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NONLINEAR, NON-GAUSSIAN, AND NONINDEPENDENT PARAMETER ESTIMATION AND CONDITIONAL CONFIDENCE REGIONS

K. E. Gustafson

Countermeasures Branch
Radio Division

October 10, 1963

U. S. NAVAL RESEARCH LABORATOR
Washington, D.C.

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ABSTRACT

A mathematical model that can accommodate non-Gaussian error distributions and correlation among contributors is derived for application to nonlinear position estimation computers to be used for HF-DF fixing. The assumption is that the physical situation is only mildly nonlinear, non-Gaussian, and nonindependent so that quadratic truncation of the analytic functions involved is sufficient.

Major improvement of confidence region reliability should be achieved by using an F test, conditional generalized net variance estimates, and multiple χ^2-rejection of bearings. Further minor improvements should result by using the derived non-Gaussian and correlated model. A variable relationship between confidence and area size is available. Implementation and testing of these models is currently underway at NRL.

PROBLEM STATUS

This is an interim report; work on this problem is continuing.

AUTHORIZATION

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NONLINEAR, NON-GAUSSIAN, AND NONINDEPENDENT PARAMETER ESTIMATION AND CONDITIONAL CONFIDENTIAL REGIONS

INTRODUCTION

Two major objectives in direction finding techniques are a best estimate of a position location and an optimal confidence region containing it. Two major tasks in this endeavor are the development of sophisticated radio interferometers (direction finders) to obtain measurements (at different locations) of certain characteristics of the incoming wave front (as bearing, great circle plane, direction) and the development of a mathematical model to incorporate these measurements into an estimate of the position of the emitter (usually assumed to be on the surface of the earth).

This second task has many aspects. This report is concerned only with the final mathematical model and not with questions relating to the numerical calculation of correlations, variance components, higher moments, ionospheric indicators, or other necessary collateral parameters. That is, it is a suggested model for accommodating these effects in obtaining position estimates (fixes).

BACKGROUND

The usual method assumes that each direction finder (station) error is independent of the others and has a normal (Gaussian) density function

$$\phi_i(\eta_i) = \frac{1}{\sqrt{2\pi}\ \sigma_i} e^{-\frac{1}{2}\left(\frac{\eta_i}{\sigma_i}\right)^2}$$

where σ_i is a known standard deviation and η_i is either the angular error at a station or the direction line (plane) displacement. Under these assumptions, the net density function is the product

$$\Phi = \prod_{i=1}^{n} \phi_i = \frac{e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\eta_i}{\sigma_i}\right)^2}}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i}$$

where i ranges through those stations that are able to obtain a measurement.

By iterative orthogonal transformations that rotate a model earth (radius 1), a linear subspace (tangent plane) can be made to touch the earth at a north pole (0, 0, 1), which is an estimate of the desired location. The underlying assumption for this approach is that the earth is locally flat, and the mathematical advantage is that calculations on, and in terms of, a plane are more efficient and that standard linear statistical tests can be applied.

The situation for a point $(r, s, 1)$ near the current north pole is as shown in Fig. 1.

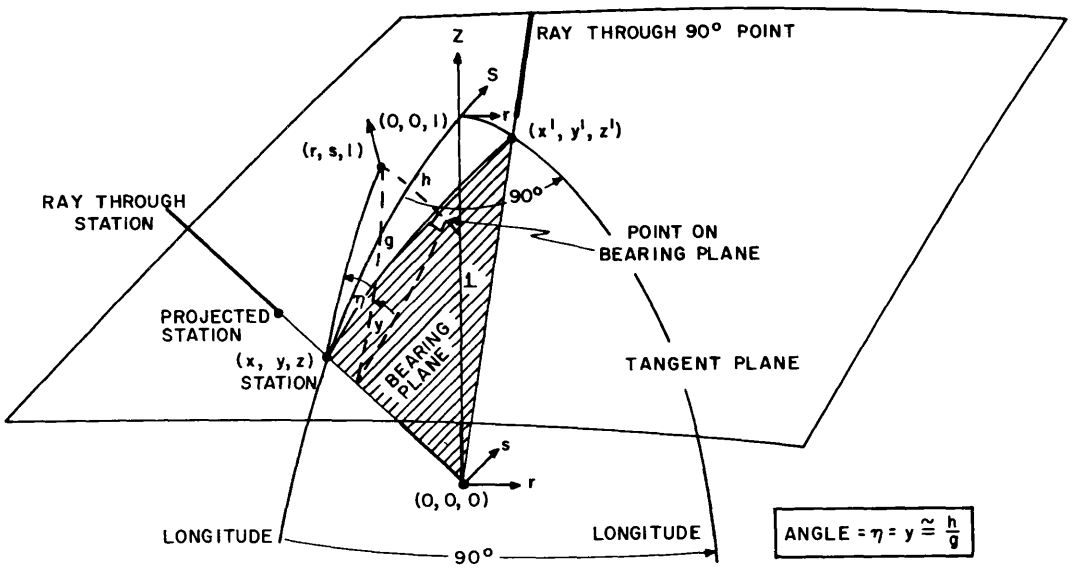


Fig. 1 - Bearing geometry

The situation on the tangent plane is as shown in Fig. 2.

The $[\text{angular error}]^2 = \eta_i^2$ between the bearing to a point which projects up to $(r, s, 1)$ and the observed bearing can be calculated in terms of r and s by

$$\eta_i^2 = y_i^2 \cong (\sin y_i)^2 = \frac{h_i^2}{g_i^2} = \sum_{p,q} a_{ipq} r^p s^q \cong \sum_{p+q \leq 2} a_{ipq} r^p s^q = Q_i(r, s)$$

as shown in the present NRL model (1).

The approximation $y \cong \sin y$ is a mathematical convenience since errors are small enough to justify it, while the validity of the truncation of the analytic expansion to a quadratic $Q_i(r, s)$ will be discussed later.

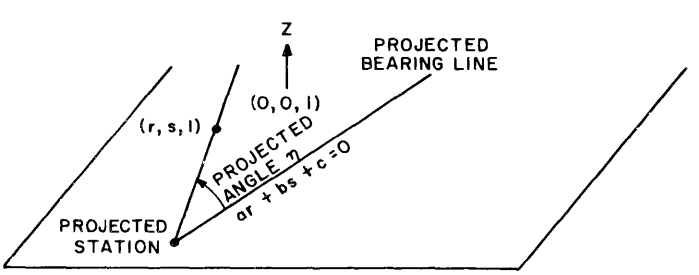


Fig. 2 - Tangent plane geometry

Hence the net density is

$$\Phi \equiv f(r,s) = \frac{e^{-\frac{1}{2}Q(r,s)}}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i}$$

where

$$Q(r,s) = \sum_{i=1}^{n} \left(\frac{Q_i(r,s)}{\sigma_i^{\,2}} \right)$$

and the classical solution (2) for the Best Point Estimate (fix) by the maximum likelihood criteria as given by the system $f_r = 0$, $f_s = 0$ is a least squares minimization of $Q(r,s)$. The confidence region (search area) in the NRL model is then determined as a contour (ellipse) on $f(r,s)$ with (when normalized to a circle) a predetermined radius dependent only on a constant probability (90% or 70%) of target inclusion (in the long run).

NONLINEAR CONFIDENCE REGIONS

We will use the "prior distribution" concept freely so that a domain space with probability distributions defined on it can be visualized (Fig. 3). We shall also assume that $Q(r,s)$ represents normal assumptions even if nonnormal effects are included, i.e., that any nonnormal effects are now incorporated into the covariance elements of $Q(r,s)$.

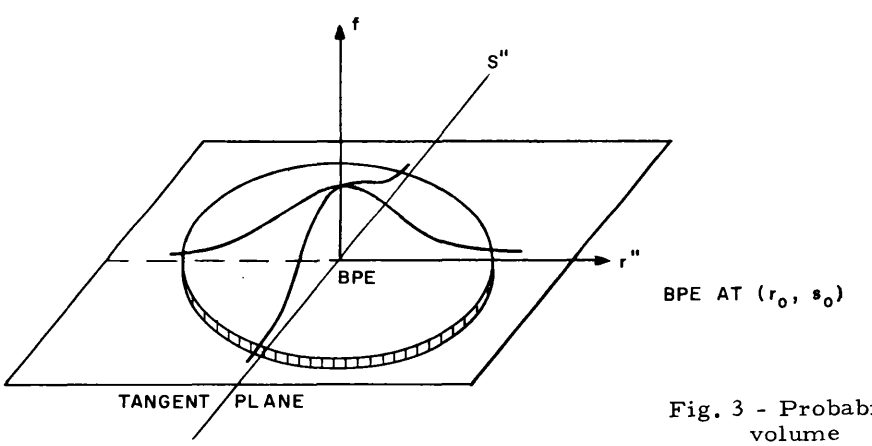


Fig. 3 - Probability volume

Duality

Confidence statements derived from the net density function $f(r,s)$ can be dually interpreted as coming from the exponent $Q''(r'',s'')$ (Fig. 4), where

$$Q(r,s) = t_{00} + t_{10}r + t_{01}s + t_{20}r^2 + t_{02}s^2$$

$$Q'(r',s') = Q(r,s) = Q(r'+r_0,\ s'+s_0) = t'_{00} + t_{20}r'^2 + t_{02}s'^2 + t_{11}r's'$$

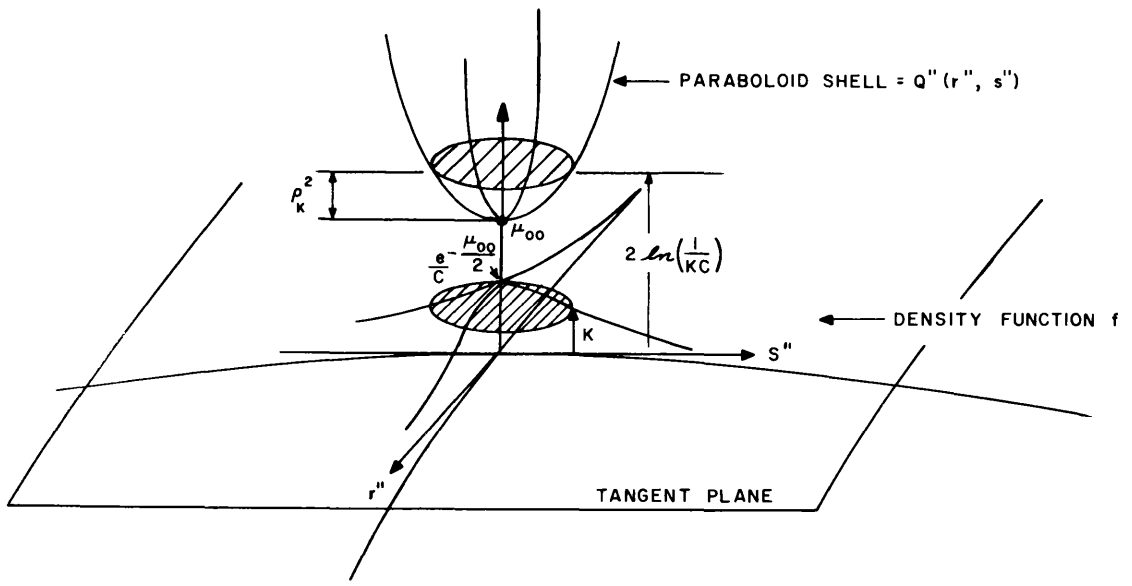


Fig. 4 - Duality

$$Q''(r'',s'') = Q'(r',s') = Q'\left(\frac{r''+ps''}{(1+p^2)^{1/2}}, \frac{s''-pr''}{(1+p^2)^{1/2}}\right) = \mu_{00} + \mu_{20}r''^2 + \mu_{02}s''^2$$

$$Q'''(r''',s''') = Q''(r'',s'') = Q''\left(\frac{r'''}{(\mu_{20})^{1/2}}, \frac{s'''}{(\mu_{02})^{1/2}}\right) = \mu_{00} + r'''^2 + s'''^2 .$$

Just as $f(r,s)$ is monotone decreasing, $Q''(r'',s'')$ is monotone increasing. $Q''(0,0) = \mu_{00}$ which should be chi-square distributed with approximately $n-2$ degrees of freedom. (The approximation is due to the nonlinearity of the model.) Hence

$$E(\mu_{00}) = n - 2 .$$

Since this duality between exponents and functions exists, most discussion concerning the search areas is more suitable in terms of $Q''(r'',s'')$, which has the chi-square distribution when minimized.

Interpretations

$\mu_{00} = Q''(0,0) = Q(r_0, s_0) = Q(BPE)$ is a fundamental quantity in the fix model. Since $Q(r_0, s_0)$ represents the sum of the normalized errors from n independent contributors, it should be distributed as chi-square, i.e., as $\chi^2(m)$, where $m = n-2$ because two parameters (r_0, s_0) had to be estimated.

Writing

$$f(r,s) = f'''(r''',s''') = \frac{1}{C}\left[e^{-\mu_{00}/2}\, e^{-(r'''^2+s'''^2)/2}\right]$$

it is illustrative to call $e^{-\mu_{00}/2}$ the "spread factor." Note that

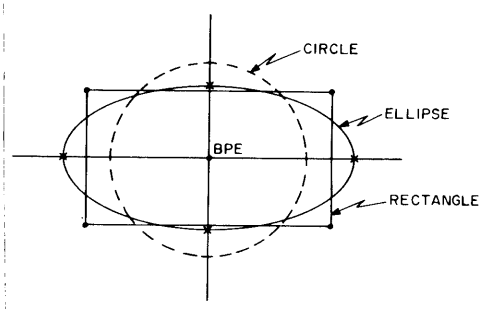
$$e^{-\mu_{00}/2} \equiv 1 \rightarrow \mu_{00} \equiv 0 \rightarrow \begin{cases} * \text{ Bullseye} \\ \text{or} \\ \text{2-station fix} \end{cases}$$

Hence the quantity μ_{00} should be interpreted on each fix as the "inherent spread" or as the measure of bearing consistency. Since μ_{00} is $\chi^2(n-2)$ and $E(\mu_{00}) = n-2$, a reliability statement is obtained for each case.

Configurations

Presently the NRL model outputs a rectangular search area (Fig. 5). Situations arise when a circular search area is preferable. The optimal search area is the ellipse, and listing the opposite ends of its axes appears as convenient as listing the corners for the rectangle. The rectangle and circle are less efficient configurations because of the greater area involved (on the average) for the same confidence statements.

Fig. 5 - Confidence configurations



These various configurations are all easily accessible in the model as are the area sizes. Moreover, a relationship between probability of inclusion P and the area A is also available.

Ellipse — After translation and rotation, we have

$$f(r,s) = f''(r'',s'') = \frac{1}{C} e^{-(\mu_{00}+\mu_{20}r''^2+\mu_{02}s''^2)/2} .$$

Let $f = K$, which determines an elliptical contour since f is monotone. Then

$$KC = e^{-(\mu_{00}+\mu_{20}r''^2+\mu_{02}s''^2)/2}$$

describes the contour, and

$$\mu_{20}r''^2 + \mu_{02}s''^2 = 2 \ln \frac{1}{KC} - \mu_{00}$$

is an ellipse, that is,

$$\frac{r''^2}{a^2} + \frac{s''^2}{b^2} = 1$$

where

$$a = \left(\frac{2 \ln \frac{1}{KC} - \mu_{00}}{\mu_{20}}\right)^{1/2}, \quad b = \left(\frac{2 \ln \frac{1}{KC} - \mu_{00}}{\mu_{02}}\right)^{1/2} .$$

Note that the stipulations $\mu_{20} > 0$, $\mu_{02} > 0$, $2\,\ell n\,(1/KC) \geqq \mu_{00} \geqq 0$ are required. In this plane,

$$A = \pi\, ab$$

$$= \frac{\pi\left(2\,\ell n\,\frac{1}{KC} - \mu_{00}\right)}{(\mu_{20}\,\mu_{02})^{1/2}}$$

in (earth radians)^2. It may be noted that in the present NRL model, the σ^{-2} are never converted to radian measure, so the μ's are too large, i.e., are in units of (radian/degree)^2; hence, they must be multiplied by (180 degrees/π radians)^2 to make them dimensionless in the above expression.

Going to the ‴ plane,

$$P = 1 - e^{-\rho_K^2/2} \text{ (where } \rho_K \text{ is the radius in the ‴ plane)}$$

$$= 1 - e^{-\left(2\,\ell n\,\frac{1}{KC} - \mu_{00}\right)/2}.$$

From the above equations we consolidate the Key Relation:

$$2\,\ell n\left(\frac{1}{1-P}\right) = \frac{A}{\pi}(\mu_{20}\,\mu_{02})^{1/2} = 2\,\ell n\,\frac{1}{KC} - \mu_{00} = \rho_K^2 = r'''^2 + s'''^2 \text{ on locus.}$$

Hence

$$a = \left[\frac{A}{\pi}\left(\frac{\mu_{02}}{\mu_{20}}\right)^{1/2}\right]^{1/2} = \left[2\,\ell n\left(\frac{1}{1-P}\right)\cdot\frac{1}{\mu_{20}}\right]^{1/2}$$

in terms of P or A, and

$$b = \left[\frac{A}{\pi}\left(\frac{\mu_{02}}{\mu_{20}}\right)^{1/2}\right]^{1/2} = \left[2\,\ell n\left(\frac{1}{1-P}\right)\cdot\frac{1}{\mu_{02}}\right]^{1/2}$$

in terms of P or A.

Rectangle — To get a Square of some area in the ‴ plane, we have as in the NRL model $4(S/2)^2 = \pi\rho^2$, from which

$$\left(\frac{S}{2}\right) = \left(\frac{\pi\rho^2}{4}\right)^{1/2}.$$

In terms of P and A,

$$\left(\frac{S}{2}\right) = \frac{1}{2}\left[A(\mu_{20}\mu_{02})^{1/2}\right]^{1/2} = \left[\frac{\pi}{2}\,\ell n\left(\frac{1}{1-P}\right)\right]^{1/2}.$$

Note that when we go back to the ″ plane,

$$4\left(\frac{S}{2}\right)^2 \rightarrow \frac{A(\mu_{20}\mu_{02})^{1/2}}{(\mu_{20})^{1/2}\,(\mu_{02})^{1/2}} = A.$$

Hence the rectangle has the same area as the ellipse that was contracted to the circle in the ‴ plane, although this rectangle will give lower reliability of inclusion due to its corners.

Circle — To get a circular search area for output, we look at the ″ plane and require $\pi ab = \pi r^2$, which gives

$$r^2 = \frac{\rho^2}{(\mu_{20}\ \mu_{02})^{1/2}} = \frac{A}{\pi} = \frac{2\ \ell n\left(\frac{1}{1-P}\right)}{(\mu_{20}\ \mu_{02})^{1/2}} .$$

Probability vs Area — From the Key Relation derived above, we see that if given P,

$$A = \frac{2\pi\ \ell n\left(\frac{1}{1-P}\right)}{(\mu_{20}\ \mu_{02})^{1/2}} ;$$

and if given A,

$$P = 1 - e^{-A(\mu_{20}\mu_{02})^{1/2}/2\pi}$$

where A here is always the optimum search area configuration, namely, the ellipse. If circular or rectangular output is desired, r and $S/2$ can be found from the above, but the probability P will be lower in the above P vs A statement.

Hence the model provides a means of arriving at approximate area vs probability statements for elliptical, rectangular, or circular search area configurations, but the probability is overstated if the ellipse is not used.

Confidence Weaknesses

Statistical studies have revealed that instead of the programmed 90-percent confidence level being attained, performance has been lower (3). Five explanations are now examined.

Variance Estimation — This is always a cause of reliability failure in estimation problems, so for some time NRL has been maintaining variance quality in a direction-finding (d.f.) operation. A large sample of fixes from a randomized situation were first processed using "questionable" variance estimates; and then, using true variances for that sample, the cases were reprocessed. Although some improvement in all aspects of the estimation was achieved, the inclusion percentage changed only from 74 to 76 percent, indicating that the preliminary variance estimates were reasonably good and that the problem was deeper.

Non-Gaussian Bearings — The possibility of non-Gaussian bearings was investigated by a factor analysis, and non-Gaussian populations were found to be the rule rather than the exception in direction finding when present equipment (4) was utilized. A mathematical way of compensating for these effects is derived later in this report. However, the deviation from Gaussian distributions in the long run is small, and, at 90-percent confidence levels, does not explain the large departure from the desired confidence level.

Nonlinearity — The model is nonlinear due to the spherical earth setting and the angular errors that are the concern of long range direction finding. When station-target distances are approximately the same for all stations, the assumption of "parallel displacement" of the bearing line is often made (5). This, along with a locally flat earth, allows development of a linear model.

The NRL model does not make these linear assumptions at the beginning, but instead assumes that quadratic truncation is sufficient for representing the model, and the tangent plane thus determined becomes the basis for confidence region estimation (1). Hence the locally flat earth is a common assumption and reasonably so for the region sizes in the application, since confidence regions calculated on the sphere and on the plane coincide up to the significance of the data (6).

It appears that the quadratic assumption is not as restrictive as the "parallel displacement" assumption even though in many cases they are equivalent.

An extensive analysis (7) of general nonlinear regression (the d.f. model is a special case) has led to the conclusion that in any reasonable d.f. application the nonlinearity is negligible, and this enforces the intuitive arguments above. Hence it seems reasonable to conclude that nonlinearity is not a major cause of confidence weakness. Two interesting studies could be made in this regard to resolve the matter.

The obvious study for the NRL model would not be to truncate at the quadratic level but instead to keep higher order terms (e.g., of the third and fourth degree). Then the magnitude of these terms could be examined theoretically for the range of (r,s) encountered to see if this remainder really is insignificant. Alternatively, position estimates could be made with the higher order polynomial and compared with the corresponding quadratic estimates.

The other study would be less direct but more feasible. It can be noted that $\mu_{20}\,\mu_{02} = |V|^{-1}$ where $|V|$ is the determinant of the covariance matrix of (r,s) when transformed to canonical form. If the model were indeed linear, we would have

$$\mu_{20}\,\mu_{02} = \sum_{k<\ell}\sum \sigma_k^{-2}\sigma_\ell^{-2}\sin^2(\theta_k - \theta_\ell)$$

where the latter value of $|V|^{-1}$ is derived in Daniels (5). But the sum above is the sum of the quick fix weights in the NRL model. It would be most interesting to compare these two quantities which are both readily available in the NRL model for each computer-processed d.f. case.

Wild Bearing Encroachment — All confidence reliability examinations to date have been performed by analysis of the results of processing on the prototype computer which has only a single normal bearing rejection scheme. Accordingly one-half of the wild bearings were considered good bearings by the computer and therefore were incorporated into the a priori estimates (BPE 2).

A special sample with no bearings in error by more than 15 degrees, i.e., not wild, and fixes in error less than 300 miles (in other words, a sample of "good" cases from "good" bearings) achieved an 88-percent confidence figure. Hence elimination of wild bearings will be a major factor in maintaining confidence strength.

The following test is recommended as a general bearing rejection scheme (Fig. 6). It is expected that it will be improved by tests currently underway, although the principal

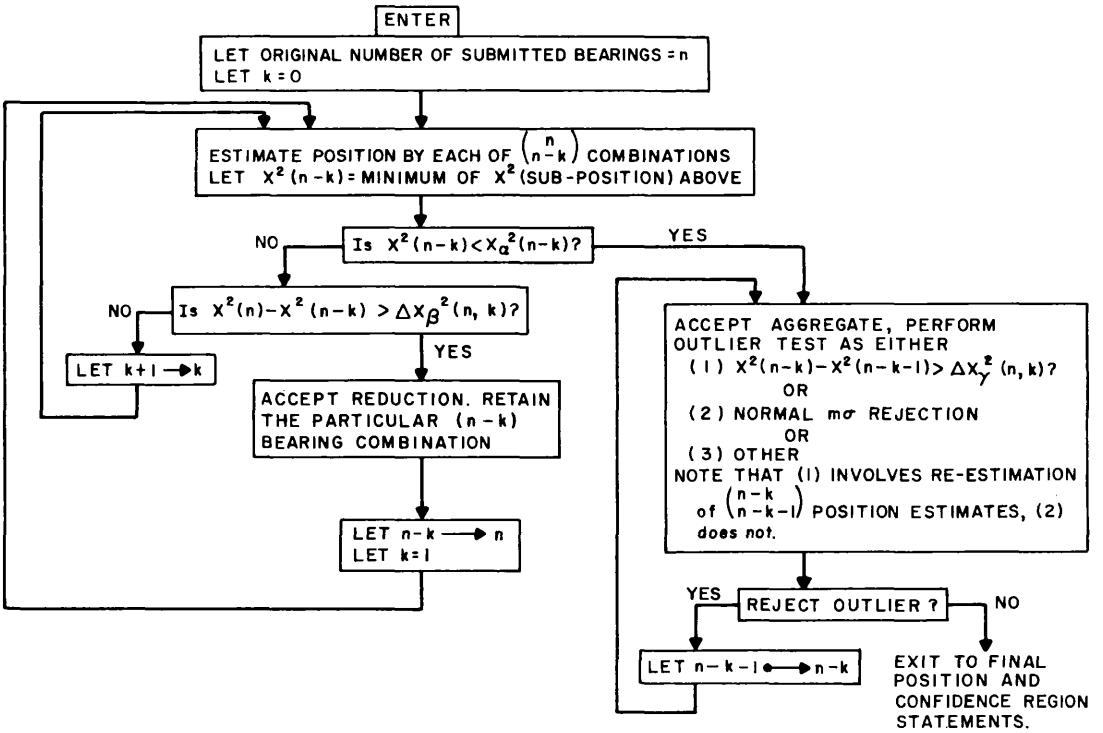


Fig. 6 - Wild bearing rejection

ideas involved will probably survive. It is entirely possible that the BPE can be used in the estimates for wild bearing rejection in place of the currently used quick fix which is no longer as relatively "quick" as it was in the prior prototype. This is pleasing theoretically, since the bearing rejection scheme and the BPE are maximum likelihood results of angular minimization, whereas the quick fix results from intersection distance minimization (in the least squares sense). Then only one quick fix would be necessary to converge the system to a local area, this being necessary because of slow BPE convergence.

Presently $\Delta\chi_\beta^2(n,k) = \chi_\beta^2(n) - \chi_\beta^2(n-k)$ is being tested. The choice of β and α can be made large (e.g., for $\beta = 0.5$ we have $\Delta\chi^2(n,k) \cong k$ independent of n) if we are willing to sacrifice more good bearings in order to reject more bad bearings. (E.g., larger $\alpha \rightarrow$ more $k > 1$ tests, which pertains to more than one wild bearing/case.)

Correlation Between Stations — Analysis of bearings (3) has shown that these correlations are of the order of 0.1 for "nice" geometrical pairs of stations, but increase to higher than 0.3 for stations located close together and looking in the same direction over a one-hop distance. This corroborates the theory of at least a short term, ionospheric gradient in a local "reflection zone." This also probably explains the different performance characteristics for different area classes as found on the factor analysis. A method of implementing these measurements is developed later in this report.

New Criteria

Even in the most well-behaved cases with no wild bearings, no correlation, and excellent variance estimates, performance was not up to the 90-percent programmed level of confidence. Also, the criterion used was absolute. It is felt that in most d.f. applications a conditional confidence region is desired: the following new criterion is therefor recommended.

The present NRL model has been using a confidence ellipse on the " plane defined by

$$\mu_{20}\, r^2 + \mu_{02}\, s^2 \leqq 2\, \ell n \left(\frac{1}{1-P}\right) .$$

The right side of the above is also

$$2\, \ell n \left(\frac{1}{1-P}\right) = 2\, \ell n \left(\frac{1}{\alpha}\right) = \chi_\alpha^2(2)$$

so that an absolute probability statement is being applied in an acceptable fashion. If, however, we assume that forming η_i^2/σ_i^2 from past variance estimates σ_i^2 only normalizes the various stations to a common (long run) variance of σ^2, then the standard Wilks-Daly confidence region for our application becomes

$$\mu_{20}\, r^2 + \mu_{02}\, s^2 \leq 2\sigma^2 F_\alpha(2, v)$$

where σ^2 is estimated with v degrees of freedom and $E(\sigma) = 1$. Note that $2F_\alpha(2,v) > \chi^2(2)$ for any finite v, and hence the F region (determined by this F test) is always larger than the χ^2 region.

To obtain a conditional confidence statement, we consider estimating the generalized net variance σ_c^2 for each case. The best estimate that can be derived from the case itself is

$$\hat{\sigma}_c^2 = \frac{\mu_{00}}{n-2}$$

so our conditional, unbiased, recommended confidence region becomes

$$\mu_{20}\, r^2 + \mu_{02}\, s^2 \leqq \frac{2\mu_{00}}{n-2} F_\alpha(2, n-2) .$$

This appears especially appropriate since it has been observed (3) that confidence declines as n decreases with the $\chi^2(2)$ model, as would be expected. The search area size is now given by

$$A = \frac{2\pi\, \mu_{00}\, F_\alpha(2, n-2)}{n-2\, (\mu_{20}\, \mu_{02})^{1/2}}$$

and $P = 1-\alpha$ will still yield P vs A statements by either consulting stored tables or calculating directly the desired quantity from F_α.

NON-GAUSSIAN ESTIMATION

Extensive observation has shown that the populations encountered in direction-finding are neither Gaussian nor erratic (4). Non-Gaussian effects are enhanced by an effective separation of bearing classes, but even the parent populations are not Gaussian.

Separation of Populations

By noting several parameters, the parent population can be partitioned into classes, whether this be done by components of variance or factor combinations. A complete factor breakdown is not feasible because of the inherently large number of factors, but partial analyses on limited parameters can be useful in analysis of variance. Some of the more reliable parameters are bearing swing, local site conditions, and distance from transmitter to receiver.

The implementation of automatic direction-finding (ADF) should provide a previously unavailable source of information concerning the ionospheric status while the bearing energy traverses its path. Other ionospheric measurements made simultaneously or within a meaningful time frame will also augment population separation.

Large samples indicate, however, that the separated bearing classes deviate even more from a Gaussian distribution. In particular, some are "steeper" than others with the same variance. For example, a class of bearings from a favorable parameter combination will have smaller variance and larger fourth cumulant (3).

Representation of Populations

The assumption of normal populations often is based upon three considerations: (a) normal curves fit many experimentally obtained statistics adequately; (b) mathematical derivations lead to the normal curve subject to only minor assumptions; and (c) it is much more difficult to work with anything other than the normal distribution curve.

Our experimental data show that the distributions are not normal (3). The mathematical assumptions in condition (b) can be weakened in our problem. The nonnormal curve will be more difficult to work with according to condition (c), but high speed computing devices should be able to handle the increased computations inherent in nonnormal statistics with no appreciable loss of time if the following model is accepted.

Mathematical Justification — Let x be the random deviate (and for specific illustration x can be thought of as the bearing error at a given station). Assume

1. $x = \lambda \epsilon_1 + \ldots + \lambda_n \epsilon_n$, that is, the x deviation is a linear function of a very large number of very small deviations, each due to a cause independent of all others.

2. $\sum_{i=1}^{n} \lambda_i^2 s_{2i}^2 < \infty$, where $s_{2i}^2 = \epsilon_i$ the variance.

3. The ϵ_i's are all of the same order of magnitude so that $\lambda_i^2 s_{2i}^2 \cong K_i/n = \mathcal{O}(1)$.

From these assumptions and some analysis (8) it follows that

$$p(x) = \left[1 + A_3 D^3 + A_4 D^4 + A_5 D^5 + \ldots\right] \frac{e^{-\frac{x^2}{2}\left(\frac{1}{\Sigma\lambda_i^2 s_{2i}^2}\right)}}{\sqrt{2\pi}\left(\Sigma\lambda_i^2 s_{2i}^2\right)^{1/2}}$$

where A_j depends on the quantities s_{ij} and λ_i^j and D is the differential operator. In a great many instances, the A_j are all negligible. Then $p(x)$ is the normal distribution $\phi(x)$.

Indications are that in direction-finding the distributions are not normal. Hence, to describe a sample distribution, a function more general than the normal is appropriate.

The use of the above series is often disputed, especially since it sometimes diverges or yields negative probabilities. On the other hand, if the given distribution appears as a distorted normal curve, the above series can be used very nicely (compared to other alternatives) for representation.

Assuming $A_j D^j = 0$, where $j \geqq 5$, analysis yields

$$p(x) = \frac{e^{-(x/\sigma)^2/2}}{\sqrt{2\pi}\,\sigma}\left[1 - \frac{\sqrt{\beta_1}}{6} H_3\left(\frac{x}{\sigma}\right) + \frac{\beta_2 - 3}{24} H_4\left(\frac{x}{\sigma}\right)\right]$$

where H_3 and H_4 are the Hermite polynomials, $H_3(u) = u^3 - 3u$, and $H_4(u) = u^4 - 6u^2 + 3$. Note that when the third and fourth moments are zero, $p(x) = \phi(x)$. The effect of $p(x)$ is to steepen or flatten and/or to skew the curve.

Representation for Estimation — Let y = bearing error at one station and $p(y)$ = the separated representation of the distribution of y for the present combination of factors.

Assume that there exists sufficient historical information to estimate the cumulants β_2, β_1, and the standard deviation σ; experiments in the last two years at NRL have yielded some of this information, and automatic direction-finding and better ionospheric sensors should improve these predictions. Hence we have the pertinent separated population given by

$$p(y) = \frac{e^{-(y/\sigma)^2/2}}{\sqrt{2\pi}\,\sigma}\left[1 - \frac{\sqrt{\beta_1}}{6} H_3\left(\frac{y}{\sigma}\right) + \frac{\beta_2 - 3}{24} H_4\left(\frac{y}{\sigma}\right)\right] .$$

In order to fit into the fixing process, the above expression needs to be put into a form more suited to numerical analysis, and then expressed as a function of a locally Euclidean 2-space, i.e., in terms of the tangent plane coordinates (r, s).

Expanding the polynomials, we have

$$p(y) = \frac{e^{-(y/\sigma)^2/2}}{\sqrt{2\pi}\,\sigma}\left\{1 + \left[y^4\left(\frac{\beta_2 - 3}{24\sigma^4}\right) + y^3\left(\frac{-\sqrt{\beta_1}}{6\sigma^3}\right) + y^2\left(\frac{3 - \beta_2}{4\sigma^2}\right) + y\left(\frac{\sqrt{\beta_1}}{2\sigma}\right) + \frac{\beta_2 - 3}{8}\right]\right\}.$$

Write

$$p(y) = \phi(y)\left[1 + g(y)\right] .$$

Since $1 + g(y) > 0$, we can let $1 + g(y) = e^{g_1(y)}$; then

$$g_1(y) = \ln\left(1 + g(y)\right) = \ln\left[(k+1) + \left(g(y) - k\right)\right] = \ln\left[1 + \frac{\left(g(y) - k\right)}{1 + k}\right] + \ln(1+k)$$

where k = the portion of $g(r,s)$ which does not depend on (r,s). So

$$p(y) = \frac{e^{-(y/\sigma)^2/2}}{\sqrt{2\pi}\,\sigma} e^{g_1(y)}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{\left[-\frac{y^2}{2\sigma^2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)}\left(\frac{g(y)-k}{1+k}\right)^{m+1} + \ln(1+k)\right]}$$

It will be shown that powers of y can be expressed approximately by polynomials of degree 2 in rectangular coordinates r and s. Then

$$p(r,s) \cong \frac{1}{\sqrt{2\pi}\,\sigma} e^{\left[w_{00} + w_{01}r + w_{10}s + w_{11}rs + w_{20}r^2 + w_{02}s^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{w(r,s)}$$

where several infinite series are truncated to yield a quadratic exponent, since r and s can be made small and the coefficients can be constrained to bounded values. This function $p(r,s)$ is more general than the normal distribution, yet reduces to a normal distribution when conditions are "normal."

The above populations lend themselves very nicely to multiple weighting of inputs. A possible (partial) listing of contributing information is given in Fig. 7.

In the future more inputs will certainly develop as the ability to monitor ionospheric conditions improves and as better descriptions of received energy (ADF + Wullenweber) become available.

Estimation

Procedure for Best Position Estimate — We have $p_i(y) = p_i(r,s)$ for the ith contributing station under a certain set of conditions for which we intend to truncate the exponent to quadratic order. Then under the assumption of independence the net density function becomes

$$f(r,s) = \prod_{i=1}^{n} p_i(r,s)$$

$$= Ke^{-(1/2)\left[W_{00} + W_{10}r + W_{01}s + W_{11}rs + W_{20}r^2 + W_{02}s^2\right]}$$

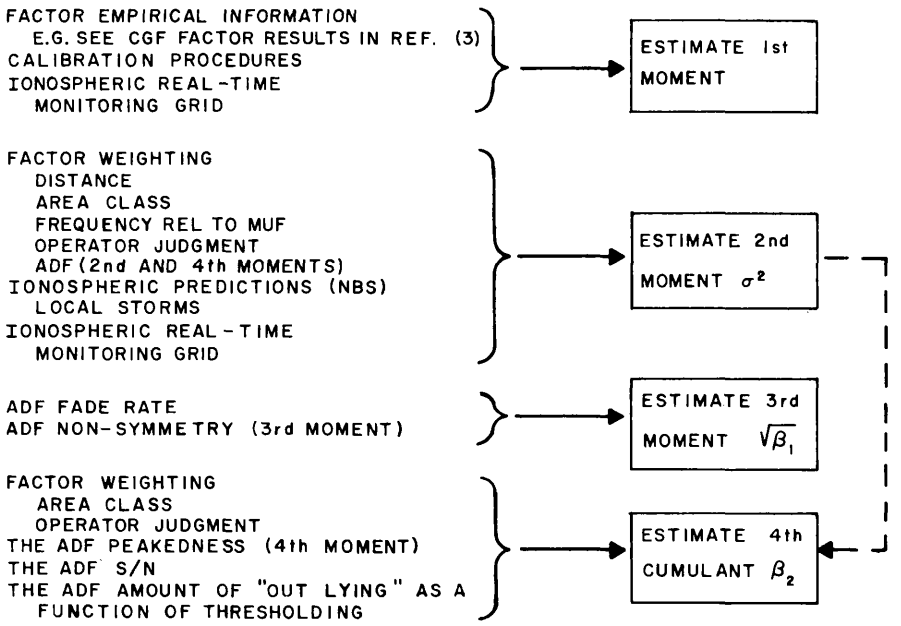


Fig. 7 - Moment estimation

where

$$W_{pq} = \sum_{i=1}^{n} w_{ipq} .$$

To maximize f and to find the search area, one now proceeds as in the present NRL model. Note that in general $W_{pq} \neq t_{pq}$, but under quadratic truncation their roles are identical.

Approximations and Expansions — Values for h and g^2 were derived previously (1). From the expression

$$g^2 = 1 - z^2 + r^2(1-x^2) + s^2(1-y^2) - 2rs\,xy - 2rxz - 2syz$$

we find

$$g^{-1} = (1-z^2)^{-1/2}\left[1 + \frac{r^2(1-x^2) + s^2(1-y^2) - 2(rsxy + rxz + syz)}{(1-z^2)}\right]^{-1/2}$$

$$\cong (1-z^2)^{-1/2}\left\{1 - \left(\frac{1}{2}\right)\left(\frac{1}{1-z^2}\right)\left[r^2(1-x^2) + s^2(1-y^2) - 2(rsxy + rxz + syz)\right]\right.$$

(Continued)

$$+ \left(\frac{3}{8}\right)\left(\frac{1}{(1-z^2)^2}\right)\Big[r^4(1-x^2)^2 + s^4(1-y^2)^2 + 4(rsxy + rxz + syz)^2$$

$$+ 2r^2s^2(1-x^2)(1-y^2) - 4(rsxy + rxz + syz)r^2(1-x^2) - 4(rsxy + rxz + syz)s^2(1-y^2)\Big]\Bigg\}$$

$$\cong (1-z^2)^{-1/2}\left\{1 + r\left[\frac{xz}{(1-z^2)}\right] + s\left[\frac{yz}{(1-z^2)}\right] + rs\left[\frac{xy}{(1-z^2)} + \frac{3xyz^2}{(1-z^2)^2}\right]\right.$$

$$\left. + r^2\left[-\frac{(1-x^2)}{2(1-z^2)} + \frac{3x^2z^2}{2(1-z^2)^2}\right] + s^2\left[-\frac{(1-y^2)}{2(1-z^2)} + \frac{3y^2z^2}{2(1-z^2)^2}\right]\right\}$$

$$= (1-z^2)^{-1/2}\left\{1 + r\left(\frac{xz}{1-z^2}\right) + s\left(\frac{yz}{1-z^2}\right) + rs\left[\frac{xy(1+2z^2)}{(1-z^2)^2}\right]\right.$$

$$\left. + r^2\left[\frac{x^2(1+2z^2) - (1-z^2)}{2(1-z^2)^2}\right] + s^2\left[\frac{y^2(1+2z^2) - (1-z^2)}{2(1-z^2)^2}\right]\right\}.$$

Since

$$h = \frac{ar + bs + c}{\Delta^{1/2}} \quad \text{where} \quad \Delta = a^2 + b^2 + c^2, \quad \text{and} \quad y = \frac{h}{g},$$

$$y = hg^{-1} = (1-z^2)^{-1/2}\left\{h + hr\left(\frac{xz}{1-z^2}\right) + hs\left(\frac{yz}{1-z^2}\right) + \frac{c}{\Delta^{1/2}}\,rs\left[xy\,\frac{(1+2z^2)}{(1-z^2)^2}\right]\right.$$

$$\left. + \frac{c}{\Delta^{1/2}}\,r^2\left[\frac{x^2(1+2z^2) - (1-z^2)}{2(1-z^2)^2}\right] + \frac{c}{\Delta^{1/2}}\,s^2\left[\frac{y^2(1+2z^2) - (1-z^2)}{2(1-z^2)^2}\right]\right\}.$$

$$y = \frac{(1-z^2)^{-1/2}}{\Delta^{1/2}}\left\{[C] + r\left[a + \frac{cxz}{(1-z^2)}\right] + s\left[b + \frac{cyz}{(1-z^2)}\right]\right.$$

$$+ rs\left[b\,\frac{xz}{(1-z^2)} + \frac{ayz}{(1-z^2)} + cxy\,\frac{(1+2z^2)}{(1-z^2)^2}\right]$$

$$+ r^2\left[a\,\frac{xz}{(1-z^2)} + c\left(\frac{x^2(1+2z^2) - (1-z^2)}{2(1-z^2)^2}\right)\right]$$

$$\left. + s^2\left[b\,\frac{yz}{(1-z^2)} + c\left(\frac{y^2(1+2z^2) - (1-z^2)}{2(1-z^2)^2}\right)\right]\right\}.$$

y^2 calculated directly is $y^2 = a_{00} + a_{01}r + a_{10}s + a_{11}rs + a_{20}r^2 + a_{02}s^2$, that is,

$$y^2 = \left\{\left[\frac{c^2}{\Delta(1-z^2)}\right] + r\left[\frac{2}{\Delta(1-z^2)^2}\left[ac(1-z^2) + xzc^2\right]\right] + s\left[\frac{2}{\Delta(1-z^2)^2}\left[bc(1-z^2) + yzc^2\right]\right]\right.$$

$$+ rs\left[\frac{2}{\Delta(1-z^2)^3}\left[ab(1-z^2)^2 + (2bcxz + 2acyz + c^2xy)(1-z^2) + 4c^2xyz^2\right]\right]$$

$$+ r^2\left[\frac{1}{\Delta(1-z^2)^3}\left[a^2(1-z^2)^2 + (4acxz + c^2x^2 - c^2)(1-z^2) + 4c^2x^2z^2\right]\right]$$

$$\left. + s^2\left[\frac{1}{\Delta(1-z^2)^3}\left[b^2(1-z^2)^2 + (4bcyz + c^2y^2 - c^2)(1-z^2) + 4c^2y^2z^2\right]\right]\right\} .$$

Write $y = b_{00} + b_{10}r + b_{01}s + b_{11}rs + b_{20}r^2 + b_{02}s^2$, where

$$b_{00} = \frac{c}{\Delta^{1/2}(1-z^2)^{1/2}}$$

$$b_{10} = \frac{a(1-z^2) + cxz}{\Delta^{1/2}(1-z^2)^{3/2}}$$

$$b_{01} = \frac{b(1-z^2) + cyz}{\Delta^{1/2}(1-z^2)^{3/2}}$$

$$b_{11} = \frac{bxz(1-z^2) + ayz(1-z^2) + cxy(1+2z^2)}{\Delta^{1/2}(1-z^2)^{5/2}}$$

$$b_{20} = \frac{2axz(1-z^2) + c\left[x^2(1+2z^2) - (1-z^2)\right]}{2\Delta^{1/2}(1-z^2)^{5/2}}$$

$$b_{02} = \frac{2byz(1-z^2) + c\left[y^2(1+2z^2) - (1-z^2)\right]}{2\Delta^{1/2}(1-z^2)^{5/2}} .$$

Then

$$y^3 = c_{00} + c_{10}r + c_{01}s + c_{11}rs + c_{20}r^2 + c_{02}s^2$$

and

$$y^4 = d_{00} + d_{10}r + d_{01}s + d_{11}rs + d_{20}r^2 + d_{02}s^2$$

where

$$c_{00} = a_{00}b_{00}$$

$$c_{10} = a_{10}b_{00} + a_{00}b_{10}$$

$$c_{01} = a_{01}b_{00} + a_{00}b_{01}$$

$$c_{11} = a_{11}b_{00} + a_{10}b_{01} + a_{01}b_{10} + a_{00}b_{11}$$

$$c_{20} = a_{20}b_{00} + a_{10}b_{10} + a_{00}b_{20}$$

$$c_{02} = a_{02}b_{00} + a_{01}b_{01} + a_{00}b_{02}$$

and where

$$d_{00} = a_{00}^2$$

$$d_{10} = 2a_{10}a_{00}$$

$$d_{01} = 2a_{01}a_{00}$$

$$d_{11} = 2(a_{11}a_{00} + a_{10}a_{01})$$

$$d_{20} = a_{10}^2 + 2a_{20}a_{00}$$

$$d_{02} = a_{01}^2 + 2a_{02}a_{00} \ .$$

Hence we now have expressions for y, y^2, y^3, and y^4 in terms of a_{pq} and b_{pq}, which are themselves calculated from the previously calculated a, b, c, x, y, z, where (x,y,z) denotes a station position. The a, b, c were originally calculated from (x,y,z) and the unique corresponding point (x',y',z') for each bearing and hemisphere.

We proceed to find a satisfactory expression for $p(y) = p(r,s)$. In order to be able to find a unique maximum for each iterative set of $[r,s]$ axes, we always truncate at the quadratic level as we did above. We now have

$$y = y(r,s)$$

$$y^2 = y^2(r,s)$$

$$y^3 = y^3(r,s)$$

$$y^4 = y^4(r,s) \ .$$

Let $g(y) = g(r,s) = A_{00} + A_{10}r + A_{01}s + A_{11}rs + A_{20}r^2 + A_{02}s^2$, where

$$A_{00} = \frac{\beta_2 - 3}{24\sigma^4} d_{00} - \frac{\sqrt{\beta_1}}{6\sigma^3} c_{00} + \frac{(3-\beta_2)}{4\sigma^2} a_{00} + \frac{\sqrt{\beta_1}}{2\sigma} b_{00} + \frac{\beta_2 - 3}{8}$$

and

$$A_{pq} = \frac{\beta_2 - 3}{24\sigma^4} d_{pq} - \frac{\sqrt{\beta_1}}{6\sigma^3} c_{pq} + \frac{(3-\beta_2)}{4\sigma^2} a_{pq} + \frac{\sqrt{\beta_1}}{2\sigma} b_{pq}, \qquad p,q \neq 0 \ .$$

Then

$$g_1(r,s) = \ln\left[1 + \frac{g(r,s) - A_{00}}{1 + A_{00}}\right] + \ln(1 + A_{00})$$

$$= \left[\frac{g(r,s) - A_{00}}{1 + A_{00}} - \frac{\left(g(r,s) - A_{00}\right)^2}{2(1 + A_{00})^2} + \cdots\right] + \ln(1 + A_{00})$$

$$\cong \frac{1}{2(1 + A_{00})^2}\left[2(1 + A_{00})(g - A_{00}) - \left(A_{10}^2 r^2 + A_{01}^2 s^2 + 2A_{10}A_{01}rs\right)\right] + \ln(1 + A_{00})$$

$$= \left[B_{10}r + B_{01}s + B_{11}rs + B_{20}r^2 + B_{02}s^2\right] + \ln(1 + A_{00}).$$

Hence

$$p(r,s) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{w(r,s)}$$

where

$$w(r,s) = w_{00} + w_{10}r + w_{01}s + w_{11}rs + w_{20}r^2 + w_{02}s^2.$$

Consolidating now,

$$w(r,s) = -\frac{y^2}{2\sigma^2} + g_1(y)$$

$$= -\left[\frac{a_{00} + a_{10}r + a_{01}s + a_{11}rs + a_{20}r^2 + a_{02}s^2}{2\sigma^2}\right] + \ln(1 + A_{00})$$

$$+ B_{10}r + B_{01}s + B_{11}rs + B_{20}r^2 + B_{02}s^2$$

$$= \left[\ln(1 + A_{00}) - \frac{a_{00}}{2\sigma^2}\right] + r\left[\frac{A_{10}}{(1 + A_{00})} - \frac{a_{10}}{2\sigma^2}\right] + s\left[\frac{A_{01}}{(1 + A_{00})} - \frac{a_{01}}{2\sigma^2}\right]$$

$$+ rs\left[\frac{A_{11}}{(1 + A_{00})} - \frac{A_{10}A_{01}}{(1 + A_{00})^2} - \frac{a_{11}}{2\sigma^2}\right]$$

$$+ r^2\left[\frac{A_{20}}{(1 + A_{00})} - \frac{A_{10}^2}{2(1 + A_{00})^2} - \frac{a_{20}}{2\sigma^2}\right]$$

$$+ s^2\left[\frac{A_{02}}{(1 + A_{00})} - \frac{A_{01}^2}{2(1 + A_{00})^2} - \frac{a_{02}}{2\sigma^2}\right]$$

(Continued)

$$w(r,s) = \ln(1+A_{00}) - \frac{a_{00}}{2\sigma^2} + \frac{1}{(1+A_{00})^2\, 2\sigma^2}\Big\{ r\Big[(1+A_{00})\Big(2A_{10}\sigma^2 - a_{10}(1+A_{00})\Big)\Big]$$

$$+ s\Big[(1+A_{00})(2A_{01}\sigma^2 - a_{01}(1+A_{00})\Big] + rs\Big[2A_{11}\sigma^2(1+A_{00}) - 2\sigma^2 A_{10}A_{01} - a_{11}(1+A_{00})^2\Big]$$

$$+ r^2\Big[2A_{20}\sigma^2(1+A_{00}) - \sigma^2 A_{10}^2 - a_{20}(1+A_{00})^2\Big]$$

$$+ s^2\Big[2A_{02}\sigma^2(1+A_{00}) - \sigma^2 A_{01}^2 - a_{02}(1+A_{00})^2\Big]\Big\} .$$

Now expressing everything in terms of a_{pq} and b_{pq}, which in turn are expressed in more fundamental symbols,

$$A_{00} = \frac{\beta_2 - 3}{24\sigma^4} a_{00}^2 - \frac{\sqrt{\beta_1}}{6\sigma^3} a_{00}b_{00} + \frac{(3-\beta_2)}{4\sigma^2} a_{00} + \frac{\sqrt{\beta_1}}{2\sigma} b_{00} + \frac{\beta_2 - 3}{8}$$

$$A_{10} = \frac{\beta_2 - 3}{24\sigma^4}(2a_{10}a_{00}) - \frac{\sqrt{\beta_1}}{6\sigma^3}(a_{10}b_{00} + a_{00}b_{10}) + \frac{(3-\beta_2)}{4\sigma^2} a_{10} + \frac{\sqrt{\beta_1}}{2\sigma} b_{10}$$

$$A_{01} = \frac{\beta_2 - 3}{24\sigma^4}(2a_{01}a_{00}) - \frac{\sqrt{\beta_1}}{6\sigma^3}(a_{01}b_{00} + a_{00}b_{01}) + \frac{(3-\beta_2)}{4\sigma^2} a_{01} + \frac{\sqrt{\beta_1}}{2\sigma} b_{01}$$

$$A_{11} = \frac{\beta_2 - 3}{24\sigma^4}(2)(a_{11}a_{00} + a_{10}a_{01}) - \frac{\sqrt{\beta_1}}{6\sigma^3}(a_{11}b_{00} + a_{10}b_{01} + a_{01}b_{10} + a_{00}b_{11})$$

$$+ \frac{(3-\beta_2)}{4\sigma^2} a_{11} + \frac{\sqrt{\beta_1}}{2\sigma} b_{11}$$

$$A_{20} = \frac{\beta_2 - 3}{24\sigma^4}(a_{10}^2 + 2a_{20}a_{00}) - \frac{\sqrt{\beta_1}}{6\sigma^3}(a_{20}b_{00} + a_{10}b_{10} + a_{00}b_{20}) + \frac{(3-\beta_2)}{4\sigma^2} a_{20} + \frac{\sqrt{\beta_1}}{2\sigma} b_{20}$$

$$A_{02} = \frac{\beta_2 - 3}{24\sigma^4}(a_{01}^2 + 2a_{02}a_{00}) - \frac{\sqrt{\beta_1}}{6\sigma^3}(a_{02}b_{00} + a_{01}b_{01} + a_{00}b_{02}) + \frac{(3-\beta_2)}{4\sigma^2} a_{02} + \frac{\sqrt{\beta_1}}{2\sigma} b_{02} .$$

The procedure following rotation to a new coordinate system would then be to calculate coefficients from

$$a_i = z_i y_i' - z_i' y_i$$

$$b_i = x_i z_i' - x_i' z_i$$

$$c_i = y_i x_i' - y_i' x_i ,$$

to determine the powers of $(1 - z_i)$, to find Δ_i and $\Delta_i^{1/2}$, and calculate the a_{ipq} as before. Now at this time the b_{ipq} would also be calculated. From the a_{ipq}, b_{ipq}, β_{i2}, $\sqrt{\beta_{i1}}$, and σ_i, the A_{ipq} are calculated. Then the w_{ipq} are calculated and then summing over n, the

$W_{pq} \sim t_{pq}$ are arrived at. Then, as before, by letting $W_{pq} = t_{pq}$, the previous procedure yields the BPE:

$$r_0 = \frac{1}{t}(t_{11}t_{01} - 2t_{02}t_{10})$$

$$s_0 = \frac{1}{t}(t_{11}t_{10} - 2t_{20}t_{01})$$

where $t = 4t_{20}t_{02} - t_{11}^2$.

NONINDEPENDENT CONTRIBUTORS

Representation of Correlation

Suppose we have (from measurement, for an approximate "area class") a total correlation matrix

$$P = (\rho_{ij}),$$

where i, j are for all stations.

From this we can find a partial correlation matrix

$$R = (r_{ij})$$

where

$$r_{ij} = \frac{-P_{ij}}{\sqrt{P_{11}P_{22}}}$$

and where P_{ij} is the ρ_{ij} cofactor.

From this we can form the net density function (assume normal contributors temporarily) of correlated stations:

$$f^*(r,s) = \frac{e^{-\frac{1}{2|R|}\left[\sum_{i=1}^{n} R_{ii}\left(\frac{y_i}{\sigma_i}\right)^2 + 2\sum_{i<j} R_{ij}\frac{y_i y_j}{\sigma_i \sigma_j}\right]}}{(2\pi)^{n/2}\prod_{i=1}^{n}\sigma_i |R|^{1/2}}$$

where R_{ij} is the r_{ij} cofactor and where y_i is the error at the ith station. The errors y_i can all be expressed in terms of (r, s) coordinates on a tangent plane and be approximated to quadratic order:

$$y = b_{00} + b_{10}r + b_{01}s + b_{11}rs + b_{20}r^2 + b_{02}s^2$$

$$y^2 = a_{00} + a_{10}r + a_{01}s + a_{11}rs + a_{20}r^2 + a_{02}s^2.$$

Correlated Estimation

Procedure for BPE — In terms of our previous $f(r,s)$ where the bearings were not correlated, we have

$$f^*(r,s) = f(r,s)\, e^{-\frac{1}{2}\left\{\ell n|R| + \frac{1}{|R|}\left[\sum_{i=1}^{n}\left(\frac{y_i}{\sigma_i}\right)^2 (R_{ii} - |R|) + 2\sum_{i<j} R_{ij}\frac{y_i y_j}{\sigma_i\sigma_j}\right]\right\}},$$

which reduces to $f(r,s)$ when the correlation vanishes. The expression for $f^*(r,s)$ assumes that the station distributions are all normal. With no theoretical justification whatever it is suggested that in the following discussion we also allow $f(r,s)$ to represent the nonnormal distributions described previously, since they are departures from Gaussian representations and their product $f(r,s)$ is represented by the usual quadratic exponent $Q(r,s)$.

Write $f^*(r,s) = f(r,s)\, e^{-1/2\, Q^*(r,s)}$, where $Q^*(r,s)$ is the quadratic exponent derived from the expression above in which $y_i(r,s)$ and $y_j(r,s)$ are substituted. More precisely, let

$$y_i = b^i_{00} + b^i_{10} r + b^i_{01} s + b^i_{11} rs + b^i_{20} r^2 + b^i_{02} s^2$$

Then

$$\begin{aligned} y_i y_j \cong\ & b^i_{00} b^j_{00} + \left(b^i_{10} b^j_{00} + b^j_{10} b^i_{00}\right) r \\ & + \left(b^i_{01} b^j_{00} + b^j_{01} b^i_{00}\right) s \\ & + \left(b^i_{00} b^j_{11} + b^j_{00} b^i_{11} + b^i_{10} b^j_{01} + b^j_{10} b^i_{01}\right) rs \\ & + \left(b^i_{10} b^j_{10} + b^i_{20} b^j_{00} + b^j_{20} b^i_{00}\right) r^2 \\ & + \left(b^i_{01} b^j_{01} + b^i_{02} b^j_{00} + b^j_{02} b^i_{00}\right) s^2 ; \end{aligned}$$

and for convenience write

$$y_i y_j = b^{ij}_{00} + b^{ij}_{10} r + b^{ij}_{01} s + b^{ij}_{11} rs + b^{ij}_{20} r^2 + b^{ij}_{02} s^2 .$$

Then $Q^*(r,s) = v_{00} + v_{10} r + v_{01} s + v_{11} rs + v_{20} r^2 + v_{02} s^2$, where

$$v_{00} = \ell n|R| + \frac{1}{|R|}\sum_{i=1}^{n}\frac{a^i_{00}}{\sigma_i^2}(R_{ii} - |R|) + 2\sum_{i<j} R_{ij}\frac{b^{ij}_{00}}{\sigma_i\sigma_j}$$

$$v_{10} = \frac{1}{|R|}\sum_{i=1}^{n}\frac{a^i_{10}}{\sigma_i^2}(R_{ii} - |R|) + 2\sum_{i<j} R_{ij}\frac{b^{ij}_{10}}{\sigma_i\sigma_j}$$

and similarly, when p and q are not both zero,

$$v_{pq} = \frac{1}{|R|} \sum_{i=1}^{n} \frac{a_{pq}^{i}}{\sigma_i^2} (R_{ii} - |R|) + 2 \sum_{i<j} R_{ij} \frac{b_{pq}^{ij}}{\sigma_i \sigma_j} .$$

Hence we have the expression

$$f^*(r,s) = f(r,s)\, e^{-Q^*(r,s)/2}$$

$$= c\, e^{-[Q(r,s)+Q^*(r,s)]/2}$$

where

$$c = \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i} .$$

Now let

$$Q^{**}(r,s) = Q(r,s) + Q^*(r,s)$$

$$= \sum W_{pq} r^p s^q + \sum v_{pq} r^p s^q$$

$$= \sum V_{pq} r^p s^q$$

where $V_{pq} = v_{pq} + W_{pq}$. Note that when conditions are "normal" $V_{pq} = v_{pq} + t_{pq}$, that when conditions are "independent" $V_{pq} = W_{pq}$, and that when conditions are both "normal" and "independent,"

$$V_{pq} = t_{pq} .$$

This illustrates how all of these mathematical derivations have been motivated by: (a) reducing to the usual form in usual circumstances, and (b) keeping quadratic regression available, and hence unique best estimates for points and plane regions.

After finding the final iterative best point estimate for (r_0, s_0), then translating to that point and rotating away the cross term, we still have for the approximate search area calculations

$$Q''(r'', s'') = \mu_{00} + \mu_{20} r''^2 + \mu_{02} s''^2$$

and we are reasonably justified in using the same confidence region criteria that we would use for the uncorrelated case. The effect of these correlations will be to widen the search area at right angles to the correlated observation paths, and to slightly increase the expected area size.

Weighting for Quick Fix — The quick fix intersection weights would now become

$$\frac{\sin^2(\theta_i - \theta_j)}{\sigma_i^2 \sigma_j^2 (1 - r_{ij}^2)} .$$

The angular weighting recently introduced (6) is geometrical, whereas the correlation weighting is ionospheric. Although presumably well correlated, these weighting effects should be independently applied.

CONCLUSIONS

Major improvement of reliability of confidence regions and point estimation should be achieved by using the unbiased F test, conditional generalized net variance estimates, and multiple χ^2-rejection of bearings. Minor improvements should result from introduction of non-Gaussian effects and correlations between contributors; these effects will be more pronounced when lower confidence statements are desired. A variable relationship between confidence and search area size is available.

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