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Ship-Track Models Based on Poisson-Distributed Port-Departure Times

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14. ABSTRACT

This report presents two models that describe the tracks of ships during an arbitrary time interval. Both models consist of a track function that describes the tracks of the individual ships and a probability law on the total number of ships en route during the interval, the initial positions of those ships, and their nominal speeds. The probability law assumes that the ship departure times are Poisson-distributed with a time-varying departure rate and that the ship speeds and the ship routes are statistically independent. For both models, it is shown that the ship-track parameters are distributed as a multidimensional, nonhomogeneous, Poisson process with a time-dependent rate function. In the first model, the tracks are deterministic functions with the constraint that a track can neither double back on itself nor intersect any other track. In this model, the track function and the track-parameter rate function are determined from probability densities describing the concentration of ship positions along the routes. In the second model, the ship tracks are obtained as realizations of a Markov process without any constraints on the track crossings.

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CONTENTS

SHIP-TRACK MODELS BASED ON POISSON DISTRIBUTED PORT-DEPARTURE TIMES

1. INTRODUCTION

A number of applications require a statistical description of the locations of ships in a region, e.g., ambient noise models and probability-based ship tracking algorithms. For many of these applications, it suffices to specify the mean number of ships on a grid of elemental longitude-latitude areas. Such descriptions, often referred to as shipping distributions, have been obtained from both direct measurements of individual ship positions and from shipping models based on port-departure times and routing data. A well-known example of the latter is the Historical Temporal Shipping (HITS) model [1,2]. This model specifies shipping distributions for different classes of commercial ships and different time periods over a large portion of the world's oceans. The resolution of these shipping distributions, i.e., the dimensions of the longitude-latitude grid, is 1° × 1°.

Many applications, however, require a description of the tracks of the individual ships over some time interval $(t_o, t_o + T]$ rather than simply the mean number of ships in elemental areas. For some of these applications, the duration of the required observation period T may exceed the time interval over which many of the ships are within the region of interest. For these applications, the ship-track model must be capable of introducing new ships into the region in a manner consistent with the underlying shipping distribution.

The requirements of the ship-track model also depend on the types of ships that are relevant to the application. For most commercial shipping, the ships simply transit from one port to another along routes that constitute a more or less well-defined shipping lane. For this type of shipping, it may suffice to provide simple smooth approximations to the actual routes traveled from the departure to the destination port. On the other hand, shipping traffic such as fishing, recreational and military vessels can travel along complicated routes as dictated by the specific mission of the vessel. For this type of shipping, more complicated descriptions of the routes may be required. In either case, the accuracy requirements of the ship tracks will depend on the specific application of the model.

This report presents two models, a deterministic and a stochastic model, each of which describe the tracks of ships en route in a region during an arbitrary time interval $(t_o, t_o + T]$. Both models consist of a track function that describes the tracks of the individual ships and a probability law on the total number of ships en route during $(t_o, t_o + T]$, the positions of those ships at the initial time t_o, and their nominal speeds. The probability law is obtained under the assumption that the times at which the ships depart each port are Poisson distributed with a time-varying departure rate and that the ship speeds and the routes that the ships travel are statistically independent. Under this assumption, it is shown that the ship-track parameters are distributed as a Poisson process with a time-dependent rate function. The rate function that specifies this process differs for the two ship-track models.

The two ship-track models provide alternate descriptions of the ship's tracks. In the deterministic model, the ship tracks are deterministic functions derived from a probability density on ship positions in

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manner analogous to that of the HITS model (see Ref. 2). As such, the individual ship tracks can neither double back on themselves nor intersect any other ship tracks. This may be an acceptable constraint for ship traffic that simply transits from one port to another, provided that detailed representations of the individual tracks are not required. In the stochastic model, the ship tracks are obtained as realizations of a Markov process without any constraints on the route crossings. Accordingly, this model is suitable for shipping where the tracks are determined by operations more complicated than port-to-port transiting. Furthermore, the stochastic nature of the tracks may be more realistic than the smooth track approximations of the deterministic model.

This report is organized as follows: Sections 2 and 3 provide the background for the model definitions. In Section 2, we present the definitions and assumptions that are used to describe the kinematics of the shipping for each ordered pair of ports that support shipping in the region. Specifically, for each port pair, we define the ship-track function in terms of a route function and a motion function. In general, these functions depend on a track parameter that specifies the distance that the ship has traveled at the initial time t_o, the specific route that the ship travels between the departure and the destination port and the ships nominal speed along that route. To account for those ships that are not en route at time t_o, but depart the port during the interval $(t_o, t_o + T]$, we allow these parameters to take on "virtual" values. In Section 3, we state the assumptions on the port-departure probability law and describe the probability law on the track parameter that results from these assumptions. It is seen that the track-parameter probability law can be viewed as the composition of two Poisson process, one representing the ships that are present at time t_o and one representing the ships that enter the region in the interval $(t_o, t_o + T]$.

The ship-track models themselves are described in Sections 4 and 5. For each of these models, we first specify the route set probability law that forms the basis of the model. We then specify the track function and the track-parameter probability law that results from the route set probability law. For both models, the track parameter can be expressed in terms of the ship position at the initial time t_o and the nominal speed. This leads to the notion of the shipping density from which the shipping distribution can be derived by integrating over elemental areas. The derivations of these results are presented in Appendixes A and B. For reference purposes we have also provided a definition of a multi-dimensional Poisson process in Appendix A. In Appendix C, we present a simplified version of the deterministic model. Finally, the results of the report are summarized in Section 6. An example of an ambient noise model that draws on a deterministic ship-track model is described in Refs. 3 and 4. Reference 5 presents an ambient noise application to the region around San Diego, California.

2. KINEMATICS

A realization of ship tracks must represent not only those ships that are present in the region during the entire time interval of interest, but also those ships that either enter the region or exit the region during that interval. The latter do so by either departing or arriving at a port or by crossing the boundary of the region. Figure 1 illustrates this process. The region of interest is bounded by the coastline segments indicated by the heavy black lines and by the dotted line segments across the access areas to the region. The ports, labeled P_1 through P_9, are classified as either "real ports" or "pseudo ports." The real ports are those that are physically located within the region. The pseudo-ports, located along the access boundaries of the region, represent the shipping from the aggregate of the ports located outside the region that provide traffic to the region. The ports P_6 and P_9 are pseudo ports; the remaining ports are real ports. The ship tracks are indicated by the line segments. The dots on the segments show the ship positions at time t_o; the arrowheads show the positions at time $t_o + T$. The ship tracks fall into two groups, ships that are en route at time t_o and ships that are in port at time t_o but depart the port during the interval $(t_o, t_o + T]$. Ship tracks in the first group have a dot at the beginning of the track; ship tracks in the second group do not have a dot at the beginning of the track. For tracks in the first group, there are a number of possibilities: (a) the ship remains en route for the entire time interval (indicated by a track with both a dot and an arrowhead); (b) the ship arrives at a port during the time interval and remains there for the rest of the time interval (indicated by a track with a dot but no arrowhead); (c) the ship arrives at a port and then departs from that port during the time interval (indicated by the dashed track at P_2); and (d) the ship arrives at and then departs from more than one port during the interval. The same possibilities exist for the ships that are at a port at the beginning of the time interval once those ships have departed the port.

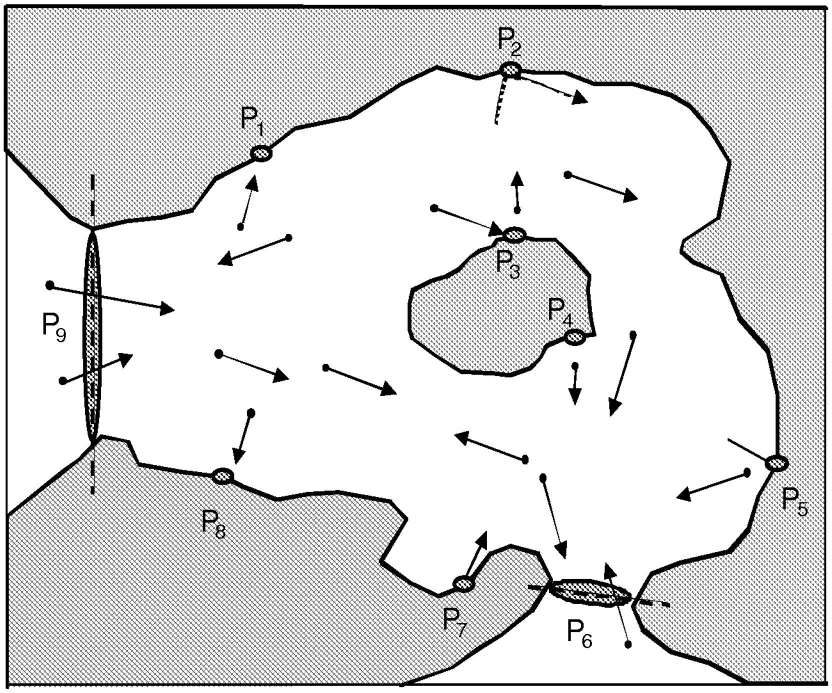

Fig. 1 — A possible set of ship tracks for a region

The ship-track model presented here does not describe the tracks of ships that pass through a port during the time interval of interest. Instead, it represents such tracks as two tracks — one associated with a ship that arrives at the port and remains there for the rest of the time interval and one associated with a ship that is at the port at the beginning of the interval and then departs at some time during the interval. Thus, the track of the ship passing through P_2 in Fig. 1 is represented as the track of one ship that enters P_2 plus the track of another ship that departs from P_2. This approximation simplifies the model considerably since it obviates the need to relate a ship's departure time to its arrival time. Furthermore, it allows the shipping between any pair of ports to be considered separately from the shipping between any other pair of ports. This approximation is reasonable if the duration of the time interval is short compared to the mean time that ships spend in a port since, in this case, the number of tracks that pass through the port in the time interval will be negligible. It is also be reasonable for longer time intervals if it is not necessary to preserve the identity of the individual ships that pass through the ports.

As a result of the independent port-pair approximation, the regional shipping model can be obtained as the composition of ship-track models for each ordered pair of ports (P_n, P_m) that support shipping within the region. The first port in the port-pair is the *departure port*; the last port is the *destination port*. We also include the port-pair (P_n, P_n) since for certain kinds of shipping (e.g., fishing and recreational) the destination port and the departure port can be the same. Furthermore, we include both the port-pairs (P_n, P_m) and (P_m, P_n), since the traffic from P_n to P_m is not necessarily the same as the traffic from P_m

to P_n. Thus, if there are N_p ports for the region, including pseudo ports, there are as many as N_p^2 port-pair models for the regional model.

To describe the kinematics for an individual port pair, it is first necessary to specify the routes that the ships travel and the motion of those ships along those routes. These quantities determine a ship-track function that specifies the tracks of the individual ships during the period of interest.

2.1 The Route Function

The routes that the ships travel are specified by a route function that describes the set of all possible routes from P_n to P_m by associating each possible route with a route parameter $\omega \in \Omega$ and expressing that route as a function of the distance x measured along that route from the departure port. The route function has the form

$$R_{\lambda,\phi}(x;\omega) = \left(R_\lambda(x;\omega), R_\phi(x;\omega)\right) , \tag{1}$$

where $R_\lambda(x;\omega)$ and $R_\phi(x;\omega)$ are the coordinate functions for the longitude coordinate λ and the latitude coordinate ϕ, respectively. For each value of the route parameter ω, we denote the length of the route by $L(\omega)$ and assume that as x increases from 0 to $L(\omega)$, the route function traces out the route ω from P_n to P_m as a continuous path in the set $\lambda \in (-\pi, \pi]$ and $\phi \in (-\pi/2, \pi/2)$. This set represents all positions on the Earth, except for the north and the south poles, with the convention that negative values of λ correspond to longitudes east of the Greenwich meridian and negative values of ϕ correspond to latitudes south of the equator.

To account for ships that depart from P_n after time t_o, we allow the distance traveled x to be negative and define the route function by

$$\begin{aligned} R_\lambda(x;\omega) &= \lambda - \pi \\ R_\phi(x;\omega) &= \phi = \omega \end{aligned} \tag{2}$$

With this definition, route segments determined for negative values of x do not describe actual routes since the coordinates (λ,ϕ) take values in the set $\Re_v = (-\infty, -\pi) \times \Omega$. We refer to this set as the virtual ship coordinates and use it to implicitly determine the departure times of the ships that leave P_n during the time interval $(t_o, t_o + T]$. We refer to the set $\Re_r = (-\pi, \pi] \times (-\pi/2, \pi/2)$ as the real ship coordinates and to the union of both sets $\Re_e = \Re_v \cup \Re_r$ as the extended ship coordinates.

In the sequel, it is convenient to define the route function and the track function in an auxiliary coordinate system and then map the results to the latitude-longitude system. The definition of the auxiliary coordinates, which are denoted here by (θ,γ), is illustrated in Fig. 2. In this definition, it is assumed that the route set is such that there is a "nominal route" ω_n with the property that, for each point (λ',ϕ') on ω_n, there is a great circle arc through that point that is orthogonal to the nominal route and that intersects every other route in the route set. The auxiliary coordinate θ is defined as the distance along the nominal route from the departure port to the point (λ',ϕ'). This distance increases from zero at the departure port to L_n at the destination port, where L_n is the length of the nominal route expressed in radians. The "cross-sectional coordinate" γ is defined as the signed distance along the great circle arc from the point (λ',ϕ') on the nominal route ω_n to the point (λ'',ϕ''). This distance is taken to be positive in the direction of the upper route envelope, denoted by $e_u(\theta)$ in Fig. 2, and negative in the direction of the lower route envelope $e_l(\theta)$. For route segments that lie in the virtual coordinate set, $(\lambda,\phi) \in \Re_v = (-\infty, -\pi) \times \Omega$, we take $\theta = \lambda + \pi$ and γ equal to a function of ϕ that is specified below.

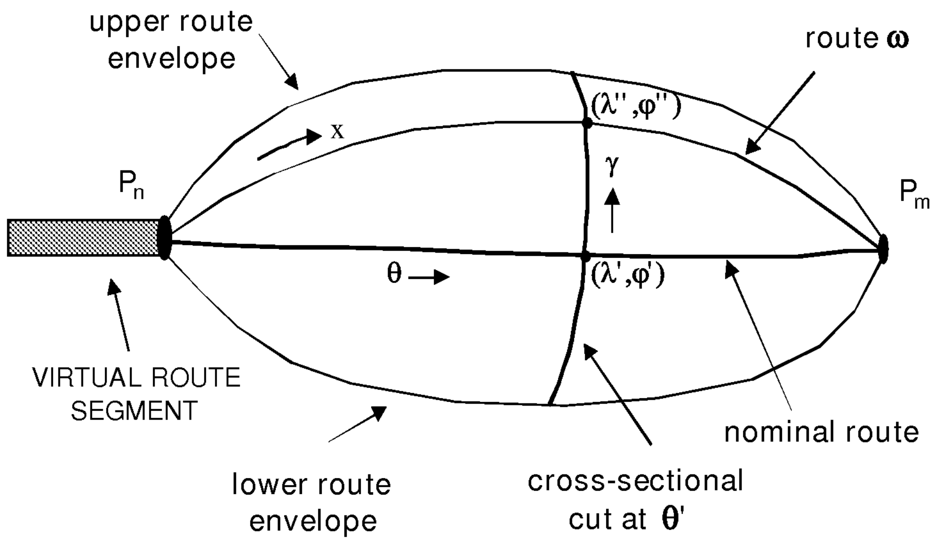

Fig. 2 — The auxiliary coordinate system

With the auxiliary coordinate system so defined, the longitude-latitude coordinates are related to the auxiliary coordinates by a one-to-one transformation. Consequently, the route function $R_{\lambda,\phi}(x;\omega)$ can be obtained by first determining this function in terms of the auxiliary coordinates (i.e., as the function $R_{\theta,\gamma}(x;\omega)$) and then using the coordinate transformation to express the results in the longitude-latitude coordinate system. Reference 5 provides an example of a one-to-one transformation between the auxiliary coordinates and the longitude-latitude coordinates for a particular choice of the nominal route.

In the deterministic model presented in Section 4, all of the routes in the route set "progress" from the departure port to the destination port without "doubling back" on themselves. For this case, there can be only one value of the cross-sectional coordinate γ for each value of the nominal route coordinate θ. Consequently, for each route ω, the cross-sectional variable can be expressed as a function $\gamma = a(\theta,\omega)$, so that the route ω is traced out as the points $(\theta, a(\theta,\omega))$ as θ increases from zero to L_n. Furthermore, the distance traveled along the route is given by

$$X(\theta;\omega) = \int_0^{\infty} \left[1 + \left(\frac{\partial a(\theta',\omega)}{\partial \theta'} \right)^2 \right]^{1/2} d\theta' \ , \tag{3}$$

which is an increasing function of the nominal route coordinate θ. It follows that θ can be expressed as a function of the distance traveled by the inverse function

$$\Theta(x;\omega) = X^{-1}(x;\omega) \ . \tag{4}$$

With these definitions, the route function for progressive route sets is given by

$$\begin{aligned} R_\theta(x;\omega) &= \Theta(x;\omega) = X^{-1}(x;\omega) \\ R_\gamma(x;\omega) &= a(\Theta(x;\omega),\omega) \end{aligned} \ . \tag{5}$$

2.2 The Ship-Motion Function

The ship-motion function describes the distance traveled by a ship along a route as a function of the elapsed time after its departure from the port. Since this motion depends on the nominal speed that the ship travels and possibly on the specific route that the ship follows, we include these quantities as parameters of the ship-motion function. Furthermore, we also allow the ship-motion function to depend on absolute time to account for environmental effects (e.g., storms). With these conventions, the distance traveled from the departure port at time t by a ship that departs at time τ and then travels the route ω with nominal speed v is represented by $x = \hat{M}(t-\tau;t,\omega,v)$. As defined, the ship-motion function takes on distance values in the interval $[0, L(\omega)]$ as t varies from the departure time τ to the arrival time at the destination port η. As with the route function, it is convenient to allow the distance traveled to take on "virtual" values by extending the definition of the ship-motion function to the whole time axis. To this end, we assume constant speed motion for $t \notin [\tau, \eta]$ and define an extended ship-motion function by

$$\hat{M}_x\left(t-\tau;t,\omega,v\right) = \begin{cases} v\left(t-\tau\right) & \text{for } t<\tau \\ \hat{M}\left(t-\tau;t,\omega,v\right) & \text{for } t\in\left[\tau,\eta\right] \\ v\left(t-\tau\right)+L\left(\omega\right) & \text{for } t>\tau \end{cases} . \tag{6}$$

With this definition, the extended motion function $\hat{M}_x(t-\tau;t,\omega,v)$ is a monotonically increasing function of elapsed time $\varsigma = t-\tau$. It follows that given any initial time t_o, there is a unique initial distance x_o obtained by setting $\varsigma = \varsigma_o = t_o - \tau$ in Eq. (5). Furthermore, given an initial distance x_o, there is a unique value of the initial elapsed time

$$\varsigma_o\left(x_o,\omega,v;t_o\right) = t_o - \tau = \hat{M}_x^{-1}\left(x_o;t_o,\omega,v\right), \tag{7}$$

corresponding to x_o. Consequently, it is always possible to express the distance traveled as a function of the initial time t_o and the initial distance x_o. To this end, we redefine the ship-motion function by

$$M\left(t-t_o;x_o,\omega,v\right) = \hat{M}_x\left(t-t_o+\varsigma_o\left(x_o,\omega,v;t_o\right);t,\omega,v\right) . \tag{8}$$

Figure 3 shows an example of these definitions. In this figure, we have illustrated the ship-motion function for two ships, each of which travels the same route ω with the same nominal velocity v. The first ship departs at time τ_1 and arrives at time η_1; the second departs at time τ_2 and arrives at time η_2. During the period when the second ship is in transit, a local storm causes it to reduce speed with the result that its total transit time $\eta_2 - \tau_2$ is longer than the transit time for the first ship. Both ships are present during the observation interval $[t_o, t_o+T]$. The first ship is en route at the initial time t_o since $t_o \in (\tau_1, \eta_1)$. This ship has an elapsed time $\varsigma(x_o, t_o;\omega, v) \ge 0$ and a "real" initial distance $x_o \in [0, L(\omega)]$. The second ship in not en route at the initial time since $t_o \notin (\tau_2, \eta_2)$. However, it is en route during the observation interval since $t_o + T \in (\tau_2, \eta_2)$. This ship has an elapsed time $\varsigma(x_o, t_o;\omega, v) < 0$ and a negative initial distance $x_o \in [vT, 0]$. The first ship arrives at the destination port in the observation interval; the second ship does not.

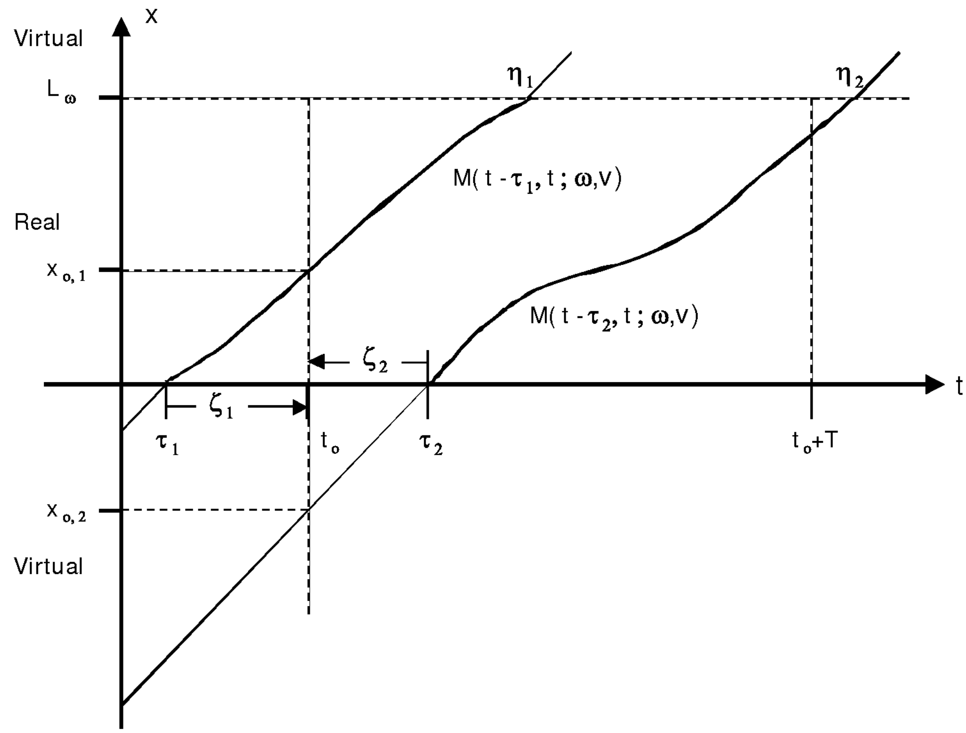

Fig. 3 — The ship-motion function

Note that in this example, the nominal speed parameter specifies a "characteristic" speed of the ship rather than the actual speed made good. Other attributes of the ship relevant to a particular application of the model (e.g., type of ship) can be included in the model by interpreting v as a vector parameter in the subsequent development. Also note that the definition of the motion function does not preclude the second ship from overtaking and passing the first ship. Strictly speaking, with the route function defined by Eq. (1), this would amount to the second ship "passing through" the first ship if both ships were traveling on the same route. However, with the specific route functions considered in this report, this is an event of zero probability and is not considered further.

2.3 The Ship-Track Function

The ship-track function $G_{\lambda,\phi}$ is obtained from the route function $R_{\lambda,\phi}$ by using the ship-motion function M to express the distance traveled by a ship as a function of the elapsed time and the ship parameter. It follows from Eqs. (1) and (7) that the longitude coordinate function G_λ and the latitude coordinate function G_ϕ of the ship-track function have the form

$$\begin{aligned} G_\lambda\left(t-t_o;x_o,\omega,v\right) &= R_\lambda\left(M\left(t-t_o;x_o,\omega,v\right);\omega\right) \\ G_\phi\left(t-t_o;x_o,\omega,v\right) &= R_\phi\left(M\left(t-t_o;x_o,\omega,v\right);\omega\right) \end{aligned} . \tag{9}$$

Note that only the route parameter component of the track parameter $\left(x_o,\omega,v\right)$ is needed to identify the particular route, whereas the complete track parameter is needed to specify the motion along that route.

A ship-track realization is obtained by specifying the number of ship tracks in the realization n and the track parameters for those ships $\{(x_{o_k},\omega_k,v_k);\, k=1,\ldots,n\}$ and then computing the track of each ship using the track function. An example of a ship track realization is shown in Fig. 4. The routes that the ships travel are shown by the thin curves. The tracks of the ships along those routes for the time interval $[t_o, t_o + T]$ are indicated by the heavy lines on these curves. For the tracks along the routes ω_1 and ω_4, the ships are en route during the whole time interval $[t_o, t_o + T]$. For the track along the route ω_3, the ship is en route at time t_o but it arrives at the destination port prior to $t_o + T$. Thus, for these three tracks, the initial-distances are positive and the initial positions, indicated by large dots, lie in the set of real ship coordinates. For the track along the route ω_2, the ship departs from P_n after time t_o. Therefore, the initial distance for this track x_{o_2} is negative and the initial position lies in the set of virtual ship coordinates. At time t_o, the ship starts at the initial distance x_{o_2} and moves along the route segment in the virtual ship coordinates with constant speed v_2 up to the departure time $\tau_2 = t_o - x_{o_2} / v_2$. At this time, it actually departs the port and travels along the route segment shown with its motion determined by $M(t - t_o; t, x_{o_2}, \omega_2, v_2)$.

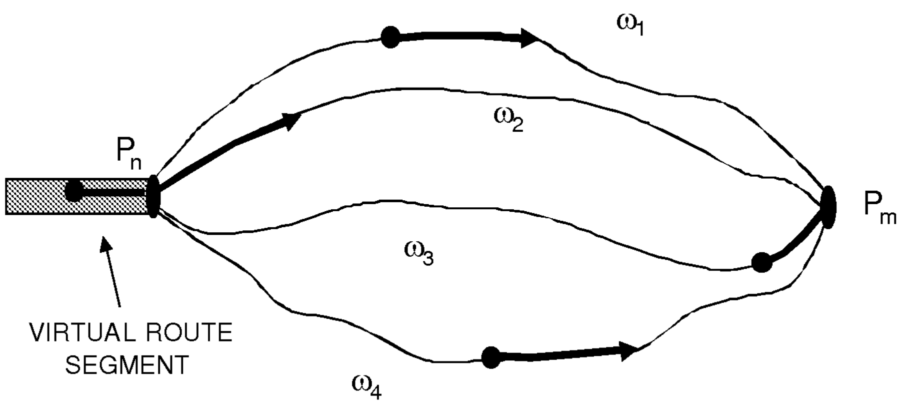

Fig. 4 — Example of a ship track realization for a single port-pair

To determine individual track realizations, it is necessary to have a mechanism for determining the track parameters for each realization. This is done in terms of the port parameter probability law defined in the following section.

3. THE PORT-DEPARTURE AND TRACK-PARAMETER PROBABILITY LAWS

To an observer located at the departure port, the shipping is described by the sequence of parameters $\{(\tau_k, \omega_k, v_k)\}$, where τ_k is the departure time of the k^{th} ship to leave the port, τ_k is its route parameter, and v_k is its nominal speed. In this report, we assume that: (1) the ship departure times are described by a Poisson process with a time-dependent rate function $\mu_\tau(t)$ that represents the mean number of ship departures per unit time; (2) the route that each ship travels and its speed are independent of its departure time and independent of one another; (3) different ships select routes independently of one another and each ship selects its route from the same probability density $p_\omega(\omega)$; and (4) the nominal speeds of all ships are independent of one another and are described by the same probability density $p_v(v)$. The rate function $\mu_\tau(t)$ can be estimated from ship departure data; the ship speed density $p_v(v)$ can be determined from the ships' register data given the identities of the departing ships. However, in general, the route parameter probability density $p_\omega(\omega)$ cannot be determined independently of the probability law describing the route set.

According to the Poisson departure time assumption, the number of ships leaving the port during the time interval $(t_o, t_o + T]$ is a Poisson random variable N_τ with a probability mass function given by

$$\Pr\left[N_\tau = n\right] = \exp\left\{-M_\tau\left(t_o,T\right)\right\}\frac{M_\tau\left(t_o,T\right)^n}{n!}, \tag{10}$$

where

$$M_\tau\left(t_o,T\right) = \int_{t_o}^{t_o+T} \mu_\tau\left(t\right)dt \tag{11}$$

is the mean number of ships to leave the port in the interval. Furthermore, given that there are n ship departures during the interval, the ship departure times $\{\tau_k\ ; k=1,...,n\}$ are independent and identically distributed with common probability density.

$$p_\tau\left(\tau_k;t_o,T\right) = \begin{cases} \mu_\tau\left(\tau_k\right)M_\tau\left(t_o,T\right)^{-1} & \text{for } \tau_k \in \left[t_o,t_o+T\right] \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

Note that since the departure rate $\mu_\tau(t)$ depends on time, the mean number of ships $M_\tau(t_o,T)$, the departure-time probability density $p_\tau(\tau_k;t_o,T)$, and the probability mass function depend on the choice of the time interval $[t_o,t_o+T]$. We allow for this time dependence to incorporate variations in the number of ship departures with time of day, day of the week, etc. For homogeneous Poisson processes, the departure rate is independent of time ($\mu_\tau(t)=\mu_\tau$); hence, the mean number of ship departures is simply $T\mu_\tau$ and the departure times are uniformly distributed on the interval $[t_o,t_o+T]$ (i.e., $p_\tau(\tau_k)=T^{-1}$).

It follows from the port-departure assumptions that the port-departure parameter (τ,ω,v) is described by a multidimensional Poisson process that is specified by a rate function $\mu_{\tau,\omega,v}(t,\omega,v)$. Moreover, from the definitions of Section 2, there is a one-to-one transformation between the port-departure parameter (τ,ω,v) and the ship-track parameter (x_o,ω,v). It follows that the ship-track parameter is also distributed as a multi-dimensional Poisson process that is specified by a rate function $\mu_{x_o,\omega,v}(x_o,\omega,v;t_o,T)$. The specification and derivation of these rate functions is presented in Appendix B (see Eqs. (B1), (B4), and (B5)). The definition of a multidimensional Poisson process is presented in Appendix A.

The Poisson process on the ship-track parameters describes the number and the distribution of the parameters for all ships present in the region during the interval $[t_o,t_o+T]$. As seen in Section 2, the ships present in the region during this interval are those that are en route at time t_o and those that depart during the half open interval $(t_o,t_o+T]$. The former have positive initial distances; the latter have negative initial distances. Consequently, by restricting the track-parameter rate function $\mu_{x_o,\omega,v}(x_o,\omega,v;t_o,T)$ to positive values of x_o, we obtain a Poisson process that describes the ships that are en route at time t_o. Similarly, by restricting $\mu_{x_o,\omega,v}(x_o,\omega,v;t_o,T)$ to negative values of x_o, we obtain a process describing the ships that depart during $(t_o,t_o+T]$. The total number of ships en route during $[t_o,t_o+T]$ is the Poisson process obtained as the composition of the two component processes. For each of these processes, the number of ships is a Poisson random variable with a probability mass function given by Eq. (10) and the track parameter for those ship are independent and identically distributed with a probability density given by an equation analogous to Eq. (12). The mean number of ships is obtained as an integral of $\mu_{x_o,\omega,v}\left(x_o,\omega,v;t_o,T\right)$ over the appropriate volume. The definition of these processes and explicit formulas for the relevant quantities are presented in the following sections.

The track-parameter rate function $\mu_{x_o,\omega,v}(x_o,\omega,v;t_o,T)$ completes the model if the route set probability density $p_\omega(\omega)$ and the route function of Eq. (1) are known. In a simple approximate model, where the traffic is described in terms of, at most, a finite number of routes, the route function might be described analytically in terms of great circle route segments and the weightings $p_\omega(\omega)$ might also be available. However, in the more realistic case, where there is a continuum of routes, the route function and the route parameter density must be determined from data on the routes that the ships follow. In principle, these data can be used to construct a probability law on the ensemble of routes from P_n and P_m. In general, such a probability law consists of the joint probability density on the coordinate positions $\{(\lambda(x_1),\phi(x_1)),...,(\lambda(x_n),\phi(x_n))\}$ for all possible samples of the distances traveled $\{x_1 < ... < x_n; n > 0\}$. To determine such a probability law requires more data than are usually available. Consequently, it is necessary to introduce approximations and assumptions that allow the route functions to be defined in terms of simplified probabilistic descriptions.

In the following section, we describe a "deterministic" model where the route functions are determined from a probability density that can be estimated from any sample of ship coordinates, regardless of whether those coordinates are organized into specific routes. In Section 5 we describe a "stochastic" model which assumes that the ship routes are described by a Markov process.

4. A DETERMINISTIC ROUTE FUNCTION MODEL

In the ship-track model presented here, both the route function and the route parameter density are determined from a probability density on the cross-sectional ship positions γ conditioned on the nominal route coordinate positions θ. This is done in such a way that each route is uniquely determined by any point on that route. As a consequence, the Poisson process describing the distribution of ship-track parameters can be expressed in terms of the initial coordinates of the ship at time t_o, rather than the initial distance and the route parameter. This simplicity is obtained by limiting the route structure to progressive route sets with the additional constraint that the individual routes can not intersect one another. To describe the model, we first define the underlying probability densities. We then specify the route and the track functions that are determined by these densities. Finally, we present the probability law on the track parameter which results from these definitions and the assumptions of Section 3.

4.1 The Route Coordinate Probability Density

The route coordinate probability density describes the concentration of the γ values of the routes as they intersect the cut in the route set at θ. We denote this probability density by $p_{\gamma|\theta}(\gamma;\theta)$, where θ specifies the location of the cut, and refer to the corresponding cumulative probability distribution

$$P_{\gamma|\theta}(\gamma;\theta) = \int_{\varepsilon_l(\theta)}^{\gamma} p_{\gamma|\theta}(\gamma';\theta)\,d\gamma' \tag{13}$$

as the route coordinate distribution function. Figure 5 shows an example of a route coordinate density and distribution function for a specific cut in the route set. In this example, the routes for the cut $\theta = \theta_o$, are concentrated near the lower envelope $e_l(\theta_o)$. The route coordinate density at other cuts in the route set need not be the same as the one shown. For example, the distribution of routes shown in the figure suggests that the route coordinate density for the cut at $\theta = \theta_1$ would reflect a more disburse concentration of the routes closer to the center of the route envelopes. A method for estimating these densities from ship route data can be found in Ref. 2 along with a number of examples.

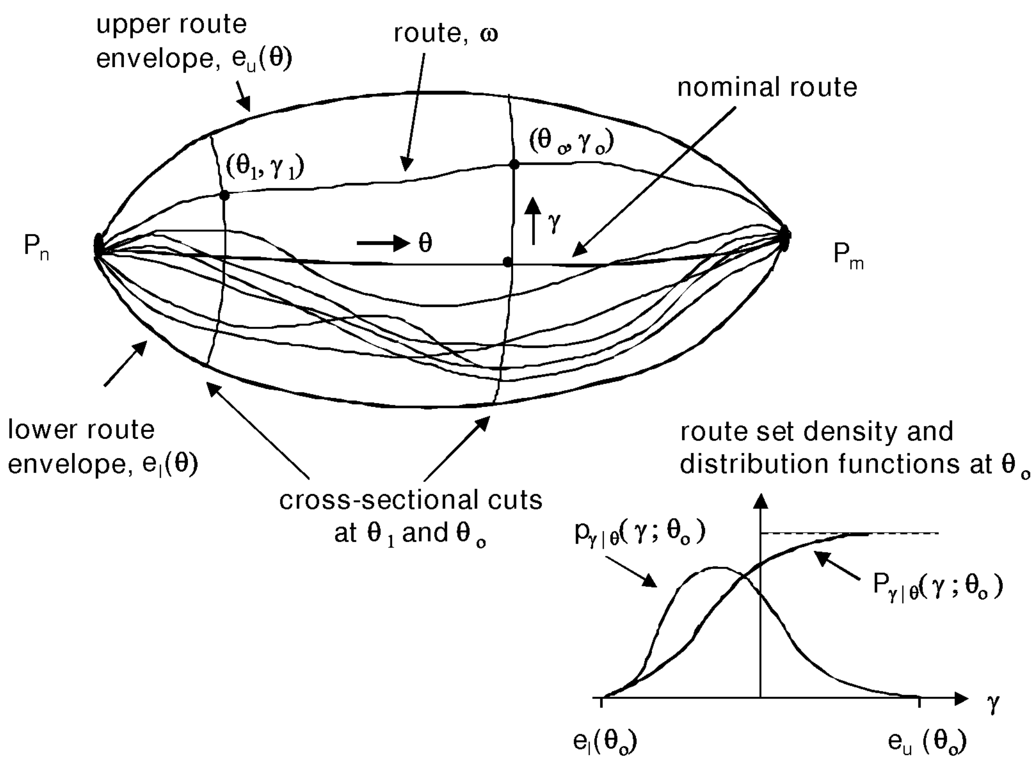

Fig. 5 — Example of a route coordinate probability density

It is important to emphasize that the route coordinate densities, in themselves, do not specify the routes that the ships travel since many different possible route functions can satisfy a given route coordinate density.

4.2 The Route Function

For progressive routes, the route sets are specified by the cross-sectional function $a(\theta,\omega)$ and the route parameter probability density $p_\omega(\omega)$. For the deterministic model, we define these quantities in such a way that the resulting set of routes is consistent with the route coordinate distribution function $P_{\gamma|\theta}(\gamma;\theta)$. To this end, we restrict ω to the interval $[0,1]$ and define the cross-sectional function $a(\theta,\omega)$ as the inverse of the route coordinate distribution. Specifically, for each $\omega \in [0,1]$, $a(\theta,\omega)$ is the function satisfying

$$P_{\gamma|\theta}\left(a(\theta,\omega);\theta\right) = \omega \ . \tag{14}$$

Clearly, $a(\theta,\omega)$ is well defined since for each value of θ, $P_{\gamma|\theta}(\gamma;\theta)$ is an increasing function of γ taking values in the interval $[0,1]$; hence, for fixed θ, there is an inverse function $P_{\gamma|\theta}^{-1}$ that maps $\omega \in [0,1]$ to the set of all possible cross-sectional coordinate values γ. For a fixed ω, the γ value corresponding to the θ value on the route ω is the value given by this inverse function. For notational convenience, we write

$$a(\theta,\omega) = P_{\gamma|\theta}^{-1}(\omega;\theta) \quad \text{for} \quad \theta > 0 \ . \tag{15}$$

Figure 6 illustrates the definition of the cross-sectional function. As seen in the figure, the two route envelopes are also routes in the route set. The lower route envelope $e_l(\theta)$ corresponds to the route parameter $\omega = 0$; the upper route envelope $e_u(\theta)$ corresponds to $\omega = 1$. For fixed θ, the cross-sectional function $a(\theta,\omega)$ yields γ values that increase over the interval $[e_l(\theta),e_u(\theta)]$ as ω increases over the interval $[0,1]$. For a fixed value of ω, $a(\theta,\omega)$ traces out the γ values along the route ω as θ increase from 0 to L_n. Note that the routes are defined so that no two can intersect one another. (If they could, Eq. (13) would not be well-defined.) Also note that since the lower route envelope and the upper route envelope are assumed to be distinct, the interval $[e_l(\theta),e_u(\theta)]$ cannot degenerate to a single point even at the departure port $\theta=0$ or at the destination port $\theta=L$.

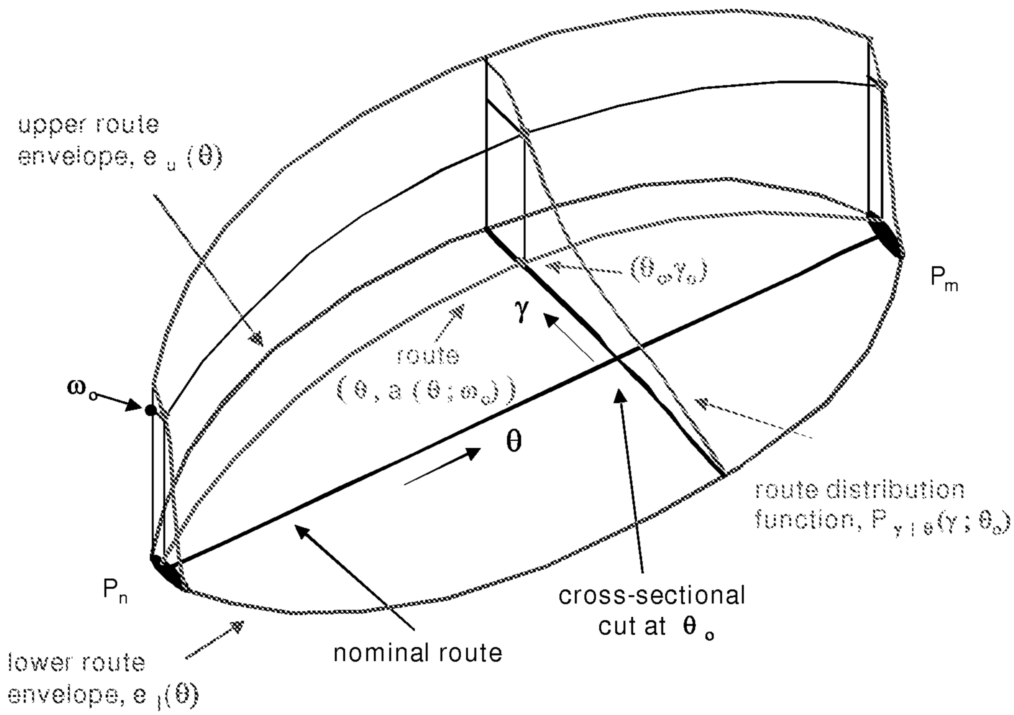

Fig. 6 — Definition of the cross-sectional route function and the route parameter

Equation (15) describes the segments of the routes that lie between the departure port and the destination port. These route segments lie in the set of real ship coordinates $\Re_r=[0,L]\times[e_l(\theta),e_u(\theta)]$. For the route segments that describe the fictitious motion of the ships before they leave the port, we take $a(\theta,\omega)$ to be the inverse route distribution function evaluated at the departure port (i.e., $a(\theta,\omega)=P_{\gamma|\theta}^{-1}(\omega;\theta=0)$) for $\theta<0$. These route segments lie in the set of virtual ship coordinates $\Re_v=(-\infty,0)\times[e_l(0),e_u(0)]$ and have γ coordinates that are independent of θ.

The definition of the cross-sectional function suffices to specify the route function $R_{\theta,\gamma}(x;\omega)$. The θ coordinate of $R_{\theta,\gamma}(x;\omega)$ is $\Theta(x_o;\omega)=X^{-1}(x;\omega)$, where the distance traveled function $X(\theta;\omega)$ is determined from $a(\theta,\omega)$ by Eq. (3). The γ coordinate of $R_{\theta,\gamma}(x;\omega)$ is $a(\Theta(x;\omega),\omega)$. The track function $G_{\theta,\gamma}(t-t_o;x_o,\omega,v)$ is obtained from the route function using the motion function $M(t-t_o;x_o,\omega,v)$ to express the distance traveled as a function of time.

An important consequence of the definition of the cross-sectional function is that there is a one-to-one transformation between the track parameter components (x_o,ω) and the initial coordinates (θ_o,γ_o). In particular, the initial coordinates are determined by $\theta_o = \Theta(x_o;\omega)$ and $\gamma_o = a(\Theta(x_o;\omega);\omega)$. The initial distance and the route parameter are determined by

$$\omega = P_o\left(\gamma_o;\theta\right) = P_{\gamma|\theta}\left(\gamma_o;\theta\right) \tag{16}$$

and

$$x_o = X_o\left(\theta_o,\gamma_o\right) = \int_0^{\theta_o}\left[1+\left(\frac{\partial P_{\gamma|\theta}{}^{-1}\left(\omega;\theta'\right)}{\partial\theta'}\right)^2\right]^{1/2} d\theta' \ . \tag{17}$$

Equation (16) follows immediately from Eq. (14) by replacing $a(\theta,\omega)$ by γ_o. Equation (17) follows from Eq. (3) by setting $\theta = \theta_o$ and by substituting for ω from Eq. (15). Note that by virtue of Eqs. (16) and (17), the route function can be expressed in terms of the initial position and the motion function and the track function can be expressed in terms of the initial position and the characteristic speed. Consequently, the ship track realizations can be obtained from the track function by specifying the track parameter sets $\{(\theta_{o_k},\gamma_{o_k},v_k);\ k=1,\ldots,n\}$. The interpretation of a ship track realization is the same as that of Fig. 3 with x_o and ω replaced by θ_o and γ_o (i.e., the initial positions (the dots) are the points (θ_o,γ_o) and the route parameter in the ship-motion function is determined from these points by Eq. (16)). The probability law on the track parameters is described in the following subsection.

To conclude these definitions, we show that the cross-sectional function is consistent with the route coordinate distribution function if the route parameter is uniformly distributed. To see this, we note that $P_{\gamma|\theta}(\gamma;\theta) = \text{Prob}\{\gamma' \in [e_l(\theta),\gamma]\} = \text{Prob}\{\gamma' \in [e_l(\theta),a(\theta;\omega)]\}$. But $\gamma' \in [e_l(\theta),a(\theta;\omega)]$ is equivalent to $\omega' \in [0,P_{\gamma|\theta}(\gamma;\theta)]$, so that $\text{Prob}\{\gamma' \in [e_l(\theta),a(\theta;\omega)]\} = \text{Prob}\{\omega \in [0,P_{\gamma|\theta}(\gamma;\theta)]\}$, which is the same as the cumulative distribution of the route parameter evaluated at $\omega = P_{\gamma|\theta}(\gamma;\theta)$. Thus, $P_{\gamma|\theta}(\gamma;\theta) = P_\omega(P_{\gamma|\theta}(\gamma;\theta))$, where $P_\omega(\omega)$ is the route parameter cumulative distribution function determined from the route parameter probability density $p_\omega(\omega)$. This equality holds if and only if the route parameter is uniformly distributed on $\left[0,1\right]$, in which case $P_\omega(\omega) = \omega$.

4.3 The Track-Parameter Process

The ship-track probability law describes the number and the distribution of the parameters of all the ships present in the region during the interval $[t_o,t_o+T]$. For the deterministic model, this probability law is obtained from the probability law of Section 3 through an invertible transformation that maps the parameter (x_o,ω,v) to the parameter (θ_o,γ_o,v). It is shown in Appendix B that this results in a Poisson process with a rate function $\mu_{\theta,\gamma,v}(\theta_o,\gamma_o,v;t_o,T)$ defined on the extended route set $\Re_e = \Re_v \cup \Re_r$. As noted in Section 3, this process can be represented as the composition of two Poisson processes, one that describes the ships that are en route at time t_o and one that describes the ships that depart during $(t_o,t_o+T]$. We refer to the former as the “en route” process and to the later as the “entry” process.

To define these processes, it suffices to specify their rate functions. The number of ships and the probability density on the parameters for those ships are then given by Eqs. (10) and (12), where the mean number of ships is obtained as an integral over the rate function. To compute this integral, it is convenient to define a shipping density $\mu_{\theta,\gamma}(\theta_o,\gamma_o;t_o)$ as the integral of the rate function with respect to ship speed. The mean number of ships is then obtained as the integral of the shipping density with respect to the ship coordinates. The shipping distribution is the integral of the shipping density over element areas. The specific equations are as follows. The derivations of these equations are presented in Appendix B.

For the en route process, the only ships that are en route at time t_o are those that have real ship coordinates. Consequently, the rate function is obtained by restricting $\mu_{\theta,\gamma,v}(\theta_o,\gamma_o,v;t_o,T)$ to the real ship coordinates $\Re_r = [0,L]\times[e_l(\theta),e_u(\theta)]$. The rate function is given by

$$\mu_{\theta,\gamma,v}\left(\theta_o,\gamma_o,v;t_o\right)=$$
$$\left[\left(\mu_\tau\left(t_o-\varsigma_o\left(x_o,\omega,v;t_o\right)\right)\left|\frac{\partial\varsigma_o\left(x_o,\omega,v;t_o\right)}{\partial x_o}\right|\right)\Bigg|_{\substack{x_o=X_o(\theta_o,\gamma_o)\\ \omega=P_o(\gamma_o,\theta o)}}\right]\left[p_{\gamma|\theta}\left(\gamma_o;\theta_o\right)\left|\frac{\partial X_o\left(\theta_o,\gamma_o\right)}{\partial\theta_o}\right|\right]p_v\left(v\right) \quad . \quad (18)$$

As noted above, the shipping density for the ships that are en route at time t_o, $\mu_{\theta,\gamma}(\theta_o,\gamma_o;t_o)$ is the integral of the rate function with respect to ship speed;

$$\mu_{\theta,\gamma}\left(\theta_o,\gamma_o;t_o\right)=\int_0^\infty \mu_{\theta,\gamma,v}\left(\theta_o,\gamma_o,v;t_o\right)dv; \qquad \left(\theta_o,\gamma_o\right)\in\Re_r . \quad (19)$$

The mean number of ships with initial coordinates (λ',ϕ') in any set $C_r \subseteq \Re_r$ is the integral of the shipping density with respect to the ship coordinates over the set C_r;

$$M_r\left(t_o;C_r\right)=\int_{C_r}\mu_{\theta,\gamma}\left(\theta',\gamma';t_o\right)d\theta' d\gamma' \ . \quad (20)$$

It follows from the definition of a Poisson process, that the number of ships en route at time t_o with coordinates $(\theta_o,\gamma_o)\in C_r\subseteq\Re_r$, $N_r(t_o;C_r)$, is a Poisson random variable with a probability mass function given by Eq. (10) with M_τ replaced by $M_r(t_o;C_r)$. Furthermore, the track parameters for those ships are independent and identically distributed with the common probability density function

$$p_r\left(\theta_o,\gamma_o,v;t_o;C_r\right)=\mu_{\theta,\gamma,v}\left(\theta_o,\gamma_o,v;t_o\right)M_r\left(t_o;C_r\right)^{-1} \quad \left(\theta_o,\gamma_o\right)\in\Re_r \ . \quad (21)$$

The total number of ships en route at time t_o is determined from these equations by taking $C_r = \Re_r$.

An interpretation of these equations is as follows. The first factor in Eq. (18) describes the mean number of ships per-unit distance along the route determined by (θ_o,γ_o). This factor incorporates the temporal variations in the rate at which ships depart the port through the time dependence on the ship departure rate $\mu_\tau(t)$ and the motion of the ships along the route through its dependence on the departure time function $\varsigma_o(x_o,\omega,v;t_o)$, and its first derivative. For the special case of constant speed motion, $\varsigma_o(x_o,\omega,v;t_o)=x_o v^{-1}$, this factor simplifies to $\mu_\tau(t_o - X_o(\theta_o,\gamma_o)v^{-1})v^{-1}$. Thus, for constant speed motion, a change in the rate at which ships depart the port simply propagates along the route with the nominal ship speed.

The second factor in Eq. (18) describes how the ships are distributed across the routes. This factor is equal to the route coordinate density weighted by the magnitude of a derivative that represents the rate of change in the distance traveled with respect to the nominal route distance. For most route sets, the routes do not depart significantly from the nominal route so that this derivative is approximately unity and hence, the second factor is approximately equal to the route coordinate density. For these "narrow" route sets and for constant speed motion, the track-parameter rate function simplifies to

$$\mu_{\theta,\gamma,v}\left(\theta_o,\gamma_o,v;t_o\right)=\mu_\tau\left(t_o-X_o\left(\theta_o,\gamma_o\right)v^{-1}\right)p_{\gamma|\theta}\left(\gamma_o;\theta_o\right)v^{-1}p_v\left(v\right);\quad \left(\theta_o,\gamma_o\right)\in\Re_r, \tag{22}$$

and the shipping density becomes

$$\mu_{\theta,\gamma}\left(\theta_o,\gamma_o;t_o\right)=p_{\gamma|\theta}\left(\gamma_o;\theta_o\right)\int_0^\infty \mu_\tau\left(t_o-X_o\left(\theta_o,\gamma_o\right)v^{-1}\right)v^{-1}p_v\left(v\right)dv;\quad \left(\theta_o,\gamma_o\right)\in\Re_r\,. \tag{23}$$

Note that if in addition the departure rate is independent of time, $\mu_\tau(\tau)=\mu_\tau$, then the rate function is $\mu_{\theta,\gamma,v}(\theta_o,\gamma_o,v;t_o,T)=\mu_\tau p_{\gamma|\theta}(\gamma_o;\theta_o)v^{-1}p_v(v)$ and the shipping density is $\mu_{\theta,\gamma}(\theta_o,\gamma_o;t_o)=\mu_\tau\overline{\beta}p_{\gamma|\theta}(\gamma_o;\theta_o)$, where $\overline{\beta}$ is the mean value of the reciprocal speed, $\overline{\beta}=E[v^{-1}]$.

For the entry process, the only ships that depart in the interval $(t_o,t_o+T]$ are those that have virtual ship coordinates. Consequently, the rate function for the entry process is obtained by restricting $\mu_{\theta,\gamma,v}(\theta_o,\gamma_o,v;t_o,T)$ to the virtual ship coordinates $\Re_v(T)=[-vT,0]\times[e_l(\theta=0),e_u(\theta=0)]$. The rate function is given by

$$\mu_{\theta,\gamma,v}\left(\theta_o,\gamma_o,v;t_o,T\right)=\mu_\tau\left(t_o-\theta_o v^{-1}\right)p_{\gamma|\theta}\left(\gamma_o;\theta_o=0\right)v^{-1}p_v\left(v\right);\quad \left(\theta_o,\gamma_o\right)\in\Re_v\left(T\right), \tag{24}$$

and the shipping density is

$$\mu_{\theta,\gamma}\left(\theta_o,\gamma_o;t_o,T\right)=p_{\gamma|\theta}\left(\gamma_o;\theta_o=0\right)\int_{|\theta_o|/T}^\infty \mu_\tau\left(t_o-\theta_o v^{-1}\right)v^{-1}p_v\left(v\right)dv;\quad \left(\theta_o,\gamma_o\right)\in\Re_v\left(T\right)\,. \tag{25}$$

Note that Eqs. (24) and (25) are the same as Eqs. (22) and (23), except that $p_{\gamma|\theta}(\gamma_o;\theta_o)$ is replaced by $p_{\gamma|\theta}(\gamma_o;\theta_o=0)$ and the lower limit of the integral in Eq. (24) depends on both (λ',ϕ') and the time interval duration T. This is not surprising since, by the definition of the route function on the virtual coordinates, the ships move at constant speed and the derivative of $X_o(\theta_o;\gamma_o)$ is unity. Also note that for the entry process, the rate function and the shipping density depend on the duration T of the interval $[t_o,t_o+T]$, as well as the initial time t_o; whereas, for the en route process, these parameters depend only on t_o.

Equations (24) and (25) determine the entry process. The mean number of the ships that depart in the interval $(t_o,t_o+T]$ with coordinates $(\theta_o,\gamma_o)\in C_v\subseteq\Re_v(T)$, $M_\tau(t_o,T;C_v)$, is obtained by integrating the shipping density over the set $C_v\subseteq\Re_v(T)$. The probability mass function on the number of these ships $N_e(t_o,T;C_v)$ is given by Eq. (9) with M_τ replaced by $M_e(t_o,T;C_v)$. The probability density on the parameters of these ships $p_e(\theta_o,\gamma_o,v;t_o,T)$ is given by Eq. (11) with $\mu_\tau(\tau_k)$ replaced by $\mu_{\theta,\gamma,v}(\theta_o,\gamma_o,v;t_o,T)$ and $M_\tau(t_o,T)$ replaced by $M_e(t_o,T:C_v)$.

Finally, the Poisson process describing the total number of ships in the interval $N_T(t_o,T)$ is the composition of the en route and the entry processes. Thus, for the closed interval $[t_o,t_o+T]$, the total number of ships is $N_T(t_o,T)=N_r(t_o)+N_e(t_o,T)$, the mean number of ships is $M_T(t_o,T)=M_r(t_o)+M_e(t_o,T)$, and the track-parameter probability density is $p_T(\theta_o,\gamma_o,v;t_o,T)=\mu_{\theta,\gamma,v}(\theta_o,\gamma_o,v;t_o,T)/M_T(t_o,T)$.

The route set assumptions for the deterministic model impose potentially important restrictions on the routes in the route set. In particular, the route set approximation can hold only if the routes have no common points and only if individual routes do not have multiple points (i.e., the routes cannot intersect one another and cannot cross over themselves). For those ships that simply transit from one port to another, the approximation that results from these restrictions may be adequate for those applications that are limited to port-to-port traffic and do not require precise descriptions of the tracks. There is, however, shipping traffic which does more than simply transit from one port to another (e.g., fishing vessels, recreational, and military). For applications where this traffic is important, the route set restrictions of the deterministic model may not be acceptable.

5. A STOCHASTIC ROUTE FUNCTION MODEL

In general, a stochastic description of the ensemble of routes requires the joint probability density on the coordinate positions $\{\theta(x_1,\gamma_1),\ldots,\theta(x_n,\gamma_n)\}$ for all possible samples of the distances traveled $\{x_1 < \ldots < x_n; n > 0\}$. In this section, we present a stochastic model based on the assumption that the route set probability law is Markov. We first specify the route set probability law, describe how the probability law is used to generate the ship tracks in terms of the track parameter (θ_o,γ_o,v), and conclude by presenting the rate function that specifies the Poisson probability law on the ship-track parameters.

5.1 The Route Set Probability Law

For a Markov process, the joint probability density is completely determined by the "first-order" probability density and the probability density transition function. The first-order probability density $p_{\theta(x_1),\gamma(x)}(\theta,\gamma;x)$ describes the coordinates of all routes in the route set that have a fixed value of the distance traveled x. In general, this density depends on x (e.g., for small x the distribution of the ship coordinates is concentrated near the departure port); as x increases, the distribution of the coordinates migrates towards the destination port and become more diffuse. Note that at the departure port ($x = 0$) the nominal route coordinate θ is zero, so that the first-order density has the form

$$p_{\theta(0),\gamma(0)}(\theta,\gamma;x=0) = p_{\gamma_d}(\gamma_d)\delta(\theta)\,. \tag{26}$$

In the following, we refer to γ_d and $p_{\gamma_d}(\gamma_d)$ as the departure cross-sectional coordinate and probability density, respectively.

The transition probability density, denoted by $p_{\theta(x),\gamma(x)|\theta(x'),\gamma(x')}(\theta,\gamma;\theta',\gamma',x,x')$, is the probability density on the coordinates at the distance $x > x'$, $(\theta(x),\gamma(x))$, given the values of the coordinates at the distance x, $(\theta(x')=\theta',\gamma(x')=\gamma')$. The first-order density for the coordinates at x is determined from the first-order density of the coordinates at x' by

$$p_{\theta(x),\gamma(x)}\left(\theta,\gamma;x\right) = \int_{\theta'}\int_{\gamma'} p_{\theta(x),\gamma(x)|\theta(x'),\gamma(x')}\left(\theta,\gamma;\theta',\gamma',x,x'\right) p_{\theta(x'),\gamma(x')}\left(\theta',\gamma';x'\right) d\theta' d\gamma'\,. \tag{27}$$

For the special case of $x' = 0$, the transition density can be written as

$$p_{\theta(x),\gamma(x)|\theta(0),\gamma(0)}\left(\theta,\gamma;\theta',\gamma'=\gamma_d,x,x'=0\right) = p_{\theta(x),\gamma(x)|\gamma_d}\left(\theta,\gamma;\theta',\gamma_d,x\right)\delta\left(\theta'\right)\,, \tag{28}$$

where $p_{\theta(x),\gamma(x)|\gamma_d}(\theta,\gamma;\theta',\gamma_d,x)$ is referred to here as the departure transition probability density. Substituting from the last equation and Eq. (26) into Eq. (27) results in

$$p_{\theta(x),\gamma(x)}(\theta,\gamma;x)=\int_{\gamma'} p_{\theta(x),\gamma(x)|\gamma_d}(\theta,\gamma;\gamma_d;x)\,p_{\gamma_d}(\gamma_d)\,d\gamma_d. \tag{29}$$

5.2 The Route Set Function

As with the deterministic model, the route and the track functions are specified in terms of the track parameter (θ_o,γ_o,v). The set of real ship coordinates $\Re_r$ and virtual ship coordinates $\Re_v$ have the same form as for the deterministic model, where the route envelopes are determined by the support of the first-order probability densities. The definition of the route function is as follows.

For the real ship coordinates, the route function is determined recursively in terms of the transition probability density. The process is as follows. At time t_o, the initial coordinates are specified by the track parameter (θ_o,γ_o,v). After a time δt, the ship has traveled a distance δx as determined by the ship-motion function. The ship coordinates corresponding to this increment in the distance traveled are distributed according to $p_{\theta(x_o+\delta x),\gamma(x_o+\delta x)|\theta(x_o),\gamma(x_o)}(\theta,\gamma;\theta',\gamma',x_o+\delta x,x_o)$. An application of a random number generator for this probability density yields specific values of the ship coordinates $(\theta(x_o+\delta x),\gamma(x_o+\delta x))$. The process is repeated to yield the sequence of ship coordinates corresponding to the sequence of distances traveled determined from the sequence of time increments.

For the virtual coordinates, we take $\gamma=\gamma_d$ and $\theta=x$ in analogy with the deterministic model. At the departure port ($x=\theta=0$), the ship coordinates are $(0,\gamma_d)$, so that the recursive process described above is initiated with the transition density $p_{\theta(x),\gamma(x)|\gamma_d}(\theta,\gamma;\gamma_d;x)$. As such the cross-sectional coordinate γ_d and the density $p_{\gamma|\gamma_d}(\gamma_d)$ correspond to the route parameter ω and its density $p_\omega(\omega)$.

5.3 The Track Parameters

The track parameter (θ_o,γ_o,v) is related to the track parameter $(x_o,\gamma_d,v)=(x_o,\omega,v)$ by the first-order probability density of Eq. (29). It follows from the probability law of Section 3 and the stochastic transformation property of Appendix A, that the track parameter is described by a Poisson process (see Appendix B). For the real ship coordinates, the rate function is given by

$$\mu_{\theta_o,\gamma_o,v}(\theta_o,\gamma_o,v;t_o)=\int_{e_l(0)}^{e_u(0)}\int_0^{L(\gamma d)} \hat{\mu}_{x_o,\gamma_d,v}(x_o,\gamma_d,v;t_o)\,p_{\theta(x),\gamma(x)|\,\gamma_d}(\theta_o,\gamma_o;x_o,\gamma_d)\,dx_o d\gamma_d\ , \tag{30}$$

where

$$\hat{\mu}_{x_o,\gamma_d,v}(x_o,\gamma_d,v;t_o)=\mu_\tau\left(t_o-\varsigma_o(x_o,\gamma_d;v;t_o)\right)\left|\frac{\partial\varsigma_o(x_o,\gamma_d;v;t_o)}{\partial x_o}\right|. \tag{31}$$

For the virtual ship coordinates,

$$\mu_{\theta_o,\gamma_o,v}(\theta_o,\gamma_o,v;t_o,T)=p_{\gamma_d}(\gamma_o)\,p_v(v)v^{-1}\mu_\tau\left(t_o-\theta_o v^{-1}\right),\quad (\theta_o,\gamma_o)\in\Re_v(T). \tag{32}$$

Note that the rate function $\hat{\mu}_{x_o,\gamma_d,v}(x_o,\gamma_d,v;t_o)$ of Eq. (31) is the same as the first factor of Eq. (18) with ω replaced by γ_d and Eq. (32) is the same as Eq. (24) with $p_{\gamma|\theta}(\gamma_o;\theta_o = 0)$ replaced by $p_{\gamma_d}(\gamma_o)$. The application of these equations to determine the shipping densities and the component probability laws is the same as that described in the preceding section.

6. SUMMARY

This report has presented two models that describe the tracks of all ships en route in a region during an arbitrary time interval $[t_o, t_o + T]$. Each of the models consists of the composition of port-pair models for all pairs of ports, pseudo as well as real, that support shipping within the region. Each port-pair model consists of a track function that describes the tracks of all ships present during $[t_o, t_o + T]$ and a probability law on the number of ships present in the interval, the initial positions of those ships, and their nominal speeds.

The probability law on the track parameters is obtained under the assumptions that the times at which the ships depart each port are distributed as a Poisson process with a time-varying departure rate and that the ship speeds and the routes that the ship travel are statistically independent. Under these assumptions, it is shown that the ship-track parameters are distributed as a space-time Poisson process with a time-dependent rate function determined by the departure rate and the route coordinate density functions. This process can be viewed as the composition of two Poisson processes, the "en route process" that describes the ships that are en route at t_o and the "entry process" that describes the ships that depart during the interval $(t_o, t_o + T]$. The en route process is defined on the set of real ship coordinates; the entry process is defined on a set of virtual ship coordinates. The rate function of the en route process determines a shipping density that represents the mean number of ships per unit area in the route set at time t_o. A shipping distribution for the region is determined by integrating the shipping density over the resolution cells in a longitude-latitude grid. For both models, the rate function specifying the Poisson process depends on the port-departure rate function and the characteristic speed probability density. For the deterministic model, the rate function also depends on the route coordinate probability density. For the stochastic model, the rate function also depends on the transition probability density of the Markov process.

For both models, the track function is determined from a route function and a ship-motion function. In the deterministic model, the route function is determined from the route coordinate probability densities in such a way that each route is uniquely determined from any point on that route. As a consequence, the model is limited to route sets where each route "progresses" from the departure port to the destination port without doubling back on itself and no two routes can cross one another. The route coordinate probability density can be estimated from any sample of ship coordinates, regardless of whether those coordinates are organized into specific routes.

In the stochastic model, the ship tracks are obtained as realizations of the Markov process on the ship route coordinates. The process is recursive. Starting from an initial position, a distance traveled is determined from the motion function for a specific time increment; the ship coordinates are then determined from the initial ship coordinates and the distance traveled using the transition probability density of the Markov process. The process is repeated to determine new coordinates by replacing the initial ship coordinates with the current ship coordinates. In the stochastic model, there are no inherent constraints on the route crossings as in the deterministic model. Furthermore, ship location data used to estimate the Markov process transition probability density must necessarily be organized into routes.

Ship location data that is not organized into routes cannot be used to estimate the Markov process.

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Appendix A
TRANSFORMATIONS OF POISSON PROCESSES

In this Appendix, we define a multidimensional, nonhomogeneous, Poisson process and derive two transformation properties that are used in the development of the probability law on the track parameter. The first of these properties applies for deterministic, invertible transformations; the second applies for stochastic transformations.

A1. THE POISSON PROCESS

Points $\{z_k \in Z\}$ are distributed as a Poisson process if, for any two disjoint subsets of Z, C_1 and C_2, the number of points in C_1 and C_2 are statistically independent Poisson random variables $N_z[C_1]$ and $N_z[C_2]$. The probability mass function for the number of points in a set $C \subseteq Z$ is given by

$$\Pr\{N_z[C]=n\} = \exp\{-M_z[C]\}\frac{M_z[C]^n}{n!} \ , \tag{A1}$$

where

$$M_z[C] = \int_C \mu_z(z)\,dz \tag{A2}$$

is the mean number of points in C and $\mu_z(z)$ is the rate function for the process. The Poisson process has the property that if there are n points in the set C, then the positions of those points $\{z_k; k=1,\ldots,n\}$ are independent, identically distributed random variables with common probability density,

$$p_{z|N_z[C]=n}(z_k) = \mu_z(z_k) M_z[C]^{-1} \ . \tag{A3}$$

Clearly, to specify a nonhomogeneous Poisson process, it suffices to specify its rate function. In the special case where the rate is independent of z, the Poisson process is said to be homogeneous. In this case, $M_z[C] = \mu_z \times \text{Volume}[C]$ and $p_{z|N_z[C]=n}(z_k) = \text{Volume}[C]^{-1}$. In the one-dimensional case, where Z is the time axis and C is a time interval, $C = [t_o, t_o + T]$, Eqs. (A1), (A2), and (A3) take the form of Eqs. (10), (11), and (12) in the text.

For later reference we note that the characteristic function of a Poisson process, $c_{N_z[C]}(u) = E\left[\exp\{iuN_z[C]\}\right]$, is given by

$$c_{N_z[C]}(u) = \exp\{M_z[C](\exp\{iu\}-1)\} \tag{A4}$$

and that a stochastic process is a Poisson process if its characteristic function has the form of Eq. (4) in the text.

A2. DETERMINISTIC TRANSFORMATION PROPERTY

Let the points in the space Z be distributed as a Poisson process with density function $\mu_Z(z)$ and let $g(z;q)$ be an invertible transformation from Z to a space Z' for each value of the parameter q. Denote the inverse of this transformation by $g^{-1}(z';q)$. Then the transformation g induces a Poisson process on Z' with a rate function given by

$$\mu_{Z'}(z') = \mu_Z\left(g^{-1}(z';q)\right)J(z';q) \ , \tag{A5}$$

where $J(z';q)$ is the Jacobian of the transformation.

The proof of the transformation property is as follows. Let $B \subseteq Z$ be any set and let $A_q = \left\{z;\, z = g^{-1}(z';q),\, z' \in B\right\} \subseteq Z$ be the pre-image of B under the transformation g. Let N' be the number of points in the set B. We show that B is a Poisson random variable with mean

$$M_{Z'} = \int_B \mu_{Z'}(z')dz' \ , \tag{A6}$$

Where $\mu_{Z'}(z')$ is given by Eq. (5). It suffices to show that the characteristic function of N', $c_{N'}(u) = E\left[\exp\{iuN'_Z\}\right]$, can be written in the form of Eq. (4) in the text. To this end, we note that N' can be written in the form

$$N' = \sum_k I_{A_q}(z_k) \tag{A7}$$

where

$$I_{A_q}(z_k) = \begin{cases} 1 & \text{if } z \in A_q \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function of the set A_q. Substituting from Eq. (A7) into the definition of the characteristic function yields

$$c_{N'}(u) = E\left[\exp\left\{iu\sum_k I_{A_q}(z_k)\right\}\right]. \tag{A8}$$

Expanding the expected value operator in Eq. (A8) yields

$$c_{N'}(u) = \sum_n E_{z_1,\ldots,z_n|N_z=n}\left[\exp\left\{iu\sum_k I_{A_q}(z_k)\right\}\right]\Pr\{N_z = n\}, \tag{A9}$$

where $\Pr\{N_z = n\}$ is given by Eq. (A1). Since the points $\{z_k; k = 1,\ldots,n\}$ are conditionally independent and identically distributed with the probability density of Eq. (A3), the expected value operator inside the summation of Eq. (A9) can be written as

$$\begin{aligned} E_{z_1,\ldots,z_n|N_z=n}\left[\exp\left\{iu\sum_k I_{A_q}(z_k)\right\}\right] &= E_{z_1,\ldots,z_n|N_z=n}\left[\prod_{k=1}^{k=n}\exp\left\{iuI_{A_q}(z_k)\right\}\right] \\ &= \prod_{k=1}^{k=n} E_{z_1,\ldots,z_n|N_z=n}\left[\exp\left\{iuI_{A_q}(z_k)\right\}\right] \text{ -} \\ &= \left(E_{z_{kn}|N_z=n}\left[\exp\left\{iuI_{A_q}(z_k)\right\}\right]\right)^n . \end{aligned} \tag{A10}$$

The expected value in the last equality on the right-hand side of Eq. (A10) can be written as

$$\begin{aligned} E_{z_{kn}|N_z=n}\left[\exp\left\{iuI_{A_q}(z_k)\right\}\right] &= \Pr\left\{A_q|N_z = n\right\}\exp\{iu\} + \left(1 - \Pr\left\{A_q|N_z = n\right\}\right) \\ &= \Pr\left\{A_q|N_z = n\right\}\left(\exp\{iu\} - 1\right) + 1. \end{aligned} \tag{A11}$$

Substituting Eq. (A11) into the last equation in Eq. (A10) and the result into Eq. (A9) yields

$$\begin{aligned} c_{N'}(u) &= \sum_n \left(\Pr\left\{A_q|N_z = n\right\}\left(\exp\{iu\} - 1\right) + 1\right)^n \frac{\exp\{-M\}M^n}{n!} \\ &= \exp\{-M\}\exp\left\{\left(\Pr\left\{A_q|N_z = n\right\}\left(\exp\{iu\} - 1\right) + 1\right)M\right\} \end{aligned}$$

or, equivalently,

$$c_{N'}(u) = \exp\left\{M\Pr\left\{A_q|N_z = n\right\}\left(\exp\{iu\} - 1\right)\right\}, \tag{A12}$$

where M is given by Eq. (A2) with A identified with A_q. Using Eq. (A3) to write

$$\Pr\left\{A_q|N_z = n\right\} = \int_{A_q} \frac{\mu_z(z)}{M}dz \tag{A13}$$

and substituting into the right-hand side of Eq. (A12) yields

$$c_{N'}(u) = \exp\left\{\left(\int_{A_q} \mu_z(z)\right) dz\left(\exp\{iu\} - 1\right)\right\}. \tag{A14}$$

Equation (A14) indicates that N' is a Poisson process with mean

$$M_{z'} = \int_{A_q} \mu_z(z)\,dz. \tag{A15}$$

Making the change of variables in the integral determined by the transformation g yields

$$M_{z'} = \int_B \mu_z\left(g^{-1}(z';q)\right) J(z';q)\,dz' \ , \tag{A16}$$

as was to be shown.

A3. STOCHASTIC TRANSFORMATION PROPERTY

Let the points in the space Z be distributed as a Poisson process with density function $\mu_z(z)$. Assume that given any sequence of points $\{z_k \in Z; k = 1,\ldots,n\}$, there is a sequence of random variables $\{z'_k \in Z'; k = 1,\ldots,n\}$ that are statistically independent and identically distributed with a common probability density $p_{z'|z}(z'_k; z_k)$, i.e.,

$$p_{z'_1,\ldots,z'_n|z_1,\ldots,z_n}(z'_1,\ldots,z'_n; z_1,\ldots,z_n) = \prod_{k=1}^{n} p_{z'|z}(z'_k; z_k) \ . \tag{A17}$$

Then the points in the space Z' are distributed as a Poisson process with a density $\mu_{z'}(z')$ given by

$$\mu_{z'}(z') = \int_Z p_{z'|z}(z'; z)\mu_z(z)\,dz \ . \tag{A18}$$

The proof of the transformation property is as follows. Let $B \subseteq Z'$ be any set and let N' be the number of points in the set B. We show that N' is a Poisson random variable with mean

$$M_{z'} = \int_B \mu_{z'}(z')\,dz', \tag{A19}$$

where $\mu_{z'}(z')$ is given by Eq. (A18). It suffices to show that the characteristic function of N', $c_{N'}(u) = E\left[\exp\{iuN'_z\}\right]$, can be written in the form of Eq. (A4) with the mean given by Eqs. (A18) and (A19). To proceed, note that a point z' is in the set B if there is a point $z \in Z$ that maps into the point z'. Thus, the random variable N' can be written in the form

$$N' = \sum_k I_{Z\times B}\left(z_k; z'_k\right) \ , \tag{A20}$$

where

$$I_{Z\times B}\left(z_k; z'_k\right) = \begin{cases} 1 & \text{if } z \in Z \text{ and } z' \in B \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function of the set $Z \times B$. But Eq. (20) is of the same form as Eq. (A7) with $I_{A_q}\left(z_k\right)$ replaced by $I_{Z\times B}\left(z_k; z'_k\right)$. Thus, proceeding in direct analogy to Eqs. (A8) through (A12), we obtain

$$c_{N'}\left(u\right) = \exp\left\{M \Pr\left\{Z \times B \middle| N_z = n\right\}\left(\exp\left\{iu\right\} - 1\right)\right\} \ , \tag{A21}$$

where M is given by Eq. (A2) with A identified with Z. Now,

$$\Pr\left\{Z \times B \middle| N_z = n\right\} = \int\limits_Z \Pr\left\{B \middle| z \in Z, N_z = n\right\} p_{z|N_z=n}\left(z\right) dz = \int\limits_Z \left[\int\limits_B p_{z|N_z=n}\left(z\right) dz'\right] \frac{\mu_z\left(z\right)}{M} dz \ , \tag{A22}$$

where we have used Eq. (A3) and the fact that

$$\Pr\left\{B \middle| z \in Z, N_z = n\right\} = \int\limits_B p_{z|N_z=n}\left(z\right) dz' \ . \tag{A23}$$

Substituting from Eq. (A22) into Eq. (A21), we obtain Eq. (A4) with $\mu_Z\left(z\right)$ given by Eq. (A19), as was to be shown.

Appendix B
DERIVATION OF THE TRACK-PARAMETER PROBABILITY LAWS

In this Appendix, we derive the probability law on the track parameter (θ_o, γ_o, v) for both the deterministic and the stochastic models. The probability law on the track parameter (λ_o, ϕ_o, v) is determined from the probability law on the track parameter (θ_o, γ_o, v) and the invertible transformation relating these parameters. To obtain these results, we first derive the probability law on the parameter (x_o, ω, v) and then use the transformation properties of Appendix A. Note that it suffices to present these derivations for the single port-pair model since the shipping between different port-pairs is assumed to be statistically independent. We conclude this Appendix with the formal definition of the shipping distribution.

B1. THE TRACK-PARAMETER PROBABILITY LAW FOR (x_o, ω, v)

The probability law on the track parameter (x_o, ω, v) is determined from the probability law on the track parameter (τ, ω, v) and the transformation defined in the text. To derive this probability law, we first show that the points $\{(\tau, \omega, v)\}$ are Poisson distributed and then use the transformation property of Eq. (A5) in Appendix A to show that the points $\{(x_o, \omega, v)\}$ are Poisson distributed and to obtain an expression for their rate function.

To this end, let Z be the set of points $\{z = (\tau, \omega, v);\ \tau \in (-\infty, \infty),\ \omega \in [0,1],\ v > 0\}$. By assumption, the route parameters $\{\omega_k\}$ and the ship speeds $\{v_k\}$ are statistically independent and identically distributed and the departure times $\{\tau_k\}$ are distributed as a Poisson process with rate function $\mu_\tau(t)$. Consequently, the points $z \in Z$ are distributed as a Poisson process with rate function

$$\mu_z(z) = \mu_\tau(\tau) p_\omega(\omega) p_v(v) . \tag{B1}$$

Next, define the space Z' to be the sets of points $\{z' = (x_o, \omega, v)\}$. The probability law on Z' is obtained from the transformation property with the transformation $g(z,q)$ of Eq. (A5) determined from Eqs. (6) and (7) in the text and the identity mappings on ω and v. Specifically, $g(z,q)$ is determined by

$$x_o = M_x(t_o - \tau; \omega, v) , \tag{B2}$$

where the initial time t_o in Eq. (B2) is the parameter q of the transformation $g(\cdot, q)$. The inverse transformation $g^{-1}(z', q)$ is determined by

$$\tau = t_o - \varsigma_o(x_o, \omega, v; t_o) \tag{B3}$$

The Jacobian of the transformation is

$$J\left(x_o,\omega,v;t_o\right)=\left|\frac{\partial\varsigma_o\left(x_o,\omega,v;t_o\right)}{\partial x_o}\right| .$$

Applying the transformation property, we conclude that the points $\left\{z'=\left(x_o,\omega,v\right)\in Z'\right\}$ are Poisson distributed with rate function given by

$$\mu_{x_o,\omega,v}\left(x_o,\omega,v;t_o,T\right)=\hat{\mu}_{x_o,\omega,v}\left(x_o,\omega,v;t_o,T\right)p_\omega\left(\omega\right)p_v\left(v\right) , \tag{B4}$$

where

$$\hat{\mu}_{x_o,\omega,v}\left(x_o,\omega,v;t_o,T\right)=\begin{cases}\mu_\tau\left(t_o-\varsigma_o\left(x_o,\omega,v;t_o\right)\right)\left|\dfrac{\partial\varsigma_o\left(x_o,\omega,v;t_o\right)}{\partial x_o},\right| & \text{for } 0<x_o<L(\omega)\\ \mu_\tau\left(t_o-x_o v^{-1}\right)v^{-1} & \text{for } -vT<x_o<0\end{cases} \tag{B5}$$

B2. THE TRACK PARAMETER PROBABILITY LAW: DETERMINISTIC MODEL

The probability law on the track parameter $\left(\theta_o,\gamma_o,v\right)$ is determined from the probability law on the track parameter $\left(x_o,\omega,v\right)$ and the transformations defined in the text. To derive this probability law, define the spaces Z' and Z'' to be the sets of points $\left\{z'=\left(x_o,\omega,v\right)\right\}$ and $\left\{z''=\left(\theta_o,\gamma_o,v\right)\right\}$, respectively. The transformation from Z' to Z'' is obtained from Eqs. (4) and (15) in the text with x replaced by x_o ;

$$\begin{aligned}\theta_o&=\Theta\left(x_o,\omega\right)\\ \gamma_o&=P_{\gamma|\theta}^{-1}\left(\omega;\Theta\left(x_o,\omega\right)\right)\end{aligned} . \tag{B6}$$

The inverse transformation is given by Eqs. (16) and (17) in the text:

$$\begin{aligned}\omega&=P_{\gamma|\theta}\left(\gamma_o;\theta_o\right)\\ x_o&=X_o\left(\theta_o,\gamma_o\right)=X\left(\theta_o,\omega\right)\Big|_{\omega=P_{\gamma|\theta}\left(\gamma_o;\theta_o\right)}\end{aligned} . \tag{B7}$$

The Jacobian of the transformation is

$$J\left(x_o,\omega,v;t_o\right)=\begin{vmatrix}\dfrac{\partial X_o\left(\theta_o,\gamma_o\right)}{\partial\theta_o} & \dfrac{\partial X_o\left(\theta_o,\gamma_o\right)}{\partial\gamma_o}\\ \dfrac{\partial P_{\gamma|\theta}\left(\gamma_o;\theta_o\right)}{\partial\theta_o} & \dfrac{\partial P_{\gamma|\theta}\left(\gamma_o;\theta_o\right)}{\partial\gamma_o}\end{vmatrix} .$$

To evaluate the Jacobian, we expand the determinant and then use the chain rule to evaluate the partial derivatives of $X_o\left(\theta_o,\gamma_o\right)$, where ω is given by Eq. (13) in the text. Finally, we use Eq. (B7) to obtain

$$J\left(x_o,\omega,v;t_o\right)=p\left(\gamma_o;\theta_o\right)\frac{\partial X_o\left(\theta_o,\gamma_o\right)}{\partial\theta_o} . \tag{B8}$$

It follows from the transformation property of Appendix A, Eqs. (B4) and (B5), and the fact that ω is uniformly distributed, that the points $\{z'' = (\theta_o, \gamma_o, v) \in Z''\}$ are Poisson distributed with the rate function given by Eq. (18) in the text. Furthermore, since the points z'' are distributed as a Poisson process, the initial ship positions (θ_o, γ_o) are distributed as a Poisson process with rate function given by Eq. (19). Equation (20) in the text follows from the general equation for the mean number of points in a set (Eq. (A1) in Appendix A) with the rate function given by Eq. (19) in the text. Eq. (21) in the text follows from Eq. (A3) in Appendix A.

To obtain the probability law for the virtual ship coordinates, we note that (a) these ships travel with constant speed motion; (b) $X_o(\theta_o, \gamma_o) = \theta_o$, and $p(\gamma_o; \theta_o) = p(\gamma_o; \theta_o = 0)$; and (c) the condition $\tau \in (t_o, t_o + T)$ is equivalent to the condition $x_o = \theta_o \in [-vT, 0)$. The results follow using the same logic as that used to obtain the probability law on the real ship coordinates.

B3. THE TRACK-PARAMETER PROBABILITY LAW: STOCHASTIC MODEL

The probability law for the stochastic model is determined from the probability law on (x_o, ω, v) and the stochastic transformation property of Eq. (A18). This follows from the fact that the track parameter for the stochastic model is just (x_o, ω, v) with ω replaced by γ_{d_o}. Thus, (x_o, γ_d, v) is Poisson-distributed with rate function given by Eqs. (B4) and (B5) with ω replaced by γ_{d_o} and $p_\omega(\omega)$ replaced by $p_{\gamma_d}(\gamma_o)$. To apply the Stochastic Transformation property, note that the probability density in Eq. (A18) is the transition probability density of Eq. (29) in the text. Substitution into Eq. (A18) completes the derivation.

B4. THE SHIPPING DISTRIBUTION FUNCTION

The shipping distribution represents the mean number of ships in each resolution cell determined by a rectangular grid defined in the (λ, ϕ) coordinate system. The formal definition of this function is as follows. Let $\Psi = \{(\lambda_i, \phi_j)\}$ be the rectangular grid of points determined by $\lambda_i = \lambda_c + \Delta\lambda(i-1); i = 1, \ldots, I$ and $\phi_j = \phi_c + \Delta\phi(j-1); i = 1, \ldots, Js$, where $\Delta\lambda$ and $\Delta\phi$ are the longitude and latitude spacings, respectively, and λ_c and ϕ_c are the coordinates of the southwest corner of the region. Further, let

$$C(\lambda_i, \phi_j) = \left[\lambda_i - \frac{\Delta\lambda}{2}, \lambda_i + \frac{\Delta\lambda}{2}\right] \times \left[\phi_j - \frac{\Delta\phi}{2}, \phi_j + \frac{\Delta\phi}{2}\right] \tag{B9}$$

be the resolution cell corresponding to the point (λ_i, ϕ_j) and let $C'(\lambda_i, \phi_j)$ be the pre-image of that resolution cell in the auxiliary coordinates obtained under the transformation between the longitude-latitude and the auxiliary coordinates. Finally, let $D_{n,m}(\lambda_i, \phi_j)$ be the shipping distribution for the single port pair $P_{n,m}(\lambda_i, \phi_j)$. Then it follows from Eq. (B11) that $D_{n,m}(\lambda_i, \phi_j)$ is given by

$$D_{n,m}(\lambda_i, \phi_j) = \int_{C'(\lambda_i, \phi_j)} \mu_{\theta,\gamma;n,m}(\theta_o, \gamma_o; t_o) d\theta_o d\gamma_o \ . \tag{B10}$$

The shipping distribution for the total region follows from the fact that an aggregate of independent Poisson processes is itself a Poisson process with a rate function and shipping density given by the sum of the rate functions and the shipping densities of the component processes. Thus, the shipping density for the region is given by

$$\mu_{\theta,\gamma}\left(\theta_o,\gamma_o;t_o\right)=\sum_{n,m=1}^{N_P}\mu_{\theta,\gamma;n,m}\left(\theta_o,\gamma_o;t_o\right), \tag{B11}$$

and, hence, the shipping distribution for the region is given by

$$D\left(\lambda_i,\phi_j\right)=\sum_{n,m=1}^{N_P}D_{n,m}\left(\lambda_i,\phi_j\right). \tag{B12}$$

Appendix C
A SIMPLIFIED DETERMINISTIC MODEL

As seen in the text, the probability law on the track parameter for the deterministic model is considerably simplified by assuming constant speed motion and by approximating $\partial x/\partial\theta_o$ by unity. In this Appendix, we describe further simplifications in the model obtained by making four additional assumptions. The first states that the departure rate is independent of time (i.e., $\mu_\tau(t)=\mu_\tau$). This assumption is often necessitated by practical considerations, since the available departure-time data may be sufficient to estimate the time-averaged departure rate but not its time dependence. The second assumption states that the route coordinate probability density $p_{\gamma|\theta}(\gamma;\theta)$ has the same form for all cuts in the route set. This assumption results in a simplified track function. The third assumption provides a simple approximation for the shipping distribution and the track-parameter probability density for a sufficiently fine grid resolution grid. Finally, the fourth assumption states that the route set itself can be approximated as a sequence of simple route sets, where the nominal route for each is a segment of a great circle arc. This assumption results in simple expressions for the transformation between the auxiliary coordinates and the longitude-latitude coordinates.

C1. CONSTANT DEPARTURE-RATE PROBABILITY LAW

The simplified track-parameter probability law for the real ship coordinates is obtained by substituting μ_τ for the time-dependent departure rate in the expressions for the track-parameter rate function and the shipping density (Eqs. (22) and (23) in the text) and then substituting the results into the expression for the mean (Eq. (20 in the text)) and the track-parameter probability density (Eq. (21} in the text)). The shipping density becomes

$$\mu_{\theta,\gamma}(\theta_o,\gamma_o)=p_{\gamma|\theta}(\gamma_o;\theta_o)\mu_\tau\overline{\beta}\ , \tag{C1}$$

where

$$\overline{\beta}=\int_0^\infty v^{-1}p_v(v)\,dv \tag{C2}$$

is the mean reciprocal ship speed. Furthermore, the mean number of ships with initial positions in a set $C_r=[\theta_1,\theta_2]\times[\gamma_1,\gamma_2]$ becomes

$$M_r(C_r)=\mu_\tau\overline{\beta}\int_{\theta_1}^{\theta_2}\left[P_{\gamma|\theta}(\gamma_2;\theta)-P_{\gamma|\theta}(\gamma_1;\theta)\right]d\theta\ . \tag{C3}$$

Note that the mean of all ships that are en route at time t_o is simply $M_r(\Re_r)=\mu_\tau\overline{\beta}L$. Finally, the track-parameter probability density becomes

$$p_e(\theta_o,\gamma_o,v;C_r)=\hat{p}_v(v)\,p_{\gamma|\theta}(\gamma_o;\theta_o)/L\ , \tag{C4}$$

where

$$\hat{p}_v(v) = (\overline{\beta}v)^{-1} p_v(v) \ . \tag{C5}$$

According to this probability density, the θ_o coordinate is uniformly distributed on the interval $[0, L]$ and, given θ_o, the γ_o coordinate has the probability density $p_{\gamma|\theta}(\gamma_o;\theta_o)$. The ship speed has the probability density $\hat{p}_v(v)$ and is statistically independent of the ship positions.

For the virtual ship coordinates, there is no appreciable simplification in the shipping density other than that obtained by replacing $\mu_\tau\left(t_o - \theta_o v^{-1}\right)$ by μ_τ in Eqs. (24) and (25) in the text. However, for sets of the form $C_v = [-\infty, 0] \times [\gamma_1, \gamma_2]$, the mean number of ships simplifies to[1]

$$M_e(T;C_v) = \mu_\tau T\left[P_{\gamma|\theta}(\gamma_2;\theta_o = 0) - P_{\gamma|\theta}(\gamma_1;\theta_o = 0)\right] \ . \tag{C6}$$

Note that the mean number of all ships that depart in the interval is simply $M_e(T;\Re_v) = \mu_\tau T$, as expected since, by assumption, the departure times are Poisson distributed with rate μ_τ. Finally, the track-parameter probability density for the ships that depart in the interval $(t_o, t_o + T]$ becomes

$$p_d(\theta_o, \gamma_o, v; t_o) = \begin{cases} (Tv)^{-1} p_v(v) p_{\gamma|\theta}(\gamma_o;\theta_o = 0)/L & \text{for } \theta_o \in [-Tv, 0] \\ 0 & \text{otherwise} \end{cases} \ . \tag{C7}$$

According to this probability density, the γ_o coordinate has the probability density $p_{\gamma|\theta}(\gamma_o;\theta_o = 0)$ and is independent of both the θ_o coordinate and the ship speed. The θ_o coordinate and the ship speed are statistically independent with the θ_o coordinate uniformly distributed on the interval $[-vT, 0)$. The ship speed has the density $p_v(v)$, rather than $\hat{p}_v(v)$.

Equations (C3) through (C7) specify the track-parameter probability law in the simplified model. For those ships en route at time t_o, the probability that there are n ships is given by Eq. (10) in the text using the mean of Eq. (C3) and the track parameters for those ships are independent and identically distributed with the common probability density of Eq. (C4). For those ships that depart during the interval $(t_o, t_o + T]$, the probability that there are n ships is also given by Eq. (10) using the mean of Eq. (C6), and the track parameters for those ships are independent and identically distributed with the common probability density of Eq. (C7).

Specific ship track realizations can be obtained using the track function with specific realization of the track parameter obtained from the probability law of Eqs. (C3) through (C6). In particular, for those ships en route at time t_o, a Poisson random number generator can be used to determine an integer n that represents the number of ships en route at time t_o. Then, for each of these n ships, random number generators for the three probability densities of (C4) can be used to determine the corresponding track parameters. A similar procedure can be used for the ships that depart in the interval $(t_o, t_o + T]$. An alternate procedure for generating track-parameter realizations is obtained under the fine resolution assumption described below.

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[1] The integral in the expression for the mean is evaluated by interchanging the order of integration.

Finally, the shipping density of Eq. (C1) determines the shipping distribution for the longitude-latitude grid of interest. For fine-resolution grids, where the cell size is small with respect to the variation in the shipping density, the shipping distribution is approximately given by Eq. (C18).

C2. THE INVARIANT ROUTE COORDINATE PROBABILITY DENSITY

The invariant route coordinate density assumption states that $p_{\gamma|\theta}(\gamma_o;\theta_o)$ has the same form for all values of θ_o. This is equivalent to the assumption that there is a "normalized" cross-sectional variable $\hat{\gamma}$ and a probability density $p_o(\hat{\gamma})$, such that, for each θ_o, the probability density $p_{\gamma|\theta}(\gamma_o;\theta_o)$ is determined from $p_o(\hat{\gamma})$ by the transformation

$$\gamma = \Delta e(\theta_o)\hat{\gamma} + e_a(\theta_o); \quad \text{for } \hat{\gamma} \in [-1,1] \ , \tag{C8}$$

where

$$\Delta e = \left(e_u(\theta_o) - e_l(\theta_o)\right)/2; \quad e_a = \left(e_u(\theta_o) + e_l(\theta_o)\right)/2 \ . \tag{C9}$$

Under this assumption, the components of the track function are given by[2]

$$\begin{aligned} G_\theta(t-t_o;\theta_o;\gamma_o;v) &= \theta(t-t_o) = v(t-t_o) + \theta_o \\ G_\gamma(t-t_o;\theta_o;\gamma_o;v) &= \left[\left(\gamma_o - e_a(\theta_o)\right)e(\theta_o)^{-1}\right]\Delta e\left(\theta(t-t_o)\right) + e_a\left(\theta(t-t_o)\right) \end{aligned} \ . \tag{C10}$$

Note that this track function is independent of the form of the route coordinate density function.

To establish Eq. (C10), we note that it follows from the transformation of Eq. (C8) that the route set density and the route set distribution function are given by

$$\begin{aligned} p_{\gamma|\theta}(\gamma;\theta_o) &= p_o\left(\frac{\gamma - e_a(\theta)}{\Delta e_a(\theta)}\right)\Delta e_a(\theta)^{-1} \\ P_{\gamma|\theta}(\gamma;\theta_o) &= P_o\left(\frac{\gamma - e_a(\theta)}{\Delta e_a(\theta)}\right) \end{aligned} \ , \tag{C11}$$

where $p_o(\hat{\gamma})$ and $P_o(\hat{\gamma})$ are the probability density and the distribution function for the normalized cross-sectional variable, respectively. Furthermore, it follows from the second equation of Eq. (C11) and the definition of the cross-sectional route function in the text that, for each route ω, we must have

$$\left[\left(\gamma - e_a(\theta)\right)/\Delta e(\theta)\right] = P_o^{-1}(\omega) \ , \tag{C12}$$

where P_o^{-1} is the inverse of the distribution function P_o. Now, Eq. (C12) must hold for all points on the route, including the initial point (θ_o,γ_o). Thus, we also have

$$\left[\left(\gamma - e_a(\theta_o)\right)/\Delta e(\theta_o)\right] = P_o^{-1}(\omega) \ . \tag{C13}$$

Comparing these last two equations yields

[2] To obtain this result, it is necessary to identify x with θ. Although this follows from the assumption that $\partial X / \partial \theta_o = 1$, it is clearly an approximation for any nondegenerate route set.

$$\gamma = \left(\gamma_o - e_a\left(\theta_o\right)\right)\left(\Delta e(\theta)/\Delta e\left(\theta_o\right)\right) + e_a(\theta). \tag{C14}$$

To obtain the track function, we identify x with θ and use the assumption of constant speed motion to write

$$\theta\left(t - t_o\right) = v\left(t - t_o\right) + \theta_o \ . \tag{C15}$$

Equation (C10) follows immediately from Eqs. (C14) and (C15).

We conclude this subsection by noting that the track function of Eq. (C10) can be further simplified by taking the lower route envelope to be the negative of the upper route envelope, i.e., $e_l(\theta) = -e_u(\theta) = -e(\theta)$. For this case, $e_a(\theta) = 0$ and $\Delta e(\theta) = e(\theta)$ so that γ component of the track function of Eq. (C10) becomes

$$G_\gamma\left(t - t_o; \theta_o; \gamma_o; v\right) = \gamma = \left[\gamma_o / e\left(\theta_o\right)\right] \Delta e\left(\theta\left(t - t_o\right)\right). \tag{C16}$$

A particularly useful choice for the normalized probability density $p_o(\hat{\gamma})$ is the beta density modified to span the interval $[-1,1]$, i.e.,

$$p_o(\hat{\gamma}) = \frac{\Gamma(\alpha+\beta)}{2\Gamma(\alpha)\Gamma(\beta)} \left(\frac{1+\hat{\gamma}}{2}\right)^{\alpha-1} \left(\frac{1-\hat{\gamma}}{2}\right)^{\beta-1} . \tag{C17}$$

The route coordinate probability density used for the San Diego ship track example in Ref. 5 is the Beta density for $\alpha = \beta = 2$. The route envelope $e(\theta)$ was taken to be a segment of a sinusoid with an amplitude that depends on the length of the route [6].

C3. THE FINE RESOLUTION ASSUMPTION

We assume that the resolution cells $C\left(\lambda_i, \phi_j\right)$ in the grid Ψ are sufficiently small that the shipping density is approximately constant over each resolution cell and equal to its value in the center of the resolution cell. Under this assumption, the port-pair shipping distribution is approximately given by

$$D\left(\lambda_i, \phi_j\right) = \mu_{\theta,\gamma}\left(\Theta\left(\lambda_i, \phi_j\right), \Gamma\left(\lambda_i, \phi_j\right)\right) A\left(\phi_j\right) , \tag{C18}$$

where $A\left(\phi_j\right)$ is the area of the resolution cell $C\left(\lambda_i, \phi_j\right)$ and $\Theta\left(\lambda_i, \phi_j\right)$ and $\Gamma\left(\lambda_i, \phi_j\right)$ are the coordinate functions in the mapping from the latitude-longitude coordinates to the auxiliary coordinates. Substituting for the shipping density from Eq. (C1) into Eq. (C18) yields

$$D\left(\lambda_i, \phi_j\right) = \mu_\tau \bar{\beta} p_{\gamma|\theta}\left(\Gamma\left(\lambda_i, \phi_j\right); \Theta\left(\lambda_i, \phi_j\right)\right) A\left(\phi_j\right) . \tag{C19}$$

For small resolution cells, $A\left(\phi_j\right)$ is approximately given by

$$A\left(\phi_j\right) = R_e^{\,2} \cos\left(\phi_j\right) \Delta\lambda \Delta\phi \tag{C20}$$

where R_e is the radius of the Earth.

To obtain an alternate means of generating the ship-track parameters, we derive the probability density on the track parameter given that the ship is in the resolution cell. This probability density is given by the rate function divided by the mean number of ships in the cell. Under the fine resolution assumption, the mean $M_r(t_o;C)$ is given by Eq. (C18) and the rate function is constant in the resolution cell with the value determined by replacing θ_o and γ_o by $\Theta(\lambda_i,\phi_j)$ and $\Gamma(\lambda_i,\phi_j)$, respectively. Thus, we have

$$p_e(\lambda_i,\phi_j,v;C) = \frac{\mu_{\lambda,\phi,v}\left(\Theta(\lambda_i,\phi_j),\Gamma(\lambda_i,\phi_j),v\right)}{\mu_{\lambda,\phi}\left(\Theta(\lambda_i,\phi_j),\Gamma(\lambda_i,\phi_j)\right)A(\phi_j)} . \tag{C21}$$

For the simplified model shipping density of Eq. (C1), this probability density becomes

$$p_e(\lambda_i,\phi_j,v;C) = \frac{\hat{p}_v(v)}{A(\phi_j)} , \tag{C22}$$

where $\hat{p}_v(v)$ is given by Eq. (C5).

According to Eq. (C22), given that a ship is in the resolution cell, its position in that cell is uniformly distributed and its speed is independent of its position and has the probability density $\hat{p}_v(v)$. Thus, under the fine resolution approximation, the ship track realizations can also be generated by first using a Poisson random number generator with the mean given by Eq. (C6) to determine the number of ships in the resolution cell and then using uniform random number generators for the probability densities of Eq. (C22) to generate the track parameters.

C4. COMPOUND ROUTE APPROXIMATION

The transformation between the auxiliary coordinates and the longitude-latitude coordinates has a particularly simple form in the special case where the nominal route is the great circle arc connecting the departure port with the destination port. However, for many port-pairs, it is not possible to choose the nominal route to be a great circle arc since the routes must bend to avoid land masses. For these port-pairs, the nominal route can be approximated as a sequence of great circle arcs connected at their endpoints (see Fig. C1). For each of these great circle arcs, the routes can be described in the auxiliary coordinates using a route function defined for the great circle arc and the simple coordinate transformation can be used to map those routes to the longitude-latitude coordinates. These "simple" route sets can then be joined together to form the complete route set between the departure port with the destination port. A method for connecting simple route sets to form a "compound route set" is described in Springer et al.[3] along with the coordinate transformations for the great circle nominal routes.

[3] P. Scrimger, R. Heitmeyer, and P. Boulon, "A Computer Model of Merchant Shipping in the Mediterranean Sea," SACLANT Undersea Research Center Report, SR-164, La Spezia, Italy, SACLANT Undersea Research Center, 1990.

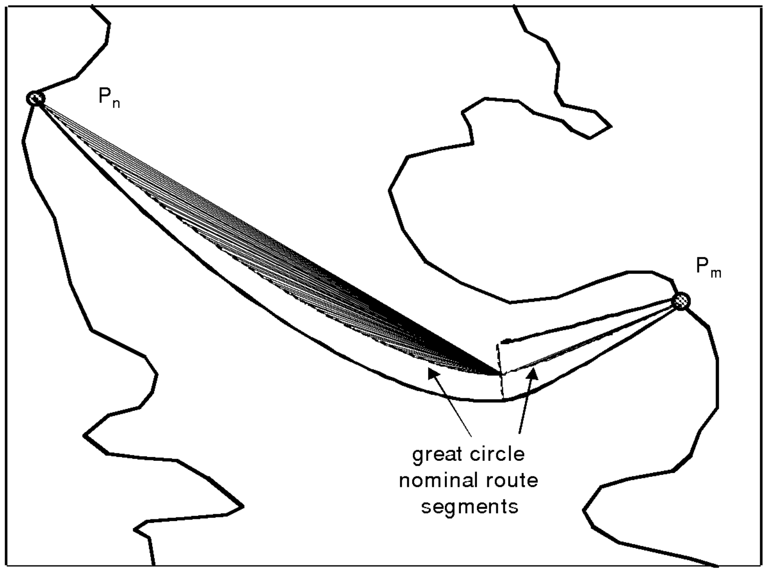

Fig. C1 — An example of a compound route set formed from two simple route sets