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PARTIAL NAVIGATION COURSES FOR A GUIDED MISSILE ATTACKING A CONSTANT VELOCITY TARGET

BY

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ABSTRACT

A kinematic study of partial navigation courses for a guided missile attacking a constant velocity target, is made in this report. A method of estimating the transverse acceleration of the missile following such a course is given. The application of the method is illustrated for a particular set of values of the various parameters. Graphical pictures of several courses are given together with curves of corresponding missile acceleration against proximity to target. The report treats only courses for which the "navigational correction constant" is 2. Other courses are at present being analysed by machine integration.

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PARTIAL NAVIGATION COURSES FOR A GUIDED MISSILE ATTCKING A CONSTANT VELOCITY TARGET

Introduction

(1) A missile is said to follow a partial, or proportional, navigation course in attacking a target, if it is guided so that the missile heading varies directly, in proportional fashion, with the missile to target bearing angle. The kinematic aspects of such courses have been considered by various authors (cf. references). A general discussion of the subject may be found in Part V of NRL Report #R2538, entitled "Guided Missile Kinematics" by H. E. Newell, Jr. The discussion referred to is principally qualitative, and is based to a large extent upon graphical analysis. It is the purpose of the present report to effect a more quantitative analysis of simple partial navigation courses.

(2) The notation adopted herein is that of Newell's report. Thus ϕ_m is used to denote missile heading, while β denotes the missile to target bearing angle, both referred to some fixed direction as prime direction (See Fig. 1). Moreover, as in the earlier report the partial navigation courses examined are precisely specified as those for which

(2.1) $\dot{\phi}_m = a\dot{\beta}$

or

(2.2) $\phi_m = a\beta + \text{const.}$

(3) The courses defined by (2.1) and (2.2) are, in a sense,

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intermediate between constant bearing courses[1], for which

(3.1) $\dot{\beta} = 0$

and pursuit courses[2], characterized by the relation

(3.2) $\phi_m = \beta$

(4) A desirable feature of constant bearing courses for missile guidance is that the required transverse accelerations of the missile seldom exceed those of the target. But, the attainment of a high degree of approximation to a constant bearing course for a missile attacking a rapidly manuevering target is a matter of some difficulty. Partial navigation courses may be regarded as offering a first approximation to constant bearing courses when the correction factor a is sufficiently large. It is important, therefore, to obtain some quantitative measure of the goodness of this approximation.

(5) For practical purposes the usefulness of partial navigation courses will probably be determined by the magnitudes of the turning accelerations the missile must experience in attacking a maneuvering target. Especial importance attaches to an investigation of these accelerations, since in view of the obvious similarity between (2.2) and (3.2), one suspects that some of the undesirable features of pursuit courses are to be found among partial navigation courses. This suspicion is actually borne out by the facts. Thus the terminal angular velocities required of the missile are in many cases very high.

[1] Cf. Newell, NRL Report #R2538, Part IV.

[2] Cf. Newell: Op. cit., Part II

(6) The turning accelerations experienced by missiles following partial navigation courses are determined below in a few specific cases for a constant velocity target.

(7) Finally the attention of the reader is directed to a recent report by O. E. Lancaster and L. H. Shornick (cf. ref. b) published during the preparation of this report, and also dealing with partial navigation courses.

General Discussion

(8) To simplify the analysis, the present investigation is confined to an examination of constant speed missiles and constant velocity targets. Such an analysis may be expected to furnish some indication of how a missile will behave in attacking a maneuvering target. It is also important to bear in mind that the spatial extents of both target and missile are disregarded for the most part, each being considered as a single point. In addition, aerodynamic and gravitational factors are completely ignored. These last factors undoubtedly have a great effect in actual physical cases, but elimination of their consideration in a preliminary, essentially kinematic analysis, such as the present one, seems desirable.

(9) With the foregoing simplifying assumptions, the problem of interest now becomes simply that of determining the essential properties of the trajectory of a constant speed point missile guided toward a constant velocity point target along a partial navigation course. Particular importance attaches to the magnitudes of the transverse

accelerations experienced by the missile.

(10) It is shown in Appendix I that there always exists a rectilinear path for the missile leading to a collision with the target. This course is obtained by launching the missile so that the ratio of the sine of the initial missile heading angle to the sine of the initial target bearing angle is equal to the target to missile speed ratio; that is:

$$\frac{\sin \gamma_o}{\sin \beta_o} = \frac{V_T}{V_m}$$

For this trajectory, the transverse acceleration of the missile vanishes, which is certainly of advantage. Plainly the path described is none other than the familiar collision course followed by torpedoes. The existence of these rectilinear courses merely bears out the rather obvious fact that in attacking constant velocity targets the principal advantage of a guided missile over an unguided one is not so much in the type of course which may be followed as in the existence of opportunity to correct initial errors in launching.

(11) The more general case, however, in which the target employs evasive action is perhaps of greater importance than that in which the target is unaccelerated. In such a case the missile must also maneuver in order to effect a collision with the target. It is plain that the target's maneuvering, is to a great extent arbitrary. The motion of the missile, on the other hand, must be related appropriately to that of the target, or no collision will occur. Thus, if the missile is constrained to follow a specific type of course, which in the present

case is to be a partial navigation course, the maneuverability required of the missile is determinate. The equations of motion for a target turning with a specified normal acceleration are at present being integrated numerically to determine the trajectory of the missile together with its turning accelerations. As stated previously in this discussion, the target is taken as unaccelerated; but the turning accelerations of the missile in following partial navigation courses other than the simple collision course, may be expected to offer some hint as to the maneuvering of which the missile should be capable in order to attack a target successfully.

(12) The rest of the report, then, concerns those partial navigation courses which are not collision courses, in which, therefore, the missile is accelerated. Moreover, the navigational correction a in Equations (2.1) and (2.2) is taken as 2. This again is for the sake of simplicity. The treatment of cases in which the navigational correction exceeds 2 will appear in a subsequent report.

(13) The analyses of Appendix I in which a is 2, shows that a constant speed missile attacking a constant velocity target may have to undergo very large or only very small accelerations, depending upon the conditions incident to the launching of the missile. In fact as in the case of simple pursuit courses, there are three separate cases which may arise; 1) the transverse acceleration of the missile vanishes as the target is neared; 2) the acceleration becomes infinite as the missile approaches the target; and finally, 3) the limiting value of the missile

acceleration is finite but not zero.

(14) <u>Case 1</u>. The transverse acceleration of the missile vanishes as the target is neared, if

(14.1) $\cos(\gamma_o + \beta_o) > -\frac{1}{p}$ or $\gamma_o + \beta_o < \arccos\left(-\frac{1}{p}\right)$

Here γ_o is the initial relative missile heading[3], β_o the initial target bearing, and p the missile to target speed ratio.

(15) Curves A of Fig. 2a and D of Fig. 3a depict paths for which (14.1) is satisfied. For path A, $\beta_o = 90^o$, $\gamma_o = 15^o$; and for D $\beta_o = 105^o$, $\gamma_o = 0^o$. In both cases $p = 2$. The shapes of the paths depend upon p, β_o, and γ_o, but not upon the initial ranges. It is convenient, therefore, to measure missile to target ranges in terms of decimal parts of the initial range. Thus, suppose that in Fig. 2a, r_o = 100,000 ft. In this case the missile following path A intercepts the target when the latter has travelled $0.6r_o$, or 60,000 ft. Since V_m is twice V_T, the length L_M of the missile trajectory is 120,000 ft. Finally, the time for interception of the target is readily computed as L_M/V_M.

(16) The variation of transverse acceleration along the trajectories A and D is shown by curves A' and D' of Figs. 2b and 3b. Curve E' of Fig. 3b exhibits the acceleration for a path in which $p = 2$, $\beta_o = 105^o$, $\gamma_o = 8^o$. It is seen that the corresponding trajectory also falls into Case 1. The graphs are plotted on logarithmic paper in order to magnify the region of small missile to target range, so

[3]As seen in Fig. 1, $\gamma = \beta - \phi_m$

that the rapidly changing acceleration in this region may be accurately represented. The inner ordinate scale is dimensionless, giving transverse acceleration in units of gV_m^2/r_o, while the inner abcissa scale shows range as decimal fractional parts of r_o. The use of such scales enables one to apply the graphs with differing values of missile speed V_m and initial range r_o. Thus, the outer scales give acceleration in units of g versus range in feet for the specific values $V_m = 1000$ ft/sec, $r_o = 100{,}000$ ft. The point marked X on curve A' represents an acceleration of 0.2g at 60 feet from the target for these specific values of V_m and r_o. At the same time its dimensionless coordinates (.0006, .002) permit interpretation for other values of V_m and r_o; for example, if $V_m = 700$ ft/sec and $r_o = 50{,}000$ ft, the acceleration is about .02g at 30 feet from the target.

(17). It is of interest to notethat in these curves the maximum acceleration is not much greater than the initial acceleration. Thus, in these cases, initial acceleration may be used as a fair estimate of the maximum. A preliminary investigation strongly suggest that this is true for all trajectories in which the ratio $V_m/V_T = \rho$ is nearly 2. However, it can be shown that as ρ decreases the ratio of maximum acceleration to initial acceleration increases.

(18) If (14.1) holds for $\rho = 2$, then $\gamma_o + \beta_o$ does not exceed 120^o. Thus, if β_o itself exceeds the "critical value" of 120^o, no positive γ_o exists which will yield a trajectory of the type described in case 1. It is plain that as ρ is decreased the range of values

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for β_0 corresponding to trajectories of case 1 becomes greater.

(19) Case 2. The transverse acceleration of the missile becomes infinite as the target is neared if

(19.1) $\cos(\gamma_0 + \beta_0) < -\frac{1}{p}$ or $\gamma_0 + \beta_0 > \arccos\left(-\frac{1}{p}\right)$,

where the symbols are those used in paragraph (14).

(20) Curves C of Fig. 2a, K of Fig. 4a, and N, Q, S, T of Fig. 5a depict paths for which (19.1) is satisfied. For path C, $\beta_0 = 90^\circ$, $\gamma_0 = 45^\circ$; for K, $\beta_0 = 120^\circ$, $\gamma_0 = 13^\circ$; for N, $\beta_0 = 135^\circ$, $\gamma_0 = 5^\circ$; for Q, $\beta_0 = 135^\circ$, $\gamma_0 = 15^\circ$; for S, $\beta_0 = 135^\circ$, $\gamma_0 = 25^\circ$; for T, $\beta_0 = 135^\circ$, $\gamma_0 = 45^\circ$. In all cases $p = 2$. As explained in the discussion of case 1, the curves may be used to determine path lengths and times of flight.

(21) The variation of transverse acceleration along each of the trajectories C, K, N, Q, S, and T is shown by the corresponding primed curve of Figs. 2b, 4b, or 5b. Curves G' of Fig. 3b and M' of Fig. 4b exhibit the accelerations for paths in which $\beta_0 = 105^\circ$, $\gamma_0 = 38^\circ$, and $\beta_0 = 120^\circ$, $\gamma_0 = 3^\circ$ respectively. Again $p = 2$. The corresponding trajectories fall into Case 2.

(22) All the curves referred to show a rapid increase in missile acceleration as the target is neared. Although one is inclined to regard such a property of a trajectory as undesirable, nevertheless this undesirability can be overemphasized. It should be kept in mind that the general discussion as given applies to point missiles and point targets, and that spatial extents of both target and missile

favor the attacking missile. Thus, a hit may occur before the missile acceleration becomes excessive.

(23) The plotted acceleration curves enable one to determine, for various initial conditions, the missile acceleration at a specified distance from the target. For any given set of values of lethal missile to target range and maximum allowable missile acceleration, there is a point determined by missile speed and initial range, such that any acceleration curve which passes below this point corresponds to a useable missile trajectory in the sense that a missile following such a trajectory can come within lethal range of the target.

(24) To illustrate, let the lethal range be 50 feet and suppose that the maximum attainable missile acceleration is 5g. The circled point in each of Figs. 2b, 3b, 4b, and 5b corresponds to these values for a missile speed of 1000 ft/sec and an initial range of 50,000 ft. All the curves except N', P', and T' pass below the encircled point, and, therefore, correspond to trajectories along which the missile acceleration does not exceed 5g until the missile is within at most 50 feet of a target.

(25) A generalization of the foregoing discussion leads to conclusions conveniently expressed by means of Fig. 6, which is based on the analysis of Appendix II. The figure is drawn for a missile speed of 1000 ft/sec, and $\rho = 2$. The ordinates show initial relative missile heading γ_0 and the abscissas show initial target bearing β_0. The entire shaded area represents the range of values for γ_0 and β_0 for which the missile can approach to within 50 ft. of the target without undergoing

an acceleration in excess of 5g. The included boxed area corresponds to trajectories exhibiting the properties of Case 1. Initial values which lead to rectilinear missile paths lie on the curve YY.

(26) Examination of Fig. 6 shows that for initial target bearings less than 120° there is a rather wide range of initial missile headings for which the missile can come close to the target without experiencing accelerations exceeding 5g. For target bearings greater than 120°, the range of such initial headings decreases rapidly to a narrow interval containing the special value corresponding to a collision course.

(27) Case 3. The transverse acceleration of the missile approaches a finite value, other than zero, as the target is neared if

(27.1) $\cos(\gamma_0+\beta_0) = -\frac{1}{p}$ or $\gamma_0+\beta_0 = \arccos\left(-\frac{1}{p}\right)$

This case is intermediate between 1 and 2.

(28) It is shown in Appendix I that in Case 3, the missile acceleration either keeps increasing or keeps decreasing throughout the entire flight. The maximum accleration accordingly remains finite and is either the initial or the terminal acceleration, and may be computed from the appropriate one of formulas (37.7) and 37.8) of Appendix I.

(29) Curves B, F, and J of Figs. 2a, 3a, and 4a respectively, depict paths for which (27.1) is satisfied. For path B, $\beta_0 = 90°$, $\gamma_0 = 45°$; for F, $\beta_0 = 105°$, $\gamma_0 = 15°$; for J, $\beta_0 = 120°$, $\gamma_0 = 0°$. As before, $p = 2$. The variation of acceleration along each of the trajectories B, F, and J is shown by the curves B', F', and J' in Figs. 2b, 3b, and 4b.

(30) If p is fixed, the trajectories of case 3 correspond to pairs (β_0, γ_0) which are the coordinates of points on a straight line such as YZ fo Fig. 6. As shown there, the straight line forms a boundary separating the points (β_0, γ_0) into regions corresponding to trajectories of cases 1 and 2 respectively.

(31) A comparison of the courses shown in Fig. 2a with those of Figs. 23a and b of Newell's report (cf. ref. a Part V) reveals a close agreement between the two sets. Both sets of trajectories were plotted for the same initial conditions, but the latter set was constructed under the assumption of discontinuous corrections applied to the missile's motion. In view of the close agreement between the two sets, one concludes that the method of Newell's report offers a simple and rapid means of approximating to the continuous correction case. It suffices merely to use a relatively small sensitivity of correction (cf. ref. a Pt. V)

Conclusions

(32) Following is a list of some of the conclusions which can be drawn from the analysis of partial navigation courses for constant speed missiles attacking constant velocity targets.

a) The straight line collision course always exists as a partial navigation course, whatever the initial relative positions and relative speeds of target and missile, and whatever the navigational corrections used. To obtain such a missile trajectory, it suffices merely to launch the missile with the appropriate heading.

b) For navigational correction constant $\underline{a}=2$, the courses may be divided into the three classes: 1) trajectories in which the terminal missile acceleration is zero; 2) trajectories in which the terminal missile acceleration is infinite; 3) trajectories in which the terminal missile acceleration is finite, but not zero.

c) Courses with zero or finite terminal acceleration exist only for initial target bearings less than a certain critical value if initial relative missile heading is non-negative. This critical value depends on the missile to target speed ratio.

d) Spatial extent of the target in general favors the missile so that trajectories of class 2 are not necessarily to be discarded as unuseable merely because of the high terminal acceleration. It may be that the missile can approach to within destructive range of the target before the accelerations become excessive.

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APPENDIX I

Mathematical Analysis of Partial Navigation Courses

(33) The purpose of this discussion is to set forth the analysis needed to determine the essential characteristics of the trajectory followed by a constant speed point missile pursuing a constant velocity point target along a partial navigation course. Let β represent target bearing: γ, relative missile heading; V_T, target speed; V_m, missile speed; and r, target to missile range. Then, referring to Figure 1:

(33.1) $r\dot{\beta} = V_m \sin\gamma - V_T \sin\beta$,

$\dot{r} = - V_m \cos\gamma + V_T \cos\beta$.

Since $\phi_m + \gamma = \beta$, the fundamental relation 2.1) may be written as

(33.2) $\dot{\gamma} = (1 - a)\dot{\beta}$.

It is readily seen that there always exists a rectilinear partial navigation course, namely, that for which $\dot{\beta} = \dot{\gamma} = 0$. This is, in fact, the familiar collision course. Letting the subscript o denote initial values, one finds for the collision course that

(33.3) $\sin\gamma_o = \dfrac{V_T}{V_m} \sin\beta_o$.

(34) The remainder of this discussion will be confined to an examination of the cases in which $a = 2$. Analysis of cases in which <u>a</u> exceeds 2 is under way and the results will appear at a later date.

(35) With $a = 2$, (33.2) integrates to

(35.1) $\gamma = -\beta + \alpha_o$ where $\alpha_o = \gamma_o + \beta_o$.

Then

(35.2)
$$\begin{cases} -r\dot{\gamma} = V_m \sin\gamma - V_T \sin(\alpha_0 - \gamma), \\ \dot{r} = -V_m \cos\gamma + V_T \cos(\alpha_0 - \gamma); \end{cases}$$

so that

$$\frac{\dot{r}}{r} = \frac{(\rho - \cos\alpha_0)\cos\gamma - \sin\alpha_0 \sin\gamma}{(\rho + \cos\alpha_0)\sin\gamma - \sin\alpha_0 \cos\gamma}\,\dot{\gamma},$$

(35.3)

where $\rho = \dfrac{V_m}{V_T}$. This has the solution

(35.4)
$$r = r_0 e^{\frac{2\rho \sin\alpha_0(\gamma_0 - \gamma)}{1 + 2\rho\cos\alpha_0 + \rho^2}} \left\{ \frac{(\rho + \cos\alpha_0)\sin\gamma - \sin\alpha_0\cos\gamma}{(\rho + \cos\alpha_0)\sin\gamma_0 - \sin\alpha_0\cos\gamma_0} \right\}^{\frac{\rho^2 - 1}{1 + 2\rho\cos\alpha_0 + \rho^2}}$$

Thus for any $\rho > 1$, r becomes zero when γ takes the finite value

(35.5)
$$\gamma_0 = \arctan\left(\frac{\sin\alpha_0}{\rho + \cos\alpha_0}\right),$$

unless

(35.6)
$$(\rho + \cos\alpha_0)\sin\gamma_0 - \sin\alpha_0\cos\gamma_0 = 0 .$$

When (35.6) is satisfied, the trajectory reduces to a collision course. In either case it is plain that the missile will overtake the target if the missile speed always exceeds that of the target.

(36) Combine (35.2) and (35.4) to obtain

(36.1)
$$\dot{\gamma} = \frac{V_T}{r_0}(\sin\beta_0 - \rho\sin\gamma_0)\, e^{\frac{2\rho\sin\alpha_0(\gamma - \gamma_0)}{\rho^2 - 1}} \left(\frac{r}{r_0}\right)^{\frac{2(1 + \rho\cos\alpha_0)}{\rho^2 - 1}}.$$

Since $\lim_{r \to 0} \gamma = \gamma_0$, one concludes that

$$\lim_{r \to 0} \dot{\gamma} = \begin{cases} 0 & \text{if } \rho\cos\alpha_0 > -1, \\ \infty & \text{if } \rho\cos\alpha_0 < -1. \end{cases}$$

The transverse acceleration of the missile A_M is given by

(36.2) $$A_M = \dot{\phi} V_m = 2\dot{\gamma} V_m$$

so that,

(36.3) $$\lim_{r \to 0} A_M = \begin{cases} 0 \text{ if } \rho\cos\alpha_0 > -1 \text{ or } \gamma_0 < \cos^{-1}(\frac{-1}{\rho}) - \beta_0, \\ \infty \text{ if } \rho\cos\alpha_0 < -1 \text{ or } \gamma_0 > \cos^{-1}(-\frac{1}{\rho}) - \beta_0. \end{cases}$$

(37) When $\gamma_0 = \arccos(-\frac{1}{\rho}) - \beta_0$ it can be shown that the missle acceleration remains finite throughout the flight. Thus, let $\rho\cos\alpha_0 = 1$ in (35.4):

(37.1) $$r = r_0 e^{\frac{2(\gamma_0 - \gamma)}{\sqrt{\rho^2 - 1}}} \left\{ \frac{\sqrt{\rho^2 - 1}\sin\gamma - \cos\gamma}{\sqrt{\rho^2 - 1}\sin\gamma_0 - \cos\gamma_0} \right\}.$$

The first of equations (33.1) becomes

(37.2) $$-r_0 e^{\frac{2(\gamma_0 - \gamma)}{\sqrt{\rho^2 - 1}}} \dot{\gamma} = \frac{V_T\sqrt{\rho^2 - 1}}{\rho} \left\{ \sqrt{\rho^2 - 1}\sin\gamma_0 - \cos\gamma_0 \right\},$$

which has the solution

(37.3) $$\gamma = \gamma_0 - \frac{\sqrt{\rho^2 - 1}}{2} \log_e \left[\frac{2V_T}{\rho r_0} \left\{ \sqrt{\rho^2 - 1}\sin\gamma_0 - \cos\gamma_0 \right\} t + 1 \right].$$

Differentiate (37.3) with respect to time:

(37.4) $$\dot{\gamma} = \frac{V_T\sqrt{\rho^2 - 1}\left\{ \cos\gamma_0 - \sqrt{\rho^2 - 1}\sin\gamma_0 \right\}}{2V_T\left\{ \sqrt{\rho^2 - 1}\sin\gamma_0 - \cos\gamma_0 \right\} t + \rho r_0}.$$

From (37.4), $\dot{r}$ is finite unless $t = \dfrac{\rho r_0}{2V_T\left(\cos\gamma_0 - \sqrt{\rho^2 - 1}\sin\gamma_0\right)} = t_\infty$

From (37.3) it is seen that the time of flight t_c is

$$(37.5)\quad t_c = \frac{\rho r_0 \left\{1 - \exp \dfrac{2(\gamma_0 - \gamma_c)}{\sqrt{\rho^2 - 1}}\right\}}{2V_T\left(\cos\gamma_0 - \sqrt{\rho^2 - 1}\,\sin\gamma_0\right)},$$

so that

$$(37.6)\quad \frac{t_c}{t_\infty} = 1 - e^{\dfrac{2(\gamma_0 - \gamma_c)}{\sqrt{\rho^2 - 1}}}.$$

Thus, collision occurs before infinite acceleration of the missile is required. Hence, in this case, the missile acceleration is bounded throughout the entire flight. It can be shown further that the acceleration either decreases continually from an initial maximum value, or increases steadily to a final maximum value. This can be seen from (37.2) which shows that $\dot{\gamma}$ is an increasing or decreasing finction of time, according as γ_c is greater or less than γ_0. If

$\gamma_c = \gamma_0$

then

$$\arctan \frac{1}{\sqrt{\rho^2 - 1}} = \arccos\left(-\frac{1}{\rho}\right) - \beta_0$$

or

$$\beta_0 = \frac{\pi}{2}.$$

Thus, if $\beta_0 > \frac{\pi}{2}$, the maximum acceleration over the path occurs at the time of collision and has the value

$$(37.7)\quad A_{max} = \left(\frac{V_m^2}{r_0}\right)\frac{2\sqrt{\rho^2 - 1}}{\rho^2}\left(\cos\gamma_0 - \sqrt{\rho^2 - 1}\,\sin\gamma_0\right) e^{\dfrac{2\beta_0 - \pi}{\sqrt{\rho^2 - 1}}}$$

If $\beta_0 < \frac{\pi}{2}$, the maximum acceleration occurs at the time of 1 launching, and is given by

$$(37.8) \quad A_{max} = \left(\frac{V_m^2}{r_0}\right) \frac{2\sqrt{p^2 - 1}}{p^2} \left(\cos \gamma_0 - \sqrt{p^2 - 1} \sin \gamma_0\right).$$

APPENDIX II

Graphical Analysis of Partial Navigation Courses

(38) The purpose of this appendix is to describe a method of graphical interpolation between acceleration curves such as those of Figs. 2b, 3b, 4b, and 5b. The curves used to illustrate the analysis are specifically those of Fig. 2b. It will be clear how the discussion applies in general.

(39) Referring to Fig. 2b, it can be seen that the curves A' and C' are for a large part approximately straight lines, the slopes of which may be obtained directly from Equation (36.1) by letting γ approach γ_c, where γ_c is the relative missile heading at collision. The slope so obtained is

(39.1) $$\frac{2}{p^2 - 1}\left(1 + p\cos\alpha_0\right),$$

p being the missile to target speed ratio. The angle α_0 is the sum of initial target bearing angle β_0 and initial relative missile heading γ_0. Setting $\frac{r}{r_0} = 1$ in equation (36.1), it is seen that the right hand intercept of an acceleration curve is given by the expression

(39.2) $$\frac{2}{pg}\left|\sin\beta_0 - p\sin\gamma_0\right|.$$

The curves of Fig. 2b were drawn for $p = 2$, $\beta_0 = 90^\circ$. Under these conditions (39.2) is

(39.3) $$\frac{1}{g}\left|1 - 2\sin\gamma_0\right|.$$

Thus, the initial value of acceleration for the trajectory with

$\gamma_0 = 30^\circ$ is zero, while the trajectory with $\gamma_0 = 0^\circ$ has initial acceleration $\left(.031 \frac{V_m{}^2}{r_0}\right)$ g for missile speed V_m and initial range r_0 . Any γ_0 between 0° and 30° yields an initial acceleration less than this value. After no more than a small increase, as $\frac{r}{r_0}$ decreases, the acceleration curve begins to follow the rectilinear decrease with slope (39.1). Thus, no partial navigation trajectory with $\underline{a} = 2$, $p = 2$, $\beta_0 = 90^\circ$, and γ_0 between 0° and 30° requires of the missile a transverse acceleration much in excess of $\left(.031 \frac{V_m{}^2}{r_0}\right)$g. Applying a similar discussion to Fig. 3b, where $\underline{a} = 2$, $p = 2$, $\beta_0 = 105^\circ$, one finds that the courses for γ_0 between 0° and 15° require accelerations not much greater than $\left(.03 \frac{V_m{}^2}{r_0}\right)$g . More exactly, as seen from curve D', the maximum acceleration does not exceed $\left(.033 \frac{V_m{}^2}{r_0}\right)$g.

(40) The curves discussed above all come under Case 1 of paragraph (13), and accordingly are characterized by zero terminal missile acceleration. For curves of Case 2 in which the terminal acceleration is infinite there is, of course, no maximum. In this case the graphs may be used to determine whether a particular set of initial conditions corresponds to a useable trajectory in the sense that a missile following such a trajectory can come within destructive range of the target without being required to undergo excessive accelerations.

(41) To illustrate[4], suppose a destructive range is 50 feet,

[4] See also paragraph (24), where the same problem is discussed.

and that the maximum attainable acceleration is 5g. The circled point in each of Figs. 2b, 3b, 4b, and 5b corresponds to these values of destructive range and maximum allowable acceleration for a missile speed of 1000 feet/second and initial range of 50,000 feet. Curve C' (Fig. 2b) representing the acceleration of the trajectory determined by $p = 2$, $\beta_0 = 90^\circ$, $\gamma_0 = 45^\circ$ is seen to pass below the circled point; this indicates that the acceleration does not exceed 5g until the missile is within at most 50 feet of the target. Keeping p and β_0 constant, and decreasing γ_0, the right hand intercept decreases as is verified by reference to (39.3). Equation (39.1) shows that the slope of the straight line portion also decreases numerically. Thus, it seems reasonable to suppose that any acceleration curve with $p = 2$, $\beta_0 = 90^\circ$, and γ_0 less than 45° passes below, the circled point, and in fact it appears that this value of 45° is conservative and may perhaps be extended as far as 50°.

(42) This estimate is used in the plotting of the upper boundary of the summary graph Fig. 6, which was discussed in paragraph (25). Other points on the upper boundary in Fig. 6 for $\beta_0 = 105^\circ$, 120°, and 135° were found in like manner from Figs. 3b, 4b, and 5b respectively. In fact that estimates for the higher values of β_0 were made in the same way with curves which are not shown in this report.

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b.) "Proportional Navigation, Its Use in Pilotless Aircraft" by Lt. Cdr. O. E. Lancaster, USNR and Ensign L. H. Shornick, USNR. A.D.R. Report T-17, BuAer, November 1945.

c.) "Homing and Navigational Courses of Automatic Target Seeking Devices" by C. L. Yuan, RCA Laboratories, Princeton, N. J. Technical Report PTR-12C, 13 December 1943.

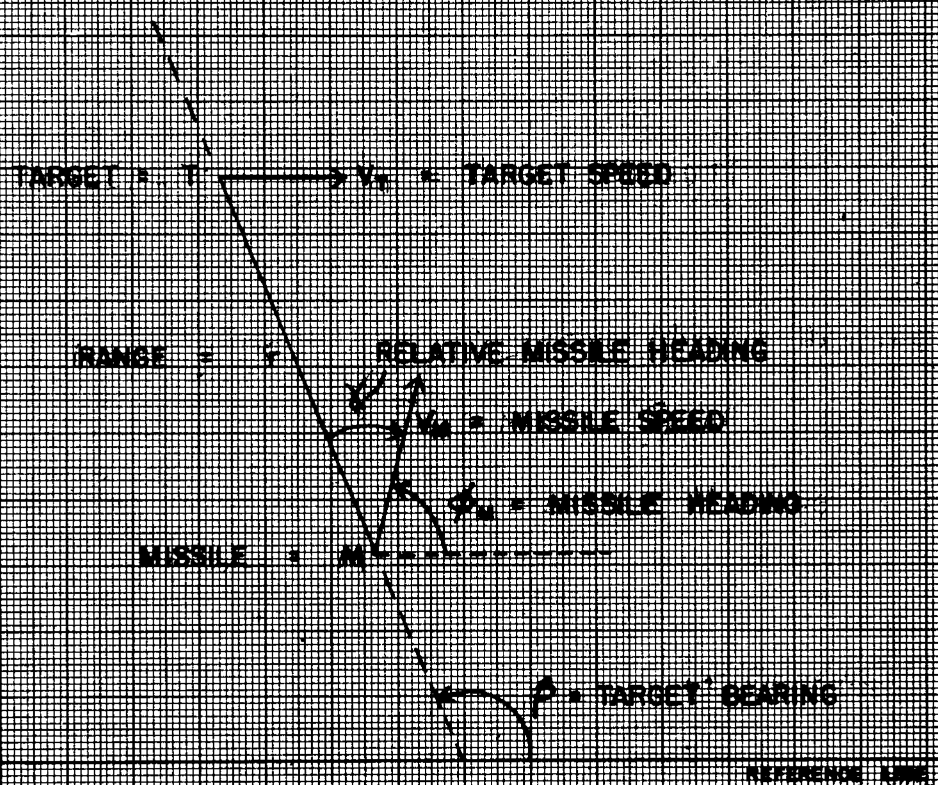

FIGURE I. NOTATION

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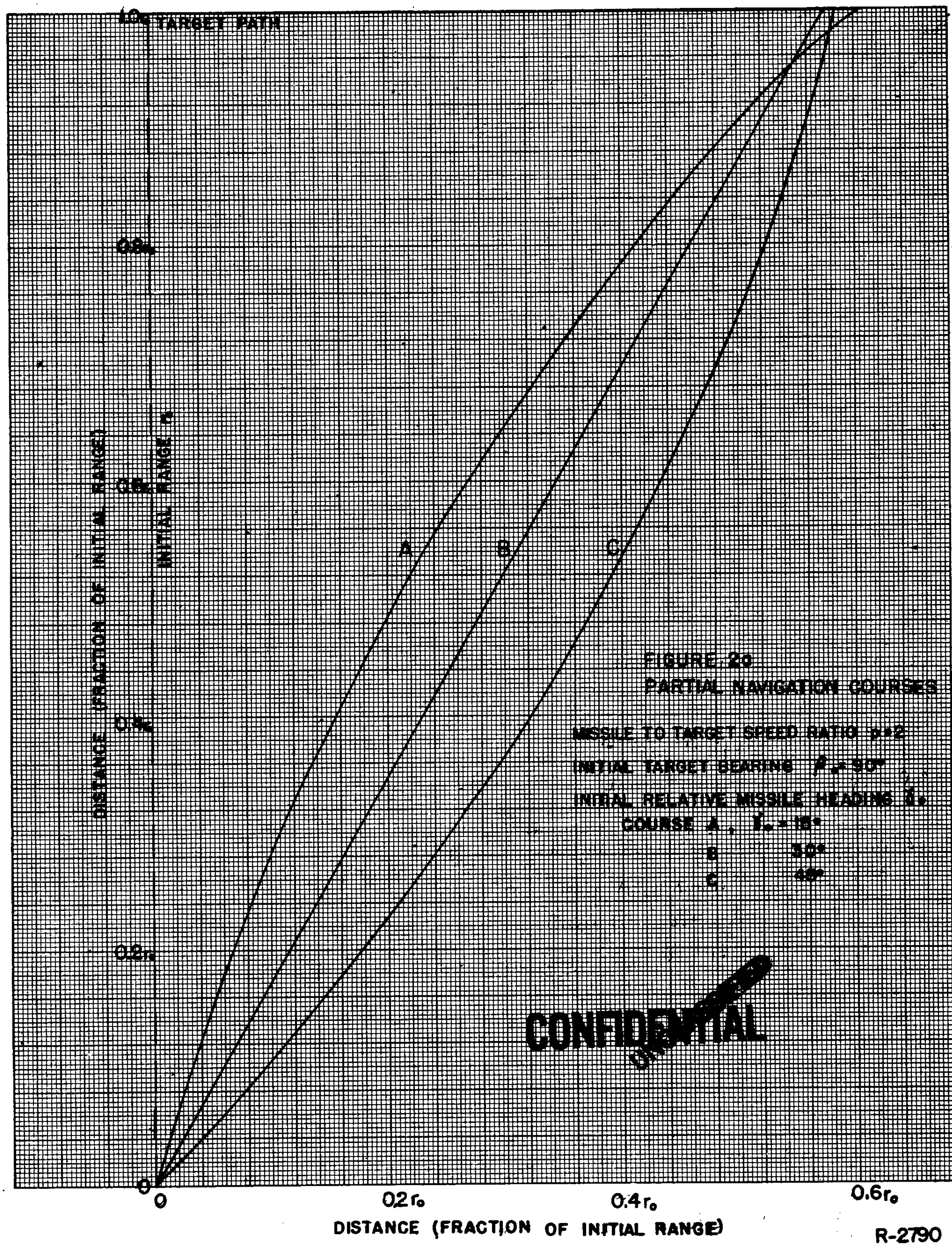

FIGURE 2c
PARTIAL NAVIGATION COURSES

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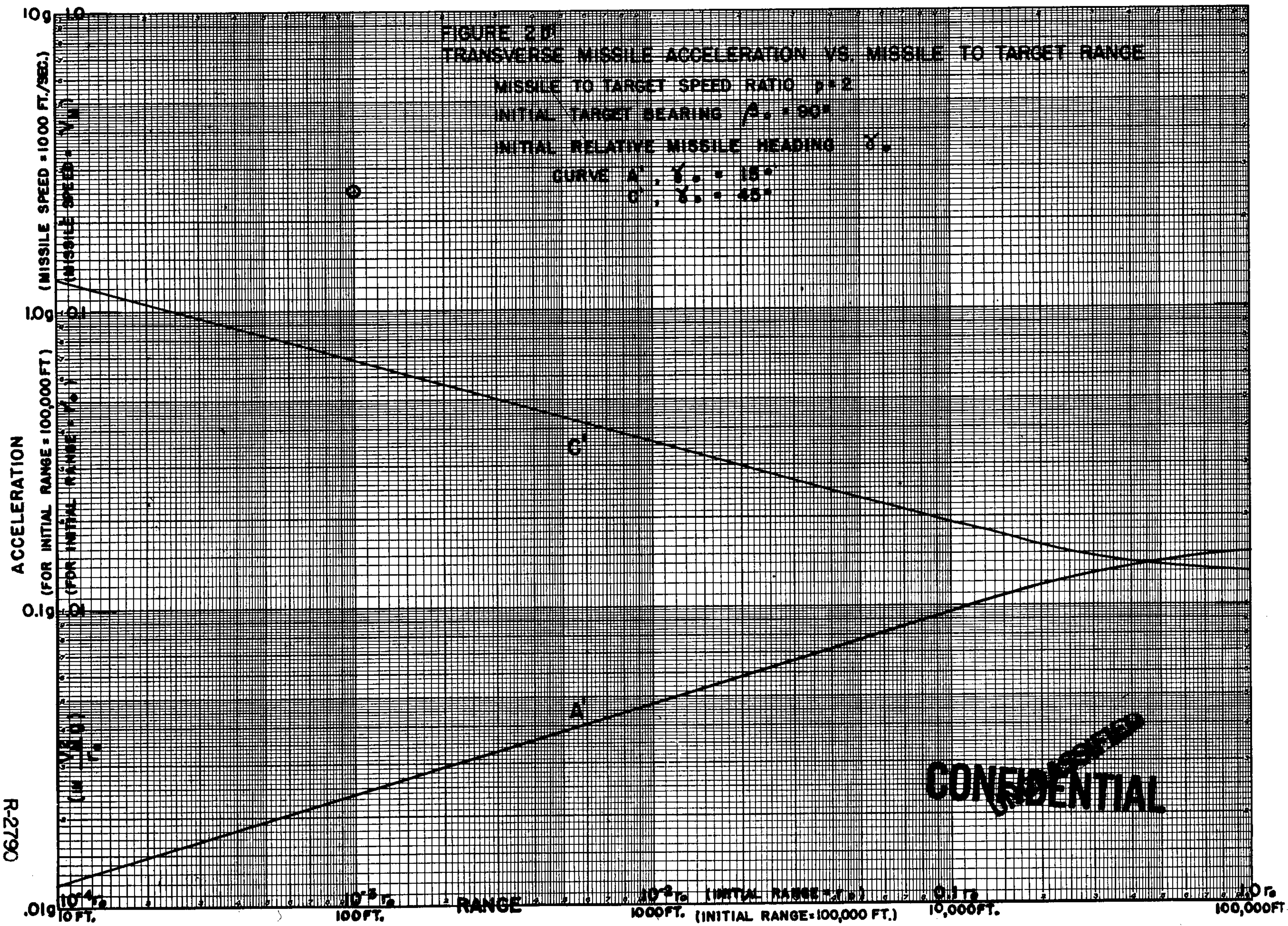

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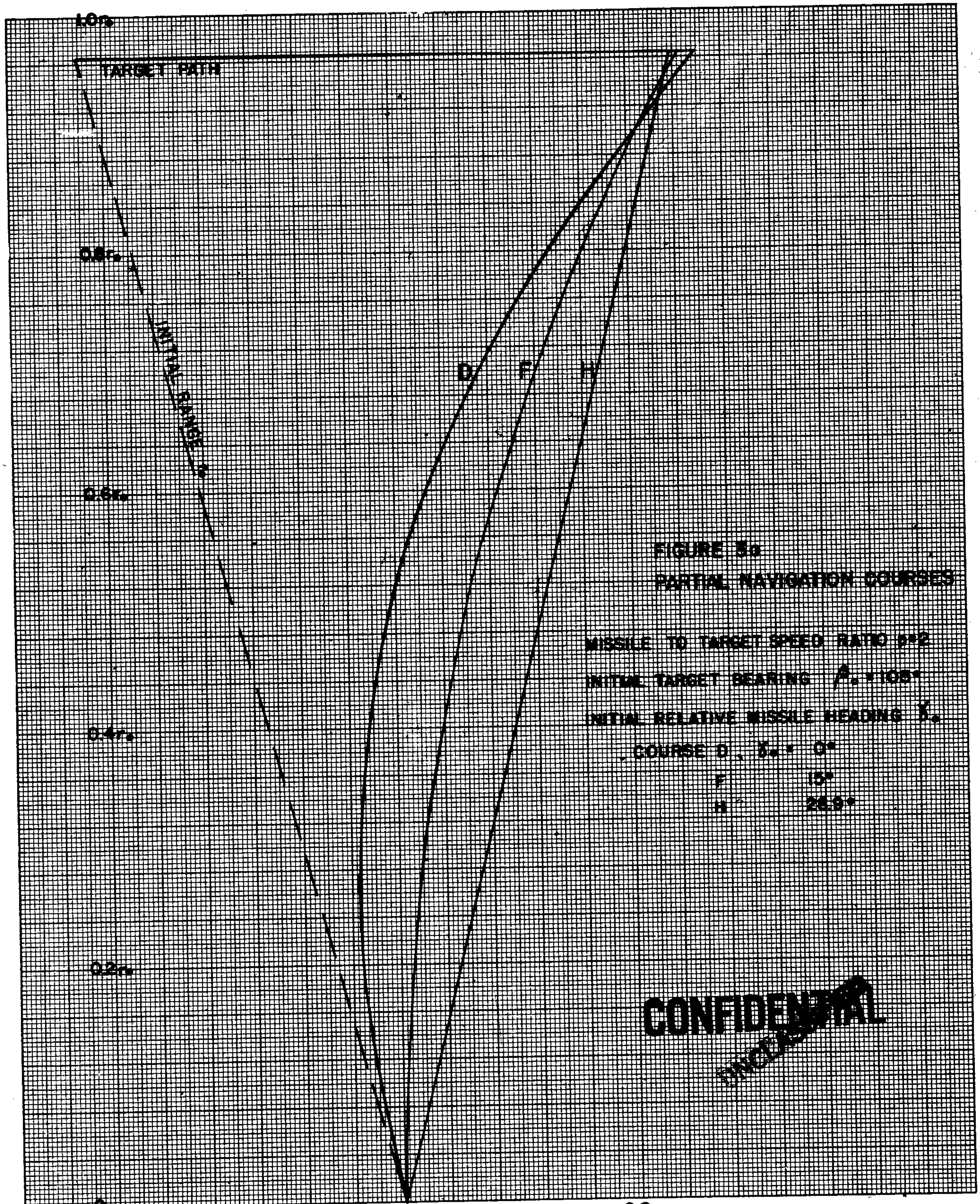

FIGURE 3a
PARTIAL NAVIGATION COURSES

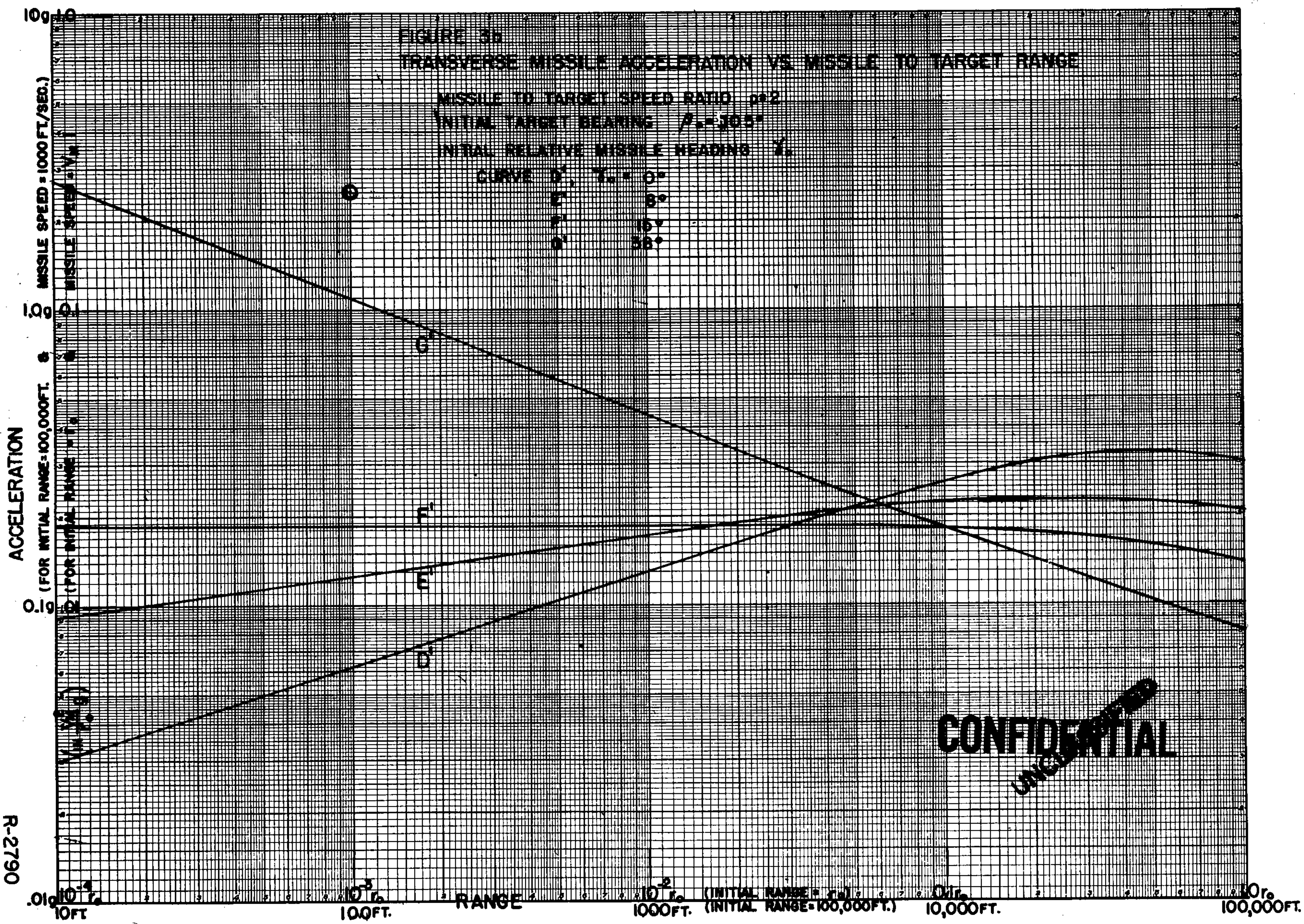
FIGURE 3b
TRANSVERSE MISSILE ACCELERATION VS. MISSILE TO TARGET RANGE
MISSILE TO TARGET SPEED RATIO ρ=2
INITIAL TARGET BEARING β₀=305°
INITIAL RELATIVE MISSILE HEADING γ₀
CURVE D', γ₀ = 0°
E' 8°
F' 16°
G' 38°
G'
F'
E'
D'
ACCELERATION
(FOR INITIAL RANGE=100,000FT.)
(FOR INITIAL RANGE = r₀)
MISSILE SPEED = 1000 FT./SEC.)
MISSILE SPEED = Vₘ
10g
1.0g
0.1g
.01g
RANGE
10FT
100FT.
1000FT.
10,000FT.
100,000FT.
(INITIAL RANGE = r₀)
(INITIAL RANGE=100,000FT.)
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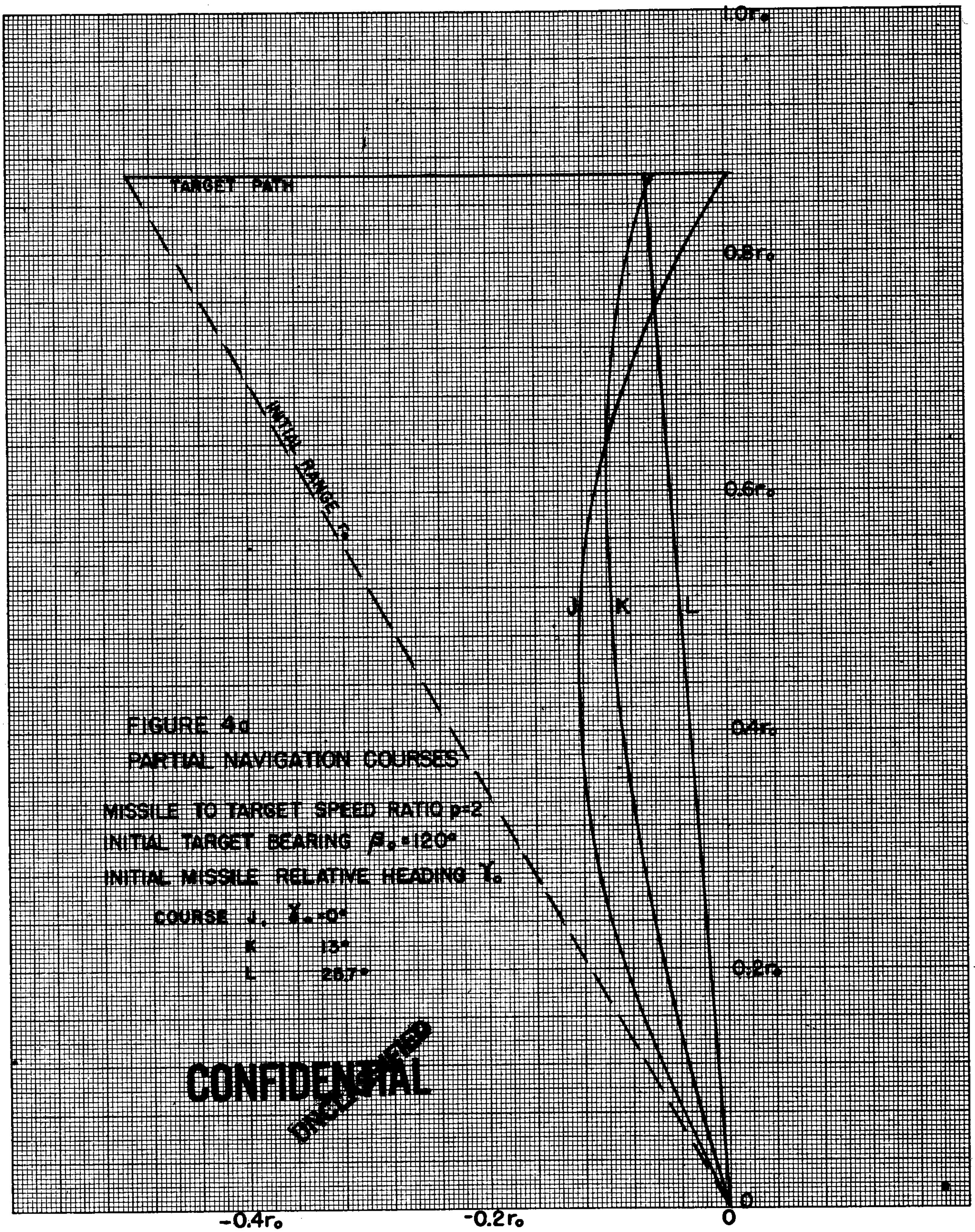
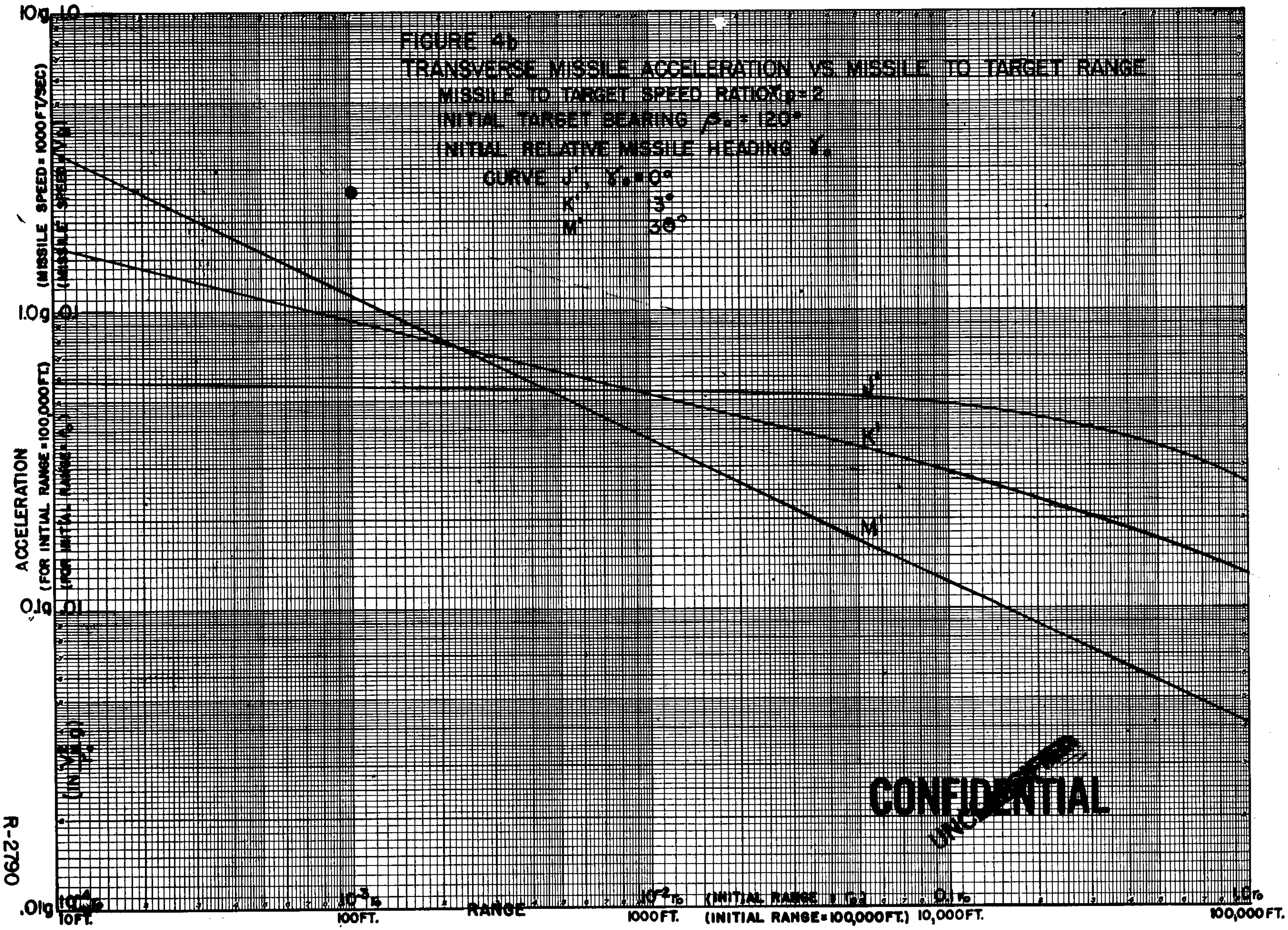


FIGURE 4a
PARTIAL NAVIGATION COURSES

MISSILE TO TARGET SPEED RATIO p=2
INITIAL TARGET BEARING β_0=120°
INITIAL MISSILE RELATIVE HEADING γ_0

COURSE	J,	γ_0=0°
	K	13°
	L	25.7°



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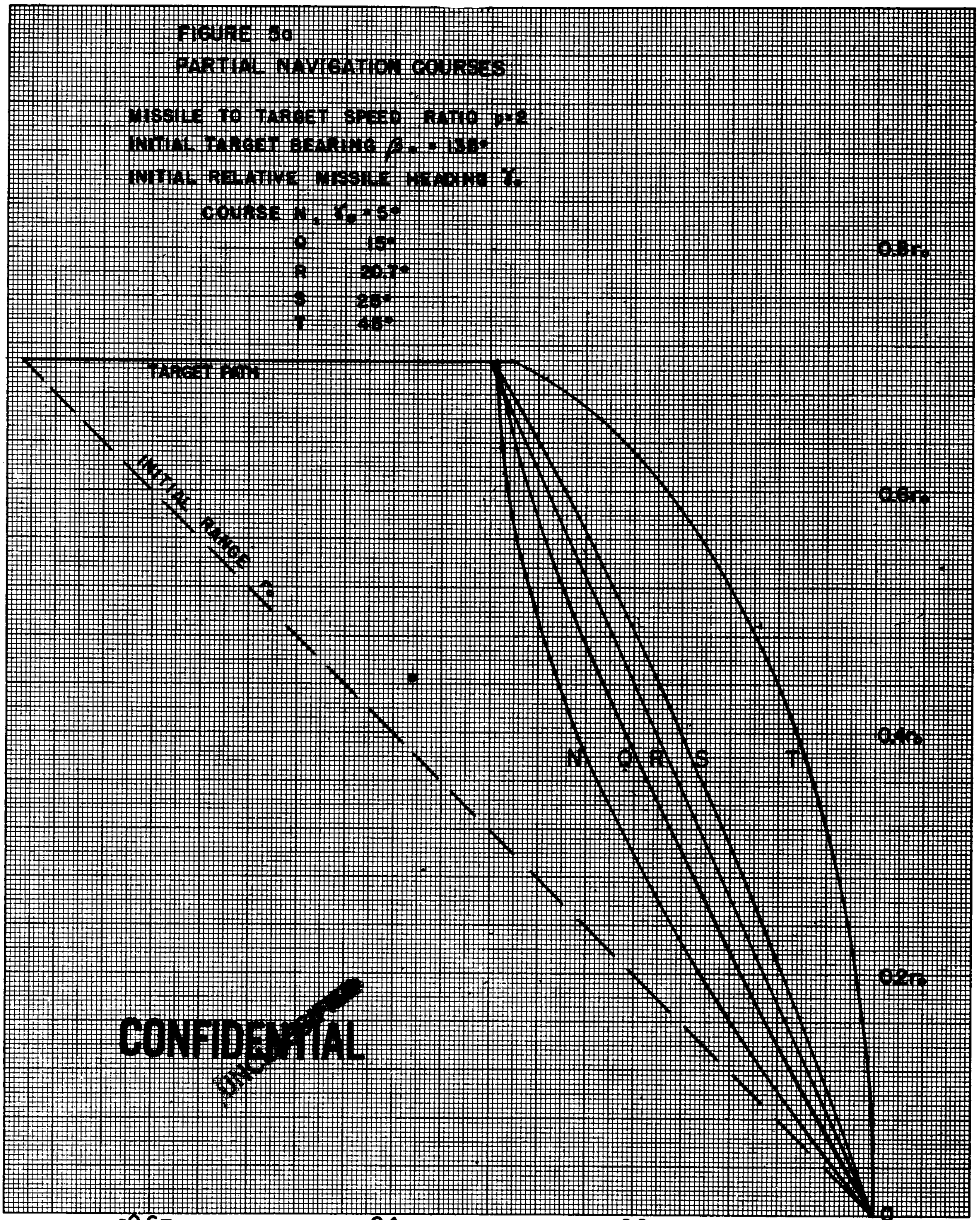
FIGURE 5a
PARTIAL NAVIGATION COURSES
MISSILE TO TARGET SPEED RATIO p=2
INITIAL TARGET BEARING β₀ = 135°
INITIAL RELATIVE MISSILE HEADING γ₀
COURSE N, γ₀ = 5°
Q 15°
R 20.7°
S 25°
T 45°
TARGET PATH
INITIAL RANGE r₀
N
Q
R
S
T
0.8r₀
0.6r₀
0.4r₀
0.2r₀
-0.6r₀
-0.4r₀
-0.2r₀
0
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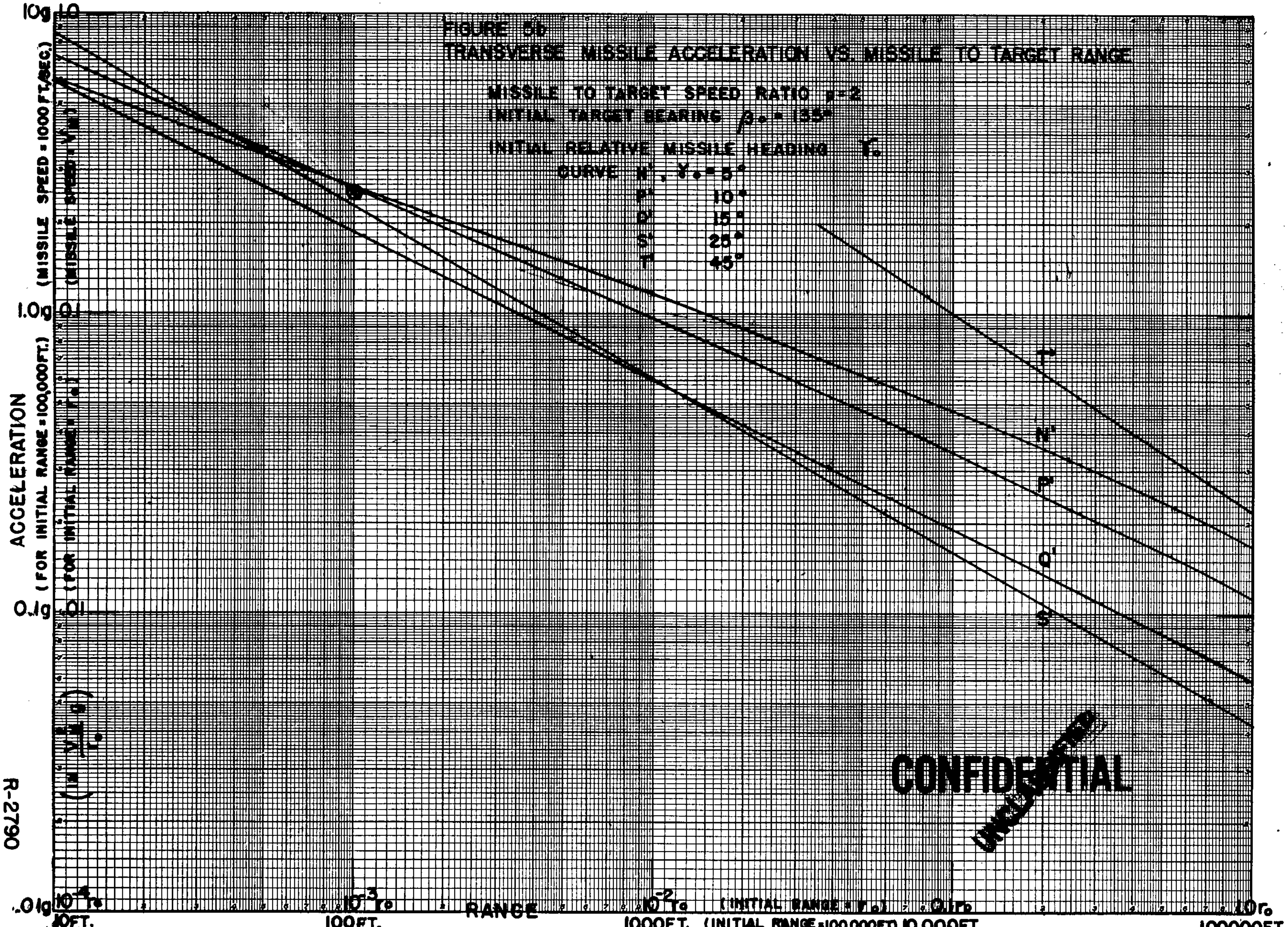

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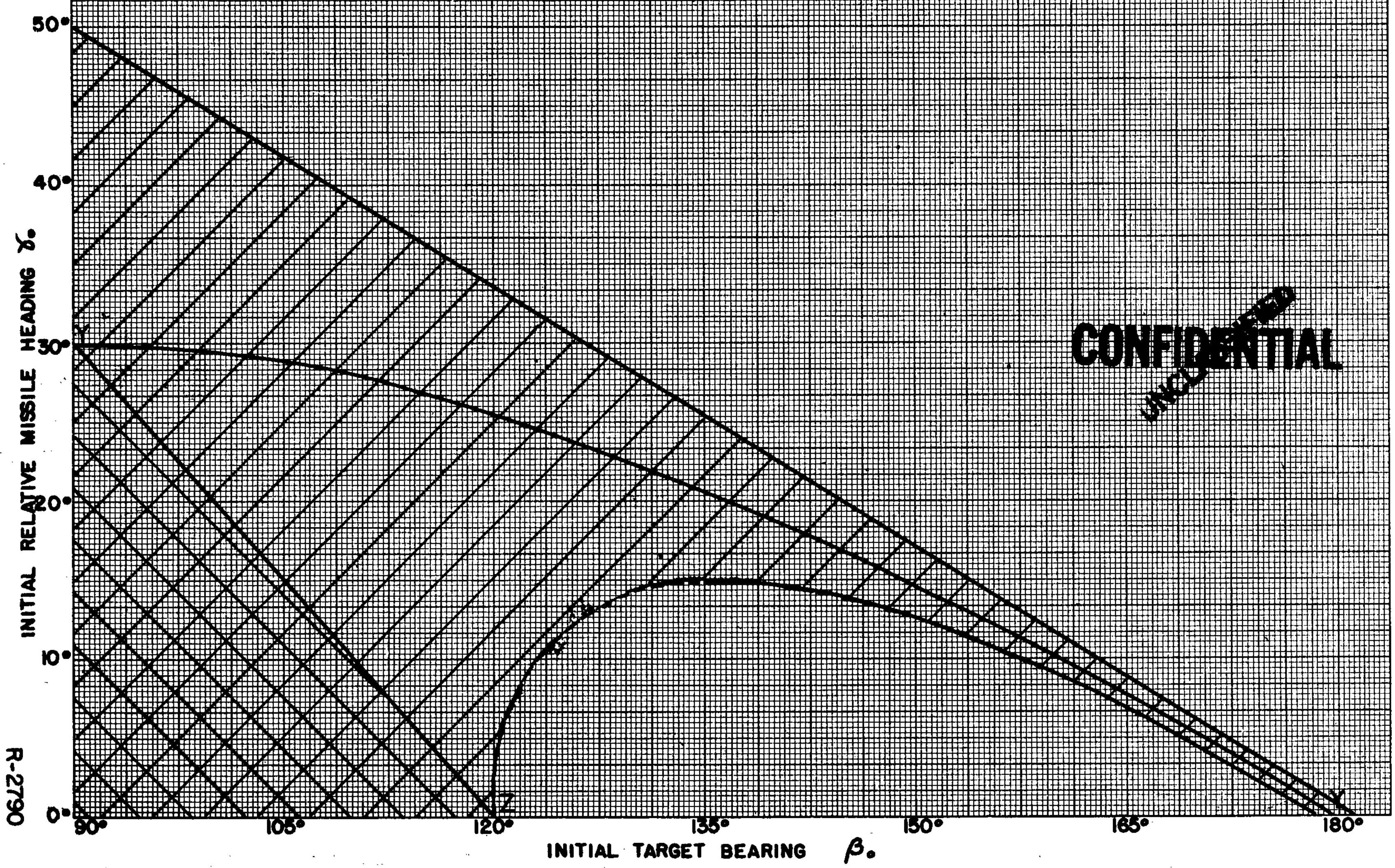
FIGURE 6.

INITIAL TARGET BEARING AND INITIAL RELATIVE MISSILE HEADING VALUES CORRESPONDING TO TRAJECTORIES ALONG WHICH MISSILE ACCELERATION DOES NOT EXCEED 5g UNTIL THE MISSILE IS WITHIN AT MOST 50 FEET OF THE TARGET. (c.f. PARAGRAPH 25.)

MISSILE SPEED = 1000 FT./SEC.
TARGET VELOCITY = 500 FT./SEC.
INITIAL RANGE = 50,000 FEET



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