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ANALYSIS OF THE AERODYNAMIC ABLATION OF A METAL SPHERE

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ABSTRACT

An approximate analytic solution has been developed for the ablated volume and shape of a metal sphere at hypersonic speeds in the early stages of aerodynamic ablation. The process of ablation, due to aerodynamic heating, is a very complicated phenomenon because of the possible existence of two layers - gas and liquid. If the metal surface is molten, then a liquid layer of molten metal is formed between the boundary layer of gas and the solid surface. The interdependent factors flight time, flight conditions, and body shape cause a variation of aerodynamic characteristics as a function of the trajectory. Furthermore, the aerodynamic ablation is responsible for a change in the shape of the body. Two cases of aerodynamic ablation were investigated. The first case considered the melting of body material alone, and the second considered melting with partial evaporation.

The analysis given in this report is useful to determine the flight conditions under which the metal sphere may travel with little or no ablation. With the assumption that the flow over the sphere is laminar, Newton's impact theory is applicable and the gas does not dissociate; the total ablated volume and shape of the sphere can be computed as a function of flight trajectory by using the analysis developed in this report.

PROBLEM STATUS

This is a final report on this phase of the project; work on this problem is continuing.

AUTHORIZATION

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ANALYSIS OF THE AERODYNAMIC ABLATION OF A METAL SPHERE

INTRODUCTION

The natural phenomenon of ablation can be observed at night when meteoroids burn because of aerodynamic heating and leave luminous trails. The problem of aerodynamic ablation is an important scientific and technical subject of many aerospace programs.

During hypersonic flight in an atmospheric environment, excessive heat is generated. Structures intended for high-speed flight must be protected from this intense heating. Ablation is a heat- and mass-transfer phenomenon, involving a physical and chemical process which protects the structure from disintegration by effective removal of heat from the body surface. When the wall temperature of a metallic body reaches the melting point, the surface material begins to melt, and a liquid layer is formed. The melt is swept downstream by aerodynamic pressure and shear forces. If the temperature of the surface increases further, the surface material evaporates and the metallic gas penetrates into the surrounding gas layer, reducing the heat flux into the solid body.

A survey of available references on aerodynamic ablation shows that the problem of ablation has been treated for the most part analytically, since experimental results are scarce. Goodman (1) developed an analysis of the liquid layer which takes into account the heat penetration into the solid body. Bethe and Adams (2) and Hidalgo (3) analyzed the ablation of glassy materials, and Scala (4), Roberts (5), and Goodman (6) solved the stagnation point ablation problem. Goodman (7) found a particular solution for the ablation of a flat plate with a constant heat flux, and Roberts (8) formulated an engineering solution for the sublimation of a blunt body.

A sphere is widely used for aerodynamic experiments, and its blunted nose absorbs a high heat flux. Some experimental data on ablation of metal spheres are available at NRL. A theoretical analysis of the ablation of a metal sphere is desired. Depending upon flight conditions, three cases of ablation may occur: ablation due to melting only, ablation due to melting and partial evaporation, and ablation due to sublimation only. Roberts (8) solved the sublimation-only case with an engineering solution.

After examination of the available analytical and experimental references on the problem, an attempt is made in this report to predict the total volume ablated and the new shape of the sphere, due to melting only, and due to melting and partial evaporation. The solution presented is limited to the early stages of ablation of a metal sphere. For the analysis, it is assumed that the gas and liquid flow are laminar, and that the gas flow is Newtonian. Although, because of the complicated process of ablation which depends on the characteristics of flow, time, and the shape of the body, only the initial condition of ablation is analyzed, the developed analytical method is useful to designers and research scientists because the flight conditions for zero ablation can be determined by this method. For example, the ambient pressure for flight with zero ablation or the time at which ablation starts can be computed. For such calculations the velocity and trajectory are assumed to be known as a function of time. Therefore, a steady-flow analysis can be

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carried out, because for small time intervals, the characteristics of flow and the shape of the body may be assumed independent of time. It is further assumed that the gas is nondissociative. The reasons for this assumption are as follows: Dissociation occurs if the temperature is high and high-energy molecules collide. For molecules of oxygen and nitrogen, of which air consists, the dissociation occurs for oxygen at 3000°K and nitrogen at 5000°K, when these molecules collide with others of the lowest minimum relative kinetic energy $\bar{D}$ (9). The values of $\bar{D}/kT$ for these elements are $59{,}300/T$ for oxygen and $113{,}100/T$ for nitrogen, where k is Boltzmann's constant (1.38×10^{-16} erg/molecule-°K). However, molecules with such high relative kinetic energies are rare, therefore, in order for dissociation to occur, a considerable time is required. This is especially true for nitrogen (9). This report deals with short-time ablation in the order of milliseconds; therefore, dissociation of the gas has been neglected. This analysis is applicable for very thin liquid layers in comparison to the radius of the sphere and in the range of Mach numbers $M_\infty > 6$, where M_∞ refers to the free stream in front of the shock wave.

THE EQUATION OF THE LIQUID LAYER

From Fig. 1, it is seen that the volume of ablated material depends on the distance h. However, because $h = s - \zeta$, in order to determine h, it is sufficient to compute s and ζ. Therefore, for the determination of the thickness of the liquid layer ζ, the equation of the liquid layer has to be derived.

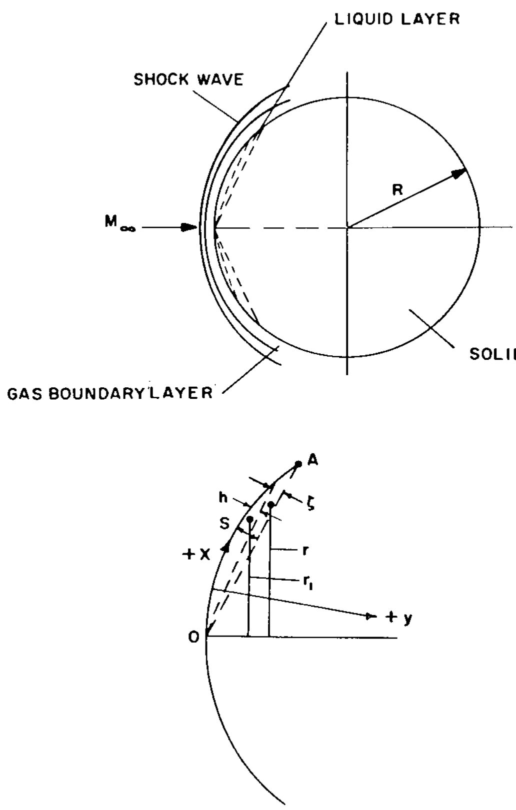

Fig. 1 - Sphere in hypersonic flight

When melting occurs, a metallic body has a definite surface of demarcation between the metal liquid and the solid, in contrast to a glassy heat-shield material which possesses no distinct solid state. The analytic solution for the total volume of material ablated and the final shape of the sphere will be presented for steady-state ablation and laminar flow.

The equations governing the liquid layer around a sphere are essentially the same as those for incompressible fluid flow around a body of revolution. Initially ablation takes place as Fig. 1 indicates, where the applicable coordinate system is also shown. The coordinate x along the surface of the sphere is measured from the stagnation point, and the coordinate y is perpendicular to x. At the early stage of ablation, the loss of mass is small and the small arc length x may be replaced by a straight segment on the gas/liquid interface. For the computation the coordinate, y is assumed to be parallel to h and s.

For steady-state incompressible flow, the governing equations are

$$\text{Continuity:} \quad \frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0$$

$$\text{Energy:}\quad \rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} - I\right)$$

$$\text{Momentum:}\quad \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$$

where I is the radiation heat flux.* Roberts (10) found that for a ballistic vehicle the heat lost by radiation is negligible compared with the heat disposed of by ablation. For simplicity, it is assumed that the dynamic viscosity μ is constant. Because the viscosity of the liquid layer is high and the inertia forces of the same layer are negligible, the terms of $\rho u(\partial u/\partial x)$ and $\rho v(\partial u/\partial y)$ in the momentum equation as well as $u(\partial p/\partial x)$ and $\mu(\partial u/\partial y)^2$ in the energy equation are neglected. The gradient of temperature $\partial T/\partial x$ is small compared to $\partial T/\partial y$, and since u and v are also small, only the term $\partial T/\partial y$ is considered in the energy equation. Then the governing equations become

$$\frac{\partial (ru)}{\partial x} = - \frac{\partial}{\partial y}(rv) \tag{1}$$

$$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = 0. \tag{2}$$

Equation (2) shows that the heat flux through the liquid layer is a constant; therefore

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}. \tag{3}$$

From Eq. (1) for the liquid layer

$$\int_h^s \frac{\partial (ru)}{\partial x}\, dy = - \int_h^s \frac{\partial (rv)}{\partial y}\, dy.$$

Because the liquid layer is constantly swept downstream and is kept thin, r, the radial coordinate from the center line to the surface, is regarded as constant. Hence

$$\int_h^s \frac{\partial (ru)}{\partial x}\, dy = - r\left[v(s) - v(h)\right] \tag{4}$$

and

$$\frac{\partial}{\partial x} \int_h^s (ru)\, dy = \int_h^s \frac{\partial (ru)}{\partial x}\, dy + u(s) r \frac{\partial s}{\partial x} - ru(h) \frac{\partial h}{\partial x}.$$

Since the velocity on the surface of the body is zero, it follows that $u(s) = 0$, and since in the outer layer the vertical velocity v consists of local and convective velocity components,

$$v(h) = \frac{\partial h}{\partial t} + u(h) \frac{\partial h}{\partial x}$$

*A list of definitions of the symbols follows the main body of this report.

and

$$\frac{\partial}{\partial x}\int_h^s (ru)\,dy = \int_h^s \frac{\partial(ru)}{\partial x}\,dy + rv(h) - r\,\frac{\partial h}{\partial t}\,.$$

From Eq. (4)

$$\frac{\partial}{\partial x}\int_h^s (ru)\,dy + r\left[v(s) - \frac{\partial h}{\partial t}\right] = 0\,. \tag{5}$$

If there exists a convective velocity caused by the change of density between liquid and solid, then the normal velocity component in the liquid side of the liquid/solid interface is given (11) by

$$v(s) = \left(1 - \frac{\rho_{sol}}{\rho_{liq}}\right)\frac{\partial s}{\partial t}\,. \tag{6}$$

Denoting $\zeta = s - h$, Eq. (5) is written as

$$\frac{\partial}{\partial x}\int_h^s (ru)\,dy + r\,\frac{\partial \zeta}{\partial t} = r\,\frac{\rho_{sol}}{\rho_{liq}}\,\frac{\partial s}{\partial t}\,. \tag{7}$$

Distribution of the Tangential Velocity

In order to solve Eq. (7), it is necessary to determine the tangential velocity u. The quantity u is obtained by using the momentum equation

$$\frac{\partial p}{\partial x} = \mu\,\frac{\partial^2 u}{\partial y^2}\,.$$

By integrating and using the boundary condition

$$\mu\,\frac{\partial u}{\partial y} = -\tau\,,$$

at $y = h$, it follows that

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\left[\frac{\partial p}{\partial x}(s-h) - \frac{\partial p}{\partial x}(s-y) - \tau\right]\,.$$

By integrating once more with respect to y and using the boundary condition $u(s) = 0$ at $y = s$, the velocity component in the x-direction becomes

$$u = \frac{1}{2\mu}\,\frac{\partial p}{\partial x}(s-y)^2 + \frac{1}{\mu}\left[\tau - \frac{\partial p}{\partial x}(s-h)\right](s-y)\,. \tag{8}$$

Governing Equation of the Liquid Layer

The thickness of the liquid layer, ζ, can be determined by using the governing equation of the liquid layer.

Substituting Eq. (8) into Eq. (7),

$$\frac{1}{\mu} \frac{\partial}{\partial x} r \left(- \frac{1}{3} \frac{\partial p}{\partial x} \zeta^3 + \frac{\tau}{2} \zeta^2 \right) + r \frac{\partial \zeta}{\partial t} = r \frac{\rho_{sol}}{\rho_{liq}} \frac{\partial s}{\partial t} . \tag{9}$$

This is the same equation as derived by Goodman (6). The right-hand side of Eq. (9) expresses the rate of melt production, which is a function of external heat flux and the rate of heat penetration into the solid. The effect of heat penetration into the solid due to aerodynamic ablation is given by Goodman (1) as

$$\rho_{sol} H \frac{\partial s}{\partial t} = q \left[1 - \frac{2}{\theta(F, \nu)} \right] . \tag{10}$$

Then Eq. (9) is reduced to

$$\frac{1}{\mu_{liq}} \frac{\partial}{\partial x} \left[r \left(- \frac{1}{3} \frac{\partial p}{\partial x} \zeta^3 + \frac{\tau}{2} \zeta^2 \right)_{liq} \right] + r \frac{\partial \zeta}{\partial t} = r \frac{q}{\rho_{liq} H} \left(1 - \frac{2}{\theta} \right) \tag{11}$$

where θ is given implicitly in terms of F and ν by

$$F = - \frac{1}{3} \left[\theta - 2 + 2(1 + \nu) \ln \frac{2(1 + \nu) - \theta}{2\nu} \right]$$

$$F = q^2 t / (\rho_{sol} H k_{sol} T_{is}) \tag{12}$$

and

$$\nu = H / (C_{sol} T_{is}) .$$

Putting

$$1 - \frac{2}{\theta} = G_1(x, t) \tag{13}$$

Eq. (11) becomes (for steady conditions, where $\partial\zeta/\partial t = 0$)

$$\frac{1}{\mu_{liq}} \frac{\partial}{\partial x} \left[r \left(- \frac{1}{3} \frac{\partial p}{\partial x} \zeta^3 + \frac{\tau}{2} \zeta^2 \right)_{liq} \right] = r \frac{q}{\rho_{liq} H} G_1(x, t) . \tag{14}$$

Equation (14) may be regarded as the final equation of the liquid layer.

AERODYNAMIC ABLATION OF A METAL SPHERE

In the case of ablation due to melting only, the liquid viscosity is much larger than that of the gas; therefore the tangential velocity of the liquid is small compared with the gas velocity. Hence the existence of a liquid layer may be ignored for the evaluation of property values of the gas layer, at the initial condition of ablation. The pressure distribution of gas layer over the liquid layer is assumed to be equal to that over the solid surface. However, if evaporation occurs in addition to melting, then the evaporated metal gas penetrates into the gas layer, and the gas layer characteristics will change. This effect is given by shear stress ratio

$$\tau_i / \tau_0 = \psi$$

and, using Reynolds analogy,

$$\tau_i/\tau_0 = q_i/q_0 = \psi \tag{15}$$

where the subscript i refers to the gas/liquid interface and the subscript o to the non-ablating body. The effect of gas-molecule dissociation due to aerodynamic heating at high speeds is neglected in the analysis. This assumption may hold for the short atmospheric flight path of 20 meters and the maximum Mach number of about 16 of the NRL test facility, because nitrogen dissociation is not expected to be serious if it occurs (12). The gas and liquid layers are assumed to be laminar.

Ablation Due to Melting Only

In order to solve Eq. (14) of the liquid layer, the liquid-flow properties contained in this equation are determined through the gas-flow properties around the sphere. Since the thin liquid layer is essentially a boundary layer, and since the static pressure of the liquid is equal to that of the ambient gas, the pressure distribution of the liquid layer can be formed by that of the gas around the sphere. The Newtonian impact theory at hypersonic velocities yields the pressure coefficient around the sphere as $c_p = 2 \sin^2 \delta$, where δ is the angle between the tangent to its local surface of the body and the stream direction. Lees (13) proposed a modified Newtonian theory, which consists in scaling down this result so as to be exact at the stagnation point of a blunt body, at which point the correct value is known. Then the modified Newtonian relation is,

$$\frac{c_p}{c_{p_s}} = \frac{p - p_\infty}{p_s - p_\infty} = \cos^2\left(\frac{x}{R}\right). \tag{16}$$

This simple equation is in a very good agreement with experimental data in the region $0 < x/R < 1.4$. Hence, the pressure distribution around the sphere is determined from Eq. (16):

$$p - p_\infty = (p_s - p_\infty) \cos^2 (x/R) .$$

Then

$$\frac{\partial p}{\partial x} = - \frac{(p_s - p_\infty)}{R} \sin\left(\frac{2x}{R}\right).$$

Shear Stress in the Liquid Layer

It is necessary to evaluate the shear stress in the liquid layer in Eq. (14). This value can be, however, expressed by the shear stress of the gas layer. The shear stress in the gas boundary layer can be computed if the thin liquid layer is ignored and the velocity profile is assumed to be the shape of the dashed curve in Fig. 2. However, such a velocity distribution deviates from the real velocity distribution in the gas boundary layer, so the real shear stress is given by

$$\tau_{rg} = \tau_{ig} K$$

where τ_{rg} and τ_{ig} are real and ideal shear stresses in the gas and correspond to the shear stresses of the velocity distributions of the gas boundary layer shown by full and broken curves respectively in Fig. 2. K is a factor to be determined. Van Driest (14) shows that for laminar gas flow, for a sphere,

$$\tau_{ig} = \left(2.64/\sqrt{Rex}\right)^{1/2} (\rho u^2)_{e_{ig}} .$$

Accounting for heat transfer, τ_{ig} may be written as

$$\tau_{ig} = \left(1.32/\sqrt{u_e\ x\ \rho^*/\mu^*}\right)(\rho u^2)_{e_{ig}}$$

$$= 1.32\left(\sqrt{\frac{u_e^3}{x}\ \frac{\mu^*}{\rho^*}\ \rho_e^2}\right)_{ig} \tag{17}$$

where the superscript * refers to the property values evaluated at the reference temperature (15)

$$T^*/T_e = 1 + 0.035\,M_e^2 + 0.45\left[(T_w/T_e) - 1\right]. \tag{17a}$$

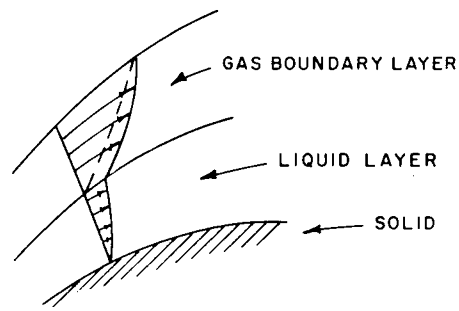

Fig. 2 - Velocity distribution

The Mach number M_e along the sphere surface behind the detached shock is approximated (16) by

$$M_e \approx \frac{3}{2}\,\frac{x}{R}\,. \tag{18}$$

Using the isentropic flow relationship in the region between the shock wave and the body surface,

$$a_e = a_s\left[1 + \frac{9}{8}(\gamma - 1)\frac{x^2}{R^2}\right]^{-1/2}.$$

Then for a diatomic gas, $\gamma = 1.4$ and

$$u_{e_{ig}}^3 \approx \frac{27}{8}\,a_s^3\,\frac{x^3}{R^3}\left(1 - \frac{27}{40}\,\frac{x^2}{R^2}\right)$$

and for diatomic gases at temperatures higher than 2000°K,

$$\frac{\mu}{\mu_{ref}} = (T/T_{ref})^{1/2}.$$

Therefore, from Eq. (17)

$$\tau_{ig} = 2.425\,(a_s/R)^{3/2}x\sqrt{\left(1 - \frac{27}{40}\,\frac{x^2}{R^2}\right)(\rho\mu)_{e,gas}\,(T^*/T_e)^{3/2}}\,. \tag{19}$$

For laminar flow, the ratio of shear stresses may be written as

$$\frac{\tau_{liq}}{\tau_{rg}} = \frac{\mu_{liq}}{\mu_{gas}}\,f$$

where

$$f = \left(\frac{\partial u}{\partial y}\right)_{liq} \Big/ \left[\left(\frac{\partial u}{\partial y}\right)_{gas} K\right].$$

The dynamic viscosity of molten metal is given by

$$\log_{10}\left(\frac{\mu_{liq}}{10^{-3}\ \text{poise}}\right) = \frac{A}{T} + B \tag{20}$$

where A and B are constants. For aluminum A = 705°K and B = 0.474, and for iron A = 1690°K and B = 0.854 (17).

Since

$$\frac{\mu_{liq}}{\mu_{gas}} = \mu_{e_{ig}} \Big/ \left[\mu_{s,gas}\left(\frac{\mu}{\mu_s}\right)_{gas}\right],$$

then

$$\tau_{liq} = \tau_{rg}\left[\frac{\mu_{liq}}{\mu_{s,gas}}\right]\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{-1/2} f .$$

For a diatomic gas

$$\tau_{liq} = 2.425\left(\frac{a_s}{R}\right)^{3/2} x \sqrt{\left(1 - \frac{27}{40}\frac{x^2}{R^2}\right)(\rho\mu)_{e,gas}\left(\frac{T^*}{T_e}\right)^{3/2}}\ \left(1 + \frac{9}{20}\frac{x^2}{R^2}\right)^{-1/2} f\,\frac{\mu_{liq}}{\mu_{s,gas}}$$

$$= 2.425\left(\frac{a_s}{R}\right)^{3/2}\sqrt{\left(\frac{\rho}{\mu}\right)_{s,gas}}\ \mu_{liq}\, f x$$

$$\times \left(1 + \frac{9}{20}\frac{x^2}{R^2}\right)^{-2}\sqrt{\left(1 - \frac{27}{40}\frac{x^2}{R^2}\right)\left(\frac{T^*}{T_e}\right)^{3/2}} = \alpha G_2(x) . \tag{21}$$

where

$$\alpha = 2.425\left(\frac{a_s}{R}\right)^{3/2}\sqrt{\left(\frac{\rho}{\mu}\right)_{s,gas}}\ \mu_{liq}\, f \tag{21a}$$

$$G_2(x) = x\left(1 + \frac{9}{20}\frac{x^2}{R^2}\right)^{-2}\sqrt{\left(1 - \frac{27}{40}\frac{x^2}{R^2}\right)\left(\frac{T^*}{T_e}\right)^{3/2}} .$$

Differential Equation of the Liquid Layer and Determination of the Liquid Layer Thickness

If in addition to the information of the preceding section the heat flux over the sphere is known, then the differential equation of the liquid layer can be derived. The heat flux around the hemisphere is given (18) by*

$$q = q_s\left\{\left[1 - \beta\sin^2\left(\frac{x}{R}\right)\right]\frac{x}{R}\sin\frac{x}{R}\right\}\Big/\left[F_a\left(\frac{x}{R}\right) + \beta F_b\left(\frac{x}{R}\right)\right]^{1/2}$$

$$= q_s g\left(\frac{x}{R}\right) \tag{22}$$

where $\beta = 1-(p'/p_s)$ and p' is the static pressure at the shoulder, and where

*The power 1/2 in Eq. (22) was erroneously omitted in Ref. 18.

$$F_a\left(\frac{x}{R}\right) = \left(\frac{x}{R}\right)^2 - \left(\frac{x}{R}\right)\sin\left(\frac{2x}{R}\right) + \sin^2\left(\frac{x}{R}\right)$$

and

$$F_b\left(\frac{x}{R}\right) = -\frac{1}{2}\left\{\frac{3}{2}\left(\frac{x}{R}\right)^2 - \left(\frac{x}{R}\right)\sin\left(\frac{2x}{R}\right)\left[\frac{3}{2} + \sin^2\left(\frac{x}{R}\right)\right] + \frac{3}{2}\sin^2\left(\frac{x}{R}\right) + \frac{1}{2}\sin^4\left(\frac{x}{R}\right)\right\} .$$

For $M_\infty > 6$, $\beta = 0.96$.

The stagnation heat flux is computed from Ref. 19:

$$q_s = \frac{0.763}{Pr^{0.6}}\left(\frac{\beta_1 D}{u_\infty}\right)^{1/2}\left(\frac{\mu_\infty}{\rho_\infty u_\infty D}\right)^{1/2}\left(\frac{\rho_e}{\rho_\infty}\right)^{-1/2}\left(\frac{\mu_e}{\mu_\infty}\right)^{1/2}\rho_\infty u_\infty c_p (T_r - T_w).$$

Taking $\beta_1 = 1.1\ (u_\infty/D)$ for a sphere,

$$q_s = \frac{0.8}{Pr^{0.6}}\left(\frac{\mu_\infty}{\rho_\infty u_\infty D}\right)^{1/2}\left(\frac{\rho_e}{\rho_\infty}\right)^{-1/2}\left(\frac{\mu_e}{\mu_\infty}\right)^{1/2}\rho_\infty u_\infty c_p (T_r - T_w) . \tag{23}$$

After substitution of Eqs. (16), (21), and (22), Eq. (14) becomes

$$\frac{1}{\mu_{liq}}\left[\frac{d\zeta}{dx}\zeta(\xi_2\zeta + \xi_4) + \zeta^2(\xi_1\zeta + \xi_3)\right] = r\,\frac{q_s g(x/R)}{\rho_{liq} H}\,G_1(x,t) \tag{24}$$

where

$$\xi_1 = \frac{(p_s - p_\infty)}{3R}\left[\cos\left(\frac{x}{R}\right)\sin\left(\frac{2x}{R}\right) + 2\cos\left(\frac{2x}{R}\right)\sin\left(\frac{x}{R}\right)\right]$$

$$\xi_2 = (p_s - p_\infty)\sin\left(\frac{x}{R}\right)\sin\left(\frac{2x}{R}\right)$$

$$\xi_3 = \frac{\alpha}{2}\left[\cos\left(\frac{x}{R}\right)G_2(x) + R\sin\left(\frac{x}{R}\right)\frac{dG_2(x)}{dx}\right]$$

$$\xi_4 = \alpha R\sin\left(\frac{x}{R}\right)G_2(x) .$$

In order to estimate the magnitude of each term, a numerical computation is carried out at the sonic point (see Appendix A). The following assumptions are made: a 4-cm-diameter aluminum sphere travels at $M_\infty = 15$ for 0.003 sec in a standard atmosphere; the thickness of the liquid layer is $\zeta = 0.001$ cm. The results show the following orders of magnitudes:

$$\xi_1 = O(10^5)\ , \quad \xi_2 = O(10^5)\ , \quad \xi_3 = O(0)\ , \quad \text{and} \quad \xi_4 = O(1)\ .$$

Since the magnitudes of the terms ξ_3 and ξ_4 are about 10% of the terms $\xi_1\zeta$ and $\xi_2\zeta$, the terms ξ_3 and ξ_4 are neglected. Physically ξ_3 and ξ_4 correspond to shear resistance; on the other hand ξ_1 and ξ_2 express the pressure distribution around the sphere. From these numerical examples it can be seen that the streamwise decreasing pressure plays a much greater role in sweeping away the molten layer than does the shear force. Equation (24) is then reduced to

$$\frac{d\zeta}{dx} + \xi(x)\,\zeta = \eta(x,t)\,\zeta^{-2} \tag{25}$$

where

$$\xi(x) = \xi_1/\xi_2$$

and

$$\eta(x,t) = \frac{R\mu_{liq}q_s}{\rho_{liq}H}\;\frac{\sin(x/R)\,g(x/R)}{\xi_2}\;G_1(x,t)\,.$$

By taking time t as constant, i.e., by fixing t, η becomes a function of x only, and Eq. (25) can be integrated. Its solution is given by

$$\zeta = \left[\frac{3}{e^{3\int\xi dx}}\left(\int e^{3\int\xi dx}\eta dx + c_1\right)\right]^{1/3}. \tag{26}$$

The constant c_1 may be determined by the boundary and initial conditions, $\zeta = 0$ at $x = 0$ and $t = 0$. The thickness of the liquid layer can be calculated by Eq. (26) at constant time t. The shape of an ablated sphere is determined by

$$h = s - \zeta$$

where s is obtained by integrating Eq. (10) as

$$s = \frac{q_s g(x/R)}{\rho_{liq}H}\left(t - 2\int\frac{dt}{\theta}\right) + G_3(x) + c_2\,. \tag{27}$$

The boundary and initial conditions are at $t = 0$, for all x, $s = 0$. These conditions can be satisfied if $G_3(x)$ is constant. Therefore

$$G_3(x) + c_2 = c_3\,.$$

Then

$$s = \frac{q_s g(x/R)}{\rho_{liq}H}\left(t - 2\int\frac{dt}{\theta}\right) + c_3\,.$$

The volume of ablated material V^* is at the fixed time t

$$V^* = 2\pi\int_0^L hr_1\,dx = 2\pi\int_0^L (s-\zeta)\left[R - \frac{(s-\zeta)}{2}\right]\sin\left(\frac{x}{R}\right)dx\,. \tag{28}$$

Thus far ζ, h, and V^* have been expressed by stagnation conditions. However, the stagnation properties are determined by the free stream values in front of the shock and by normal shock relations. Hence, the total ablated volume and shape of the sphere are evaluated by the trajectory at any specified time t.

The physical meaning of the equations derived is summarized as follows: If the velocity of the sphere is zero, but a heat flux exists, the molten layer s increases with time t and the thickness ζ is equal to s, since no ablation occurs under such conditions. However, if the velocity of the sphere is not zero, then the molten layer is swept away because of the pressure decrease in the streamwise direction, and h increases with time.

In other words ζ will decrease. As Eq. (28) shows, with decreasing thickness of the liquid layer, the ablated volume increases accordingly. The thickness ζ can be determined by solving Eq. (24) as a closed form of Eq. (26) as a function of x, if the time is fixed. Therefore, it is possible to calculate numerically, using Eq. (28), the total ablated volume at any fixed time of the trajectory. The determination of the ablated mass at a fixed time is in practice very important, for example, for problems of heat transfer, design of high speed vehicles, impact of bodies traveling at hypersonic speed, and intensities of luminous trails and ionization behind bodies.

Ablation Due to Melting with Partial Evaporation

The occurrence of evaporation may be determined by the wall pressure or conduction parameter. For an ice sphere immersed in high-speed air, it is known that for a given recovery temperature there exists a value of wall pressure at which the wall temperature is just at the melting point of the ice. For wall pressures above this critical pressure, the ice melts, but if the pressure at the wall is less than the critical pressure, then the ice does not melt but evaporates (20). The dimensionless conduction parameter χ is given (8) by

$$\chi = \frac{(\rho C k)_b^{1/2}\,(T_{sub} - T_\infty)}{\left[m/(C_D A R)\right]^{1/2}}$$

where the subscript b refers to body. If $\chi > 2\times10^3$, then no sublimation occurs. Bethe and Adams (2) worked out the theory for aerodynamic ablation with partial evaporation, and their formulations are applied here.

For partial evaporation, ψ, the ratio of shear stress or heat flux at the gas/liquid interface to that for a nonablating body, Eq. (15), is an important parameter. In a strong evaporation

$$\psi \approx \frac{h_v}{h_v + 0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right)} = f_1(x)\,. \tag{29}$$

The metal vapor pressure at the gas/liquid interface is

$$p_{vi} = \frac{p_s}{1 + \dfrac{q_s\, g(x/R)}{\rho_{1iq} v_i \left(h_{1_s} - h_{1_i}\right) H} - 0.68\, M_1^{-0.74}}\,. \tag{30}$$

The quantity v_i, the normal velocity of evaporated metal perpendicular to the sphere surface at the gas/liquid interface, is determined as follows: Since, in addition to Eq. (29),

$$\psi \approx 1 - 0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right) \rho_{1iq} v_i / q_s\, g(x/R) \tag{31}$$

the two equations for ψ are equated to give

$$v_i = \frac{1 - h_v\Big/\left[h_v + 0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right)\right]}{0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right) \rho_{1iq}/q_s\, g(x/R)}\,. \tag{32}$$

Substituting Eq. (32) into Eq. (30),

$$p_{vi} = \frac{p_s}{1 + \dfrac{0.68\, M_1^{0.26}}{H\left[1 - \dfrac{h_v}{h_v + 0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right)}\right]} - 0.68\, M_1^{-0.74}} \tag{33}$$

Thus p_{vi} is a function of x.

With this information, Eq. (14) is reduced to the following form if small terms are neglected:

$$\frac{d\bar{\zeta}}{dx}\bar{\xi}_2 + \bar{\xi}_1 \bar{\zeta} = \frac{\sin(x/R)}{\bar{\zeta}^2} \frac{\mu_{liq} \psi\, q_s\, g(x/R)}{\rho_{liq} H} \left(\frac{2}{\theta} - 1\right)$$

where

$$\bar{\xi}_1 = -\left[\frac{1}{3R} \frac{\partial p_{vi}}{\partial x} \cos\left(\frac{x}{R}\right) + \frac{1}{3} \sin\left(\frac{x}{R}\right)\left(\frac{\partial^2 p_{vi}}{\partial x^2}\right)\right]$$

$$\bar{\xi}_2 = -\sin\left(\frac{x}{R}\right)\frac{\partial p_{vi}}{\partial x}$$

and a bar over a symbol refers to ablation with partial evaporation. Then it follows that

$$\frac{d\bar{\zeta}}{dx} + \bar{\xi}\,\bar{\zeta} = \bar{\eta}\zeta^{-2} \tag{34}$$

where

$$\bar{\xi} = \frac{\bar{\xi}_1}{\bar{\xi}_2} = \frac{1}{3R}\cot\left(\frac{x}{R}\right) + \frac{1}{3} \frac{\partial^2 p_{vi}}{\partial x^2} \bigg/ \frac{\partial p_{vi}}{\partial x}$$

and

$$\bar{\eta} = \frac{R\mu_{liq}}{\rho_{liq} H} \frac{\psi\, q_s\, g(x/R)}{\dfrac{dp_{vi}}{dx}} \left(\frac{2}{\theta} - 1\right).$$

For a fixed time, the solution of Eq. (34) is

$$\bar{\zeta} = \left[\frac{3}{e^{3\int\bar{\xi}dx}} \left(\int e^{3\int\bar{\xi}dx}\, \bar{\eta}dx + c_4\right)\right]^{1/3}. \tag{35}$$

The aerodynamic heat flux is balanced by the energy of evaporation and the heat capacity of the liquid and solid phases. Thus

$$\psi q_o = q_{absorption} + q_{evaporation} = \rho_{liq}(v_w h_t + v_i h_v). \tag{36}$$

Using Eq. (32),

$$q_{\text{evaporation}} = \rho_{\text{liq}} v_i h_v$$
$$= \frac{q_s\, g(x/R)\, h_v}{0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right)}\left[1 - \frac{h_v}{h_v + 0.68\, M_1^{0.26}\left(h_{1_s} - h_{1_i}\right)}\right].$$

The absorbed heat can be computed by

$$q_{\text{absorption}} = \psi q_o - q_{\text{evaporation}}.$$

The mass loss is caused by evaporation and absorption, so

$$\bar{s} = \bar{s}_{\text{evaporation}} + \bar{s}_{\text{absorption}}$$

$$\bar{s}_{\text{evaporation}} = \frac{q_{\text{evaporation}}}{\rho_{\text{liq}} H_{\text{evaporation}}}\left(t - 2\int \frac{dt}{\theta}\right) + c_5$$

$$\bar{s}_{\text{absorption}} = \frac{q_{\text{absorption}}}{\rho_{\text{liq}} H_{\text{absorption}}}\left(t - 2\int \frac{dt}{\theta}\right) + c_6 .$$

Hence, the shape of an ablated sphere with partial evaporation is determined by

$$\bar{h} = \bar{s} - \bar{\zeta} = \left(\frac{t - 2\int \frac{dt}{\theta}}{\rho_{\text{liq}}}\right)\left(\frac{q_{\text{evaporation}}}{H_{\text{evaporation}}} + \frac{q_{\text{absorption}}}{H_{\text{absorption}}} + c_7\right) - \bar{\zeta}. \tag{37}$$

The total ablated volume $\bar{V}^*$ is

$$\bar{V}^* = 2\pi \int_0^L \bar{h}\, r_1\, dx = 2\pi \int_0^L (\bar{s} - \bar{\zeta})\left[R - \frac{(\bar{s} - \bar{\zeta})}{2}\right] \sin\left(\frac{x}{R}\right) dx . \tag{38}$$

Hence, the total ablated volume and shape can be computed based upon the trajectory at any specified time.

DISCUSSION

Aerodynamic ablation is a very complex phenomenon, because the change of body shape due to ablation also causes changes in gas and liquid flow characteristics along the body. The approximate analysis presented here is for steady-state aerodynamic ablation of a sphere, assuming that the flow characteristics are those on the sphere and neglecting the change of shape. Hence, this analysis is applicable for a flight condition with small ablation. This method is especially useful to determine the flight conditions under which the sphere can travel without any ablation. However, if considerable ablation occurs, the front part of the sphere deforms to a cone shape (Fig. 3). No attempt has been made to confirm the deformation to a cone by analysis. The computation of change of shape due to ablation will be possible only when a high-speed computer is used. If the moving boundaries for melting and subliming are considered, then the problem is more complex, and Goodman's "heat-balance integral" (21) may be useful to solve the problem of ablation with changing shape.

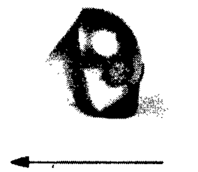

Fig. 3 - Recovered ablated aluminum sphere (original diameter, 0.635 cm; initial velocity, 16,000 ft/sec)

The unsteady aerodynamic ablation due to the thermal impulse has been treated by Goodman (7), who found that total ablation will be increased if the unsteady ablation is considered. Photographs taken at early stages of flight at NRL show a luminous trail as seen in Fig. 4. This trail is caused by burning and excitation of material which is ablated from the surface. This material collides with air molecules, and oxidation results (22).

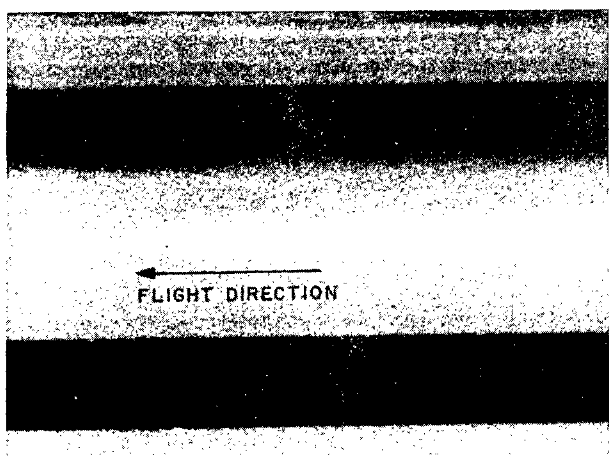

Fig. 4 - Luminous trail of an aluminum sphere during hypersonic flight

There is an indication that this luminous trail is connected with the unsteady heat-impulse phenomenon. An estimate of time to reach a quasi-steady state of ablation during flight in an environment (6) is given by the following order of magnitude:

$$t_{qs} = O\left[\frac{H\rho_{liq}\,\rho_{sol}\,k_{sol}\,T_{is}^{2}\,C_{sol}}{\frac{1.32}{\mu_{liq}}\sqrt{(\rho\mu)_{gas}\,u_1^{3}}\;q^{3}}\right]$$

where

$$u_1 = \frac{3u_2}{R}\,(1 - 0.252\,M_2^2 - 0.0175\,M_2^4)\,.$$

The subscript 2 refers to the values immediately behind the normal shock. Test conditions at NRL show that during the first 3 milliseconds the sphere is subject to unsteady ablation due to thermal impulse. This time amounts to the order of 10 percent of the total flight time. Hence, the analysis presented may yield a total ablated mass which is less than the measured value.

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SYMBOLS

A	Reference area for drag
c	Specific heat
c_D	Drag coefficient
c_f	Skin friction coefficient
c_p	Specific heat at constant pressure
D	Diameter of the sphere
F	Nondimensional time
H	Latent heat of fusion
H_{evap}	Latent heat of evaporation
h	Distance between the original surface and the gas/liquid interface ($h \perp$ gas/liquid interface)
h_1	Enthalpy
h_t	Energy absorbed by heat capacity (c_pT)
h_v	Energy absorbed by evaporation
I	Radiation heat flux
k	Thermal conductivity
L	Rear endpoint of the ablated layer
M	Mach number
M_1	Molecular weight ratio (gas to metal vapor)
m	Mass
p	Static pressure
p'	Static pressure at the shoulder of the sphere
Pr	Prandtl number
q	Heat flux
R	Radius of the sphere

R_e Reynolds number

r Radial coordinate of the body of revolution

r_1 Radial coordinate of the body for the center of h (see Fig. 1)

s Distance between the original surface and the ablated surface ($s \perp$ solid surface)

T Absolute temperature

T_{is} Initial temperature of solid below melting temperature

T_o Stagnation temperature

T_r Recovery temperature

T_{sub} Sublimation temperature

T_w Wall temperature

t Time

u Velocity component in x-direction

V^* Volume

v Velocity component in y-direction

v_w Ablation velocity (normal to wall)

x Coordinate along the surface, measured from the forward stagnation point

y Coordinate normal to the surface

β $\beta = 1 - p'/p_s$

β_1 Stagnation velocity gradient parameter

γ Ratio of specific heats

ζ Liquid layer thickness ($\zeta = s - h$)

μ Dynamic viscosity

ν Nondimensional parameter $H/(T_{is} c_{sol})$

ρ Density

τ Shear stress

χ Conduction parameter

ψ Ratio of heat transfer with and without mass injection

SUBSCRIPT

s	Stagnation point
e	Edge of the gas boundary layer
0	Nonablating body
ref	Reference
∞	Free stream in front of the shock
2	Flow conditions immediately behind the normal shock
b	Body
liq	Liquid
sol	Solid
i	Gas-liquid interface

SUPERSCRIPT

$-$	Ablation due to melting and partial evaporation

Appendix A

ESTIMATES OF THE VALUES OF ξ_1, ξ_2, ξ_3, AND ξ_4

In order to estimate the values of ξ_1, ξ_2, ξ_3, and ξ_4 which are found in Eq. (24), the following conditions are assumed. An aluminum sphere 4 cm in diameter travels at $M_\infty = 15$ under standard atmospheric conditions. At the sonic point ($M_e = 1$), the thickness of the liquid layer is $\zeta = 0.001$ cm. For the estimate this thickness is regarded as constant over x. The flight time is 0.003 sec.

CALCULATION OF ξ_1 AND ξ_2

Initially the values of ξ_1 and ξ_2 are computed. From Eq. (24),

$$\xi_1 = \frac{(p_s - p_\infty)}{3R}\left[\cos\left(\frac{x}{R}\right)\sin\left(\frac{2x}{R}\right) + 2\cos\left(\frac{2x}{R}\right)\sin\left(\frac{x}{R}\right)\right]$$

and

$$\xi_2 = (p_s - p_\infty)\sin\left(\frac{x}{R}\right)\sin\left(\frac{2x}{R}\right) .$$

The value of x/R at the sonic point ($M_e = 1$) from Eq. (18) is

$$\frac{x}{R} \approx \frac{2}{3} .$$

The pressure at the stagnation point p_s can be computed by using normal shock relationships. Then

$$\xi_1 = 0.561 \times 10^5 \tag{A1}$$

and

$$\xi_2 = 1.74 \times 10^5 . \tag{A2}$$

ESTIMATION OF ξ_3 AND ξ_4

From Eq. (24)

$$\xi_3 = \frac{\alpha}{2}\left[\cos\left(\frac{x}{R}\right)G_2(x) + R\sin\left(\frac{x}{R}\right)\frac{dG_2(x)}{dx}\right] \tag{A3}$$

$$\xi_4 = \alpha R\sin\left(\frac{x}{R}\right)G_2(x) \tag{A4}$$

where

$$G_2(x) = x\left(1 + \frac{9}{20}\frac{x^2}{R^2}\right)^{-2}\sqrt{\left(1 - \frac{27}{40}\frac{x^2}{R^2}\right)\left(\frac{T^*}{T_e}\right)^{3/2}} .$$

In order to estimate the values of ξ_3 and ξ_4 it is necessary to compute α from Eq. (21a):

$$\alpha = 2.425\left(\frac{a_s}{R}\right)^{3/2}\sqrt{\left(\frac{\rho}{\mu}\right)_{s,gas}}\;\mu_{liq} f . \qquad \text{(A5)}$$

The value of f is estimated as follows:

$$\frac{\tau_{liq}}{\tau_{rg}} = \left(\frac{\mu_{liq}}{\mu_{rg}}\right)f = \left(\frac{\mu_{liq}}{\mu_{ig}}\right)f$$

or

$$\frac{\tau_{liq}}{\tau_{rg}} = \frac{c_{f_{liq}}}{c_{f_{rg}}}\;\frac{\rho_{liq}}{\rho_{ig}}\left(\frac{u_{liq}}{u_{ig}}\right)^2 K$$

where c_f is the skin friction coefficient. In laminar flow

$$c_f = \frac{c}{\sqrt{R_e}} = c\sqrt{\frac{\mu}{u\ell\rho}}$$

where c is a constant and ℓ is the length of the gas boundary layer or liquid layer. Therefore,

$$f = \left(\frac{\ell_{ig}}{\ell_{liq}}\right)^{1/2}\left(\frac{u_{liq}}{u_{ig}}\right)^{3/2}\left(\frac{\rho_{liq}}{\rho_{ig}}\right)^{1/2}\left(\frac{\mu_{ig}}{\mu_{liq}}\right)^{1/2} K . \qquad \text{(A6)}$$

For the estimation of f, the right-hand terms of this equation are computed as follows:

$$\frac{\mu_{ig}}{\mu_\infty} = \left(\frac{T_e}{T_\infty}\right)^{1/2} = \left[\frac{T_s\,(1 + 0.2)^{-1}}{288}\right]^{1/2} = 6.21 .$$

From Eq. (20)

$$\mu_{liq} = 1.704 \text{ centipoise.}$$

Since $\mu_\infty = 0.018$ centipoise,

$$\left(\frac{\mu_{ig}}{\mu_{liq}}\right)^{1/2} = 0.2565$$

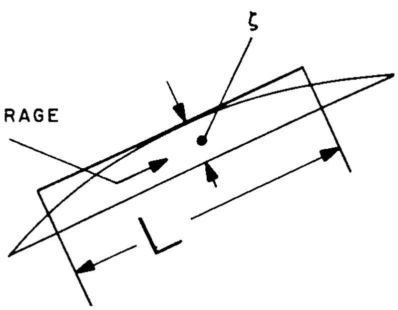
$$\rho_{ig} = \rho_{s,gas}\left(1 + 0.2\,M_e^2\right)^{-5/2} = 4\rho_\infty .$$

Therefore,

$$\left(\frac{\rho_{liq}}{\rho_{ig}}\right)^{1/2} = 23.75 \, .$$

By assuming the total length of molten surface L to be $L = 5\zeta$,

$$L = (u_{liq})_{average} t = 5\zeta$$



or, for $t = 0.003$ sec and $\zeta = 0.001$ cm,

$$(u_{liq})_{average} = \frac{5\zeta}{t} = 1.667 \text{ cm/sec}.$$

Also

$$u_{ig} = M_e a_e = a_s \left(1 + 0.2\, M_e^2\right)^{-1/2}$$

$$= 2.09 \times 10^{-5} \text{ cm/sec} \, .$$

Therefore

$$\left(\frac{u_{liq}}{u_{ig}}\right)^{3/2} = 7.75 \times 10^{-9} \, .$$

For further computation of average values, the length of the molten surface is assumed to be

$$\ell_{liq} = \frac{L}{2} \, .$$

Then

$$f = 1.09 \times 10^{-7} K \, .$$

Because the average value of u_{liq} is very small, the liquid layer can be considered to be frozen, so $K \to 1$ and

$$f = 1.09 \times 10^{-7} \, . \tag{A7}$$

Another term of Eq. (A5) to be evaluated is

$$\sqrt{\left(\frac{\rho}{\mu}\right)_{s,\,gas}} = \sqrt{\frac{\rho_\infty}{\mu_\infty}\,\frac{\rho_s}{\rho_\infty}\,\frac{\mu_\infty}{\mu_s}} = \sqrt{\frac{6.3}{\left(\frac{\mu_\infty}{\rho_\infty}\right)}\left(\frac{T_\infty}{T_s}\right)^{1/2}}$$

Since $\mu_\infty/\rho_\infty = 14.9 \times 10^{-2}\, \text{cm}^2/\text{sec}$ at 15°C,

$$\sqrt{\left(\frac{\rho}{\mu}\right)_{s,\,gas}} = 2.5 \text{ sec}^{1/2}/\text{cm} \, . \tag{A8}$$

Finally

$$\alpha = 2.425 \times (3.89 \times 10^{7}) \times 2.5 \times 0.01704 \times (1.09 \times 10^{-7}) = 0.438. \tag{A9}$$

For computation of the values of ξ_3 and ξ_4, the terms of

$$G_2(x) = x\left(1 + \frac{9}{20}\,\frac{x^2}{R^2}\right)^{-2} \sqrt{\left(1 - \frac{27}{40}\,\frac{x^2}{R^2}\right)\left(\frac{T^*}{T_e}\right)^{3/2}}$$

and $dG_2(x)/dx$ are calculated.

Using Eq. (17a):

$$\frac{T^*}{T_e} = 1 + 0.035\,M_e^2 + 0.45\left(\frac{T_w}{T_e} - 1\right)$$

and, assuming T_w is equal to the melting temperature of aluminum, 933°K,

$$\left(\frac{T^*}{T_e}\right)^{3/2} = 0.493\,.$$

Then

$$G_2(x) = 0.544 \quad \text{and} \quad \frac{dG_2(x)}{dx} \approx -0.351\,.$$

From Eqs. (A3) and (A4) and the values computed in this section,

$$\xi_3 \to 0$$

$$\xi_4 = 0.294\,.$$

* * *