distribution P_x is fixed over the baffle in region I, and a specific velocity distribution V_x is fixed over the baffle in region *II*. In place of Eqs. (10a) and (10b) we may write the following boundary conditions for the inside and outside of the mathematical surface $r = b$:

$$p_I(b, \theta, \phi) = p_{II}(b, \theta, \phi) + P_x(b, \theta, \phi) \tag{21a}$$

$$V_{II}(b, \theta, \phi) = V_I(b, \theta, \phi) + V_x(b, \vartheta, \phi). \tag{21b}$$

In these equations we have applied the principle of superposition. The pressure $p_{II}(b, \theta, \phi)$ corresponds to a finite pressure in region *II* at $r = b$ for the boundary where no baffle exists. Elsewhere on the surface $r = b$ the value of p_{II} is zero. Now we can specify $P_x(b, \theta, \phi)$ to have a value zero in region *I* at $r = b$ for the boundary where no baffle exists (*i.e.*, just at the boundary where p_{II} is not

zero) and to have a fixed distribution elsewhere, *i.e.*, over the baffle.

The analytic representation of P_x which is conformable to the problem may be derived by expanding P_x in a Fourier-type series of tesseral harmonics over the sphere $r = b$. Now the pressure P_x may be expressed as the product of the velocity $V_I(b, \theta, \phi)$ and the specific acoustic impedance pc. In this event the expansion of P_x will involve the same expansion constants as V_I. If however $P_x(b, \theta, \phi)$ is known and fixed, the expansion constants can be computed, and the total expression $P_x(b, \theta, \phi)$ will enter Eq. (21a) as a constant. Similar remarks can be deduced for $V_x(b, \theta, \phi)$ in Eq. (21b).

NOTES ON THE NUMERICAL COMPUTATION OF EQ. (15a)

From the experience of other authors (3) it seems likely that the inversion of a 100 by 100 matrix will be required to achieve reasonable convergence for the solutions of Eq. (15a). Using moderate-speed electronic digital computers we estimate that approximately 6 hours will be required to find a reasonably complete pressure distribution in the region between the spherical transducer and the rotationally symmetric concentric baffle. For the case of unsymmetric baffles (*i.e.*, $m \neq 0$) the time required for solution of n modes will increase at least $(2n + 1)$-fold. It thus appears that the calculation of all but the simplest sphere-and-baffle combination is not feasible at the present time.

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