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13. ABSTRACT

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CONTENTS

Abstract ii
Problem Status ii
Authorization ii

1. INTRODUCTION 1

2. HAAR FUNCTIONS 1

3. CONVERGENCE OF HAAR SERIES FOR CONTINUOUS FUNCTIONS 7

4. A CONVERGENCE ESTIMATE 16

5. CONVERGENCE OF HAAR SERIES FOR DISCONTINUOUS FUNCTIONS 18

6. COMPARISON WITH TRIGONOMETRIC FOURIER SERIES 20

ACKNOWLEDGMENT 22

REFERENCES 22

ABSTRACT

This report is primarily a review of Haar functions. The functions are defined, and it is shown that they form an orthonormal basis of $L^2[0,1]$. A basic property of Haar series is then proved (the proof given is different from Haar's). This property states that the Nth Haar-series approximation to a function $f(x) \in L^2[0,1]$ is a step function of 2^N steps, each of width $1/2^N$. The value of the approximation on each step is the mean value of the function $f(x)$ in the interval covered by the step. This property is taken as the basis for an investigation of the convergence properties of Haar series for various classes of functions. Estimates of approximation accuracy are derived. Haar-series convergence is compared with the convergence of trigonometric Fourier series. The class of functions for which uniform convergence of Haar series can be proved is larger than that class of functions whose trigonometric Fourier series is uniformly convergent to the given function.

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CONVERGENCE PROPERTIES OF HAAR SERIES

1. INTRODUCTION

The importance in several branches of pure and applied mathematics of functions which are elements of the space L^2 has led to the development of theories of various orthogonal sequences which form bases of L^2. Most well-known bases of L^2 are sequences of continuous functions. This is related to the fact that such sequences usually comprise the general solution of a certain differential equation.

However, there exist an infinite number of bases of L^2 which are sequences of discontinuous functions, as was pointed out by Walsh (1) when he introduced one such sequence, now known as Walsh functions. Walsh functions may be defined by taking linear combinations of another discontinuous orthogonal sequence first introduced by the Hungarian mathematician Alfred Haar (2) in 1909. Haar functions have properties which may make them convenient for application in a number of areas such as communications, image coding, and pattern recognition (3). They are also an example of a generalized Fourier series with convergence properties that are particularly straightforward and intuitive.

This report is primarily a review of Haar functions. Much of the material is available elsewhere (2,4,5), although in most cases the presentation is different. The report is motivated by the increasing interest in engineering applications of Walsh functions and their relatives (3,6,7,8).

In his published work on the subject, Haar shows that these functions form an orthonormal basis of $L^2[0, 1]$ and proves that the Haar series expansion of a continuous function $f(x) \in L^2[0, 1]$ is uniformly convergent to the function on this interval. He also extends these results to what is best described as a set of generalized Haar functions. Section 2 of this report follows Haar's work in showing that Haar functions are an orthonormal basis of $L^2[0, 1]$. In Section 3 we show that the Haar series expansion of a continuous function is uniformly convergent to that function. The proof is different from that given by Haar. A new convergence estimate is given in Section 4. The convergence of Haar series for discontinuous functions is investigated in Section 5; some of the results in this section have been presented differently by Faber (4). In Section 6 the convergence properties of Haar functions are compared with those of the more familiar trigonometric Fourier series.

2. HAAR FUNCTIONS

The Haar orthonormal sequence is defined on $[0, 1]$ and is composed of functions labeled by two indices:

$$\{\varphi_n^m\} = \varphi_0; \varphi_1^1; \varphi_2^1, \varphi_2^2; \ldots; \varphi_k^1, \ldots, \varphi_k^{2^{k-1}}; \ldots . \tag{1}$$

The first Haar function is defined as

$$\varphi_0(x) = 1, \qquad x \in [0, 1]. \tag{2}$$

All other Haar functions φ_n^m are zero except on one of 2^{n-1} equal subintervals of $[0, 1]$, namely $[(m-1)/2^{n-1}, m/2^{n-1}]$. In particular

$$\varphi_n^1(x) \equiv \begin{cases} +2^{(n-1)/2}, & x \in \left[0, \dfrac{1}{2^n}\right) \\ -2^{(n-1)/2}, & x \in \left(\dfrac{1}{2^n}, \dfrac{1}{2^{n-1}}\right) \\ -2^{(n-3)/2}, & x = \dfrac{1}{2^{n-1}} \\ 0, & \text{all other points} \end{cases} \tag{3}$$

$$\varphi_n^{2^{n-1}}(x) \equiv \begin{cases} +2^{(n-3)/2}, & x = \dfrac{2^{n-1}-1}{2^{n-1}} \\ +2^{(n-1)/2}, & x \in \left(\dfrac{2^{n-1}-1}{2^{n-1}}, \dfrac{2^{n-1}-\frac{1}{2}}{2^{n-1}}\right) \\ -2^{(n-1)/2}, & x \in \left(\dfrac{2^{n-1}-\frac{1}{2}}{2^{n-1}}, 1\right] \\ 0, & \text{all other points} \end{cases} \tag{4}$$

and

$$\varphi_n^m(x) \equiv \begin{cases} +2^{(n-3)/2}, & x = \dfrac{m-1}{2^{n-1}} \\ +2^{(n-1)/2}, & x \in \left(\dfrac{m-1}{2^{n-1}}, \dfrac{m-\frac{1}{2}}{2^{n-1}}\right) \\ -2^{(n-1)/2}, & x \in \left(\dfrac{m-\frac{1}{2}}{2^{n-1}}, \dfrac{m}{2^{n-1}}\right) \\ -2^{(n-3)/2}, & x = \dfrac{m}{2^{n-1}} \\ 0, & \text{all other points} \end{cases} \tag{5}$$

where in the last equation $m = 2, 3, \ldots, 2^{n-1} - 1$.

We note that the Haar functions form an unbounded sequence. The first few Haar functions are shown in Fig. 1. The complexity of the definition reflects the careful choice of the function value on each discontinuity, with choices made so that the convergence properties to be derived later hold for all points on [0, 1].

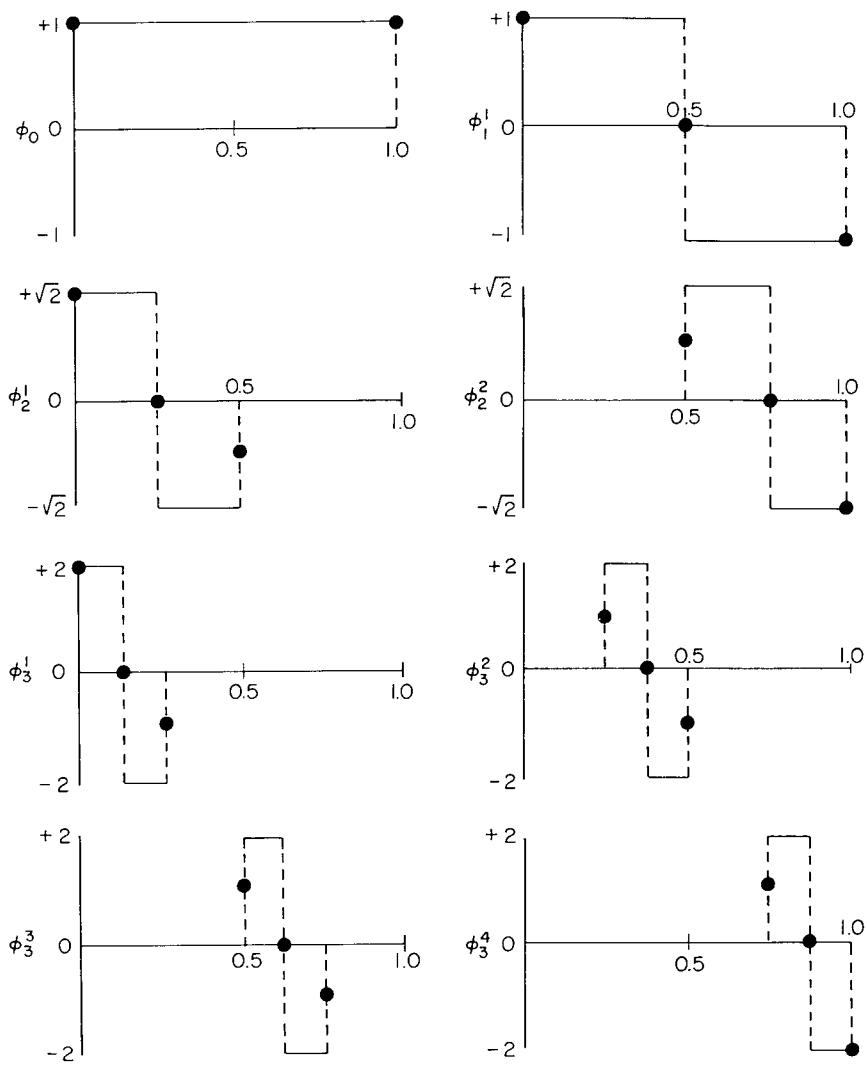

Fig. 1—The first eight Haar functions

Before continuing, we introduce some definitions that will be used throughout the report.

DEFINITION. *Let* f, g $\in L^2[0, 1]$; *then*

$$\langle f|g\rangle = \langle g|f\rangle \equiv \int_0^1 f(x)g(x)\,dx. \tag{6}$$

Thus the projection of f *onto any Haar function is given by* $\langle f|\varphi_n^m\rangle$.

DEFINITION. *Let* $f \in L^2[0, 1]$; *then for* $m \leqslant 2^{n-1}$,

$$S_n^m(x) \equiv C_0 + \sum_{k=1}^{n-1} \sum_{\ell=1}^{2^{k-1}} C_k^\ell \varphi_k^\ell(x) + \sum_{j=1}^{m} C_n^j \varphi_n^j(x) \tag{7}$$

is the (n, m) *partial sum of the expansion of* f *in terms of Haar functions, where* $C_k^\ell = \langle f|\varphi_k^\ell\rangle$; $S_n^m(x)$ *is defined on* $[0, 1]$.

DEFINITION. *Let* $f \in L^2[0, 1]$; *then*

$$S_N(x) \equiv C_0 + \sum_{k=1}^{N} \sum_{\ell=1}^{2^{k-1}} C_k^\ell \varphi_k^\ell(x) \tag{8}$$

is the Nth *associated partial sum of* f *expanded in terms of Haar functions;* $S_N(x)$ *is defined on* $[0, 1]$.

We note that

$$S_N(x) = S_N^{2^{N-1}}(x)$$

and

$$S_n^m(x) = S_{n-1}(x) + \sum_{k=1}^{m} C_n^k \varphi_n^k(x).$$

DEFINITION. *The interval* $((m - 1)/2^n, m/2^n)$ *will be denoted by* i_n^m.

We note that $[0, 1]$ is subdivided into 2^n equal parts labeled $i_n^1, i_n^2, \ldots, i_n^{2^n}$ and that $\varphi_k^p(x)$ is nonzero in the intervals i_k^{2p-1} and i_k^{2p}, i.e. in the interval i_{k-1}^p.

DEFINITION. *Let* $f \in L^2[0, 1]$ *and* $0 \leqslant a < b \leqslant 1$; *then*

$$\bar{f}(a, b) \equiv \frac{1}{b - a} \int_a^b f(x)\, dx \tag{9}$$

is the mean value of f *in the interval* (a, b). *The notation*

$$\bar{f}(i_n^m) = \bar{f}\left(\frac{m-1}{2^n}, \frac{m}{2^n}\right) = 2^n \int_{i_n^m} f(x)\, dx \tag{10}$$

will also be used.

DEFINITION. *If* x $\in$ [0, 1] *satisfies for some* k *and* M

$$x = \frac{k}{2^M}, \tag{11}$$

where k = 0, 1, 2, ..., 2^M, *then it is called a binary rational, in particular an* M*th order dyadic. If* k *is odd,* x *may be called an odd* M*th order dyadic and similarly for* k *even. If* x *does not satisfy Eq. (11) for any* M, *then it is called a binary irrational.*

We note that any Mth-order dyadic is also an $(M + P)$th-order dyadic for any positive integer P.

The last definition is motivated by the fact that every binary rational on [0, 1] is a point of discontinuity of one or more Haar functions. These points are of special concern in the convergence investigations to follow.

We now show that $\{\varphi_n^m\}$ is an orthonormal basis of $L^2[0, 1]$. The Haar functions are normalized to unity, since

$$\left\langle \varphi_n^m | \varphi_n^m \right\rangle = \int_0^1 \varphi_n^m(x)\varphi_n^m(x)\, dx = 2^{n-1}\left(\frac{m}{2^{n-1}} - \frac{m-1}{2^{n-1}}\right) = 1. \tag{12}$$

Now, φ_0 is orthogonal to all other φ_n^m, since

$$\left\langle \varphi_0 | \varphi_n^m \right\rangle = \int_0^1 \varphi_n^m(x)\, dx = 0. \tag{13}$$

Two different Haar functions with the same subscript are orthogonal, since

$$\varphi_n^m(x)\varphi_n^k(x) = 0, \tag{14}$$

where $m \neq k$. For any two functions with different subscripts, either

$$\varphi_n^m(x)\varphi_k^p(x) = 0, \tag{15}$$

or

$$\varphi_n^m(x)\varphi_k^p(x) = \pm 2^{(n-1)/2}\varphi_k^p(x),$$

where $k > n$. In the latter case,

$$\left\langle \varphi_n^m | \varphi_k^p \right\rangle = \pm 2^{(n-1)/2} \int_0^1 \varphi_k^p\, dx = 0. \tag{17}$$

The Haar functions therefore compose an orthonormal sequence. That this sequence is complete in $L^2[0, 1]$ can be proved by the following: if there exists no element $f \in L^2[0, 1]$ which is orthogonal to all $\{\varphi_n^m\}$ and at the same time satisfies $f \neq 0$, then $\{\varphi_n^m\}$ is complete in $L^2[0, 1]$. We assume that f is orthogonal to all φ_n^m, i.e., $\langle f|\varphi_n^m\rangle = 0$, and show that $f = 0$ necessarily.

Defining

$$F(x) \equiv \int_0^x f(x')\,dx', \tag{18}$$

we have

$$F(1) - F(0) = \int_0^1 f(x')\,dx' = \langle f|\varphi_0\rangle = 0. \tag{19}$$

Hence, $F(1) = F(0) = 0$. Also

$$\begin{aligned}\langle f|\varphi_1^1\rangle &= \left[F\left(\frac{1}{2}\right) - F(0)\right] - \left[F(1) - F\left(\frac{1}{2}\right)\right] \\ &= 2F\left(\frac{1}{2}\right) = 0.\end{aligned} \tag{20}$$

By induction,

$$F\left(\frac{k}{2^n}\right) = 0 \tag{21}$$

for $k = 0, 1, \ldots, 2^n$; $n = 1, 2, \ldots$.

Since $F(x)$ is continuous and the binary rationals are dense in $[0, 1]$, $F(x) = 0$ everywhere, and $f(x) = F'(x) = 0$ almost everywhere. This proves completeness of the Haar orthonormal sequence in $L^2[0, 1]$.

As a consequence, by a general property of complete orthonormal sequences, every element in $L^2[0, 1]$ has an expansion in terms of Haar functions by which it is approximated in the mean. That is, for every $f \in L^2[0, 1]$, the sequence of partial sums $S_n^m(x)$ satisfies

$$||f - S_n^m|| \to 0 \quad \text{as} \quad k = m + 2^{n-1} \to \infty, \tag{22}$$

where

$$||g|| \equiv \langle g|g\rangle^{1/2}. \tag{23}$$

Furthermore

$$||f||^2 = C_0^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} (C_n^m)^2 \tag{24}$$

(Parseval's formula).

3. CONVERGENCE OF HAAR SERIES FOR CONTINUOUS FUNCTIONS

Perhaps the most surprising property of Haar functions is, despite the fact that they are discontinuous, for a certain class of functions in $L^2[0, 1]$, we can prove a much stronger convergence than mean convergence, namely uniform convergence. In fact, as will be discussed later, this class of functions is larger than that class of functions whose trigonometric Fourier series is uniformly convergent to the given function.

We begin with a theorem that describes an important property of Haar expansions which is the basis of their possible utility. This theorem states that the Nth associated partial sum of a function expanded in terms of Haar functions has 2^N equal-length steps and $2^N - 1$ discontinuities between adjacent steps. The value of $S_N(x)$ on each of these steps is simply the mean value of $f(x)$ in the interval covered by the step. The value of $S_N(x)$ at a discontinuity between adjacent steps is halfway between the two steps. As an example, Fig. 2 shows six successive associated Haar approximations to the function

$$f(x) = 100x^2e^{-10x},$$

each superimposed on the function itself. The effect of additional terms is simple and intuitive, unlike the effect of additional terms when the function is expanded as a trigonometric Fourier series (or as a series in terms of other continuous bases of $L^2[0, 1]$).

THEOREM 1. *Let* $f(x) \in L^2[0, 1]$ *be expanded as a Haar series, so that the* P*th associated partial sum is given by*

$$S_P(x) = C_0 + \sum_{n=1}^{P} \sum_{k=1}^{2^{n-1}} C_n^k \varphi_n^k(x), \tag{25}$$

where $C_n^k = \langle f | \varphi_n^k \rangle$. *Then for any point in the interval* (0, 1) *that is not a* P*th-order dyadic,*

$$S_P(x) = \bar{f}\left(\frac{m-1}{2^P}, \frac{m}{2^P}\right) = \bar{f}(i_P^m), \tag{26}$$

where m *satisfies*

$$x \in i_P^m. \tag{27}$$

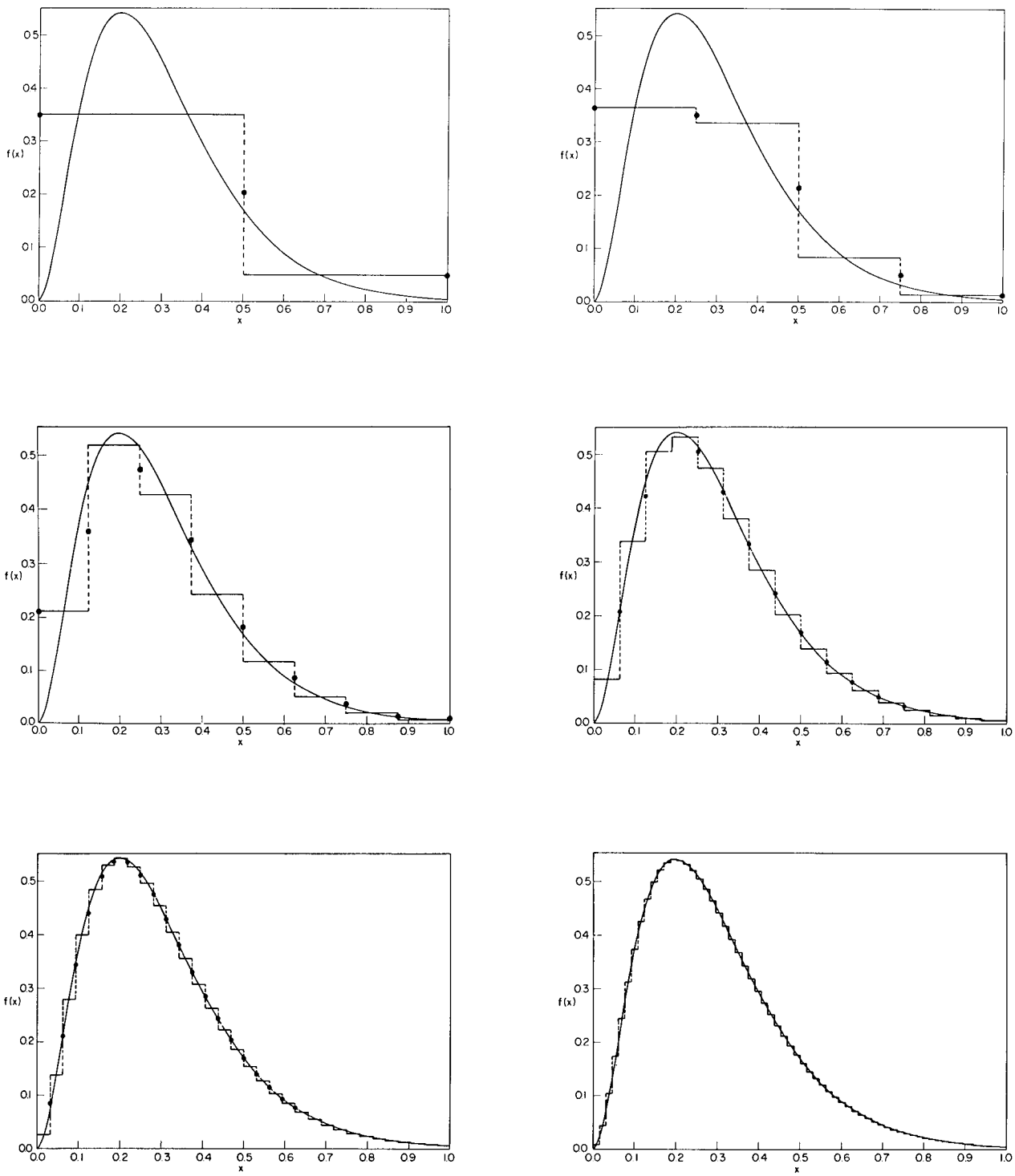

Fig. 2—Six associated Haar partial sums in the expansion of $f(x) = 100x^2e^{-10x}$

For P*th-order dyadics in* (0, 1),

$$S_P(x) = S_P\left(\frac{n}{2^P}\right) = \frac{1}{2}\left[\bar{f}\left(\frac{n-1}{2^P}, \frac{n}{2^P}\right) + \bar{f}\left(\frac{n}{2^P}, \frac{n+1}{2^P}\right)\right]$$

$$= \frac{1}{2}\left[\bar{f}(i_P^n) + \bar{f}(i_P^{n+1})\right], \tag{28}$$

where n = 1, 2, 3, ..., $2^P - 1$. *Finally*

$$S_P(0) = \bar{f}\left(0, \frac{1}{2^P}\right) = \bar{f}(i_P^1) \tag{29}$$

$$S_P(1) = \bar{f}\left(\frac{2^P - 1}{2^P}, 1\right) = \bar{f}(i_P^{2^P}). \tag{30}$$

Proof. We prove Eq. (26) first and then Eqs. (28) - (30). Both proofs are by induction.

For $N = 0$

$$S_0(x) = C_0 = \int_0^1 f(x)\,dx = \bar{f}\,[0, 1]. \tag{31}$$

Assuming that the theorem is true for the Nth associated partial sum, we have

$$S_N(x) = \bar{f}(i_N^m), \tag{32}$$

where m satisfies $x \in i_N^m$. If $P = N + 1$,

$$S_P(x) = S_N(x) + \sum_{u=1}^{u=2^{P-1}} C_P^u \varphi_P^u(x). \tag{33}$$

For any particular value of x, only one term contributes to the summation in Eq. (33) (see Fig. 1). In fact

$$S_P(x) = \bar{f}(i_N^m) + C_P^m \varphi_P^m(x), \tag{34}$$

since $x \in i_N^m$ and φ_P^m is the only φ_P^u that is nonzero in this interval. Now

$$C_P^m = \left\langle f | \varphi_P^m \right\rangle = 2^{(P-1)/2} \left[\int_{i_P^{2m-1}} f(x)\, dx - \int_{i_P^{2m}} f(x)\, dx \right]$$

$$= 2^{-(P+1)/2} \left[\bar{f}(i_P^{2m-1}) - \bar{f}(i_P^{2m}) \right]. \tag{35}$$

Substituting for C_P^m and φ_P^m in Eq. (34), we obtain

$$S_P(x) = \bar{f}(i_N^m) + \frac{1}{2} sgn(m, P, x) \left[\bar{f}(i_P^{2m-1}) - \bar{f}(i_P^{2m}) \right], \tag{36}$$

where

$$sgn(m, P, x) = \begin{cases} +1, & x \in i_P^{2m-1} \\ -1, & x \in i_P^{2m} \\ 0, & x = \dfrac{2m-1}{2^P}. \end{cases} \tag{37}$$

Since

$$\bar{f}(i_N^m) = \frac{1}{2} \left[\bar{f}(i_{N+1}^{2m-1}) + \bar{f}(i_{N+1}^{2m}) \right]$$

$$= \frac{1}{2} \left[\bar{f}(i_P^{2m-1}) + \bar{f}(i_P^{2m}) \right], \tag{38}$$

we have

$$S_P(x) = \bar{f}(i_P^v) \tag{39}$$

when $x \in i_P^v$, where we have written $v = 2m - 1$. Likewise

$$S_P(x) = \bar{f}(i_P^u) \tag{40}$$

when $x \in i_P^u$, where we have written $u = 2m$. Theorem 1 is therefore proved for points in (0, 1) that are not Pth-order dyadics.

All Pth-order dyadics in (0, 1) are points of discontinuity of $S_P(x)$. For odd-Pth-order dyadics, these discontinuities are occurring for the first time. That is, $S_{P-1}(k/2^P)$ is not discontinuous for $k = 1, 3, 5, \ldots, 2^P - 1$. These points correspond to the case $sgn(m, P, x) = 0$ in Eq. (36) since the Pth-order dyadic $(m - 1)/2^P$ is not an Nth-order dyadic ($N = P - 1$). From Eqs. (36) - (38) we have

$$S_P\left(\frac{2m-1}{2^P}\right) = \bar{f}(i_N^m)$$

$$= \frac{1}{2}\left[\bar{f}(i_P^{2m-1}) + \bar{f}(i_P^{2m})\right]. \tag{41}$$

Thus Eq. (28) is satisfied at every new discontinuity (odd-Pth-order dyadic). Any higher partial sum $S_R(x)$, for $R > P$, will also be discontinuous at these points, which then become even-Rth-order dyadics. To continue our proof by induction, we assume that Eq. (28) is still satisfied at these points, i.e.,

$$S_R\left(\frac{k}{2^R}\right) = \frac{1}{2}\left[\bar{f}(i_R^k) + \bar{f}(i_R^{k+1})\right] \tag{42}$$

for $k = 2, 4, \ldots, 2^R - 2$. From the derivation of Eq. (41), we know that Eq. (42) also holds for odd values of k. If $T = R + 1$,

$$S_T(x) = S_R(x) + \sum_{u=1}^{2^{T-1}} C_T^u \varphi_T^u(x), \tag{43}$$

and at $x = k/2^R$, two terms survive the summation, namely those containing the Haar functions which are nonzero over the intervals i_R^k and i_R^{k+1}. Thus from Eqs. (3) - (5),

$$S_T\left(\frac{k}{2^R}\right) = S_R\left(\frac{k}{2^R}\right) - 2^{(T-3)/2}C_T^k + 2^{(T-3)/2}C_T^{k+1}. \tag{44}$$

Now

$$C_T^k = \left\langle f|\varphi_T^k\right\rangle = 2^{-(T+1)/2}\left[\bar{f}(i_T^{2k-1}) - \bar{f}(i_T^{2k})\right], \tag{45}$$

and

$$C_T^{k+1} = \left\langle f|\varphi_T^{k+1}\right\rangle = 2^{-(T-1)/2}\left[\bar{f}(i_T^{2k+1}) - \bar{f}(i_T^{2k+2})\right], \tag{46}$$

so that

$$S_T\left(\frac{k}{2^R}\right) = S_R\left(\frac{k}{2^R}\right) + \frac{1}{4}\left[\bar{f}(i_T^{2k}) - \bar{f}(i_T^{2k-1}) + \bar{f}(i_T^{2k+1}) - \bar{f}(i_T^{2k+2})\right]. \tag{47}$$

Substituting Eq. (42) and using Eq. (38), we finally obtain

$$S_T\left(\frac{2k}{2^T}\right) = \frac{1}{2}\left[\bar{f}(i_T^{2k}) + \bar{f}(i_T^{2k+1})\right] \tag{48}$$

or, writing $v = 2k$,

$$S_T\left(\frac{v}{2^T}\right) = \frac{1}{2}\left[\bar{f}(i_T^v) + \bar{f}(i_T^{v+1})\right] \tag{49}$$

for $v = 2, 4, \ldots, 2^T - 2$.

From the derivation of Eq. (41) we know that Eq. (49) also holds for odd v. This completes the proof of Eq. (28) by induction. The remaining parts of Theorem 1, Eqs. (29) and (30), should now be clear. Detailed proof is left to the reader.

The properties of S_N described by Theorem 1 also hold for the partial sums S_n^m, for which it is now easy to prove the following Theorem.

THEOREM 2. *Let* $f(x) \in L^2[0, 1]$ *be expanded as a Haar series, so that the* (n, m) *partial sum is given by*

$$S_n^m(x) = C_0 + \sum_{v=1}^{n-1} \sum_{k=1}^{2^{v-1}} C_v^k \varphi_v^k(x) + \sum_{k=1}^{m} C_n^k \varphi_n^k(x), \tag{50}$$

where $C_v^k = \left\langle f | \varphi_v^k \right\rangle$ *and* $1 \leqslant m < 2^{n-1}$. *Then for any point in the interval* $(0, m/2^{n-1})$ *that is not an* n*th-order dyadic,*

$$S_n^m(x) = \bar{f}(i_n^k), \tag{51}$$

where k *satisfies* $x \in i_n^k$. *For any point in the interval* $(m/2^{n-1}, 1)$ *that is not an* $(n - 1)$*th-order dyadic,*

$$S_n^m(x) = \bar{f}(i_{n-1}^p), \tag{52}$$

where p *satisfies* $x \in i_{n-1}^p$.

For n*th-order dyadics in* $(0, m/2^{n-1})$,

$$S_n^m(x) = S_n^m\left(\frac{v}{2^n}\right) = \frac{1}{2}\left[\bar{f}(i_n^v) + \bar{f}(i_n^{v+1})\right], \tag{53}$$

where $v = 1, 2, 3, \ldots, 2m - 1$. *For* $(n - 1)$*th-order dyadics in* $(m/2^{n-1}, 1)$,

$$S_n^m(x) = S_n^m\left(\frac{w}{2^{n-1}}\right) = \frac{1}{2}\left[\bar{f}(i_{n-1}^w) + \bar{f}(i_{n-1}^{w+1})\right], \tag{54}$$

where $w = m + 1, m + 2, \ldots, 2^{n-1} - 1$.

At the end points of $[0, 1]$

$$S_n^m(0) = \bar{f}(i_n^1) \tag{55}$$

and

$$S_n^m(1) = \bar{f}(i_{n-1}^{2^{n-1}}). \tag{56}$$

Finally, at the dyadic $x = m/2^{n-1}$,

$$S_n^m\left(\frac{m}{2^{n-1}}\right) = \frac{1}{2}\left[\bar{f}(i_n^{2m}) + \bar{f}(i_{n-1}^{m+1})\right]. \tag{57}$$

By analogy with Theorem 1, Theorem 2 states that $S_n^m(x)$ consists of $2m$ steps of width $1/2^n$ followed (at $x = m/2^{n-1}$) by $2^{n-1} - m$ steps of width $1/2^{m-1}$. The value of $S_n^m(x)$ on each of these steps is simply the mean value of $f(x)$ in the interval covered by the step. The value at a discontinuity between adjacent steps is halfway between the two steps.

Proof. We note that

$$S_n^m(x) = S_{n-1}(x) + \sum_{k=1}^{m} C_k^m \varphi_k^m(x) \tag{58a}$$

$$= \begin{cases} S_n(x), & x < \dfrac{m}{2^{n-1}} \\ S_{n-1}(x), & x > \dfrac{m}{2^{n-1}} \end{cases} \tag{58b}$$

Equations (51) - (56) then follow directly from Theorem 1. That Eq. (57) is also true can be seen from the proof of Eq. (28). When $x = m/2^{n-1}$, only one term survives the summation in Eq. (58a). This corresponds to Eq. (44) without the last term. We then obtain, using Eq. (45),

$$S_n^m\left(\frac{m}{2^{n-1}}\right) = S_{n-1}\left(\frac{m}{2^{n-1}}\right) + \frac{1}{4}\left[\bar{f}(i_n^{2m}) - \bar{f}(i_n^{2m-1})\right]$$

$$= \frac{1}{2}\left[\bar{f}(i_{n-1}^{m}) + \bar{f}(i_{n-1}^{m+1})\right] + \frac{1}{4}\left[\bar{f}(i_n^{2m}) - \bar{f}(i_n^{2m-1})\right], \tag{59}$$

which reduces to Eq. (57) through the use of Eq. (38).

Two important properties of Haar series follow directly from Theorem 1 or 2 and are stated now as corollaries.

COROLLARY 1. *Let* $f(x) \in L^2[0, 1]$ *be expanded as a Haar series in the form of Eq. (50). Then in any subinterval* $(k/2^n, v/2^n) \in [0, 1]$, *where* $k, v = 0, 1, 2, 3, \ldots, 2m, 2m + 2, 2m + 4, \ldots, 2^n$ *and* $k < v$, *the convergence behavior of* $S_n^m(x)$ *and of all higher*

partial sums is related only to the behavior of f(x) *within the same subinterval. For large* n *the convergence behavior at any point depends only on* f(x) *in the neighborhood of that point.*

COROLLARY 2. *Let* $f(x) \in L^2[0, 1]$ *be expanded as a Haar series in the form of Eq. (50). If* $f(x) = C$, *a constant, within the subinterval* $(k/2^n, v/2^n) \in [0, 1]$, *where* $k, v = 0, 1, 2, \ldots, 2m, 2m + 2, 2m + 4, \ldots, 2^n$ *and* $k < v$, *then the partial sum* $S_n^m(x)$ *satisfies* $S_n^m(x) = f(x) = C$ *within this subinterval, as do all higher partial sums.*

THEOREM 3. *Let* $f(x) \in L^2[0, 1]$ *be continuous over the interval* $[0, 1]$. *Then the sequence of Haar partial sums (Eq. (7) is uniformly convergent to* f(x).

Proof. We first prove that Theorem 3 holds for the sequence of associated partial sums (Eq. (8)). We must show that for any $\epsilon > 0$, there exists an integer M such that $|S_N(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$ whenever $N > M$. For points that are not Nth-order dyadics, we have from Theorem 1 for $x \in i_n^m$,

$$S_N(x) = \bar{f}(i_N^m) = f(\xi_N^m), \tag{60}$$

where ξ_N^m is some point in i_N^m. The last step in Eq. (60) is by the mean-value theorem for integrals. Thus for $x \in i_n^m$

$$|S_N(x) - f(x)| = |f(\xi_N^m) - f(x)|. \tag{61}$$

Since $f(x)$ is continuous over $[0, 1]$, it is uniformly continuous, so that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \epsilon \tag{62}$$

whenever

$$|y - x| < \delta. \tag{63}$$

Let us choose a fixed integer M such that $1/2^M < \delta$. Then for any $x \in i_M^k$ we have

$$|\xi_M^k - x| < \frac{1}{2^M} < \delta \cdot$$

Therefore

$$|f(\xi_M^k) - f(x)| = |S_M(x) - f(x)| < \epsilon \tag{65}$$

for $x \in i_M^k$. This inequality holds for all non-Mth-order dyadics in $[0, 1]$ as well as, by virtue of Eqs. (29) and (30), for the two end points. For interior Mth-order dyadics, we consider the point $x_k = k/2^M$. Defining, for positive α,

$$f(x+) \equiv \lim_{\alpha \to 0} f(x + \alpha) \tag{66}$$

and

$$f(x-) \equiv \lim_{\alpha \to 0} f(x - \alpha), \tag{67}$$

we have

$$|S_M(i_M^k) - f(x_k-)| < \epsilon \tag{68}$$

and

$$|S_M(i_M^{k+1}) - f(x_k+)| < \epsilon . \tag{69}$$

Thus

$$|S_M(i_M^k) - f(x_k-) + S_M(i_M^{k+1}) - f(x_k+)| < 2\epsilon , \tag{70}$$

and since f is continuous,

$$\left| \frac{1}{2}\left[S_M(i_M^k) + S_M(i_M^{k+1}) \right] - f(x_k) \right| = |S_M(x_k) - f(x_k)| < \epsilon , \tag{71}$$

where we have used Eq. (28). We have therefore shown that for every $\epsilon > 0$, there exists an integer M such that

$$|S_M(x) - f(x)| < \epsilon \tag{72}$$

for all $x \in [0, 1]$. That this holds for all $N > M$ can be seen by going back to Eq. (64) and noting that for $x \in i_N^m$, $|\xi_N^m - x| < 1/2^M < \delta$ also. The steps leading to Eq. (72) therefore follow for $N > M$.

Uniform convergence of the associated partial sums implies uniform convergence of the general partial sums, since (Eqs. (57) and (58))

$$S_n^m(x) = \begin{cases} S_n(x), & x < \dfrac{m}{2^{n-1}} \\ S_{n-1}(x), & x > \dfrac{m}{2^{n-1}} \\ \dfrac{1}{2}\left[S_n(x-) + S_{n-1}(x+)\right], & x = \dfrac{m}{2^{n-1}} . \end{cases} \tag{73}$$

We state explicitly the following corollary

COROLLARY 3. *Let* f(x) $\in L^2[0, 1]$ *be continuous over the interval* $[0, 1]$. *Then the sequence of associated Haar partial sums (Eq. (8)) is uniformly convergent to* f(x).

Recalling Theorem 2 and Corollary 1, we also have another corollary.

COROLLARY 4. *Let* $f(x) \in L^2[0, 1]$ *be continuous over the subinterval* (a, b). *Then the sequences of Haar partial sums and associated Haar partial sums are uniformly convergent to* f(x) *in* (a, b).

4. A CONVERGENCE ESTIMATE

The rather simple expansion property described by Theorems 1 and 2 and shown in Fig. 2 suggests that simple convergence estimates can be found. One such estimate is given in the next Theorem.

THEOREM 4. *Let* $f(x) \in L^2[0, 1]$ *be continuous in the interval* i_N^m *with a first derivative* $f'(x)$ *that is bounded and exists everywhere in* i_N^m. *Then*

$$|S_N(x) - f(x)| \leqslant \frac{\Xi(i_N^m)}{2^N} \tag{74}$$

for $x \in i_N^m$, *where* $\Xi(i) = \sup_{x \in i} |f'(x)|$, i *a subinterval of* [0, 1].

Proof. By the mean-value theorem for integrals,

$$|S_N(x) - f(x)| = |f(\xi_N^m) - f(x)| \tag{75}$$

for $x \in i_N^m$, where ξ_N^m is a particular point in i_N^m. By the mean-value theorem,

$$|f(\xi_N^m) - f(x)| = |f'(\alpha_N^m)[\xi_N^m - x]|, \tag{76}$$

where α_N^m is a particular point in (ξ_N^m, x) or (x, ξ_N^m). But

$$|f'(\alpha_N^m)[\xi_N^m - x]| \leqslant \frac{|f'(\alpha_N^m)|}{2^N} \leqslant \frac{\Xi(i_N^m)}{2^N}, \tag{77}$$

which proves Eq. (74).

COROLLARY 5. *If* f(x) *is continuous with a bounded first derivative existing everywhere on* [0, 1],

$$|S_N(x) - f(x)| \leqslant \frac{\Xi(i_0^1)}{2^N} \tag{78}$$

for $x \in [0, 1]$.

Eq. (78) follows directly from Eq. (74). We note that the corollary says something definite about the rate of convergence of Haar series, namely that the accuracy of the

estimate S_{N+1} is bounded by a value which is half that which bounds the estimate of S_N.

When dealing with well-behaved functions, the following theorem is useful.

THEOREM 5. *If* f(x) *is linear* (f(x) = ax + b) *in the intervals* i_N^m *and* i_N^{m+1}, *then for* $x \in i_N^m, i_N^{m+1}$,

$$|S_N(x) - f(x)| \leqslant \frac{|f'(x)|}{2^{N+1}}, \tag{79}$$

and

$$\left| S_N\left(\frac{m}{2^N}\right) - f\left(\frac{m}{2^N}\right) \right| = 0. \tag{80}$$

Proof. If $f(x)$ is linear in i_N^m and i_N^{m+1}, $f(x) = xf'(x) + b$, then the range in either interval is

$$\Delta f = \frac{|f'(x)|}{2^N}. \tag{81}$$

For $x \in i_N^m$,

$$S_N(x) = \bar{f}(i_N^m) = f\left(\frac{m - \frac{1}{2}}{2^N}\right), \tag{82}$$

so that

$$|S_N(x) - f(x)| = f\left(\frac{m - \frac{1}{2}}{2^N}\right) - f(x) \leqslant \frac{\Delta f}{2}. \tag{83}$$

Similarly, for $x \in i_N^{m+1}$,

$$S_N(x) = \bar{f}(i_N^{m+1}) = f\left(\frac{m + \frac{1}{2}}{2^N}\right), \tag{84}$$

and

$$|S_N(x) - f(x)| = \left| f\left(\frac{m + \frac{1}{2}}{2^N}\right) - f(x) \right| \leqslant \frac{\Delta f}{2}. \tag{85}$$

Thus

$$|S_N(x) - f(x)| \leqslant \frac{|f'(x)|}{2^{N+1}} \tag{86}$$

in both intervals. Finally, at $x = m/2^N$,

$$S_N\left(\frac{m}{2^N}\right) = \frac{1}{2}[\bar{f}(i_N^m) + \bar{f}(i_N^{m+1})] = \frac{1}{2}\left[f\left(\frac{m-\frac{1}{2}}{2^N}\right) + f\left(\frac{m+\frac{1}{2}}{2^N}\right)\right] = f\left(\frac{m}{2^N}\right), \tag{87}$$

where we have used Theorem 1 and the fact that f is linear.

We note that in the expansion of well-behaved functions, Eqs. (79) and (80) will be approximately true for large N except at points where $f'(x) = 0$.

5. CONVERGENCE OF HAAR SERIES FOR DISCONTINUOUS FUNCTIONS

Since Haar functions are discontinuous themselves, it is not surprising that for certain classes of discontinuous functions the Haar-series convergence is stronger than in the mean. As might be expected from the developments in Section 3, the convergence properties in the neighborhood of a discontinuity depend on whether it is at a binary-rational or binary-irrational point.

Consider first the case of discontinuities at binary-rational points. Suppose that $f(x) \in L^2[0, 1]$ has an isolated discontinuity of the first kind at an Mth-order dyadic $x_1 = k/2^M$. From Theorem 1 we know that $S_N(x_1)$ is discontinuous for all $N \geqslant M$. In particular

$$S_N(x_1) = \frac{1}{2}[\bar{f}(i_N^m) + \bar{f}(i_N^{m+1})], \tag{88}$$

where $m = 2^{N-M}k$. This may also be written as

$$S_N(x_1) = \frac{1}{2}[S_N(x_1-) + S_N(x_1+)], \tag{89}$$

which shows that

$$\lim_{N\to\infty} S_N(x_1) = \frac{1}{2}[f(x_1-) + f(x_1+)]. \tag{90}$$

If $[a, b]$ is a subinterval of $[0, 1]$ that contains a finite number of binary-rational discontinuities, we see that

$$S_N(x) \to \frac{1}{2}[f(x-) + f(x+)] \tag{91}$$

for $x \in [a, b]$. By Corollary 4, the convergence is uniform. Applying Eq. (73), we have proved the following theorem.

THEOREM 6. *Let* $f(x) \in L^2[0, 1]$ *have a finite number of binary-rational discontinuities of the first kind in the subinterval* $[a, b]$. *The sequence of Haar partial sums is uniformly convergent to* $[1/2][f(x-) + f(x+)]$ *on* $[a, b]$.

It is easy to see that this theorem does not hold if $f(x)$ has jump discontinuities at binary-irrational points. We consider the following example:

$$f(x) = \begin{cases} 0, & x < \frac{1}{3} \\ \frac{1}{2}, & x = \frac{1}{3} \\ 1, & x > \frac{1}{3} \end{cases} \tag{92}$$

which has a binary-irrational jump discontinuity at $x = 1/3$. Let us calculate $S_N(1/3)$ for a few values of N:

$$\begin{aligned} S_0\left(\frac{1}{3}\right) &= \int_0^1 f(x)\,dx = \frac{2}{3} \\ S_1\left(\frac{1}{3}\right) &= 2\int_0^1 f(x)\,dx = \frac{1}{3} \\ S_2\left(\frac{1}{3}\right) &= 4\int_{1/4}^{1/2} f(x)\,dx = \frac{2}{3} \\ S_3\left(\frac{1}{3}\right) &= 8\int_{1/4}^{3/8} f(x)\,dx = \frac{1}{3} \\ S_4\left(\frac{1}{3}\right) &= 16\int_{5/16}^{3/8} f(x)\,dx = \frac{2}{3} \end{aligned} \tag{93}$$

In general, if x_1 is a binary-irrational jump discontinuity, $S_N(x_1)$ will be discontinuous for no value of N. Thus $S_N(x_1) = \bar{f}(i_N^m)$ always, where $x_1 \in i_N^m$. As N changes, x_1 falls at a different relative position within i_N^m, resulting in a different value for $\bar{f}(i_N^m)$. The sequence does not approach a limit, and $S_N(x)$ does not converge at x_1.

However, in any interval containing a finite number of binary-irrational jump discontinuities, it is easy to see that $S_N(x)$ is pointwise convergent, except at these discontinuities. We consider the two nearby points x_1 and x_2 and assume that $f(x)$ has a jump discontinuity at the binary-irrational point x_1 and that it is continuous at x_2. For a large enough value of N, $x_1 \notin i_N^k$ while $x_2 \in i_N^k$, i.e., $f(x)$ is continuous in i_N^k. By Corollary 4, $S_N(x)$ converges uniformly to $f(x)$ in i_N^k, so that

$$\lim_{N\to\infty} |S_N(x_2) - f(x_2)| = 0. \tag{94}$$

Using Eq. (73) once again, we therefore have the following theorem.

THEOREM 7. *Let* f(x) $\in L^2[0, 1]$ *have a finite number of isolated jump discontinuities at binary-irrational points in the subinterval* (a, b). *Then the sequence of Haar partial sums is pointwise convergent to* f(x) *everywhere in* (a, b), *except at the binary-irrational jump discontinuities, where there is no pointwise convergence.*

6. COMPARISON WITH TRIGONOMETRIC FOURIER SERIES

We have seen that the Haar series of any continuous function $f(x) \in L^2[0, 1]$ is uniformly convergent to the given function. This is not the case for trigonometric Fourier series. In fact there are continuous functions whose Fourier series diverges at an infinite number of points. We have also seen that for functions with a finite number of jump discontinuities, the Haar series is still uniformly convergent as long as all the discontinuities are at binary-rational points. However, *no* functions with jump discontinuities have uniformly convergent trigonometric Fourier series. For functions with isolated jump discontinuities at binary-irrational points, the Haar series is pointwise convergent, except at the discontinuities. However, there are functions with jump discontinuities whose Fourier series diverges at an infinite number of points.

These differences in convergence behavior are related to two important properties of the Haar sequence:

- Each Haar function contains all frequencies in the trigonometric Fourier sense.
- As n increases, the coefficients C_n^m of the Haar expansion depend on a smaller and smaller portion of the function being expanded.

The behavior of the Haar series expansion in the vicinity of a binary-irrational jump discontinuity of $f(x)$ makes an interesting comparison with the behavior of the trigonometric Fourier series near the same point.

Let P_n be the nth partial sum of the trigonometric expansion of $f(x)$ on $[0, 1]$. Then

$$P_n(x) = \frac{a}{2} + \sum_{k=1}^{n} [a_k \cos(2\pi kx) + b_k \sin(2\pi kx)], \tag{95}$$

where

$$a_k = 2 \int_0^1 f(x) \cos (2\pi kx)\, dx \tag{96}$$

$$b_k = 2 \int_0^1 f(x) \sin (2\pi kx)\, dx . \tag{97}$$

We note that the nth Fourier partial sum has $2n + 1$ terms. The Nth associated Haar partial sum has 2^N terms. We consider the following function which has a binary-irrational jump discontinuity at $x_1 = 2/3$:

$$f(x) = \begin{cases} 1 - x, & 0 \leqslant x \leqslant \frac{2}{3} \\ \frac{5}{3} - x, & \frac{2}{3} < x \leqslant 1 \end{cases} \tag{98}$$

For this function, both expansions are pointwise convergent, except that the Haar series does not converge at the jump discontinuity, whereas the trigonometric Fourier series converges to $[1/2][f((2/3)-) + f((2/3)+)]$. For $n = 32$ (65 terms) the trigonometric Fourier expansion of $f(x)$ is plotted in Fig. 3, superimposed on the function itself. This expansion exhibits the well-known Gibbs overshoot phenomenon, present at every jump discontinuity. For comparison the Haar expansion of the same function is plotted in Fig. 4 for $N = 6$ (64 terms).

First of all, we note the absence of any overshoot phenomenon. In fact, the behavior of S_N at a binary-irrational discontinuity is better described as an undershoot. We recall from the development of Theorem 6 that this undershoot cannot be eliminated by

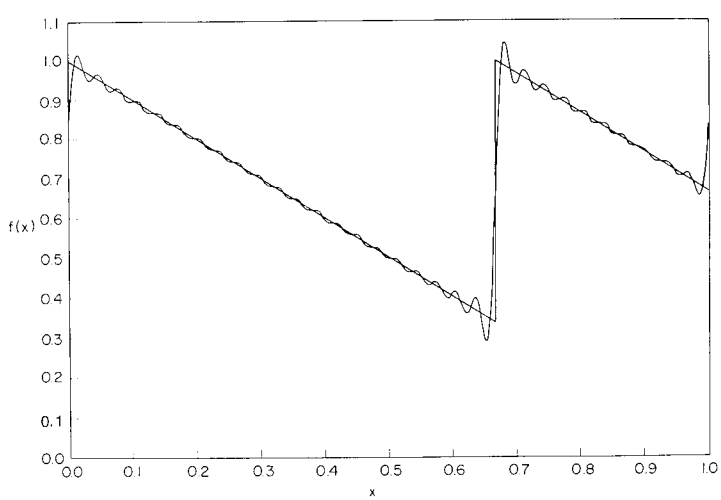

Fig. 3—Fourier series P_{32} (65 terms) of a function with a binary-irrational jump discontinuity

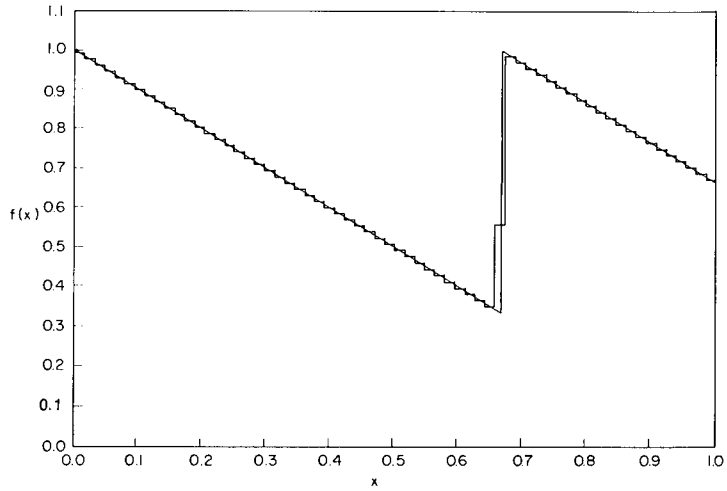

Fig. 4—Associated Haar series S_6 (64 terms) of a function with a binary-irrational jump discontinuity

increasing N. However, the width of this undershoot region may be made as narrow as desired, the undershoot width being given by $w = 1/2^N$. This is directly analogous to a one-half wavelength of the highest frequency component of P_n. In fact, the magnitudes are roughly the same.

Figure 4 also illustrates the fact that the effect of a jump discontinuity on the convergence of S_N is entirely limited to this undershoot region, a direct result of Corollary 1. The opposite is true for the trigonometric Fourier expansion P_n in which the effect of the Gibbs overshoot is felt throughout the entire interval of expansion. This relates to the fact that each term in P_n contributes to the expansion throughout the entire interval. On the other hand, as previously mentioned, each successive term in S_N contributes to the expansion in a smaller and smaller portion of the interval.

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