NRL Report 8432

Performance of Optimum and Suboptimum Detectors for Spread Spectrum Waveforms

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December 30, 1980

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NRL Report 8432	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PERFORMANCE OF OPTIMUM AND SUBOPTIMUM DETECTORS FOR SPREAD SPECTRUM WAVEFORMS		5. TYPE OF REPORT & PERIOD COVERED Interim report on a continuing NRL problem. 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Donald G. Woodring		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, DC 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62721N; ZF21-222-001; 75-0203-0-0
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE December 30, 1980 13. NUMBER OF PAGES 50
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution limited to U.S. Government agencies only; test and evaluation; December 1980. Other requests for this document must be referred to the Commanding Officer, Naval Research Laboratory, Washington, DC 20375

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Optimum detectors
Filter bank combiner
Low probability of intercept
Frequency-hopped waveforms

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The vulnerability of spread spectrum waveforms to unintended interception is examined. Optimum and suboptimum intercept receivers are derived for a general class of frequency hop waveforms. Alternative techniques for analyzing the performance of these detectors are presented along with specific examples of calculations. In each case, these alternative techniques are shown to give equivalent results. The discussions are intended to provide some insight into the detection process.

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S/N 0102-014-6601

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CONTENTS

PERFORMANCE OF OPTIMUM AND SUBOPTIMUM DETECTORS FOR SPREAD SPECTRUM WAVEFORMS

INTRODUCTION

Signal detection offers an enemy the opportunity to gain information on platform existence, location, identification, and perhaps message content, depending on the sophistication he employs. Consequently, detection techniques are a key issue during waveform selection for covert communication systems. Selection must be based on an evaluation of detectability by postulated threats and tradeoffs between detectability, system cost, and required threat investment. This report presents a variety of techniques for evaluating waveform performance against several detection models.

The performance of a signal detector is best described by the carrier signal power-to-noise density ratio required at the detector input for a specified probability of detection, P_D, and probability of false alarm, P_{FA}. Detection and false alarm probabilities can be specified independent of signal structure, detector strategy, and implementation and are strictly a matter of operational doctrine. In general, the level of the listener's effort in responding to an alarm will determine the maximum number of false alarms he can tolerate within a given time. On the other hand, the value he places upon detection of a transmission or a transmitting platform will determine the maximum number of valid transmissions he is willing to miss, and consequently the minimum percentage he can expect to detect. Once P_{FA} and P_D are specified, the performance of any signal against any detector postulated can be completely described by the input signal power-to-noise density ratio required, $(C/N_o)_{req.}$, to achieve these probabilities.

The determination of optimum detectors for signals with unknown parameters in Gaussian noise is based upon the likelihood ratio criteria which are detailed in Appendix B and essentially follow from Peterson [1]. This report outlines techniques utilized in the calculation of the performance of optimum and suboptimum detectors for the general class of spread-spectrum signaling techniques. A more detailed treatment of frequency-hopped signal detectability can be found in a separate report [2].

RADIOMETRIC DETECTOR PERFORMANCE

For an unknown signal in additive white Gaussian noise occupying a bandwidth W and time interval T, the optimum detector is a simple energy detector (or radiometer) as shown in Fig. 1. The statistics describing the output of such a device are well known [3]. With noise only at the input, the output follows a chi-square density function with $2TW$ degrees of freedom. With a signal present, the output has a noncentral chi-square density function with $2TW$ degrees of freedom and a noncentrality parameter, $2E/N_o$, where N_o is the one-sided noise power density, and

$$\frac{E}{N_o} = \left(\frac{C}{N_o}\right) T \tag{1}$$

is the predetection energy-to-noise density ratio.

Manuscript submitted May 28, 1980.

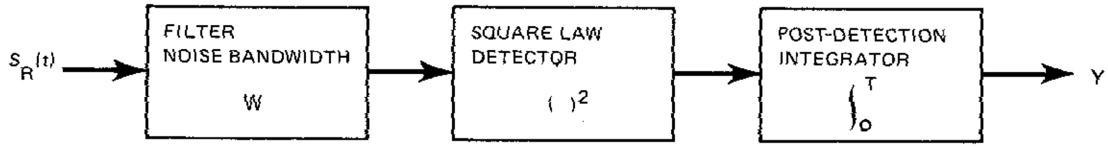

Fig. 1 — *Simple energy detector*

Performance Calculation by Gaussian Approximation

For large TW products, the output statistics for the system in Fig. 1 may be assumed Gaussian and the detector performance can be completely characterized by d^2. Defined in Appendix B as the square of the difference in the means of the output densities under noise and signal-plus-noise conditions, d^2 is a measure of the postdetection or output signal-to-noise power ratio of the detector. For the radiometer, this can be shown to yield [4]

$$d^2 = [Q^{-1}(P_{FA}) - Q^{-1}(P_D)]^2 = \frac{1}{TW}\left\{\frac{E}{N_o}\right\}^2_{\text{req.}} = \frac{T}{W}\left\{\frac{C}{N_o}\right\}^2_{\text{req.}} \tag{2a}$$

or

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = d\sqrt{\frac{W}{T}} = [Q^{-1}(P_{FA}) - Q^{-1}(P_D)]\sqrt{\frac{W}{T}} \tag{2b}$$

where Q^{-1} is the inverse normal cumulative distribution function, $(C/N_o)_{\text{req.}}$ is the carrier power-to-noise density ratio required at the input to the intercept receiver for the specified P_D and P_{FA}, and d is the output signal-to-noise *voltage* ratio, a quantity which is directly proportional to the input C/N_o. The quantity d is plotted in Fig. A-1 as a function of P_{FA} and P_D, and its utility is illustrated in Example 1.

EXAMPLE 1

Consider a frequency-hopped signal with the following characteristics:

T = message duration = 4 sec

and

W = spread bandwidth = 2 GHz.

The time-bandwidth product (TW) is large, so Eq. (2) may be used. For a performance criteria of $P_D = 0.1$ and $P_{FA} = 10^{-6}$, the postdetection SNR, d^2, is found from Fig. A-1:

$$d^2 = [Q^{-1}(10^{-6}) - Q^{-1}(0.1)]^2 = 10.8\,\text{dB}.$$

Thus from Eq. (2)

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = 48.9\,\text{dB-Hz}.$$

For small TW products the Gaussian approximation to the chi-square distribution will yield results which are generally pessimistic in the predicted covertness of the waveform (*i.e.*, the calculated $(C/N_o)_{\text{req.}}$ will be less than the true value). The difference between the C/N_o computed from the chi-square statistics and the Gaussian approximation is plotted in Fig. A-2 (as a function of TW) in terms of a correction factor, η_{TW}, where

$$\eta_{TW} \triangleq \frac{(C/N_o)_{\text{req.}}(\text{assuming } \chi\text{-square statistics})}{(C/N_o)_{\text{req.}}(\text{assuming Gaussian statistics})}.$$

Therefore, Eq. (2b) can be rewritten in the general case

$$\left(\frac{C}{N_o}\right)_{\text{req.}} = \eta_{TW} d \sqrt{\frac{W}{T}} = \eta_{TW} \left[Q^{-1}(P_{FA}) - Q^{-1}(P_D)\right] \sqrt{\frac{W}{T}}. \tag{3}$$

The Gaussian approximation may then be corrected by merely adding the correction factor (in dB) to the value of C/N_o determined from Eq. (2) as shown in Example 2.

EXAMPLE 2

Consider the signal of Example 1, except that now we wish to compute the detectability of a single hop or pulse. In this case,

$$T = T_P = \text{pulse duration} = 500\ \mu\text{sec}$$

and

$$W = W_P = \text{pulse bandwidth} = 2000\ \text{Hz}.$$

The TW product is now one, and Eq. (3) must be used. Again, the postdetection SNR is

$$d^2 = 10.8\ \text{dB (from Fig. A-1)}.$$

The chi-square correction factor for $TW = 1$ is

$$\eta_{TW} = 3.3\text{dB (from Fig. A-2b)}.$$

Thus, for a single pulse, Eq. (3) yields

$$\left(\frac{C}{N_o}\right)_{\text{req.}} = 41.7\ \text{dB-Hz}.$$

An Alternate Technique for Performance Calculation

An alternative technique for calculating the performance of the radiometric energy detector is based on sampling theory. A narrowband-limited process of duration T seconds can be represented by a series of TW pairs of samples, each containing amplitude and phase information on the process during the sampling period. These pairs of samples are either the inphase and quadrature samples or envelope and phase samples, and may be considered samples of a pulse with duration $T = 1/W$. Thus, the detection of the entire signal can be treated as the sequential detection of unit time-bandwidth product pulses, followed by postdetection noncoherent combining of TW of these pulses. This model for the radiometric detector is shown in Fig. 2.

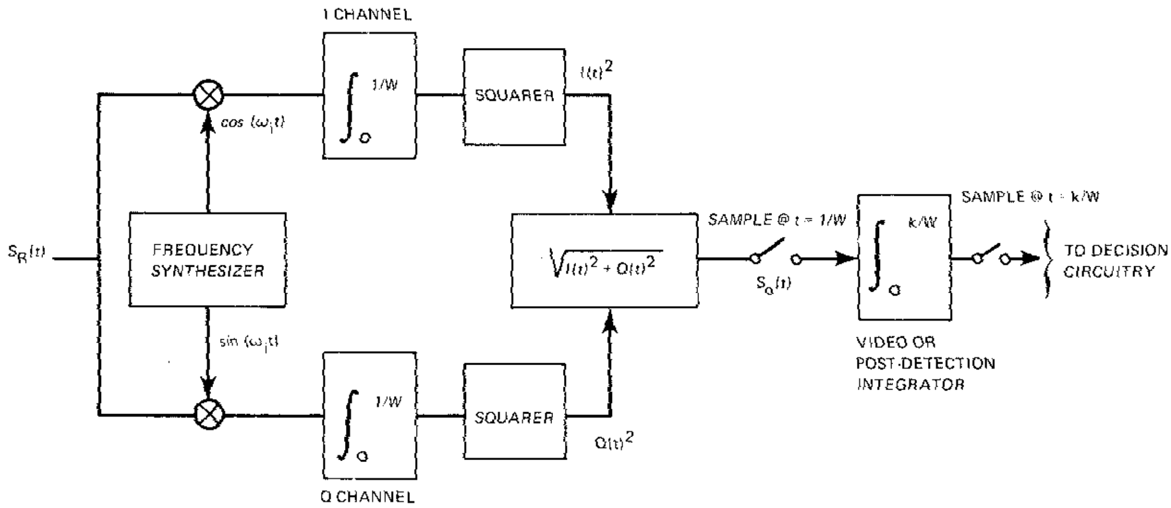

Fig. 2 — Noncoherent radiometric detection

This problem then reduces to the classic radar detection problem of a nonfluctuating, noncoherent pulse train of fixed length T. The postdetection signal-to-noise ratio per pulse, $(S/N)_{\text{pulse}}$, is just the input energy-to-noise density ratio divided by the number of pulses:

$$\left\{\frac{S}{N}\right\}_{\text{pulse}} = \left\{\frac{E}{N_o}\right\} \cdot \frac{1}{TW} = \left\{\frac{C}{N_o}\right\}\frac{1}{W}.$$

It can be shown [3] that the noncoherent sum of TW pulses has a noncentral chi-square probability density function with TW degrees of freedom and noncentrality parameter $d_\chi(TW)$, the RMS output signal-to-noise ratio, where

$$d_\chi^2\,(TW) = TW\left\{\frac{S}{N}\right\}^2_{\text{pulse}}. \tag{5}$$

Thus the performance can be computed directly from the incomplete Toronto function which is plotted in Marcum [3].

A somewhat simpler approach, however, has been suggested by DiFranco and Rubin [4]. If coherent addition of the TW samples is considered, the output statistics can be expressed in terms of the more familiar Rayleigh and Rician density function for the noise-only and signal-plus-noise cases, respectively. (These functions are special cases of the chi-square and noncentral chi-square functions for two degrees of freedom.) Therefore, the performance can be computed for a specified P_{FA} and P_D:

$$P_{FA} = \int_K^\infty xe^{-x^2/2}dx = e^{-K^2/2}, \tag{6a}$$

and

$$P_D = \int_K^\infty xe^{-(x^2+2d_\chi)/2}\, I_o\,(x\sqrt{2\,d_\chi})\,dx, \tag{6b}$$

where $K = \sqrt{2\ln(1/P_{FA})}$ is the detection threshold and d_χ is the required output SNR for the coherent combining case. Note that

$$d_X = d_X(TW) \text{ for } TW = 1.$$

The ROC curves determined from Eqs. (6a) and (6b) are plotted in Fig. A-3. It should be pointed out that these curves can be obtained from the known signal ROC by adding the chi-square correction factor η_{TW} for $TW = 1$, or

$$d_X = \eta_1 d = \eta_1 [Q^{-1}(P_{FA}) - Q^{-1}(P_D)]. \tag{7}$$

For the same specified performance, then, the required SNR for the coherent pulse train will be less than that of the noncoherent case. This performance degradation is usually referred to as the integration loss, or noncoherent combining loss (NCL), $L_{TW,}$ where the subscript indicates the number of pulses or samples being combined, and

$$L_{TW} = \frac{d_X(TW)}{d_X}. \tag{8}$$

This function is plotted in Fig. A-4 as a function of TW, and with d_X as a parameter.

Thus, from Eqs. (4), (5), and (8),

$$d_X(TW) = \left(\frac{C}{N_o}\right) T = L_{TW} d_X \tag{9}$$

so that the required input (C/N_o) is

$$\left(\frac{C}{N_o}\right)_{\text{req.}} = \frac{1}{T} d_X L_{TW}. \tag{10}$$

Example 3 illustrates the correspondence between Eqs. (10) and (3).

It is interesting to compare the two techniques for determining the performance of the radiometric detector as described in Eqs. (3) and (10). The noncoherent combining loss can be related to the chi-square correction factor, η_{TW}, by

$$L_{TW} = \frac{\eta_{TW}}{\eta_1} \sqrt{TW}. \tag{11}$$

Substituting the above expression for L_{TW}, and $\eta_1 d$ for d_X (Eq. (7)) in Eq. (10) yields

$$\left(\frac{C}{N_0}\right)_{\text{req.}} = \frac{1}{T} (\eta_1 d) \frac{\eta_{TW}}{\eta_1} \sqrt{TW} = \eta_{TW}\, d \sqrt{\frac{W}{T}}, \tag{12}$$

which is the same as Eq. (3). For large TW products, η_{TW} approaches one and L_i can be approximated by

$$L_{TW} \approx \frac{\sqrt{TW}}{\eta_1}; TW \sim \text{Large} \tag{13}$$

EXAMPLE 3

The signal in Example 1 can be analyzed by use of Eq. (10). The required postdetection SNR, d_χ, for a pulse of unknown phase is found from Fig. A-3:

$$d_\chi^2 = 17.4 \text{ dB for } P_D = 0.1,\ P_{FA} = 10^{-6}.$$

Note that $d^2 = 10.8$ dB and $\eta_{TW} = 3.3$ dB so that from Eq. (7),

$$d_\chi = d\eta,$$

and

$$8.7 \text{ dB} = 5.4 \text{ dB} + 3.3 \text{ dB}.$$

The noncoherent combining loss, L_{TW}, for $TW = 8 \times 10^9$ is determined from Fig. A-4:

$$L_{TW} = 11.7 \text{ dB, for } TW - 10^3 \text{ and } d_\chi = 8.7 \text{ dB}.$$

For $TW > 10^3$, the slope of the L_{TW} curves is $\sqrt{TW}$, so that the additional loss for $TW > 10^3$ is given by $\sqrt{TW/10^3}$. Thus,

$$L_{TW} = L_{1000}\sqrt{\frac{8 \times 10^9}{10^3}} = 11.7 \text{ dB} + 34.5 \text{ dB} = 46.2 \text{ dB}.$$

Finally, Eq. (10) yields (for $T = 4$ sec)

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = -6\text{dB-Hz} + 8.7 \text{ dB} + 46.2 \text{ dB} = 48.9 \text{ dB-Hz},$$

which is the same result obtained in Example 1.

and Eq. (13) becomes

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = d\sqrt{\frac{W}{T}}, \tag{14}$$

which gives a result identical to Eq. (2). Under the Gaussian assumption then, d in Eq. (2) is the approximation to the single-pulse, postdetection, signal-to-noise *voltage* ratio and $\sqrt{TW}$ is the approximation to L_{TW}, the noncoherent combining loss.

OPTIMUM DETECTION OF SPREAD SPECTRUM SIGNALS

The wideband radiometer discussed in the previous section is optimum, in the maximum-likelihood sense, for any signal whose only known characteristics are the total bandwidth W and duration T. This is true of waveforms employing frequency-hopping, direct sequence pseudo-noise (PN) modulation, pulsed transmission, or combinations of these techniques to achieve a spread spectrum signal of this bandwidth and duration. In practice, however, many characteristics of the signal such as hopping rate, modulation, instantaneous bandwidth, and pulse duration, must be treated as known, while only secure generating functions, patterns, or codes remain completely unknown to the interceptor. These known characteristics can be exploited by the interceptor in designing an optimum detector.

The optimum detector for spread spectrum waveforms is the likelihood ratio receiver which utilizes the structure and statistics of all known signal parameters. The derivation of a general likelihood ratio detector for frequency-hopped, PN-spread, pulsed waveforms and their hybrids is given in Appendix B.

Spread Signals with Unity Time-Bandwidth Products

Assume a frequency-hopped signal of duration T occupying a bandwidth W which consists of N_p pulses of duration T_p, each occurring in one of M channels* of bandwidth W_P, where

$$T_p = \frac{T}{N_P}$$

and

$$W_P = \frac{W}{M} = \frac{1}{T_P}. \tag{15}$$

This signal has a TW product per pulse equal to one.

Although the optimum detector for this frequency-hopped signal can be defined (Fig. 3), the distribution function of the output statistic has not been determined and exact expressions for the performance of the receiver cannot be obtained. However, for a large number of pulses ($N_p > 100$) the output statistic of the equivalent log likelihood ratio detector can be assumed to have a Gaussian density function in both the noise-only and the signal-plus-noise cases.

The performance of this detector can be approximated by the parameter d^2, the postdetection SNR, provided that the variances of the output statistics are approximately equal under noise-only and signal-plus-noise inputs. This is found from Appendix B to be given by

$$d^2 = N_p \ln\left\{1 + \frac{1}{M}\left[I_o\left(\frac{2S}{N}\right) - 1\right]\right\}, \tag{16}$$

where, under the assumption of Gaussian statistics, the required postdetection SNR, d^2, for the specified P_D and P_{FA} is given by Eq. (2) and can be found from Fig. A-1. The term S/N is the input signal-to-noise power ratio in a single radiometer bandwidth W_P, or equivalently, for $T_P W_P = 1$, the single pulse predetection signal energy-to-noise density ratio, E_p/N_o. Thus, the C/N_o required to detect can be computed by

$$\left(\frac{C}{N_o}\right)_{\text{req.}} = W_P\left(\frac{S}{N}\right) = W_P \frac{1}{2} I_o^{-1}[1 + M(e^{d^2/N_p} - 1)] \tag{17}$$

where $I_o^{-1}(\ )$ is the inverse of the modified Bessel function plotted in Fig. A-5.

Example 4 illustrates the performance of this detector for a simple frequency-hopping waveform.

*M is the number of radiometer channels required to cover the signal bandwidth and is not necessarily the number of signaling channels or tones.

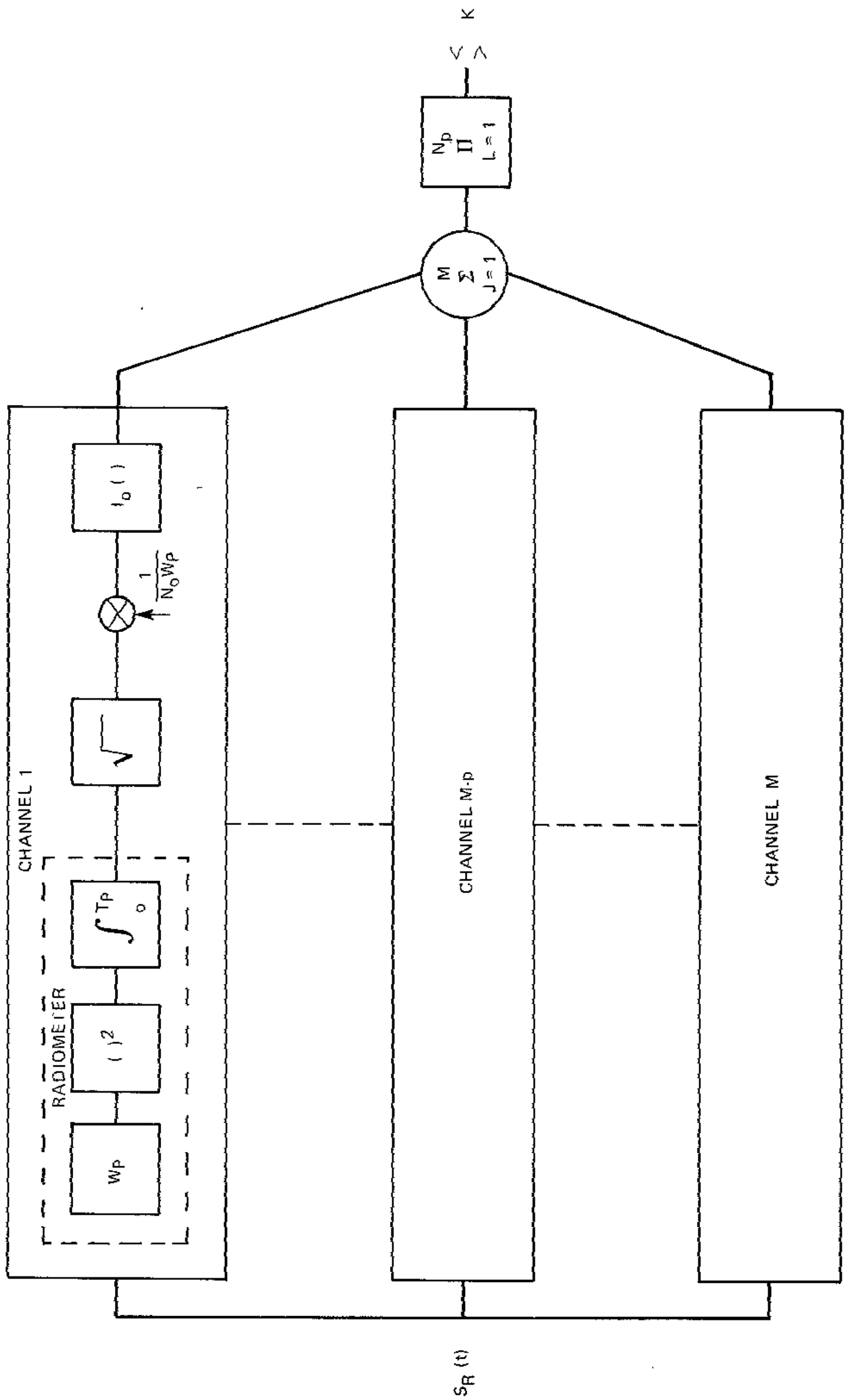

Fig. 3 — Optimum detector for frequency-hopped waveform

EXAMPLE 4

Consider the frequency-hopped signal of Example 1, with a TW product per pulse equal to one:

$$W_P = \text{pulse bandwidth} = r_H = 2000 \text{ Hz},$$

$$T_P = \text{pulse duration} = \frac{1}{r_H} = 500\ \mu\text{sec},$$

$$N_p = \text{number of pulses (hops)} = \frac{T}{T_P} = 8000,$$

and

$$M = \text{number of radiometer channels} = \frac{W}{W_P} = 10^6.$$

The performance of the optimum detector for this signal is computed by using Eq. (17) and Fig. A-5 as follows:

$$\left(\frac{C}{N_o}\right)_{\text{req.}} = W_P \cdot \frac{1}{2} \cdot I_o^{-1}[1 + M(e^{d^2/N_P} - 1)]$$

$$= 2000 \cdot \frac{1}{2} \cdot I_o^{-1}[1 + 10^6(e^{.0015} - 1)]$$

$$= 1000 \cdot I_o^{-1}(1502)$$

$$= 39.7 \text{ dB-Hz}.$$

This example illustrates the performance improvement for message detection, over 9 dB, when the detector is optimized to the waveform.

Spread Spectrum Signals with Time-Bandwidth Products Greater than One

The optimum detector for direct sequence spread spectrum waveforms, such as straight PN modulation, has been shown [3] to be a radiometer matched to the instantaneous spread bandwidth or the PN rate, r_{PN}, and the message duration T. The performance of each radiometer was treated in the preceding section for a time-bandwidth product equal to $T_P r_{PN}$.

The use of PN in conjunction with frequency-hopping and/or pulsed waveforms increases the bandwidth of each pulse, or hop, to the PN rate, so that the time-bandwidth product of each pulse becomes

$$n \underline{\Delta} T_P W_P = T_P r_{PN} > 1.$$

The optimum detector in this case consists of a bank of radiometers with time-bandwidth product of $T_P W_P$. The structure of this receiver is the same as that shown in Fig. 3 with the operation $I_o(\ )$ replaced by the function $K_n(\ )$ which is defined as

$$K_N(X) \triangleq \left[\frac{I_{N-1}(NX)}{(NX)^{N-1}}\right] 2^{N-1}\Gamma(N). \tag{18}$$

The performance of this detector can be determined by the same methods used for the previous case of the frequency-hopped, pulsed waveform, except that the time-bandwidth product is now $n = T_P W_P > 1$. The postdetection SNR is given in Appendix B as

$$d^2 = N_p \ln\left\{1 + \frac{1}{M}\left[K_n\left\{\frac{2S}{N}\right\} - 1\right]\right\}, \tag{19}$$

where again S/N is the predetection signal-to-noise power ratio in the individual radiometer bandwidth W_P and is related to the input C/N_o by

$$\frac{C}{N_o} = W_P\left\{\frac{S}{N}\right\}. \tag{20}$$

Note that for $TW = 1$,

$$\frac{S}{N} = \frac{E_p}{N_o}\,\frac{1}{T_P W_P} = \frac{E_p}{N_o}.$$

By assuming Gaussian output statistics and equal noise and signal-plus-noise variances, the required postdetection SNR, d^2, for a specified P_D and P_{FA} can be approximated by Eq. (2), or

$$d^2 = [Q^{-1}(P_{FA}) - Q^{-1}(P_D)]^2$$

and can be found from Fig. A-1. From Eqs. (19) and (20), the required C/N_o, given by

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = \frac{W_P}{2} K_{TW}^{-1}[1 + M(e^{d^2/N_p} - 1)] \tag{21}$$

can be calculated as shown in Example 5 by using the function $(1/2)\,K_{TW}^{-1}(\;)$, plotted for convenience in Fig. A-6.

For large time-bandwidth products, a good approximation to the predetection signal-to-noise power ratio is

$$\frac{S}{N} = \frac{1}{2} K_{TW}^{-1}(\gamma) \approx \sqrt{\frac{\ln(\gamma)}{TW}}. \tag{22}$$

This approximation is quite accurate for $S/N < 1/4$ (−6 dB), which corresponds to $TW \gtrsim 200$ for the range of values plotted in Fig. A-6. For this case, Eq. (21) can be written

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = \{\ln[1 + M(e^{d^2/N_P} - 1)]\}^{1/2}\sqrt{\frac{W_P}{T_P}};\; T_P W_P > 200. \tag{23}$$

It is interesting to note that for a continous wave (CW) spread spectrum signal ($M = 1$), Eq. (23) reduces to

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = \frac{d}{\sqrt{N_P}}\sqrt{\frac{W}{T_P}} = d\sqrt{\frac{W}{T}};\; TW > 200, \tag{24}$$

EXAMPLE 5

Consider a hybrid frequency hopping and pseudonoise modulation (FH-PN) signal which has the following characteristics:

$$T = 4\,\text{sec},$$

$$W = 2\,\text{GHz},$$

$$r_H = 25\,\text{kHz},$$

$$r_{PN} = PN \text{ chip rate} = 200\,\text{kHz},$$

$$T_P = \text{pulse duration} = \frac{1}{r_H} = 40\,\mu\text{sec},$$

and

$$W_P = \text{pulse bandwidth} = r_{PN} = 200\,\text{kHz}.$$

For this case, $T_P W_P = 8$, and from Eq. (15),

$$N_P = \text{number of pulses} = \frac{T}{T_P} = 10^5$$

and

$$M = \text{number of frequencies} = \frac{W}{W_P} = 10^4.$$

Since n is very large, the Gaussian approximation can be used and the required postdetection SNR for $P_D = 10^{-1}$ and $P_{FA} = 10^{-6}$ is

$$d^2 = 10.8 \text{ dB}.$$

Then the carrier-to-noise density ratio required for detection can be computed from Eq. (21) and Fig. A-6:

$$\left(\frac{C}{N_o}\right)_{\text{req.}} = \frac{W_P}{2} K_{TW}^{-1} [1 + M(e^{d^2/N_p} -)]$$

$$= 2 \times 10^5 \cdot \frac{1}{2} K_{TW}^{-1} [1 + 10^4 (e^{12/10^5} - 1)]$$

$$= 48.1 \text{ dB-Hz}.$$

where $W_P = W$ and $T_P = T$, which is the same as in Eq. (2) for the wideband radiometer.

Alternative Techniques for Computing Performance

The performance of the optimum detector can also be computed using standard radar detection curves. The individual radiometer channel outputs in the optimum detector are again modeled as the noncoherent sum of $T_P W_P$ sample pulses on each hop (Fig. 2), each sample having a duration of $1/W_P$, and consequently a TW product of one.

If on a given transmitted hop or pulse the signal is first assumed to be a single sample-pulse of duration T_P and time bandwidth product one ($W_P = 1/T_P$), the input SNR can be found from Eq. (17).

$$\left\{\frac{S}{N}\right\}' = \frac{S'}{N_o W_P'} = \left\{\frac{E}{N_o}\right\}' = \frac{1}{2} I_o^{-1}[1 + M(e^{d^2/N_p} - 1)], \tag{25}$$

where $\left\{S/N\right\}'$ and $\left\{E/N_o\right\}'$ are the equivalent single sample-pulse predetection SNR and energy per pulse-to-noise density ratio, respectively.

However, since the signal during the period T_P is actually composed of $T_P W_P$ sample pulses, in order to achieve the same performance as the single pulse case, the required input signal or carrier power, S, must be increased over that predicted by Eq. (25) by an amount equal to the noncoherent combining loss for $T_P W_P$ samples for an optimum detector.

Unfortunately, these losses are not readily computed, but a good approximation is to use the non-coherent combining losses, L_{TW}, given in Fig. A-4. There curves were computed for the square law detector of Fig. 1, which is a small-signal approximation to the optimum detector considered here. Marcum [3] has shown that the maximum difference in performance between the square law and optimum detectors is less than 0.19 dB. Therefore, an approximate expression for the increased signal power is

$$S \approx S' L_{TW}.$$

Therefore, as illustrated in Example 6,

EXAMPLE 6

Considering the signal structure and detection criterion used in Example 5, the $(C/N_o)_{\text{req.}}$ can be calculated from Eq. (26) and Fig. A-4 after finding the input signal-to-noise ratio per pulse from Eq. (25) and Fig. A-5:

$$\left\{\frac{S}{N}\right\}' = \frac{1}{2} I_o^{-1}[1 + M(e^{d^2/N_p} - 1)]$$

$$= \frac{1}{2} I_o^{-1}(2.2) = 0.97 = -0.13 \text{ dB}$$

and

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} \approx \frac{1}{T_P} \left\{\frac{S}{N}\right\}' L_{TW}$$

$$\approx \frac{1}{4 \times 10^{-5}} \cdot 0.97 \cdot L_8$$

$$\approx 44.0 \text{ dB-Hz} - 0.13 \text{ dB} + 4.2 \text{ dB}$$

$$\approx 48.1 \text{ dB-Hz}.$$

As expected, this result is the same as that found in Example 5.

$$\left(\frac{C}{N_o}\right)_{\text{req.}} \approx \frac{S'}{N_o} L_{TW} = \frac{1}{T_P}\left(\frac{S}{N}\right)' L_{TW} = W_P \left(\frac{S}{N}\right)' \frac{1}{G_{TW}}, \tag{26}$$

where $(S/N)'$ is determined from Eq. (25) and L_{TW} is found from Fig. A-4 for $TW = T_P W_P$ and the single pulse SNR = $(S/N)'$.*

Pulsed Waveforms

The waveforms treated thus far have all been assumed to be continuous wave (CW) signals. Non-CW or pulsed signals, such as time shift keying (TSK), burst transmissions, and time hopped (TH), are characterized by a transmission duty cycle, α, which is the ratio of the "on" time to the "off" time during the message duration. For pulsed signals, the quantity of interest is usually the average $(C/N_o)_{\text{req.}}$ which is related to the $C/N_o)$ for a single pulse by

$$\left(\frac{C}{N_o}\right)_{\text{avg.}} = \alpha \left(\frac{C}{N_o}\right)_{\text{peak}}. \tag{27}$$

For the radiometer detector, the performance $(C/N_o)_{\text{req.}}$, as determined from Eqs. (2), (3), or (10), is the average value or

$$\left(\frac{C}{N_o}\right)_{\text{req.}} \equiv \left(\frac{C}{N_o}\right)_{\text{avg.}}.$$

For either optimum or hop detectors, the performance is computed on a per-pulse basis so that

$$\left(\frac{C}{N_o}\right)_{\text{req.}} \equiv \left(\frac{C}{N_o}\right)_{\text{peak}}.$$

Equations (17), (21), and (26) can be used to compute the required peak C/N_o if the parameter M is considered to be the total number of orthogonal signals possible or if M is replaced by M/α (as derived in Appendix B) and N_p is the total number of pulses transmitted,

$$N_p = \alpha \left(\frac{T}{T_P}\right). \tag{28}$$

As shown in Example 7, the average C/N_o is calculated for the optimum detector by

$$\left(\frac{C}{N_o}\right)_{\text{avg.}} = \alpha \left(\frac{C}{N_o}\right)_{\text{req.}}. \tag{29}$$

The choice of average or peak (C/N_o) to describe detector performance is often a source of considerable confusion. In general, the required peak (C/N_o) for a pulsed waveform $(\alpha < 1)$ will be

*The actual noncoherent gain for the optimum detector, G_{TW}, can be found by solving Eqs. (26) and (21) for $G_{TW} = K_I^{-1}(\,)/K_{TW}^{-1}(\,)$.

EXAMPLE 7

The third type of signal to be analyzed is a hybrid FH/PN/TH waveform, which will be characterized by a pulse duty cycle α. The specific example considered is a FH/PN signal with m-ary TSK modulation. In this case, the duty cycle $\alpha = 1/m$. The waveform parameters are as follows

$$T = 4\,\text{sec} \qquad r_H = 25\,\text{kHz}$$
$$W = 2\,\text{GHz} \qquad r_{PN} = 2\,\text{MHz}$$
$$\alpha = \text{pulse duty cycle} = \frac{1}{16}.$$

For this example, $T_P W_P = \dfrac{r_{PN}}{r_H} = 80$, since

$$T_P = \frac{1}{r_H} = 40\,\mu\text{sec}$$
$$W_P = r_{PN} = 2\text{ MHz}.$$

The radiometer performance for the detection criterion of Example 1 is computed from Eq. (2)

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = \left\{\frac{C}{N_o}\right\}_{\text{avg.}} = 48.9\text{ dB-Hz},$$

the same result as Example 1.

With these parameters, the $(C/N_o)_{\text{req.}}$ for the optimum detector from Eq. (21) and Fig. A-6 is

$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = W_p \cdot \frac{1}{2} K_{TW}^{-1} [1 + M(e^{d^2/N_p} - 1)]$$
$$\left\{\frac{C}{N_o}\right\}_{\text{req.}} = 2 \times 10^6 \cdot \frac{1}{2} K_{80}^{-1} [1 + 16\,000\,(e^{12/6250} - 1)]$$
$$= 56.2\,\text{dB-Hz}.$$

Finally, the $(C/N_0)_{\text{avg.}}$ is found from Eq. (29):

$$(C/N_0)_{\text{avg.}} = \alpha\,(C/N_0)_{\text{req.}} = 44.2\,\text{dB-Hz}.$$

higher than for a CW waveform with the same parameters. At the same time, the required average (C/N_0) for a pulsed waveform is lower than that required for the CW signal. Thus, a pulsed waveform will result in either better or worse detector performance, depending on which measure is employed (average or peak (C/N_o) in characterizing this performance.

This apparent contradiction is easily resolved when computing the vulnerability of a communication signal to detection. Detector performance is only one factor in assessing this vulnerability. Edell [2] had proposed the ratio of the (C/N_o) required to detect to the (C/N_o) required to communicate as a measure of this vulnerability. With this measure, the problem of peak *vs* average (C/N_o) is solved by maintaining consistency between the communication and detector (C/N_o). If the average (C/N_o)

required to detect is used, then the communication (C/N_o) is given by $(E_b/N_o)R_b$ (energy per bit-to-noise density ratio times the data rate). If the peak (C/N_o) required to detect is desired, then the communication (C/N_o) must account for the duty factor, α. In this case the communication (C/N_o) is given by $(E_b/N_o)\,R_b\,(1/\alpha)$. It can be shown [2] that this ratio is reduced (signal is more vulnerable to detection) by lowering the duty cycle. When computing detector performance, therefore, it is more instructive to use average (C/N_o), which will reflect this degradation since it includes the duty cycle, α.

SUBOPTIMUM DETECTORS FOR SPREAD SPECTRUM SIGNALS

The optimum detector structure described in the previous section suffers from two significant shortcomings. The first is the complexity of the detector, particularly for TW products greater than one, which may make the detector impractical to implement. The second and perhaps more troublesome problem is that performance cannot be expressed exactly; the performance measure, d, is based upon two assumptions: the output statistics are Gaussian, and the variance of the output is equal under both signal-plus-noise and noise-only hypotheses. In view of these considerations, a suboptimum version of the optimum, multichannel receiver for frequency-hopped spread spectrum signals will be examined for which the performance can be computed exactly, and which is more practical to implement.

Filter Bank Combiner Detector

This receiver, which is often referred to by DiFranco and Rubin [4] and Dillard [5] as a Binary Moving Window (BMW) detector or a Filter Bank Combiner (FBC) with individual thresholds, is shown in Fig. 4. Essentially, the receiver is again a bank of radiometers matched to the signal pulse, one for each of the M possible channels or slots which the signal is expected to occupy. The output of each radiometer on each hop is detected, and a decision is made in each channel. These decisions are logically OR'd and summed over the signal duration. At the end of the signal duration, the sum is compared to a threshold L, an integer number determined from the required P_{FA} and P_D.

It is not difficult to show that the optimum detector output after each hop or pulse is often dominated for useful values of SNR by the output of the one channel containing the signal, due to the single pulse postdetection weighting of the channel output. The suboptimum detector approximates this performance characteristic by reducing the postdetection processing to a simple threshold decision on each channel, which is equivalent to a binary weighing of the output. Thus, if the single-pulse SNR in the channel containing the signal is sufficient to cause that channel output to exceed its threshold on a particular hop, then the suboptimum detector output for that hop is wholly determined by the output of that one channel.

The performance of the FBC detector is discussed in more detail by Edell [2]. Two thresholds determine the P_{FA} and P_D: the individual channel thresholds, K_i, and the integer threshold L. The latter is a threshold on the number of hops or pulses for which at least one individual channel radiometer threshold has been exceeded. The optimum value of this threshold cannot be found directly and is determined from the inverse of the binomial distribution function by an iterative computation. Fortunately, a threshold of $L = 1$ will yield results which are only about 1 to 2 dB high in required input C/N_o, but which can be computed directly.

The individual radiometer thresholds are identical for each channel and are determined from the required P_{FAI} and P_{DI}, where P_{FAI} is the probability of false alarm for an individual radiometer on a single hop or pulse, for the required message or signal P_{FA}, and P_{DI} is the probability of detection for an individual radiometer on a single hop or pulse for the required message P_D.

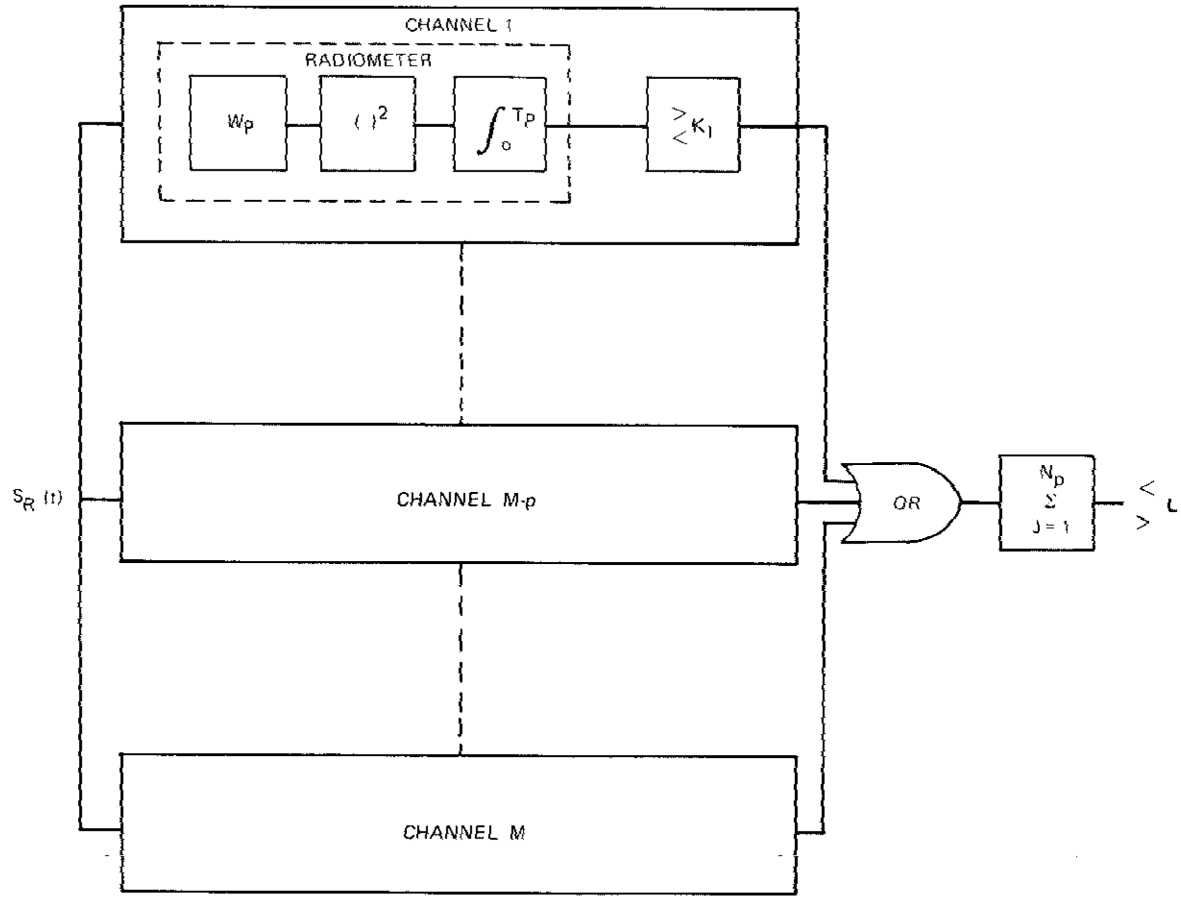

Fig. 4 — Filter bank combiner

For the message threshold of $L = 1$, the individual P_{FAI} and P_{DI} can be simply related to the required P_{FA} and P_D, and an exact expression is given in Appendix C. For reasonable values of P_D and P_{FA}, this can be expressed (for $N_p = T \cdot r_H \cdot \alpha$) as

$$P_{FAI} \approx \frac{\alpha}{MN_p} P_{FA};\ P_{FA} << 1$$

$$P_{DI} \approx \frac{P_D}{N_p};\ \frac{P_D}{N_p} << 1;\ P_{FAI} << P_{DI}. \tag{30}$$

The C/N_0 required can be computed from the P_{FAI} and P_{DI} by any of the techniques for determining the performance of a radiometer detector. For the case considered here, Eq. (3) becomes

$$\left(\frac{C}{N_0}\right)_{\text{req.}} = \eta_{TW}\, d_H \sqrt{\frac{W_P}{T_P}};\ TW = T_P W_P, \tag{31}$$

where d_H is the Gaussian approximation for the single channel, single pulse, postdetection, voltage SNR, and

$$d_H = Q^{-1}(P_{FAI}) - Q^{-1}(P_{DI}), \tag{32}$$

which can be found from Fig. A-1. The chi-square correction factor, η_{TW}, is plotted in Fig. A-2. As an alternative, one may utilize Eq. (10).

$$\left(\frac{C}{N_o}\right)_{req.} = \frac{1}{T_P} d_{\chi_H} L_{TW}, \tag{33}$$

where $TW \approx T_P W_P$, and d_χ, for a single pulse, is found from Fig. A-3. In this case, using P_{FAI} and P_{DI}, the noncoherent combining loss, L_{TW}, is found from Fig. A-4 for $TW = T_P W_P$ and SNR = d_{χ_H}. Example 8 illustrates the performance calculation using both of these methods.

EXAMPLE 8

For the signal in Example 1, recall (see Example 4)

$$T_P = 500\,\mu\text{sec},$$

$$W_P = 2000\,\text{Hz},$$

$$N_p = 8000,$$

and

$$M = 10^6.$$

Therefore, from Eq. (30),

$$P_{FAI} = \frac{P_{FA}}{MN_p} = 1.25 \times 10^{-16}$$

and

$$P_{DI} = \frac{P_D}{N_p} = 1.25 \times 10^{-5}.$$

Utilizing Eq. (32),

$d_H = Q^{-1}(1.25 \times 10^{-16}) - Q^{-1}(1.25 \times 10^{-5}) = 6.0\,\text{dB}$ (from Fig. A-1), and for $TW = 1$, Fig. A-2; gives approximately

$$\eta = 3.5\,\text{dB}$$

Therefore, Eq. (31) becomes

$$\left(\frac{C}{N_o}\right)_{req.} = 3.5\text{ dB} + 6.0\text{ dB} + 33\text{ dB-Hz} = 42.5\text{ dB-Hz}.$$

Thus, the FBC for an $L = 1$ threshold performs 7 dB better than the radiometer and about 2 dB worse than the optimum detector (see Examples 1 and 4).

For a pulsed waveform the false alarm and detection probabilities are given by Eq. (30), which uses the expression for N_p given in Eq. (28). The C/N_o computed is then the peak value, where (Eq. (27))

$$\left\{\frac{C}{N_o}\right\}_{avg.} = \left\{\frac{C}{N_o}\right\}_{peak} = \left\{\frac{C}{N_o}\right\}_{req.}.$$

Fractional Bandwidth Detectors

Optimum maximum-likelihood ratio (MLR) detectors for frequency-hopped waveforms utilize a separate channel for each possible instantaneous hop frequency, or frequency slot, within the total spread bandwidth (see Fig. 3). The same is true of the filter bank combiner (FBC) receiver. For very large spread bandwidths, the number of frequencies, and consequently the number of detector channels, can be enormous, and it may therefore be argued that the detector is impractical to implement.

"Optimum" Partial Band Detector

The suboptimum detectors of interest in this case are those that use only a fraction, f, of the total spread bandwidth. Thus, the number of channels is reduced to fM, where M is the total number of frequency slots to which the signal may hop.

It will often be the case that near-optimum performance can be achieved with a greatly reduced number of channels by judicious choice of the fraction, f. This case has been analyzed by Niessen [6] and is treated in Appendix B. Again, it is necessary to assume Gaussian output statistics when computing the performance for the log-likelihood ratio detector. This appproximation is valid for signals with a large number of pulses or hops.

For a train of N_p pulses each with time-bandwidth product TW, the performance of a maximum-likelihood ratio detector which covers a fraction, f, of the bandwidth, W, is given by

$$K_{TW}\left\{\frac{2S}{N}\right\} = 1 + \frac{M}{f\alpha}[e^{d^2/N_p} - 1].$$

Thus

$$\left\{\frac{C}{N_o}\right\}_{req.} = W_P \ \frac{1}{2} K_{TW}^{-1} \left[1 + \frac{M}{f\alpha}(e^{d^2/n} - 1)\right]. \tag{34}$$

For a pulse waveform, the number of pulses, N_p, is given by Eq. (28), where α is the pulse duty cycle. Calculation of the required peak C/N_o can be accomplished with either of the techniques utilized for the simple energy detector.

Partial Band Filter Bank Combiner

A filter bank combiner uitilizing a fraction, f, of the total number of frequencies, M, will have fM channels. The formulas for computing the probability of detection and false alarm for the individual channels given in Eq. (30) can be modified with the restrictions that $fM \geqslant 1$ and $fN_p \geqslant 1$:

$$P_{FAI} \approx \frac{\alpha}{fN_p M} P_{FA}; \ P_{FA} << 1$$

$$P_{DI} \approx \frac{P_D}{fN_p}; P_D >> P_{FA}. \tag{35}$$

The first condition requires that the minimum detector bandwidth be at least as great as the instantaneous signal bandwidth. If this condition does not hold, the predicted $(C/N_o)_{\text{req.}}$ must be increased to account for the lost signal energy outside the detector bandwidth. The second condition ensures that the probability of detection in the individual channels on a per-hop basis, P_{DI}, is not required to be greater than the probability of detection of the multichannel filter bank detector on a per-message basis.

The degradation in performance sacrificed by implementing a single channel detector is calculated in Example 9 for both the optimum partial band and FBC detectors. The result of trading off system simplicity (reducing the number of detector channels) for detection threshold, illustrated by Fig. 5, emphasizes the small degradation in performance at a significant savings in cost. Thus, the argument that the FBC detector is an unrealistic threat due to the large number of channels required in the full band detector is an unreliable assumption. At least in the examples shown, the number of channels can be reduced to a manageable size by restricting the total bandwidth covered while retaining a performance advantage over the wideband radiometer.*

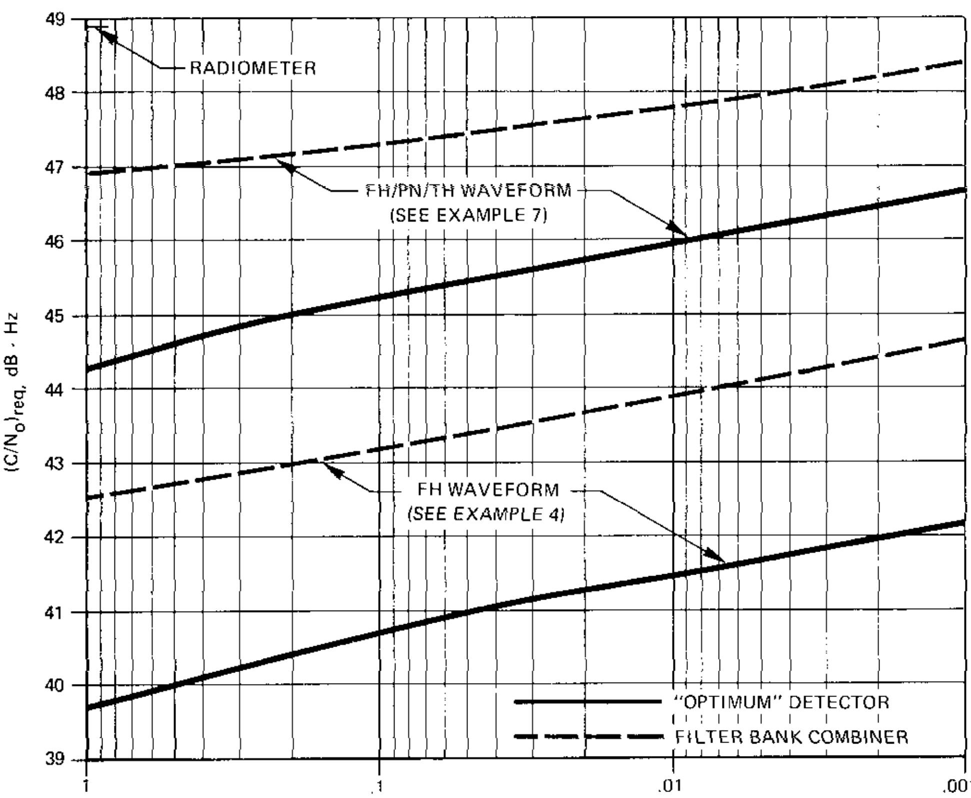

Fig. 5 — Waveform detectability as a function of detector bandwidth

*There is a lower limit to the fraction of the band that must be covered. It can be shown that the fraction, f, must satisfy

$$f \geqslant \frac{1}{N_p} = \frac{T_P}{T}.$$

This restriction is of concern therefore for long pulses (slow hops) and/or short message times.

EXAMPLE 9

Considering the signal structure and detection criterion used in Example 7, the performance of an optimum partial band and FBC detectors with a single radiometer channel can be calculated from Eqs. (34) and (31), respectively, with the use of Eq. (35) to determine pulse detectability criterion.

Using Eq. (34) with $f = 1/M$, we calculate the performance of the optimum single channel detector:

$$\left\{\frac{C}{N_o}\right\}_{req.} = W_P \approx \frac{1}{2} K_{TW}^{-1}\left[1 + \frac{M^2}{\alpha}\left(e^{d^2/N_p} - 1\right)\right]$$

$$= 2 \times 10^6 \frac{1}{2} K_{80}^{-1}\,[1 + 16 \times 10^6(e^{12/6250} - 1)]$$

$$= 58.7 \text{ dB-Hz}.$$

This is the peak C/N_o required for detection, and the average is found by reducing this value by the duty cycle:

$$\left\{\frac{C}{N_o}\right\}_{avg.} = \alpha\left\{\frac{C}{N_o}\right\}_{req.} = 46.7 \text{ dB-Hz}.$$

Thus the degradation that results from using a single channel instead of the 1000 channel detector is 2.5 dB.

To calculate the performance of a single channel ($fM = 1$) filter bank combiner, the probabilities of detection and false alarm per pulse must be established:

$$P_{FAI} \approx \frac{\alpha}{fN_pM} P_{FA} = \frac{\alpha}{N_p} P_{FA} = 10^{-11}$$

and

$$P_{DI} \approx \frac{P_D}{fN_p} = \frac{10^{-1}}{6.25} = 1.6 \times 10^{-2}.$$

The required C/N_o for detection can be calculated from Eq. (31):

$$\left\{\frac{C}{N_o}\right\}_{req.} = \eta_{TW}\, d_H \sqrt{\frac{W_P}{T_P}}$$

$$= \eta_{80}[Q^{-1}(10^{-11}) - Q^{-1}(1.6 \times 10^{-2})]\sqrt{\frac{2 \times 10^6}{40 \times 10^{-6}}}$$

$$= 0.4 \text{ dB} + 6.5 \text{ dB} + 53.5 \text{ dB-Hz} = 60.4 \text{ dB-Hz}.$$

This equates to an average C/N_o of

$$\left\{\frac{C}{N_o}\right\}_{avg.} = \alpha\left\{\frac{C}{N_o}\right\}_{req.} = 48.4 \text{ dB},$$

which is only about 1.5 dB worse than the full-band FBC.

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6. C.W. Niessen, "The Detectability of Frequency-Hopped Communication Systems (U)," Technical Note 1973-45, Lincoln Laboratory, MIT, 5 September 1973 (Secret). AD-527 358

ACKNOWLEDGMENTS

The author wishes to thank J. Edell, now with SHAPE Technical Center, who participated in the preparation of early drafts of this report, and who contributed a number of curves and figures from an earlier report.

Appendix A
USEFUL DETECTABILITY CALCULATION CURVES

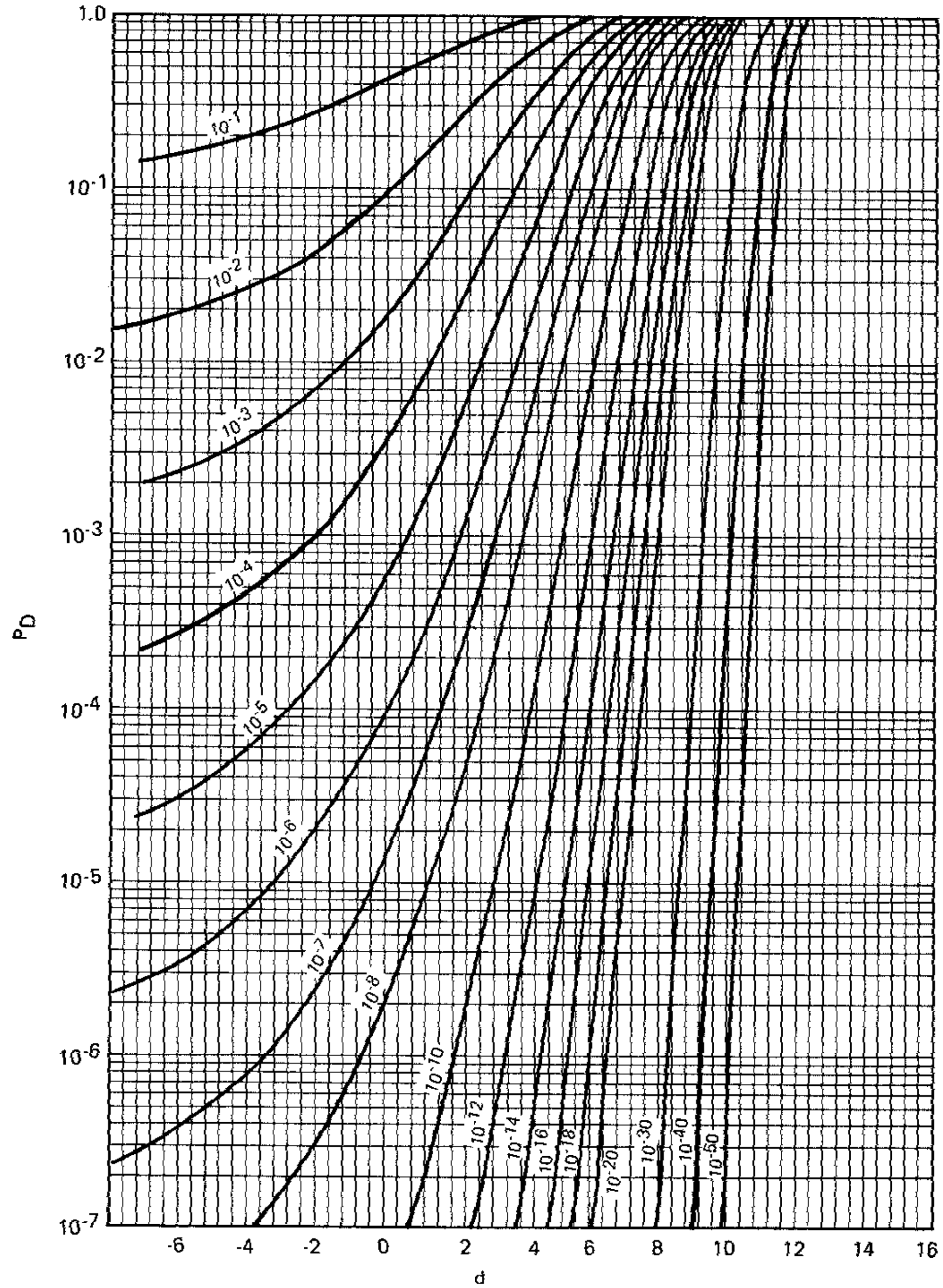
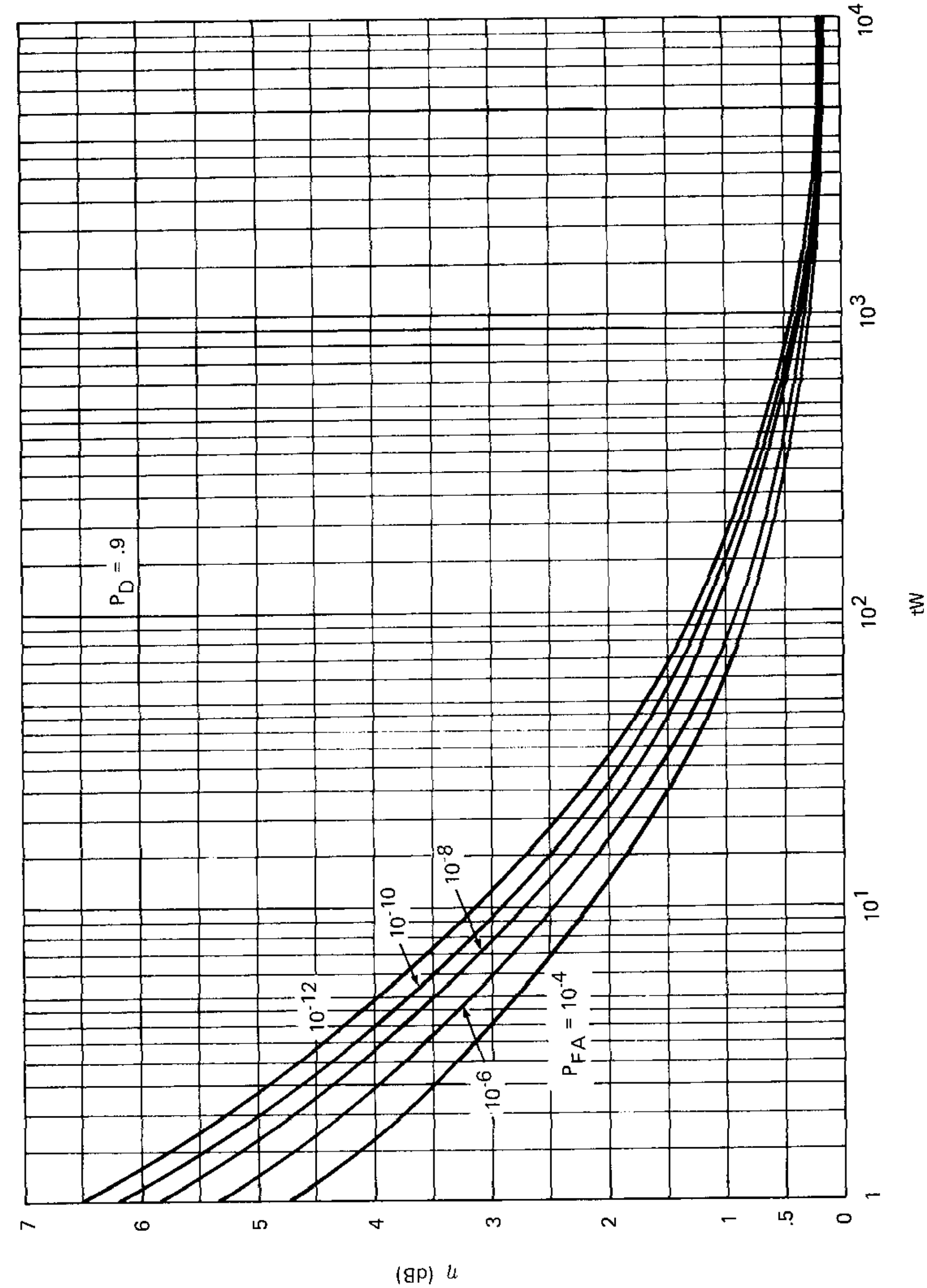
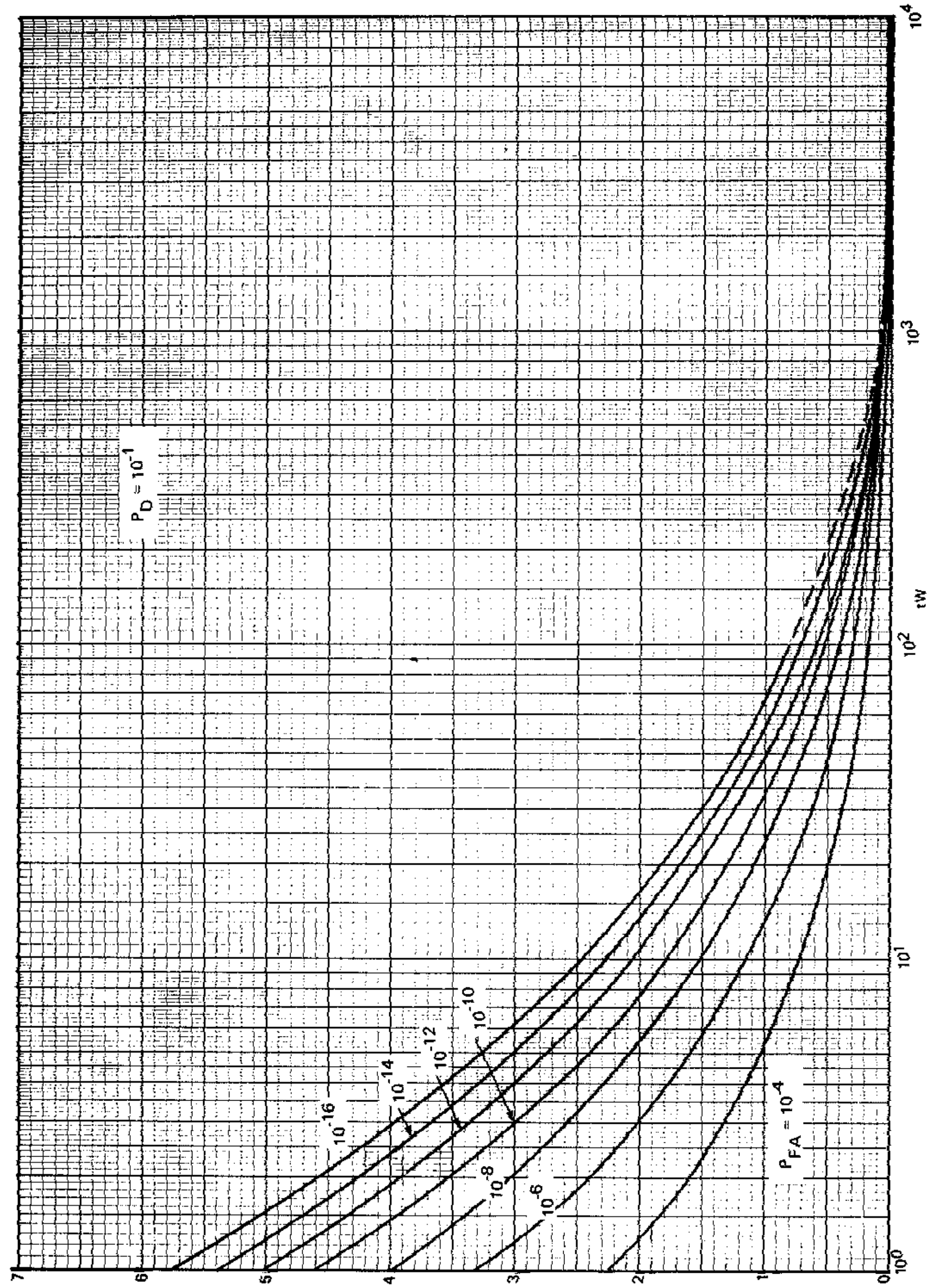


Fig. A-1 — Radiometer probability of detection P_D and probability of false alarm P_{FA} *vs* parameter $d = \left[Q^{-1}\left\{P_{FA}\right\} - Q^{-1}\left\{P_D\right\}\right]$



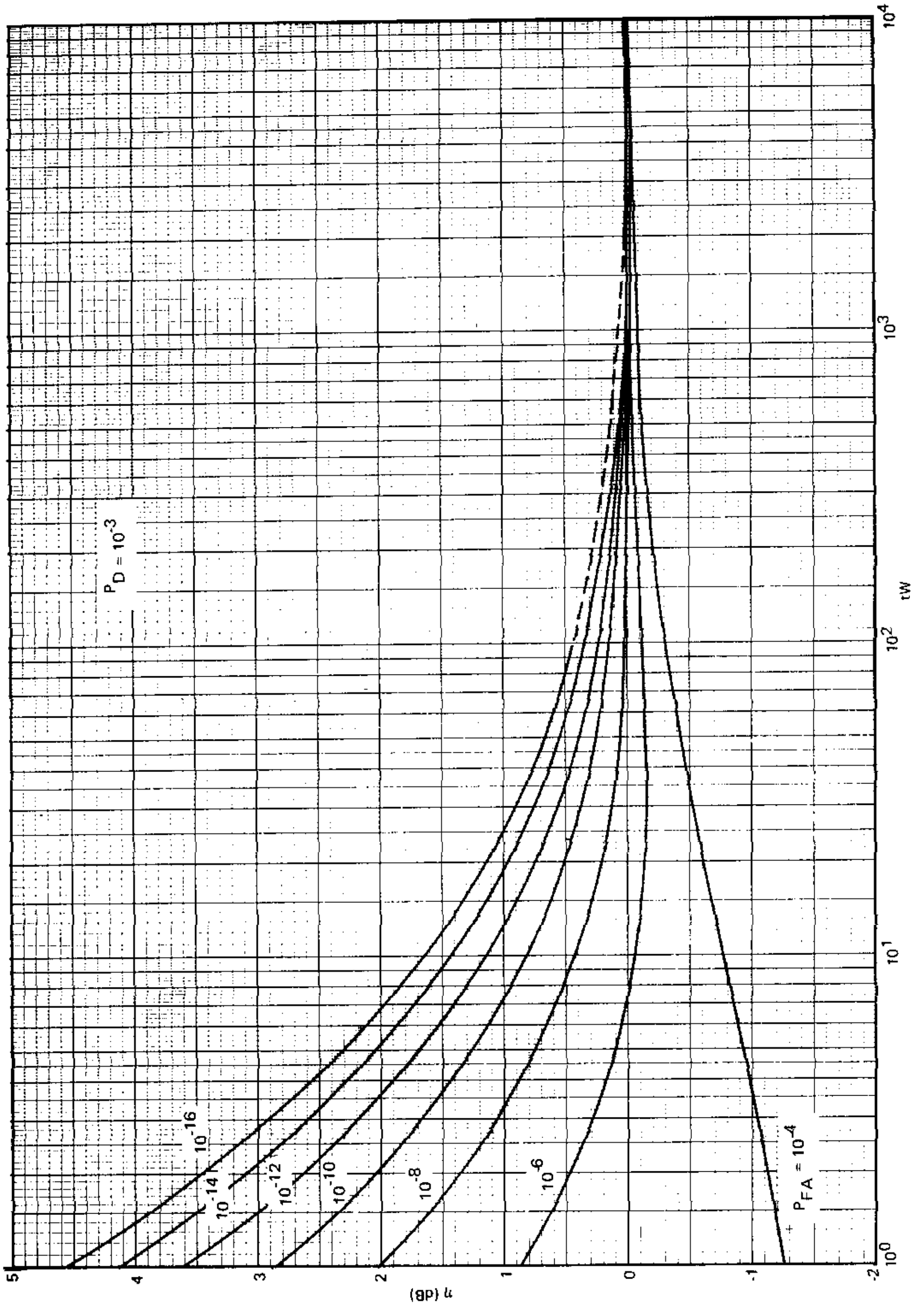
(a) Correction factor η for Gaussian approximation as a function of time-bandwidth product TW ($P_D = 0.9$)

Fig. A-2 — Chi-square correction factor curves



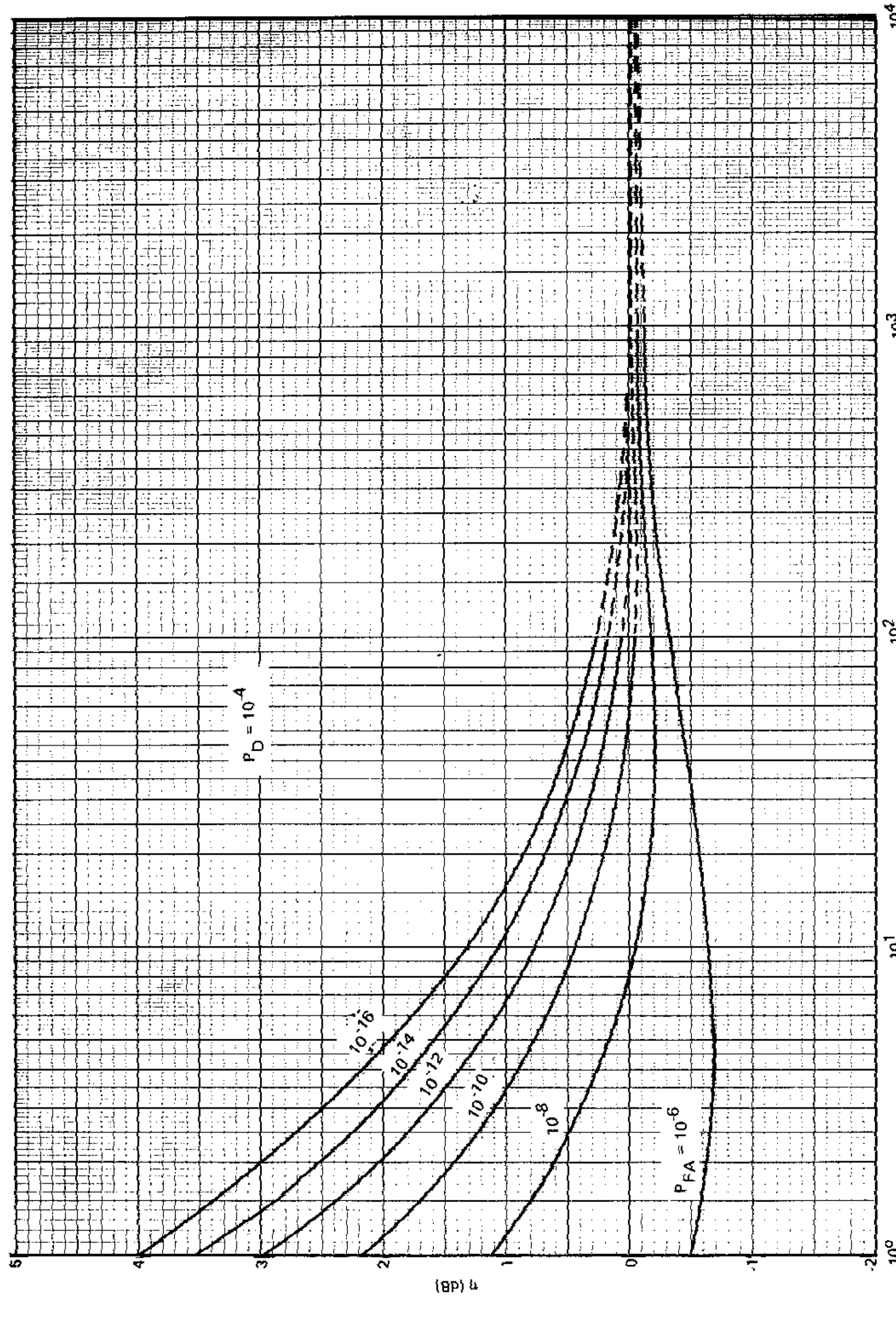
(b) Correction factor η for Gaussian approximation as a function of time-bandwidth product TW ($P_D = 10^{-1}$)

Fig. A-2 (Continued) — Chi-square correction factor curves



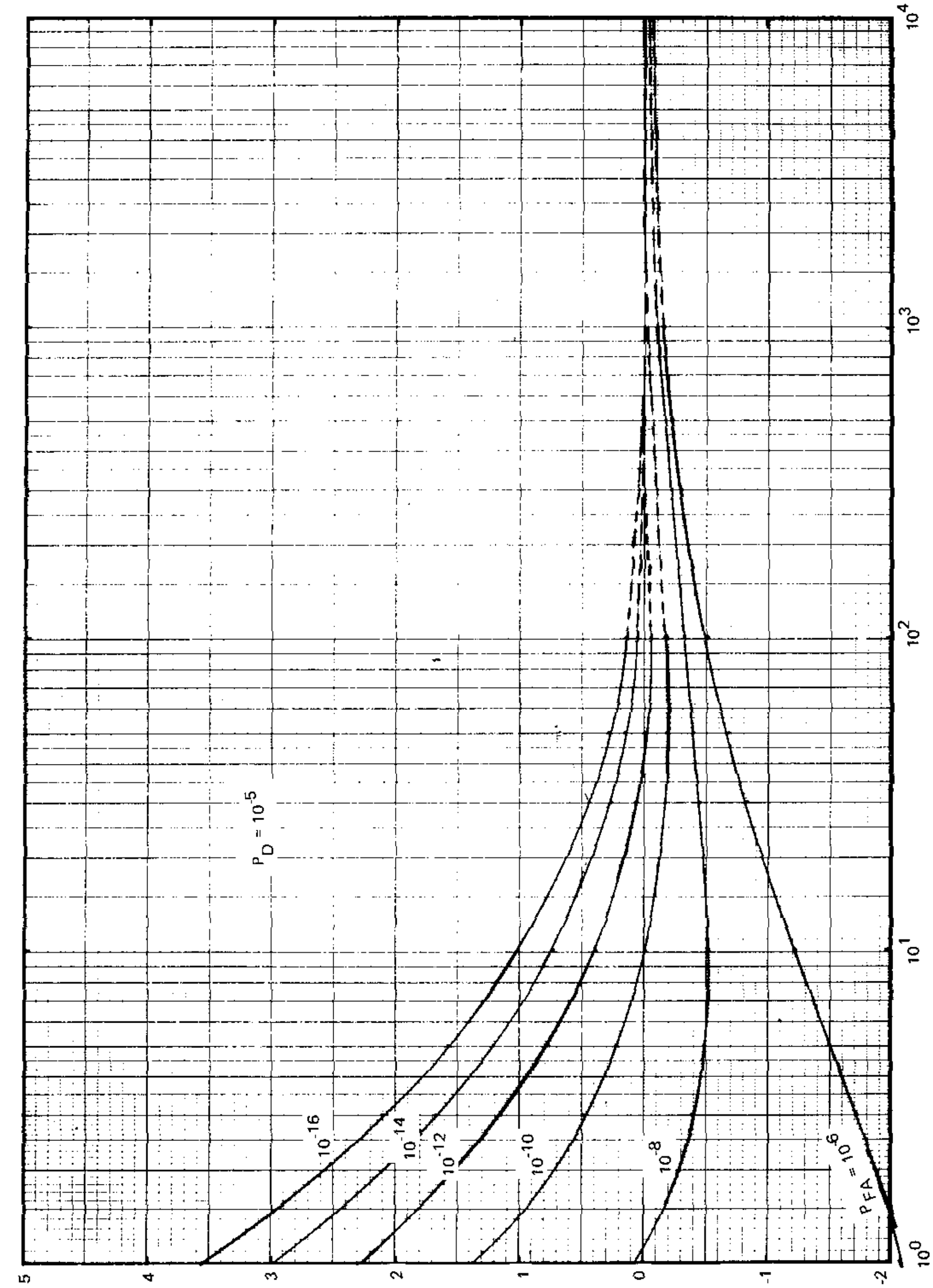
(c) Correction factor η for Gaussian approximation as a function of time-bandwidth product TW ($P_D = 10^{-3}$)

Fig. A-2 (Continued) — Chi-square correction factor curves



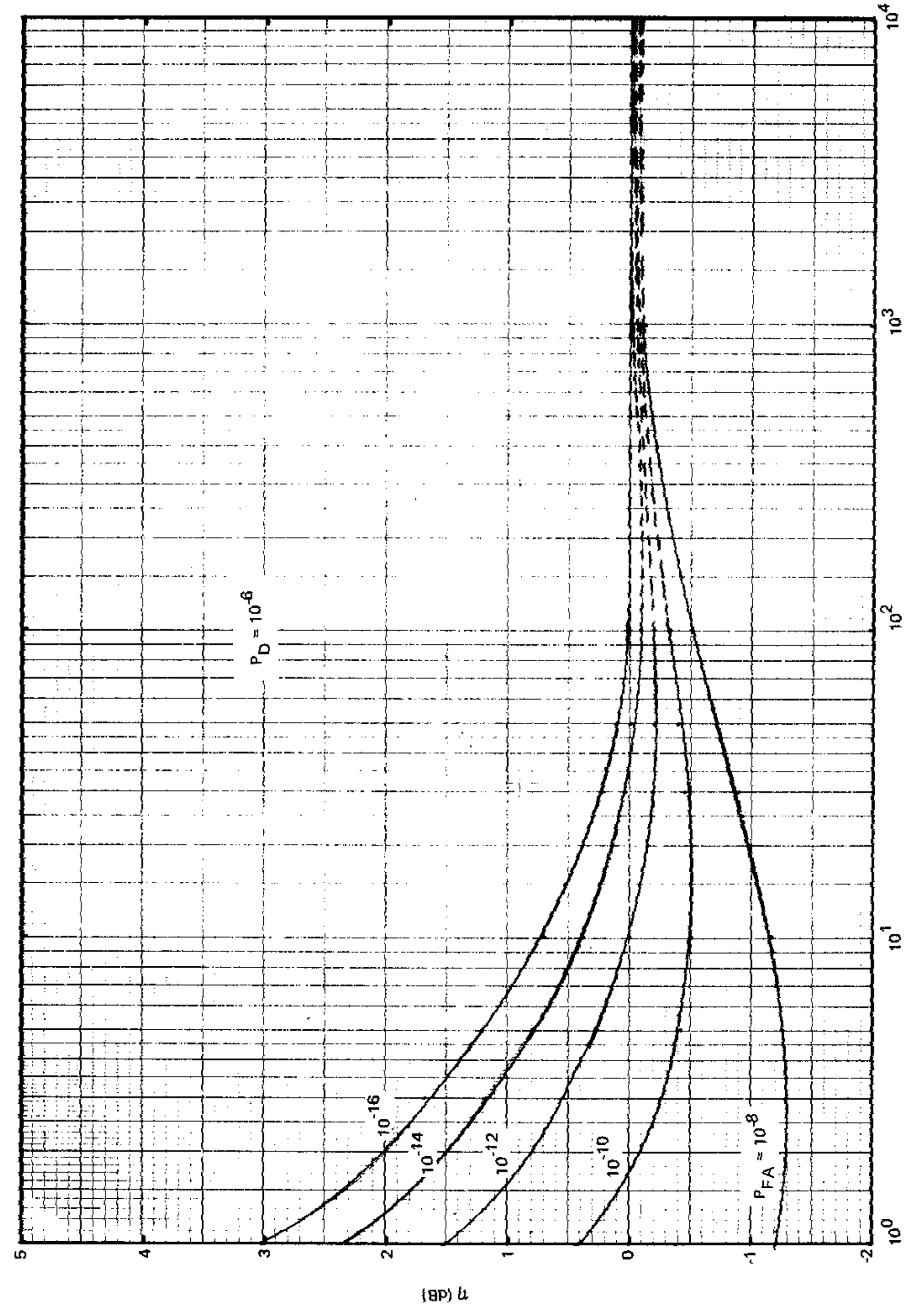
(d) Correction factor η for Gaussian approximation as a function of time-bandwidth product TW ($P_D = 10^{-4}$)

Fig. A-2 (Continued) — Chi-square correction factor curves



(e) Correction factor η for Gaussian approximation as a function of time-bandwidth product TW ($P_D = 10^{-5}$)

Fig. A-2 (Continued) — Chi-square correction factor curves



(f) Correction factor η for Gaussian approximation as a function of time-bandwidth product TW ($P_D = 10^{-6}$)

Fig. A-2 (Continued) — Chi-square correction factor curves

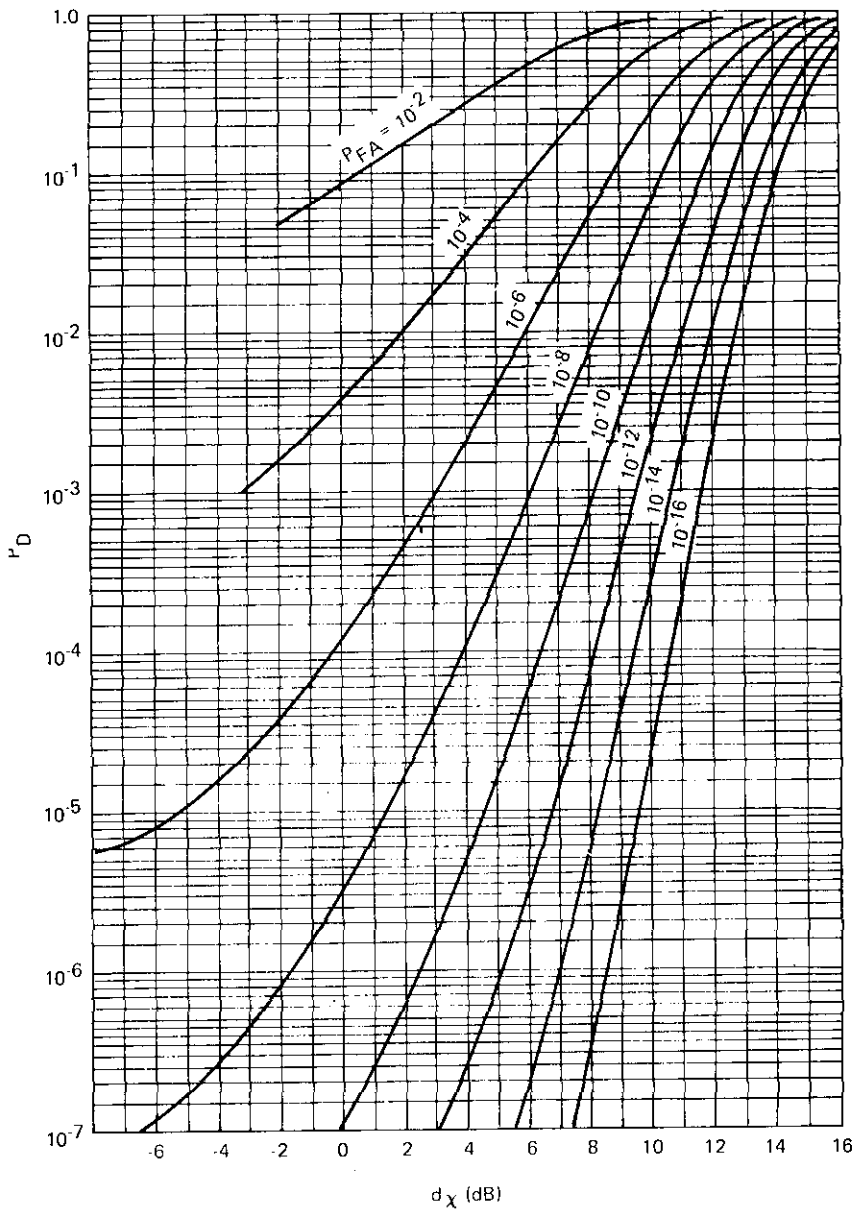


Fig. A-3 — P_{FA} and P_D *vs* parameter d_χ, the postdetection SNR

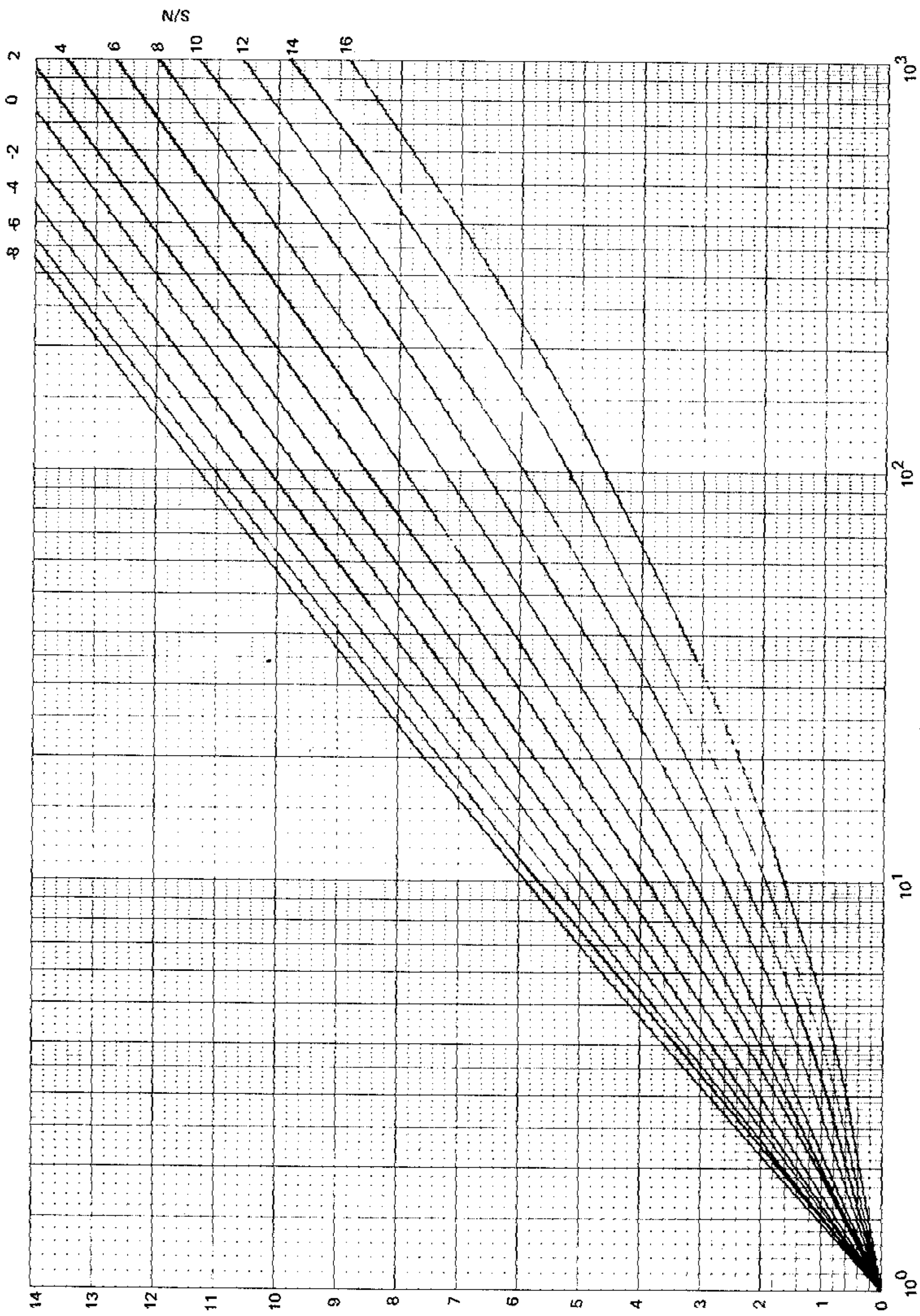


Fig. A-4 — Noncoherent combining loss L_{TW}, as a function of the number of pulses integrated N, for various values of postdetection signal-to-noise ratio (S/N)

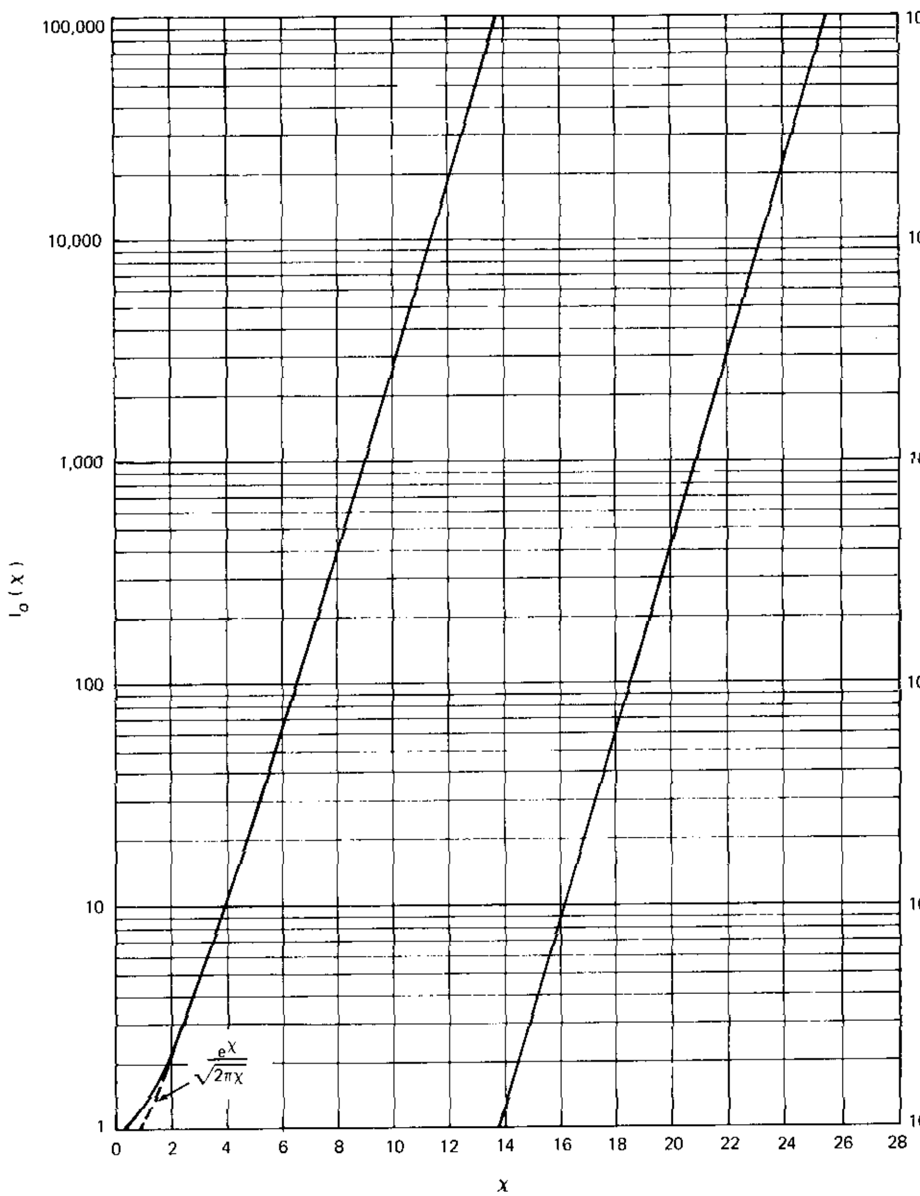


Fig. A-5 — Modified Bessel function $I_o(\chi)$

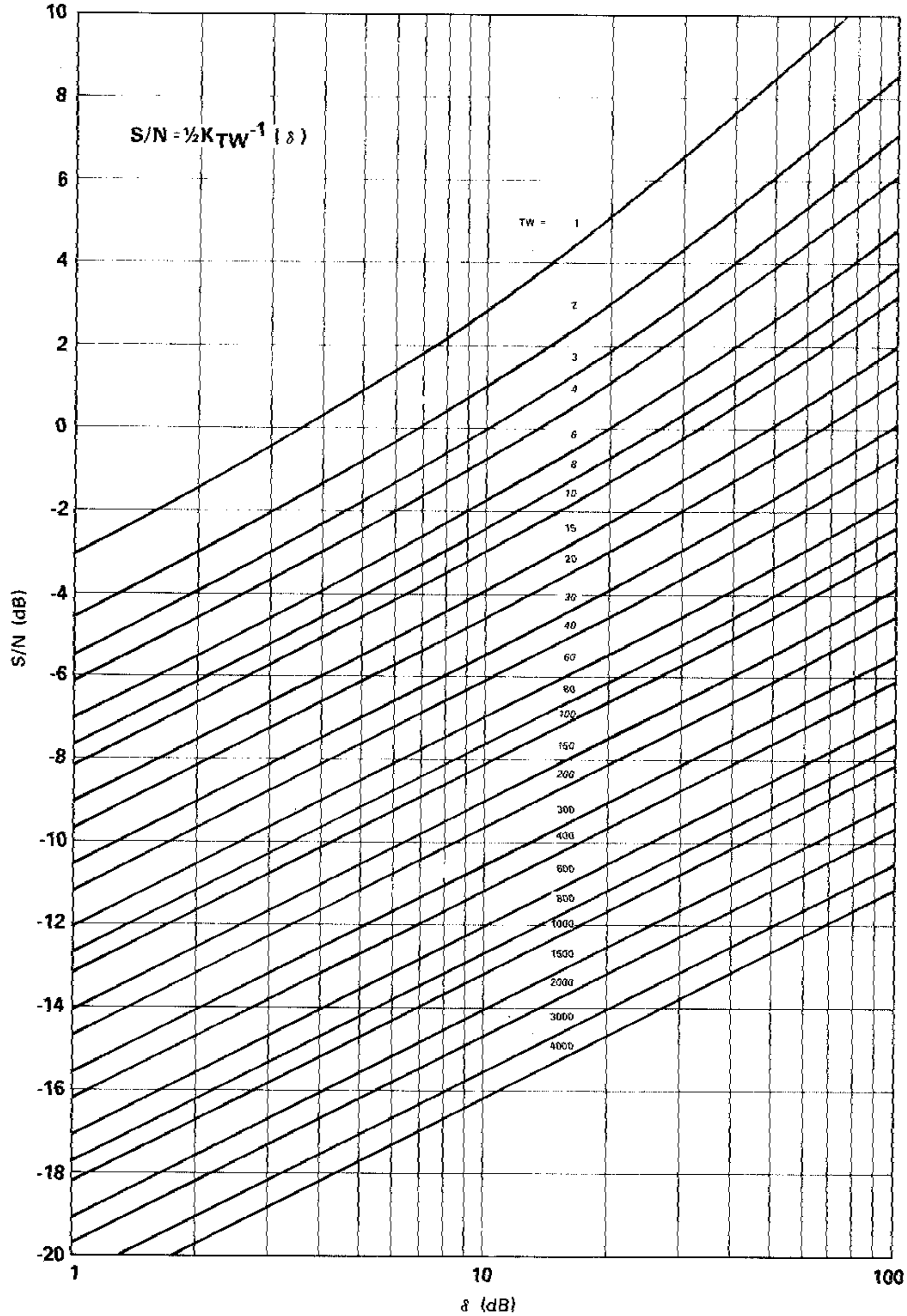


Fig. A-6 — Optimum receiver channel weighting factor for several time-bandwidth products

Appendix B

DETECTION STRATEGIES AND PERFORMANCE

The performance of signal detection schemes depends on the decision-making strategy that is implemented. Although many strategies are available, maximum likelihood detection provides an optimum means of determining a transmitted message on the basis of a received signal. Optimum in this case means that the probability of doing this correctly is maximum. In this appendix, the likelihood ratio and the detection performance measure are derived for both known and unknown signals.

DETECTION OF KNOWN SIGNALS

Likelihood Ratio

For an exactly known signal, $s(t)$, in white Gaussian noise, Peterson [B1] gives the likelihood ratio as

$$\ell_s(x) = \exp\left[-\frac{E(s)}{N_o}\right]\exp\left[\frac{2}{N_o}\int_0^T x(t)\,s(t)\right]dt, \tag{B-1}$$

where $E(s)$ is the average signal energy, $N_o/2$ is the noise power density per hertz, $x(t)$ is the observable signal-plus-noise, T is the observation interval, and $s(t)$ is the exactly known signal.

In this case, the optimum receiver computes $\ell_s(x)$ and compares it to a threshold:

$$\ell_s(x) = \exp\left[-\frac{E(s)}{N_o}\right]\exp\left[\frac{2}{N_o}\int x(t)\,s(t)\,dt\right] \gtrless K. \tag{B-2a}$$

Equivalently, one can use any monotonic function of the likelihood ratio. A convenient function in this case is the natural logarithm, so that

$$\ln\left[\ell_s(x)\right] = \frac{2}{N_o}\int x(t)\,s(t)\,dt - \frac{E(s)}{N_o} \gtrless \ln K = K'. \tag{B-2b}$$

Since $x(t)$ is normally distributed, the logarithm of the likelihood ratio for the exactly known signal is also normally distributed with mean m and variance d^2.

It can be shown [B1] that the mean and variance of the likelihood ratio for noise only are given by

$$E_N\left\{\ell_s(x)\right\} = \exp\left\{\frac{d^2}{2} + m\right\}$$

and

$$VAR_N\left\{\ell_s(x)\right\} = \exp\left[2\,(d^2 + m)\right] - \exp\left[d^2 + 2\,m\right]. \tag{B-3}$$

However, from the properties of the maximum likelihood ratio, it is known that

$$E_N\left\{\ell_s(x)\right\} = 1 = \exp\left\{\frac{d^2}{2} + m\right\}$$

so that $m = -d^2/2$. Therefore,

$$VAR_N\left\{\ell_s(x)\right\} = \exp\left[2\left\{d^2 - \frac{d^2}{2}\right\}\right] - 1 = \exp(d^2) - 1.$$

Solving for d^2,

$$d^2 = \ln\left[1 + VAR_N\left\{\ell_s(x)\right\}\right]. \tag{B-4}$$

Performance Measure

The performance of the equivalent log likelihood ratio detector can be found exactly by using the Gaussian density functions with variance d^2 and means m_n and m_{n+s} under noise-only and signal-plus-noise cases, respectively. Thus, the probability of false alarm and the probability of detection are given by

$$P_{FA} = P\left[\ell_s(x) > K'\right] \text{ with noise only}$$

$$= \int_{K'}^{\infty} \frac{1}{\sqrt{2\pi d^2}} \exp\left[\frac{-(x - m_n)^2}{2d^2}\right] dx \tag{B-5a}$$

and

$$P_D = P\left[\ell_s(x) > K'\right] \text{ with signal-plus-noise}$$

$$= \int_{K'}^{\infty} \frac{1}{\sqrt{2\pi d^2}} \exp\left[\frac{-(x - m_{n+s})^2}{2d^2}\right] dx. \tag{B-5b}$$

By defining a function Q (the complement of the Gaussian distribution function with zero mean and unit variance),

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\alpha^2/2}\, d\alpha. \tag{B-6}$$

P_{FA} and P_D can be expressed

$$P_{FA} = Q\left\{\frac{K' - m_n}{d}\right\}$$

$$P_D = Q\left\{\frac{K' - m_{n+s}}{d}\right\}.$$

Solving for K' and equating yields

$$\left[Q^{-1}(P_{FA}) - Q^{-1}(P_D)\right]^2 = \frac{(m_{n+s} - m_n)^2}{d^2} = \frac{2E}{N_0} = d^2, \tag{B-7}$$

where

$$m_n = -\frac{E}{N_o},$$

$$m_{n+s} = +\frac{E}{N_o},$$

and

$$d^2 = -2\, m_n = \frac{2E}{N_o}.$$

Therefore, d^2 completely defines the performance of the log likelihood ratio detector for the known signal case.

DETECTION OF SIGNALS WITH UNKNOWN PARAMETERS

A more useful case than the known signal is the signal with unknown or random parameters with known probability density functions. Consider a signal $s(t)$ which is expressed in terms of n random variables,

$$s(t) = s\,(t; \underset{\sim}{s}_1, \underset{\sim}{s}_2, \ldots, \underset{\sim}{s}_n),$$

where the $\underset{\sim}{s}_i$ are independent random variables. The likelihood ratio for this signal can be expressed by

$$\ell_s(x) = \int_{s_1} \int_{s_2} \ldots \int_{s_n} \ell_s\ (x;\ \underset{\sim}{s}_1, \underset{\sim}{s}_2 \ldots \underset{\sim}{s}_n) p_{s_1} p_{s_2} \ldots p_{s_n}\, ds_1\, ds_2 \ldots ds_n, \qquad \text{(B-8)}$$

where $\ell_s\,(x; \underset{\sim}{s}_1, \underset{\sim}{s}_2, \ldots, \underset{\sim}{s}_n)$ is the likelihood ratio for an exactly known signal with independent* parameters $\underset{\sim}{s}_1 = s_1, \underset{\sim}{s}_2 = s_2, \ldots \underset{\sim}{s}_n = s_n$. This equation is very convenient since it gives the likelihood ratio for a general signal class with any number of random parameters.

In the remainder of this appendix, detection of a general class of spread spectrum signals is considered. The likelihood ratio and performance measure for signals that employ frequency hopping, time hopping, pseudonoise spreading or any combination of these techniques is derived for an optimal partial-band detector. Also, the case of a full-band optimum detector for signals with a unity time-bandwidth product is discussed.

Consider a signal which consists of a train of N_p frequency-hopped pulses of duration τ occurring on the average every T_o with a bandwidth W_p such that the message may occupy a total bandwidth W_m. In addition, the observable signal space in the frequency domain is assumed to consist of a fraction f of the total signaling bandwidth W_m. For a frequency-hopped signal, this is equivalent to observing fM out of M possible signal frequencies. On any given hop, the probability that the transmitted signal is within the observation space is f. If the signal is outside this space, the observed waveform can be represented as a signal with amplitude $a = 0$ and otherwise, amplitude $a = A$. Thus, in general, the observed signal can be expressed by

$$s_k(t) = s_k\ (t; \underset{\sim}{a}, \underset{\sim}{\omega}, \underset{\sim}{\theta}, \underset{\sim}{n}) = \underset{\sim}{a}_k \cos\,(\underset{\sim}{\omega}_k\, t + \underset{\sim}{\theta}_k),$$

where the kth pulse occurs during the random time slot defined by

$$(k-1)\, T_o + (\underset{\sim}{n}_k - 1)\,\tau < t < (k-1)\, T_o + \underset{\sim}{n}_k\,\tau,$$

*Independence is not necessary but is convenient since the joint density functions are often unwieldy.

where

$$\underline{n}_k = \{1, 2, \ldots T_o/\tau\}$$

is one of T_o/τ time slots, $\underline{\omega}_k$ can be any one of fM frequencies, and $\underline{a}$ is a random variable that can take the values 0 or A. The density functions for these random variables can be defined by:

$$p(\underline{n}_k) = \tau/T_o = \alpha = \text{duty cycle}$$

$$p(\underline{\theta}_k) = \frac{1}{2\pi}$$

$$\begin{aligned} p(\underline{a}_k) &= f \text{ for } \underline{a}_k = A \\ &= 1 - f \text{ for } \underline{a}_k = 0 \end{aligned}$$

$$\begin{aligned} p(\underline{\omega}_k \mid \underline{a}_k) &= \frac{1}{fM} \text{ for } \underline{a}_k = A \\ &= \frac{1}{(1-f)M} \text{ for } \underline{a}_k = 0. \end{aligned}$$

Likelihood Ratio

Following the work of Peterson [B1], the receiver for this signal is shown in Fig. B-1 and the likelihood ratio for a single pulse, $\ell_k(x)$, can be expressed by

$$\begin{aligned} \ell_k(x) &= \int d\theta_k\, p(\theta_k) \int dn_k\, p(n_k) \int da_k\, p(a_k) \\ &\quad \int d\omega_k\, p(\omega_k | a_k)\, \ell_s(x; a_k, n_k, \omega_k, \theta_k) \\ &= \int_0^{2\pi} \frac{1}{2\pi}\, d\theta \sum_{j=1}^{1/\alpha} \alpha \Bigg[f \sum_{i=1}^{fM} \frac{1}{fM} \ell_s(x; A, n_k, \omega_k, \theta_k) \\ &\quad + (1-f) \sum_{i=1}^{(1-f)M} \frac{1}{(1-f)M} \ell_s(x; 0, n_k, \omega_k, \theta_k) \Bigg] \\ &= \frac{\alpha}{M} \sum_{i=1}^{fM} \sum_{j=1}^{1/\alpha} \ell_{ijk}(x) + (1-f). \end{aligned} \qquad \text{(B-9)}$$

In this last expression, $\ell_{ijk}(x)$ is the likelihood ratio for a single channel of the detector based on an observation of duration $\alpha T_o = \tau$.

Within each channel of the detector, the likelihood ratio can be expressed in terms of the probability density functions for noise-only and signal-plus-noise cases:

$$\ell_{ijk}(x) = \frac{p_{s+n}(x)}{p_n(x)}.$$

For this general class of signals, within each channel the density functions can be expressed by

$$p_{s+n}(x) = \frac{1}{2^N} \left\{ \frac{4x}{N\beta} \right\}^{(N-1)/2} \exp\left\{ -\frac{x + N\beta}{2} \right\} I_{N-1} \sqrt{N\beta x}$$

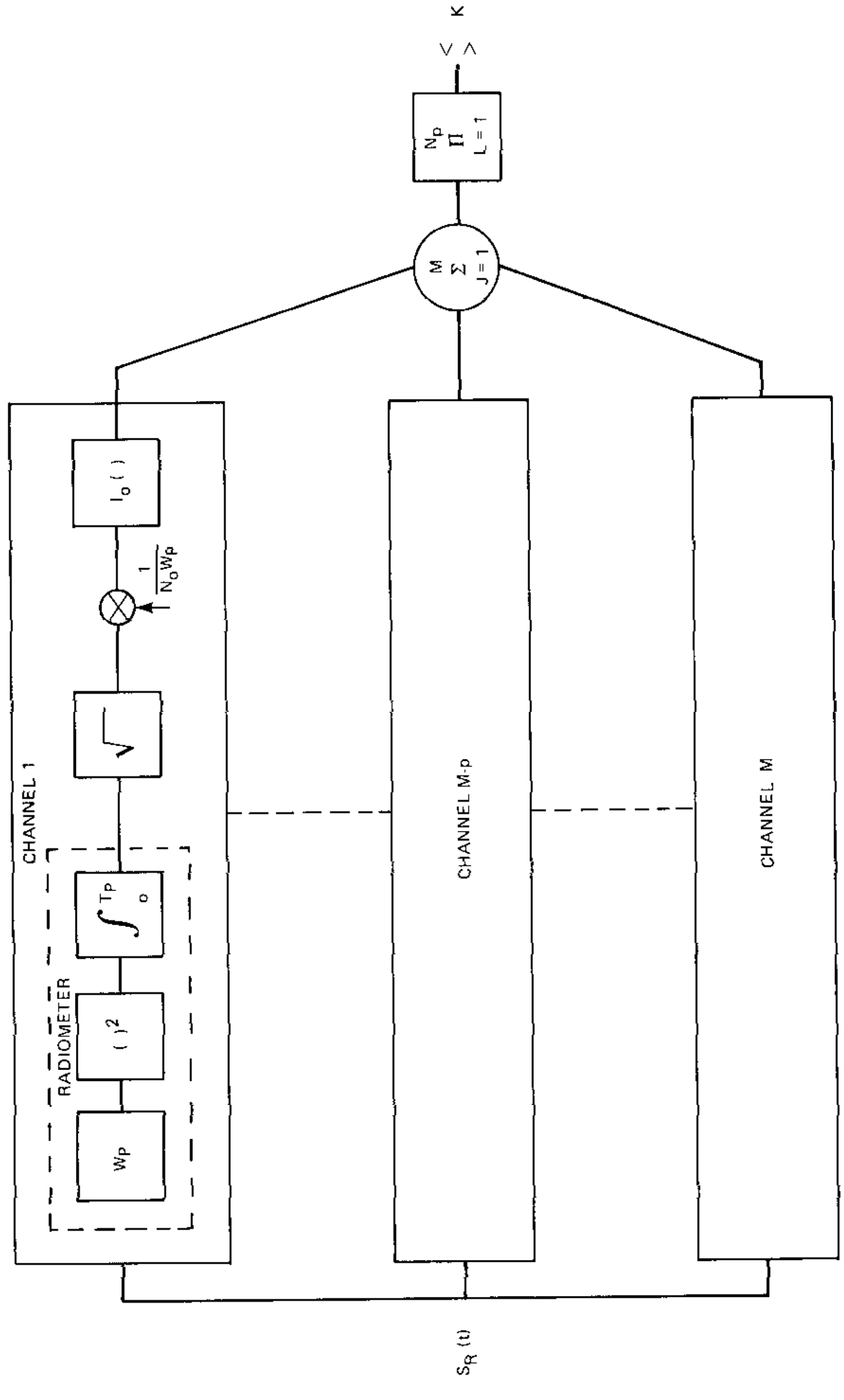


Fig. B-1 — Optimum detector for frequency-hopped waveform

$$p_n(x) = \frac{x^{N-1}\, e^{-x/2}}{2^N \Gamma(N)}, \tag{B-10}$$

where N is the TW product ($\tau\, W_p$), and β is the peak carrier-to-noise ratio in the radiometer bandwidth

$$\beta = \frac{2C}{N_o\, W_p}.$$

The likelihood ratio can then be expressed by

$$\ell_{ijk}(x) = \frac{2^{N-1}\, e^{-N\beta/2}}{(N\beta)^{(N-1)/2}}\, \Gamma(N)\, \frac{I_{N-1}(\sqrt{N\beta x})}{x^{(N-1)/2}}. \tag{B-11}$$

The likelihood ratio for the entire signal, a train of N_p pulses, can be expressed by taking the product of the likelihood ratios for the pulses:

$$\ell(x) = \prod_{k=1}^{N_p} \ell_k(x) = \prod_{k=1}^{N_p} \left[\frac{\alpha}{M} \sum_{i=1}^{fM} \sum_{j=1}^{1/\alpha} \ell_{ijk}(x) + (1-f) \right]. \tag{B-12}$$

Performance Measure

The performance of the likelihood ratio detector is measured by the parameter d which is given by Peterson [B1] for the exactly known signal as

$$d^2 = \ln\,[1 + VAR_N\{\ell(x)\}], \tag{B-13}$$

where the variance is taken under the noise-only condition. This measure is valid if the logarithm of the likelihood ratio has a normal probability density function, as is the case for the known signal receiver.

When the density is not normal, d will approximate the performance of the likelihood receiver whenever the log of the likelihood ratio has a limiting distribution function which is Gaussian due to the central limit theorem, and the variances are approximately equal under both noise and signal plus noise. For the case considered here, from Eq. (B-12),

$$\ln \ell(x) = \sum_{k=1}^{N_p} \ln \left[\frac{\alpha}{M} \sum_{i=1}^{fM} \sum_{j=1}^{1/\alpha} \ell_{ijk}(x) + (1-f) \right], \tag{B-14}$$

where the logarithm of the product has been expressed as the sum of logarithms of each term. Since each term in the outermost summation is an independent random variable (signal and noise are independent from hop to hop), the central limit theorem can be applied and the distribution function tends to Gaussian as N_p becomes large. From the properties of the likelihood ratio

$$VAR_N\,\{\ell(x)\} = E_N\,\{\ell^2(x)\} - 1, \tag{B-15}$$

Eq. (B-13) becomes

$$d^2 = \ln E_N\,\{\ell^2(x)\}. \tag{B-16}$$

Substituting the expression for $\ell(x)$ given in Eq. (B-12), we can reduce the resulting equation to

$$d^2 = N_p \ln \left[1 + \frac{f\alpha}{M} \, VAR_N \, \{\ell_{ijk}(x)\} \right], \tag{B-17}$$

where use has been made of the fact that the inner terms, $\ell_{ijk}(x)$, in Eq. (B-12) are statistically independent and identically distributed random variables. Thus, it remains only to find the variance of $\ell_{ijk}(x)$ from the definition of expected value:

$$E_N\{\ell_{ijk}^2(x)\} = \int \ell_{ijk}^2(x) \, p_n(x) \, dx. \tag{B-18}$$

By substituting Eqs. (B-10) and (B-11), the integral can be written in standard form

$$E_N \left\{ \ell_{ijk}^2(x) \right\} = \left[\frac{2}{N\beta} \right]^{N-1} \Gamma(N) \, e^{-N\beta} \int_0^\infty e^{-t} I_{N-1}^2 \; [2^{1/2} (N_\beta)^{1/2} \, t^{1/2}] \, dt, \tag{B-19}$$

which is solved by Bateman [B2]. From Eq. (B-15), the variance becomes

$$VAR_N\{\ell_{ijk}\} = \left[\frac{2}{N\beta} \right]^{N-1} \Gamma(N) \; I_{N-1} \; (N\beta) - 1, \tag{B-20}$$

which can be substituted into Eq. (B-17) to yield

$$d^2 = N_p \; \ln \left\{ 1 + \frac{f\alpha}{M} \left[K_{TW} \left(\frac{2C}{N_o W_p} \right) - 1 \right] \right\}, \tag{B-21}$$

where

$$K_N(\gamma) \triangleq I_{N-1} \; (N\gamma) \left(\frac{2}{N\gamma} \right)^{N-1} \Gamma(N). \tag{B-22}$$

This expression can now be solved for the required input carrier-to-noise density ratio as a function of d^2, the specified output SNR,

$$\frac{C}{N_o} = \frac{W_p}{2} \; K_{TW}^{-1} \left[1 + \frac{M}{\alpha f} \; (e^{d^2/N_p} - 1) \right], \tag{B-23}$$

where $K^{-1}(\;)$ denotes the inverse function.

A special case results when the time-bandwidth product for the radiometers is unity. The result obtained when this constraint is imposed is found directly from Eqs. (B-21) and (B-23). With an additional assumption of full band detection ($f = 1$) and a full duty-cycle waveform ($\alpha = 1$), the carrier-to-noise density simplifies to the following, familiar expression:

$$\frac{C}{N_o} = W_p \cdot \frac{1}{2} \, I_0^{-1} \; [1 + M \; (e^{d^2/N_p} - 1)]. \tag{B-24}$$

REFERENCES

B1. W.W. Peterson *et al.*, "The Theory of Signal Detectability," *IRE Trans.* **PGIT-4**, 171-212 (1954).

B2. A. Erdélyi, ed., "California Institute of Technology, Bateman Manuscript Project," in *Tables of Integral Transforms* (McGraw-Hill, New York, 1954), Vol. 1, pp. 4.16-21.

Appendix C

PERFORMANCE OF THE FILTER BANK COMBINER DETECTOR

Analysis of the performance of the detector shown in Fig. 4 is straightforward. For the i th radiometer channel, the parameters shown in Fig. C-1 are defined as follows:

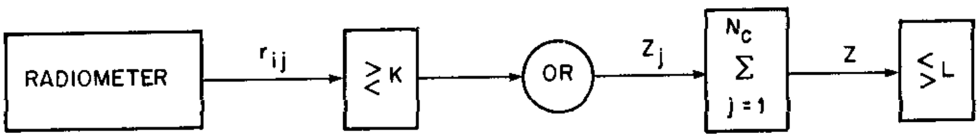


Fig. C-1 — i th channel of FBC

r_{ij} = output of ith radiometer on j th hop

K = individual radiometer threshold

Z_j = union of all radiometer decisions on the j th hop

$$= \begin{cases} 0; \ r_{ij} \leqslant K \quad \forall \ i \\ 1; \text{ otherwise} \end{cases}$$

Z = sum of Z_j over N_p hop

L = filter bank combiner detector threshold.

Thus, the probability of detection, P_D, and false alarm, P_{FA}, can be defined as

$$P_D = P_{\text{ROB}} \, (Z \geqslant L | H_{1m}) \tag{C-1}$$

$$P_{FA} = P_{\text{ROB}} \, (Z \geqslant L | H_{0m}),$$

where H_{1m} represents the hypothesis that a message was transmitted during the observation time, and H_{0m}, the hypothesis that no message was transmitted. Since $Z = \Sigma \ Z_j$, these probabilities can be obtained from the binomial distribution

$$P(Z > L) = \sum_{j=L}^{N_c} \begin{Bmatrix} N_c \\ j \end{Bmatrix} P(Z_j = 1)^j \, P(Z_j = 0)^{N_c - j} \tag{C-2}$$

where $N_c = T/T_{\text{HOP}}$, the number of hop periods in a message.

Thus, P_D and P_{FA} are given by

$$P_D = \sum_{j=L}^{N_c} \begin{Bmatrix} N_c \\ j \end{Bmatrix} P(Z_j = 1 | H_{1m})^j \, P(Z_j = 0 \mid H_{1m})^{N_c - j} \tag{C-3}$$

$$P_{FA} = \sum_{J=L}^{N_c} \begin{Bmatrix} N_c \\ j \end{Bmatrix} P(Z_j = 1 | H_{0m})^j \, P(Z_j = 0 | H_{0m})^{N_c - j}.$$

For a pulsed waveform with duty cycle α and an FBC that has N_R radiometers matched to the pulse bandwidth W_P which cover a fraction f of the total spread bandwidth W,

$$f = \frac{N_R\, W_P}{W}, \tag{C-4}$$

On any given hop period there is a probability $(1 - \alpha)$ that a pulse is not transmitted during the observation time, and a probability $(1 - f)$ that the hop frequency is outside the bandwidth of the FBC. Thus, the presence or absence of a pulse within the radiometer during a message transmission must be accounted for. This is accomplished by defining the following hypotheses which apply to each hop observation interval.

H_{1h} = pulse present within FBC bandwidth during hop period;

H_{0h} = pulse absent during hop period.

For the j th hop period, therefore, the probabilities in Eq. C-3 must be expanded,

$$\begin{aligned} P(Z_j = 0 \mid H_{1m}) &= P(Z_j = 0 \mid H_{1h})\, P(H_{1h} \mid H_{1m}) + P(Z_j = 0 \mid H_{0h})\, P(H_{0h} \mid H_{1m}) \\ P(Z_j = 0 \mid H_{0m}) &= P(Z_j = 0 \mid H_{1h})\, P(H_{1h} \mid H_{1m}) + P(Z_j = 0 \mid H_{0h})\, P(H_{0h} \mid H_{1m}) \end{aligned} \tag{C-5}$$

and

$$\begin{aligned} P(Z_j = 1 \mid H_{1m}) &= 1 - P\,(Z_j = 0 \mid H_{1m}) \\ P(Z_j = 1 \mid H_{0m}) &= 1 - P\,(Z_j = 0 \mid H_{0m}). \end{aligned} \tag{C-6}$$

The dependent probabilities are determined as follows:

$P(H_{1h} \mid H_{1m})$ = P (a pulse is present within the FBC bandwidth given that a message was transmitted)
= P (a pulse transmitted during hop observation period) $\cdot$ P (hop frequency within FBC bandwidth
$= \alpha \cdot f$ (C-7)

$P(H_{0h} \mid H_{1m})$ = P (pulse not present given message transmitted)
= P (pulse not transmitted during observation period + P (pulse transmitted but hop frequency outside of FBC bandwidth)
$= (1 - \alpha) + \alpha(1 - f)$
$= 1 - \alpha f.$ (C-8)

$P(H_{1h} \mid H_{0m})$ = P (pulse present given no message transmitted)
$= 0.$ (C-9)

$P(H_{0h} \mid H_{0m})$ = P (no pulse present given no message transmitted)
$= 1.$ (C-10)

From the definition of Z_j, $P(Z_j = 0)$ is the probability that no radiometers exceeded threshold on the j th hop period. For the H_{0h} hypothesis,

$$\begin{aligned} P(Z_j = 0 \mid H_{0h}) &= P(r_{ij} < K \text{ for } i = 1, 2, \ldots N_R \text{ given that no signal energy is present}) \\ &= \prod_{i=1}^{N_R} P(r_{ij} < K \mid H_{0h}). \end{aligned}$$

Since outputs are independent from hop to hop,

$$P(Z_j = 0 \mid H_{0h}) = P(r_{ij} < K \mid H_{0h})^{N_R}. \tag{C-11}$$

In the H_{1h} case,

$$P(Z_j = 0 \mid H_{1h}) = P(r_{ij}\, k \text{ for } i = 1, 2, \ldots N_R \text{ given that a pulse is present in one and only one individual radiometer})$$

$$= P(r_{ij} < K \mid H_{1h}) \prod_{i=1}^{N_R-1} P(r_{ij} < K \mid H_{0h})$$

$$= P(r_{ij} < K \mid H_{1h}) \left[P(r_{ij} < K \mid H_{0h})\right]^{N_R-1}. \tag{C-12}$$

The performance of the individual radiometers can be described by the required P_{DI} and P_{FAI} where

P_{DI} is the probability of detection of an individual radiometer on a single hop or pulse for the required filter bank message P_D;

P_{FAI} is the probability of false alarm for an individual radiometer on a single hop or pulse, for the required filter bank message P_{FA};

and

$$P_{DI} = P(r_{ij} > K \mid H_{1h}) = 1 - P(r_{ij} < K \mid H_{1h}) \tag{C-13}$$

$$P_{FAI} = P(r_{ij} > K \mid H_{0h}) = 1 - P(r_{ij} < K \mid H_{0h}).$$

Substituting Eq. C-13 in Eqs. C-11 and C-12 yields

$$P(Z_j = 0 \mid H_{0h}) = (1 - P_{FAI})^{N_R}$$

and

$$P(Z_j = 0 \mid H_{1h}) = (1 - P_{DI})\ (1 - P_{FAI})^{N_R-1}. \tag{C-14}$$

Combining Eq. C-14 with Eqs. C-7 through C-10 in Eq. C-5

$$P(Z_j = 0 \mid H_{1m}) = (1 - P_{DI})\ (1 - P_{FAI})^{N_R-1}\ \alpha f + (1 - P_{FAI})^{N_R}\ (1 - \alpha f) \tag{C-15}$$

$$P(Z_j = 0 \mid H_{0m}) = (1 - P_{FAI})^{N_R}.$$

Equations C-6, C-15, and C-3 now relate P_{DI} and P_{FAI} to P_D and P_{FA}. Solving Eq. C-3 for P_{FAI} and P_{DI}, however, involves the inverse of the binomial distribution and must be done iteratively to determine the optimum value of the threshold L. For $L = 1$, Eq. C-3 reduces to

$$P_D = 1 - P(Z_j = 0 \mid H_{1m})^{N_c} \tag{C-16}$$

$$P_{FA} = 1 - P(Z_j = 0 \mid H_{0m})^{N_c}.$$

Substituting Eq. C-15 and solving for P_{DI} and P_{FAI} yields

$$P_{FAI} = 1 - (1 - P_{FA})^{1/N_c N_R} \tag{C-17}$$

$$P_{DI} = P_{FAI}\left(1 - \frac{1}{\alpha f}\right) + \left(\frac{1}{\alpha f}\right) - \frac{1}{\alpha f}\,(1 - P_D)^{1/N_c}\ (1 - P_{FAI})^{-(N_R-1)}.$$

These equations can be simplified by using the approximation

$$(1 - X)^N \approx 1 - NX\,; NX << 1.$$

Applied to Eq. C-17, this approximation yields

$$P_{FAI} \approx \frac{P_{FA}}{N_c N_R} = P_{FA}\left\{\frac{\alpha}{N_P fM}\right\} \tag{C-18}$$

and

$$P_{DI} \approx \frac{P_D}{f\alpha N_c} = P_D\left\{\frac{1}{fN_P}\right\},$$

where $\alpha N_c = N_P$ and $N_R = fM$.

Appendix D

SYMBOLS AND ACRONYMS

BMW	Binary Moving Window (type of detector)
C/N_o	Carrier power-to-noise density ratio
$(C/N_o)_{avg.}$	Time average of C/N_o required to meet detection criterion
$(C/N_o)_{peak}$	Instantaneous maximum C/N_o required to meet detection criterion
$(C/N_o)_{req.}$	Either $(C/N_o)_{avg.}$ or $(C/N_o)_{peak}$ depending upon detection strategy
CW	Continuous wave
d	Output signal-to-noise *voltage* ratio assuming Gaussian statistics
$d_\chi(TW)$	d for detector with TW product greater than 1 and chi-square statistics $(d_\chi = d_\chi(1))$
d_{χ_H}	d_χ for a single hop $(TW = 1)$
E/N_o	Predetection signal energy-to-noise density ratio
E_p/N_o	E/N_0 per single pulse
f	Fraction of total spread bandwidth
FBC	Filter Bank Combiner (type of detector)
FH	Frequency hopped
G_{TW}	Noncoherent processing gain
$I_0(\ )$	Modified bessel function of order zero
$I_0^{-1}(\ )$	Inverse function of $I_0(\ )$
K	Detector channel threshold
$K_n(\ )$	Detector channel weighting factor for channels having non-unity TW products $(n = TW)$
$K_n^{-1}(\ )$	Inverse function of $K_n(\ )$
L	Detection threshold, number of channels or pulses summed
L_{TW}	Noncoherent integration loss; NCL
M	Number of radiometer channels required to cover the signal bandwidth
MLR	Maximum likelihood ratio (type of detection)
N_p	Number of pulses per message
n	Time-bandwidth product; TW
NCL	Noncoherent combining loss
P_D	Probability of detection per message
P_{DI}	P_D on a single hop or pulse
P_{FA}	Probability of false alarm per message period
P_{FAI}	P_{FA} on a single hop or pulse period
PN	Pseudo noise
$Q^{-1}(\ )$	Inverse function of the normal probability distribution
r_H	Signal hopping rate
r_{PN}	PN rate
RMS	Root mean square
ROC	Receiver operating characteristics
S/N	Input signal-to-noise power ratio

$(S/N)_{pulse}$	S/N for a single pulse
SNR	Signal-to-noise ratio
T	Message period
T_P	Pulse duration
TH	Time hopped
TSK	Time shift keyed
W	Total signal bandwidth
W_P	Single pulse bandwidth
α	Transmission duty cycle during a message
η_{TW}	Chi-square correction factor to Gaussian approximation, a function of ROC and TW