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APPLICATION OF A WIENER-HOPF TECHNIQUE TO CERTAIN DIFFRACTION PROBLEMS

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CONTENTS

ABSTRACT

In applying the Wiener-Hopf technique to practical problems, the central difficulty is in properly factoring h(w), the Fourier transform of the kernel. This factorization is sometimes made practicable by replacing the variable w with a new variable, z, in such a manner that the form of h(w(z)) is factorable by 'inspection' in the z-plane. The Introduction contains a more general statement of this idea, and Section I, an application to the well-known problem of straight-edge diffraction.

In Section II, a rough-surface reflection problem is formulated. The surface consists in randomly spaced, parallel, conducting half-planes, with edges lying in a 'reflecting' plane on which is incident a plane wave with electric vector parallel to the edges. A self-consistent formulation of the problem of finding the current-distribution in a typical half-plane results in a Wiener-Hopf integral equation. The general method of Section I is applied, and a formal process for 'inspection' leads to factors having the suitable analytic properties.

Section III deals with some results valid when the half-planes are perpendicular to the reflecting plane. When the grazing angle, μ, is small, and when S, the average distance between edges, measured in wavelengths, is large, the approximate reflection coefficient R of the reflecting surface is given by

$$R = -\{\sqrt{1-2iz^2} - z + iz\}^2 \tag{A}$$

where $z^2 = \frac{1}{2}\pi\mu^2 S$ is a measure of the number of Fresnel zones entering a physical-optics calculation of the field illuminating the edge of a typical half-plane. A highly implausible alternate derivation of (A) is also presented, as well as a formula for back-scattered power density.

The problem considered in Section IV is that of diffraction by a conducting half-plane which lies in the plane interface between two different media. The present factorization method is applied to obtain, in principle, the factors required in solving this problem by the Wiener-Hopf method.

PROBLEM STATUS

This is a final report on one phase of the general problem of wave propagation over a rough surface; other work continues.

AUTHORIZATION

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INTRODUCTION

The now familiar Wiener-Hopf method has been found a powerful tool in the solution of certain diffraction problems. (See the paper of Carlson and Heins [1] for an example and a comprehensible discussion of the method.) Generally speaking, the Wiener-Hopf method is used to solve an integral equation of the Faltung type having the following structure:

$$\int_{-\infty}^{\infty} F(y)\, H(x-y)\, dy = \begin{cases} \Phi(x), & x > 0 \\ \Psi(x), & x < 0 \end{cases} . \tag{1}$$

Here F(x) vanishes in $x < 0$ and is unknown in $x > 0$, $\Phi(x)$ is known in $x > 0$ and vanishes in $x < 0$, $\Psi(x)$ is unknown in $x < 0$ and vanishes in $x > 0$, and H is known. Taking Fourier transforms of each side by multiplying by $\exp(-iwx)$ and integrating over $-\infty < x < \infty$, we get

$$2\pi f(w)\, h(w) = \varphi(w) + \psi(w). \tag{2}$$

Here the transforms f(w) and $\varphi(w)$ are R in L (regular, zero-free, and of limited growth in Im $(w) < 0$), $\psi(w)$ is R in U (regular, zero-free and of limited growth in Im$(w) > 0$), and h(w) generally has singularities in both half-planes, but (for simplicity) none on Im$(w) = 0$. Then one writes

$$h(w) = h_L(w)\, h_U(w) \tag{3}$$

where h_L is R in L, h_U is R in U. If the singularities of φ are simple poles in U then $\varphi(w)$ can be expanded in partial fractions as

$$\varphi(w) = \Sigma_i \frac{c_i}{w - p_i}\ ; \quad \text{Im}\,(p_i) > 0\ . \tag{4}$$

From (2) we readily obtain

$$2\pi f(w)\, h_L(w) - \Sigma_i \frac{c_i}{(w-p_i)\, h_U(p_i)} = \frac{\psi(w)}{h_U(w)} + \Sigma_i \frac{c_i}{w-p_i}\left[\frac{1}{h_U(w)} - \frac{1}{h_U(p_i)}\right]. \quad (5)$$

Here the left side is R in L, the right R in U. Now, if it is assumed or somehow proved that neither f(w) nor ψ(w) have singularities on Im(w) = 0, then the two sides have no common singularities, and there is a 'common strip of regularity' containing Im(w) = 0, in which neither side, or in fact no term on either side, has a singularity. The two sides are then equal in this strip, and therefore both represent the same function z(w). But z(w) has no singularities and grows too slowly with $|w| \to \infty$ to be other than a constant C. Thus both sides of (5) may be equated to C, giving two equations, the first containing f(w), the second ψ(w). One then evaluates C by a discussion of the asymptotic behavior of either of the two equations, or otherwise, and solves the two equations for the Fourier transforms f(w) and ψ(w). Taking inverse transformations then yields F and Ψ having the originally prescribed properties.

In applying this Wiener-Hopf method to a physical problem, we are entitled to dispense with analytic rigor in performing the various steps, provided that the final formulation of the solution is rigorously shown to obey the required physical conditions. In using the method to suggest the form of the answer to a physical problem, the only step that is not routine is the factorization of h(w) into $h_L h_U$. Given the 'common strip of regularity,' each factor can be expressed through a contour integral containing a logarithm in the integrand.[2] Generally speaking, these integrals seem difficult. Perhaps some study should be given to their asymptotic evaluation, which may be all that is required in certain problems. Certain forms of h(w) can be factored by inspection, or by development as an infinite product (as in Reference (1)).

The diffraction problems to follow are formally of the Wiener-Hopf type, but the factorization problems appear practically insoluble by any of the stated methods. An alternative procedure will be presented. One starts, conceptually, at the end of the problem, where the inverse transformation is being taken to obtain F(x). For this purpose, one multiplies f(w) by exp(ixw) and integrates along the real w-axis, in the common strip of regularity. One is entitled to transform the integration into a contour integral in a complex z-plane through the substitution w = w(z), dw = w'(z) dz. The form of w(z) can be chosen so that h(w(z)) has a convenient behavior as function of z. One can perhaps factor h in the z-plane, so that the z-equivalent of (3) is at hand; but then the problem is to show that the z-integral, equivalent to the inverse Fourier transformation, has the correct properties as function of x.

As w(z) is at least partly determined through the singularities of h(w), we tend to lose the common strip of regularity as a domain for analytic argument, and must use other properties to show that the results have the desired analytic or physical behavior. In the examples to follow, odd and even properties of various z-integrands will be used to guarantee the correct analytic behavior of the resulting expressions.

Without the common strip of regularity, there is further difficulty in proving analytically that, in the z-analog of Equation (5), the two sides may be equated to a constant. An alternative is presented by the physical origin of the subsequent problems. One sets the left side of the analog of (5) equal to some constant C, solves algebraically for f(w) and ψ(w), and shows that the resulting F(x) and Ψ(x) satisfy all the requirements set by the physics of the problem.

SECTION I - DIFFRACTION BY A HALF-PLANE

Now we formulate the first of three two-dimensional diffraction problems which can in principle be solved by the Wiener-Hopf method. This is the well-known problem of the perfectly conducting half-plane, introduced both in order to have a solved problem and as a limiting case of the two problems to follow. We consider a metal half-plane lying in the region $y = 0$, $x > 0$ of a rectangular coordinate system, and plane wave, $\varphi_0(x, y) = A \exp(ikx \cos\theta_0 - iky \sin\theta_0)$, $I_m(k) > 0$, incident on this half-plane from $x < 0$, $y > 0$. (The time factor $\exp(-iwt)$ is suppressed here and in the formulations to follow.) With the electric vector of this wave assumed parallel to the z-axis, i.e., to the edge of the half-plane, there is a current density $F(x)$ induced in the half-plane. Suppose the total field radiated by all currents is $\varphi_1(x, y)$. Then,

$$\varphi_1(x, y) = \int_0^\infty F(x') H_0^{(1)} \left[k \sqrt{(x - x')^2 + y^2} \right] dx' . \tag{1.1}$$

To satisfy the boundary conditions, we must have $\varphi_0(x, 0) + \varphi_1(x, 0) = 0$ in $x > 0$. Thus, with $F(x) = 0$ in $x < 0$,

$$\int_{-\infty}^\infty F(x') H_0^{(1)} (k|x - x'|) dx' = \begin{cases} -A e^{ikx\cos\theta_0} \equiv \Phi(x), & x > 0 \\ \Psi(x), & x < 0 \end{cases} . \tag{1.2}$$

It is well-known that

$$H_0^{(1)} (k\sqrt{x^2 + y^2}) = K \int_{-\infty}^\infty \frac{e^{iwx} e^{i\sqrt{k^2 - w^2}\,|y|}}{\sqrt{k^2 - w^2}} dw \tag{1.3}$$

and that, with $\Phi(x) = 0$ in $x < 0$

$$\Phi(x) = \frac{iA}{2\pi} \int_{-\infty}^\infty \frac{e^{iwx}}{w - \mu} dw \tag{1.4}$$

where $\mu = k \cos\theta_0$, and the constant K is nonessential. Letting

$$\Psi(x) = \int_{-\infty}^\infty e^{iwx} \psi(w) dw \tag{1.5}$$

$$F(x) = \int_{-\infty}^\infty e^{iwx} f(w) dw \tag{1.6}$$

we have from (1.2) and (1.4) an algebraic relation among the Fourier transforms,

$$\frac{2\pi K f(w)}{\sqrt{k^2 - w^2}} = \frac{iA}{2\pi(w - \mu)} + \psi(w) . \tag{1.7}$$

With $\mathrm{Im}(k) > 0$, there are no singularities on the real w-axis, and the previously described procedure leads to the correct solution without difficulty, since $h(w) = (k^2 - w^2)^{-1/2} \equiv (k - w)^{-1/2} (k + w)^{-1/2} = h_L h_U$ and $h(w)$ is split into two factors with the required analytic properties. We thus obtain the equivalent of Equation (5),

$$\frac{2\pi K f(w)}{\sqrt{k - w}} - \frac{iA\sqrt{k + \mu}}{2\pi(w - \mu)} = \sqrt{k + w}\ \psi(w) + \frac{iA}{2\pi}\left[\frac{\sqrt{k + w} - \sqrt{k + \mu}}{w - \mu}\right]. \tag{1.8}$$

For $f(w)$, the result of equating each side of (5) to C is

$$2\pi K f(w) = \frac{iA}{2\pi}\ \frac{\sqrt{k + \mu}\ \sqrt{k - w}}{w - \mu} + C\ \sqrt{k - w}\ . \tag{1.9}$$

When this $f(w)$ is substituted into (1.6), the integral vanishes for $x < 0$. Here the contour of integration can then be removed to infinity across the $\mathrm{Im}(w) < 0$ half-plane, where the integrand has no singularities; for $x < 0$, the integrand vanishes with $\mathrm{Im}(w) \to -\infty$, and $F(x) = 0$, $x < 0$ follows.

On the other hand, when $x > 0$, the integrand increases exponentially with $\mathrm{Im}(w) \to -\infty$, but the integral can be evaluated as a residue and a branch-line integral in $\mathrm{Im}(w) > 0$ (in U).

With $C \neq 0$ the C-dependent term of $F(x)$ is proportional to

$$C e^{ikx} \int_0^{\infty} e^{-ux} \sqrt{u}\ du = C e^{ikx}\ \Gamma(3/2)\ x^{-3/2}.$$

The similar term in $\Psi(x)$ is

$$C e^{ikx} \int_0^{\infty} e^{-u|x|} \frac{du}{\sqrt{u}} = C e^{ikx}\ \Gamma(1/2)\ x^{-1/2}\ .$$

Thus $\Psi(x)$, the scattered field, is infinite at the edge of the plane when $C \neq 0$, a physically unacceptable conclusion. Secondly, if $C \neq 0$ and $A = 0$, we have a source-free solution of the problem, with the total field (calculated from (1.1)) properly "outgoing" and vanishing on the sides of the plate but not vanishing for $y = 0$, $x \nearrow 0-$. This standing wave is not excited by φ_0, and would soon radiate away if present at any time. Hence we may take $C = 0$ in (1.8) and (1.9).

Now let us examine the case $\mathrm{Im}(k) \searrow 0$. Here the contour in the w-plane is deformed according to the scheme of Figures 1a, 1b, into portions of the real axis connected by three semicircular arcs, one lying in U and centered on $w = -k$, two lying in L and centered on $w = k$ and $w = \mu$. This contour can be used in both (1.5) and (1.6). The singularities are now <u>on</u> $\mathrm{Im}(w) = 0$: a pole of φ at $w = \mu$ and branch points of h at $w = \pm k$. Here we have no 'common strip of regularity,' but continuations of the solutions F, Ψ, already obtained from an argument using a 'common strip,' must be solutions over the new contour in the limit $\mathrm{Im}(k) \searrow 0$.

For guidance in what is to follow, let us assume that we have obtained the transform relation (1.7) and the deformed contour applying to $\mathrm{Im}(k) = 0$, but do not know how to factor $h(w) = (k^2 - w^2)^{-1/2}$. If we let $w = k \sin\theta$, all of the known functions of w in (1.7) transform into periodic functions of θ for which the only singularities in the finite θ-plane are simple poles. Thus the transformation

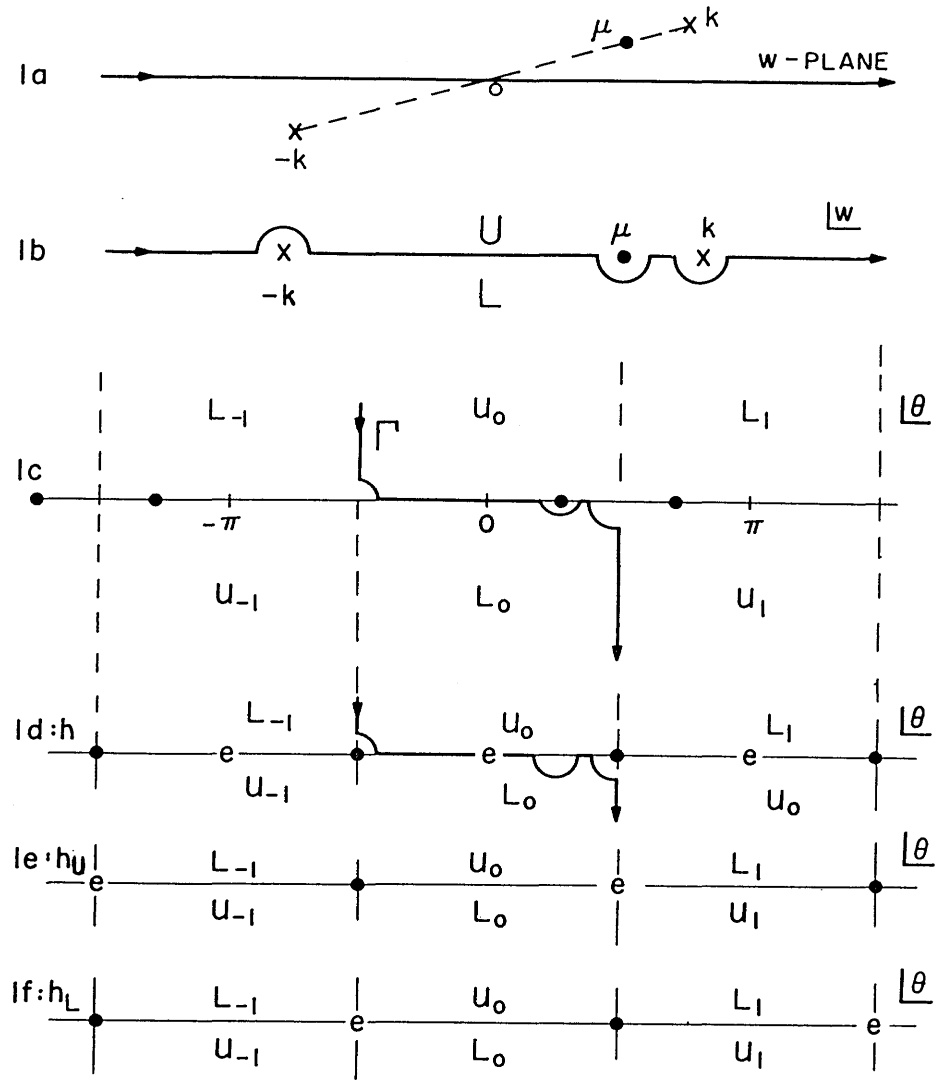

Figure 1 - Contours and singularities for ~~sing-6~~ SINGLE-plate diffraction

achieves an apparent simplification in removing branch-points. In making the inverse Fourier transformation to find F and ψ, we would now multiply 'solutions' of the factorization problem in the θ-plane by $\exp(ikx \sin \theta) \cos \theta \, d\theta$ and integrate, in the θ-plane, over the contour Γ which corresponds to the deformed contour in the w-plane. The contour and the location of the known poles (•) of $\varphi(k \sin \theta)$ are shown in Figure 1c. Here the regions in the θ-plane for which $\text{Im}(w) = k\text{Im}(\sin \theta) > 0$ are marked U, and the regions for which $\text{Im}(w) = k\text{Im}(\sin \theta) < 0$ are marked L.

Momentarily overlooking the singularities marked in Figure 1c, let us find some θ-analog of the following statement: if g(w) is R in U, and has no singularities on $\text{Im}(w) = 0$, then $\int_{-\infty}^{\infty} \exp(iwx)\, g(w)\, dw = 0$, $x > 0$. The conclusion follows because the integration contour can be removed across U, and the exponential factor vanishes in the process. In the θ-plane, the integral becomes $k \int_\Gamma \exp(ikx \sin \theta)\, g(k \sin \theta) \cos \theta \, d\theta$. When $x > 0$, Γ can be deformed across U_0 into $\theta = \frac{1}{2}\pi + it$, t real. As function of t on the new contour, $\sin \theta$ is even, $\cos \theta$ is odd, and $d\theta$ is even. Thus the integral vanishes when g = constant, as the integrand is an odd function of t. If g(w) is R in U and on the real w-axis, then $g(k \sin \theta)$ is even in t. But the integral (assumed to converge) vanishes if g has more general properties, i.e., if $g(k \sin \theta)$ is R in $-\frac{1}{2}\pi < \mathrm{Re}(\theta) \leq \frac{1}{2}\pi$, $\mathrm{Im}(\theta) > 0$, and is an <u>even</u> function of t on $\theta = \frac{1}{2}\pi + it$ (here we say that g is R in U_0). Thus the properties of g in other U-regions of the θ-plane are not important, even though $g(k \sin \theta)$ may not be R except in U_0 (and therefore in U_1, by the even property). Of course, the same argument can be carried through in the w-plane if the original contour is thought of as lying on a certain sheet of a Riemann surface, g(w) is R in U on this sheet, and the contour avoids the singularities of g on $\mathrm{Im}(w) = 0$ by semicircles lying in U. By passing to the θ-plane we unfold the Riemann surface and perhaps have a clearer idea of the structure of the integrands than if we tried to argue from a Riemann surface in the w-plane.

In applying this analysis to (1.7), we start from

$$\frac{2\pi K f(k \sin \theta)}{k \cos \theta} = \frac{iA}{2\pi} \frac{1}{k \sin \theta - \mu} + \psi(k \sin \theta) . \tag{1.10}$$

Here the denominator $k \cos \theta$ is an odd function about both $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$; by the foregoing argument, $\psi(k \sin \theta)$ must be even about $\theta = -\frac{1}{2}\pi$. The known function $\varphi(k \sin \theta)$ is even about both $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The factorization problem becomes one of splitting $h = \cos \theta^{-1}$ into two factors, one even about $-\frac{1}{2}\pi$, the other even about $\frac{1}{2}\pi$. The structure of the product $\cos \theta^{-1}$ is indicated in Figure 1d, where the symbol e marks 'even points' of $\cos \theta^{-1}$, about which $\cos \theta$ is even and where the θ-derivative vanishes, and (●) indicates a pole. The structure of the two factors is indicated with the same symbolism in Figures 1e, 1f. One can obviously choose periodic factors with period 4π, whereas $\cos \theta$ has period 2π. Factors of $(k \cos \theta)^{-1}$ with the requisite zeros, even points, and periodicity are

$$h_L = \left[\sqrt{2k} \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]^{-1} \quad h_U = \left[\sqrt{2k} \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]^{-1} . \tag{1.11}$$

The w-equivalents are respectively $(k - w)^{-\frac{1}{2}}$, $(k + w)^{-\frac{1}{2}}$. Carrying out the separation of (1.10) into two equations we get:

$$\frac{2\pi K f(k \sin \theta)}{\sqrt{2k} \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} - \frac{iA \sqrt{2k} \sin\left[\frac{\pi}{4} + \frac{1}{2} \sin^{-1}\left(\frac{\mu}{k}\right)\right]}{2\pi(k \sin \theta - \mu)}$$

$$= \psi(k \sin \theta) \sqrt{2k} \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \frac{iA\sqrt{2k}\left\{\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \sin\left[\frac{\pi}{4} + \frac{1}{2}\sin^{-1}\frac{\mu}{k}\right]\right\}}{2\pi(k \sin \theta - \mu)} . \tag{1.12}$$

Here the heuristic argument is that the two sides are representations of the same function of period 4π, and since the regions of regularity of the two sides overlap in the period, the function must be a constant. By a version of the previous argument, this constant must be zero, so that each side of

(1.12) can be equated to zero. The two resulting equations can then be solved for $f(k \sin \theta)$, $\psi(k \sin \theta)$. For f, the result is

$$2\pi K f(k \sin \theta) = \frac{iAk \sin\left[\frac{\pi}{4} + \frac{1}{2} \sin^{-1}\left(\frac{\mu}{k}\right)\right] \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\pi(k \sin \theta - \mu)}. \tag{1.13}$$

The constant-determining argument is based on the rate of growth of the part of ψ which is <u>odd</u> about $-\frac{1}{2}\pi$. We assume that $\Psi(x)$ must be bounded in the limit $x \nearrow 0$. The part of ψ which is even about $-\frac{1}{2}\pi$ produces no contribution to Ψ, and the behavior of Ψ is determined primarily by the behavior of the odd part of ψ near $t = \pm\infty$ on the vertical line $\theta = it - \frac{1}{2}\pi$. Here the odd part must vanish with $t \to \infty$ at such a rate that $\lim_{a \searrow 0} \int_0^\infty e^{-a \cancel{\sin k}\, \sinh t}\, \psi(\theta)\, dt$ exists. This is seen to be true (or at least possible) only if the constant, to which the right side of (1.12) is equated, is zero. The argument is analytically slipshod, and we should show that we have been led to a correct solution of the physical problem by examining the resulting electromagnetic field. This verification is straightforward, and will not be reproduced here.

For this half-plane problem, Sommerfeld's original solution[3] was a contour integral in a similar θ-plane. A good reason for the fact that the present contour cannot be deformed into that used by Sommerfeld is that he started with an exponent $(ixk \sin \theta_0 + iyk \sin \theta_0)$, whereas the corresponding exponent in (1.3) is not analytic in y at $y = 0$.

SECTION II - DIFFRACTION BY RANDOMLY SPACED, PARALLEL HALF-PLANES

We consider diffraction over the 'rough surface' shown in Figure 2. The surface consists in a randomly spaced array of parallel, perfectly conducting half-planes (plates) lying in $x > 0$, the edges lying in $x = 0$, parallel to the z-axis. The plates make an angle β with the positive y-axis. Let the plane wave

$$w_0 = \exp\left(ipy + ix\sqrt{k^2 - p^2}\right) \tag{2.1}$$

be incident on the array. Let u measure distance from the edge of a plate, and assume that the induced current-density in the plate with edge at y is given by $\exp(ipy)I(u)$. That is, the same current is induced in each plate except for the phase factor, $\exp(ipy)$, determined by the location of the plate's edge on the y-axis. The currents between u and $u + du$ in all strips radiate a field $W(u, du)$ which, at least on the average, is expressible as an integral,

$$W(u, du) = N I(w) du \int_{-\infty}^{\infty} dy'\, e^{ipy'}\, H_0^{(1)}(k\rho)$$

$$\rho = \left[(x - u \sin \beta)^2 + (y - y' - u \cos \beta)^2\right]^{\frac{1}{2}} \tag{2.2}$$

where N is the average number of plates per unit length of y-axis. Now, with

$$H_0^{(1)}\left(k\sqrt{x^2+y^2}\right) = K\int_{-\infty}^{\infty} \frac{e^{i\alpha x}\, e^{i\sqrt{k^2-\alpha^2}\,|y|}}{\sqrt{k^2-\alpha^2}}\, d\alpha \qquad (2.3)$$

where K is a nonessential constant, and with $I(u) = \int_{-\infty}^{\infty} f(w)\, e^{iwu}\, dw$ we have

$$w(u, d\dot{u}) = du\, 2\pi NK \int_{-\infty}^{\infty} \frac{f(w)\, dw\, e^{iwu}\, e^{ip(y - u\cos\beta)}\, e^{i\sqrt{k^2-p^2}\,|x - u\sin\beta|}}{(k^2-p^2)^{1/2}}\,. \qquad (2.4)$$

If we take $x < 0$ and integrate this expression over the length of the plates, we get the average reflected field $R\exp(ipy - i\sqrt{k^2-p^2}\,x)$, where R is the effective specular reflection coefficient of the 'rough surface,' regarded as lying in the $x = 0$ plane:

$$R = 4\pi^2 NKf(p\cos\beta - \sqrt{k^2-p^2}\,\sin\beta)\,(k^2-p^2)^{-1/2}. \qquad (2.5)$$

Thus the reflection coefficient R is directly obtained through the Fourier transform $f(w)$ of the current density, and the current itself becomes of secondary interest.

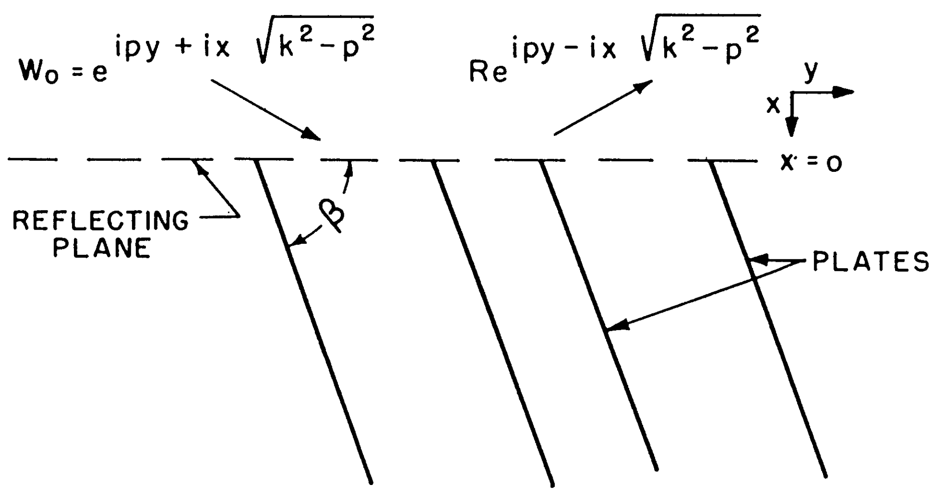

Figure 2 - Parallel-plate surface

Hence we are directly interested in finding $f(w)$. Toward this end, we need to know the average field W_1 scattered to a point u in a plate with edge at y. Here we assume that W_1 is independent of the presence of the particular plate, and is the average field found at the particular point in the absence of the plate:

$$W_1(u)\,e^{ipy} = \int_{-\infty}^{\infty} \frac{2\pi NKf(w)\,e^{ipy}\,e^{iwu}\,2i\sin\beta\,dw}{k^2\sin^2\beta - p^2 + 2pw\cos\beta - w^2}\,. \tag{2.6}$$

Finally, we assume that the currents in this plate radiate fields which cancel the total incident field $W_0 + W_1$ on the surface of the plate. Using the fact that, in $x > 0$

$$W_0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i(wu + py)}\,dw}{w - (p\cos\beta + \sqrt{k^2 - p^2}\,\sin\beta)} \tag{2.7}$$

we have an algebraic relation among the Fourier transforms:

$$2\pi K f(w)\left[\frac{1}{\sqrt{k^2 - w^2}} + \frac{2Ni}{k^2\sin^2\beta - p^2 + 2pw\cos\beta - w^2}\right] -$$

$$\frac{i}{2\pi(w - p\cos\beta - \sqrt{k^2 - p^2}\,\sin\beta)}\,, \equiv 2\pi K f(w)h(w) + \varphi(w) = \psi(w) \tag{2.8}$$

where $\psi(w)$ is the transform of the unknown field scattered by the plate along its continuation into $u < 0$. (We set $N = 0$ to verify that Equation (2.8) becomes essentially the same as Equation (1.7) of Section I.)

Now $f(w)$ must be R in L_0 as there are no currents in $u < 0$, and $\psi(w)$ is similarly R in U_0. To find f and ψ with these analytic behaviors, the problem is to factor $h(w)$, the coefficient of $f(w)$.

By combining terms in h, one sees that to find the roots of the resulting numerator requires the solution of a fairly general quartic equation in w, unless $\beta = \frac{1}{2}\pi$. We continue with the detailed discussion of the latter case only.

If $N > 0$, $h(w)$ gains two additional poles, owing to the denominator of the second term, and <u>four</u> additional zeros, as one sees by combining the terms and regarding both signs of $(k^2 - w^2)^{1/2}$ as possible. Of course, two of the added zeros of h lie on the 'wrong' leaf of the two-sheeted Riemann surface required by the presence of the radical in the first term of $h(w)$. All zeros must be accounted for in a factorization.

To locate zeros and poles of h, we first find the poles due to the second term, namely, those at $w = \pm(k^2 - p^2)^{½}$. Taking $p = k \cos \mu (\mu > 0)$, say, as befits the plane incident wave, we see that these poles lie symmetrically with respect to the origin on the line segment connecting $w = k$ and $w = -k$. Thus the contour (the real w-axis) passes below the pole on the right, above that on the left, as in Figure 3a. Now let $\mathrm{Im}(k) \searrow 0$, and let $w = k \sin \theta$, as in Section I. In the θ-plane the contour, Γ, avoids the poles (●) as shown in Figure 3b. The form of h is now (with $Q = N/k$)

$$h = \frac{\cos^2 \theta + 2iQ \cos \theta - \cos^2 \mu}{k \cos \theta (\cos \theta - \cos \mu)(\cos \theta + \cos \mu)} . \tag{2.9}$$

The zeros of h are found through

$$\cos \theta = -iQ \pm \sqrt{\cos^2 \mu - Q^2} . \tag{2.10}$$

(These zeros are indicated by (0) on Figure 3b.) For small Q, the zeros lie near the poles of the original second term of h. In the factorization, each zero must be contained in the same factor as the nearby pole, so that when $N \searrow 0$, the zeros move toward the poles and cancel them in each factor, and the factorization of Section I results. By examining (2.10) one verifies this behavior.

(The argument for $\beta \neq ½\pi$ is more complicated. It will, however, be assumed that the discussion applies to this case, i.e, that the zeros and poles of h, migrating in the θ-plane under change of N and β, always lie with respect to the contour as shown symbolically in Figure 3b. This cannot be proved, apparently, without a formal discussion. Here one determines signs of such forms as $(N^2 - p^2)^{½}$ by the fact that $N > 0$ and $\mathrm{Im}(p) \searrow 0$.)

Now we factor h into two factors, $h = h_U h_L$, where h_U is R in U_0 and even about $½\pi$, and h_L is R in L_0 and even about $-½\pi$. (The expressions R in U_0 and R in L_0 will be understood henceforth to include the foregoing evenness propeıties.) First (in Figure 3c) we label the poles on the real θ-axis, those with the notation U belonging in h_U, and those with the symbol L belonging with h_L. This can be done almost without thought. Then we label the zero in U_0 with the symbol L, since this zero cannot belong to h_U. Similarly, the zero in L_0 is labeled U. Now there is no zero in U_{-1} symmetric at the image point (with respect to $-½\pi$) of the zero in U_0; we place a zero (0) here, and label it L, since such a zero is required in h_L. Since h lacks this zero in U_{-1}, we cancel it with a pole (∞), which must belong to h_U, for if this pole is in h_L, a similar pole must cancel the the zero in U_0. (The dashed arrow represents the argument connecting the new zero and pole with the zero in U_0, and is a first step in a 'zig-zag' argument specifying the poles and zeros required by the presence of the zero in U_0.) If the new pole in U_{-1} belongs to h_U, a similar pole must be found at a point symmetric with respect to $½\pi$, i.e., in U_2. Placing a pole of h_U here, we find a new zero of h_L required to cancel it, etc. (Thus there will be a <u>double</u> zero of h_L at the point at the end of the second dashed arrow, since there was already a zero of h at this point.)

In Figure 3c the zero of h in L_1 must belong to h_L, since otherwise h would require a new zero in L_0; similarly the zero of h in U_{-1} belongs to h_U. The zeros in L_2 and U_{-2} are assigned in the same way, except that one step in a zig-zag argument is required. We then carry out the zig-zag arguments, starting from each of the above zeros. The resulting structure of h_L is shown in Figure 3d; rotating the structure 180^0 about the origin gives the structure of h_U (Figure 3e). In fact, we may set

$$h_U(\theta) = h_L(-\theta) \tag{2.11}$$

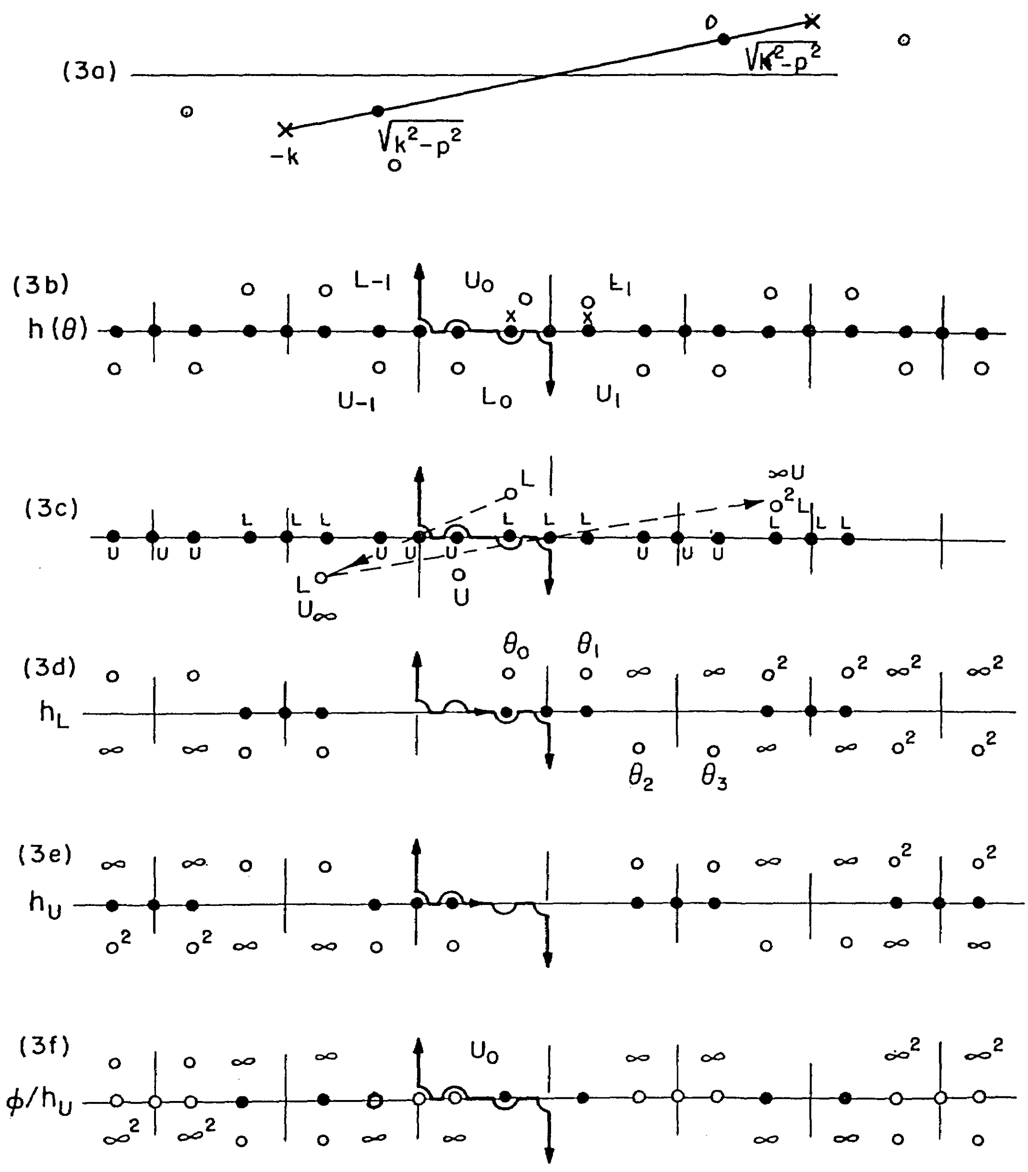

Figure 3 - Contours and singularities for parallel-plate diffraction

after making proper choice of constant factors (or, more generally, exponential factors). One readily verifies that the product of h_U and h_L is h (assuming correct 'constant' factors) and that the behavior of each is correct when $N \searrow 0$.

From (2.8) we now have, with $w = k \sin \theta$,

$$2\pi K f h_U h_L - \frac{i}{2\pi k(\sin \theta - \sin \mu)} \equiv 2 K f h_U h_L + \varphi = \psi . \qquad (2.12)$$

The next step is to divide through by h_U, which results in

$$2\pi K f h_L + \frac{\varphi}{h_U} = \frac{\psi}{h_U} \quad . \tag{2.13}$$

The first term on the left is R in L_0, the term on the right is R in U_0, but the second term is of a mixed character. Its structure is shown in Figure 3f.

We note that φ is even about both $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; φ is R in L_0 but not in U_0, having a pole at $\theta = \mu$ on the U_0 side of the integration contour. The form φ/h_U is even about $\frac{1}{2}\pi$, but retains the pole at $\theta = \mu$, and hence is not R in U_0. We now write

$$\frac{\varphi(\theta)}{h_U(\theta)} = \varphi(\theta)\left[\frac{1}{h_U(\theta)} - \frac{1}{h_U(\mu)}\right] + \frac{\varphi(\theta)}{h_U(\mu)} \quad . \tag{2.14}$$

Here the first term on the right is even about $\frac{1}{2}\pi$ and contains no pole in U_0, and hence is R in U_0 (zeros in U_0 may now be disregarded). The second term is R in L_0, since it differs from φ by a constant factor. We can therefore set

$$2\pi K f h_L(\theta) - \frac{\varphi}{h_U(\mu)} = \frac{\psi}{h_U(\theta)} + \varphi\left[\frac{1}{h_U(\theta)} - \frac{1}{h_U(\mu)}\right] \quad . \tag{2.15}$$

The left side being R in L_0, the right R in U_0, the function represented by each side is even about both $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, and hence has period 2π, and has no singularities in U_0, U_1, L_0, and L_{-1}, or on the boundary, Γ, between these regions. Thus the periodic function represented has no finite singularities in one full period, and therefore has no singularities in the finite θ-plane. Finally, if the function grows sufficiently slowly with $\mathrm{Im}(\theta)$, the function can be only a constant. As to these rates of growth, one sees from Figure 3e that poles and zeros of h_U may be paired in such a way that one pole is left over at each of the points $\pi(2n - \frac{1}{2})$. Therefore $1/h_U$ must grow like $\cos(\theta/2)$ (the 'exponential factor' being a constant). The same growth holds for $1/h_L$ by virtue of (2.11); hence $1/h$ grows like $\cos\theta$, as may be verified by inspection of (2.9). Since φ vanishes for $\mathrm{Im}(\theta) \to \pm\infty$, the rate of growth of the right side of (2.15) is 'sufficiently slow' if ψ grows more slowly than $\cos(\theta/2)$. The ψ of (2.16) will be seen acceptable in this regard.

Again we determine the constant (represented by both sides of (2.15)) by appeal to the physics, concluding that the constant must be such that the rate of growth with $t > 0$ of $\psi(\frac{1}{2}\pi + it) - \psi(\frac{1}{2}\pi - it)$ is minimal. The part of $\psi(-\frac{1}{2}\pi + it)$ which is even in t produces no contribution to the part of the scattered field Ψ found from ψ by integration over $-\infty < t < \infty$. Owing to its poles at $-\frac{1}{2}\pi + 2n\pi$, h_U is essentially odd as function of large $t > 0$; the remaining poles and zeros pair up to produce an asymptotically even behavior as function of t. Equating the right side of (2.15) to a constant K'' and solving for ψ, we have, using the fact that φ is even in t:

$$\text{Part of } \psi \text{ which is odd in t} \simeq \frac{\varphi(\theta)\, h_U(\theta)}{h_U(\mu)} + K'' h_U(\theta) \quad .$$

Since $\varphi \to 0$ with $t \to \infty$, we obtain the minimal rate of growth by setting $K'' = 0$. We therefore have

$$2\pi K f(k \sin \theta) = \frac{\varphi(\theta)}{h_U(\mu) h_L(\theta)}, \quad \psi = \varphi \left[\frac{h_U(\theta)}{h_U(\mu)} - 1 \right]. \tag{2.16}$$

The poles of φ and h_L at $\theta = \mu$ cancel, and the zero of h_L in the interior of U_0 becomes the pole of f which determines the magnitude of the currents at large x. When $N \searrow 0$, this zero migrates to $\theta = \mu$, and the current distribution in the single plate of Section I is achieved in the limit.

With φ given in (2.12), we can say that the problem is formally solved by substituting an analytic expression for the h_U, h_L of Figures 3d and 3e in (2.16). Toward this end, we can write (see Figure 3b)

$$h_L(\theta) = \frac{c}{\sin\left[\frac{\theta - \mu}{2}\right] \sin\left[\frac{\theta - (\pi - \mu)}{2}\right] \sin\left[\frac{\theta - \frac{1}{2}\pi}{2}\right]} \cdot$$

$$\prod_{n=0}^{\infty} \left\{ \frac{\Gamma\left[\frac{\theta_0 - \theta}{2\pi} + n + \frac{1}{2}\right] \Gamma\left[\frac{\theta_1 - \theta}{2\pi} + n + \frac{1}{2}\right] \Gamma\left[\frac{\theta_2 - \theta}{2\pi} + n + \frac{1}{2}\right] \Gamma\left[\frac{\theta_3 - \theta}{2\pi} + n + \frac{1}{2}\right]}{\Gamma\left[\frac{\theta_0 - \theta}{2\pi} + n\right] \Gamma\left[\frac{\theta_1 - \theta}{2\pi} + n\right] \Gamma\left[\frac{\theta_2 - \theta}{2\pi} + n\right] \Gamma\left[\frac{\theta_3 - \theta}{2\pi} + n\right]} \cdot \frac{\Gamma\left[\frac{\theta_0 + \theta}{2\pi} + n + 1\right] \Gamma\left[\frac{\theta_1 + \theta}{2\pi} + n + 1\right] \Gamma\left[\frac{\theta_2 + \theta}{2\pi} + n + 1\right] \Gamma\left[\frac{\theta_3 + \theta}{2\pi} + n + 1\right]}{\Gamma\left[\frac{\theta_0 + \theta}{2\pi} + n + \frac{1}{2}\right] \Gamma\left[\frac{\theta_1 + \theta}{2\pi} + n + \frac{1}{2}\right] \Gamma\left[\frac{\theta_2 + \theta}{2\pi} + n + \frac{1}{2}\right] \Gamma\left[\frac{\theta_3 + \theta}{2\pi} + n + \frac{1}{2}\right]} \right\}$$

The convergence of the infinite product is required only in the strip $-\pi/2 \leq \mathrm{Re}(\theta) \leq -\frac{1}{2}\pi$; we do not explore this question, but it seems apparent that rapid convergence can be achieved by properly grouping factors after removing certain of the gamma functions with low n-indices from the product. That is, after removal (from the denominator) of the gamma functions containing the zeros of h_L in U_0, U_{-1}, L_1, and L_{-2}, the remaining gamma functions may be grouped four at a time in a manner producing factors rapidly approaching unity in the strip of interest.

To evaluate the remaining constant c, we set $h_U(\theta) = h_L(-\theta)$, equate the product $h_U h_L$ to the h of (2.9), and find, using the identities

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad \sin\left[\frac{x+y}{2}\right] \sin\left[\frac{x-y}{2}\right] = \frac{\cos y - \cos x}{2},$$

that $c^2 = 2\pi^4/k$.

Through the logarithmic integrals, Bazer and Karp[3] were able to factor an expression having many of the difficulties presented by the h(w) of (2.8) in the present case $\beta = \frac{1}{2}\pi$. Their factorization may possibly be adapted to the present case, but the case $\beta \neq \frac{1}{2}\pi$ would probably present additional difficulties.

We conclude with some remarks about the uniqueness of the factorization. The zig-zag argument seems to lead to unique factors insofar as the zeros and poles are concerned. For if we introduced a zero of h_L at some new point in the θ-plane, there would have to be a compensating pole in h_U. Furthermore, there would be a zero and a pole at points symmetric with respect to $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, respectively, and these new singularities would require compensating zeros and poles, etc. Eventually a pole or zero would have to be located in the strip $-\frac{1}{2}\pi \leq \mathrm{Re}(\theta) \leq \frac{1}{2}\pi$. Such a pole or zero, not being found in h, would have to be compensated in a manner resulting in (say) a pole in h_U and a zero in h_L at the same point in this strip. Thus the factors could not have the desired analytic properties in U_0 or L_0, and the introduction of new poles or zeros is impossible. We then have the option of multiplying the present h_U by $e^{q(\theta)}$ and h_L by $e^{-q(\theta)}$. Here q must be an entire function and even about both $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; q is therefore periodic with period 2π and must be of the form $q'(\sin\theta)$, where q' is some entire function. If q' is not identically a constant, $q'(\sin\theta)$ grows at least as fast as $q_0 \ |\sin\theta|$, for some finite constant q_0 along some vertical line in the foregoing strip. Here the location of the contour integrals would be determined by q_0 rather than x when $|kx| < q_0$; this is physically inacceptable and we must have $q' =$ constant. Such a constant would not appear explicitly in the present results, and so it may be taken as zero.

SECTION III - SOME QUANTITATIVE RESULTS

This section is devoted to the consideration of some of the reflecting properties of the parallel-plate medium of Section II (Figure 2). The plane $x = 0$ may be regarded as a rough surface, a surface which can be described only statistically. The complete statistical description of the surface is contained in the following statement. Let L be the distance (along the y-axis) between the edges of consecutive plates. The probability that L lies between L_1 and $L_1 + dL$ is given by $\exp(L_1/S)\,dL$, where the average separation S is connected with the N of Section II through $S = 1/N$. The fact that only $Q = N/k$ (or, equivalently, S) appears in the results of Section II might suggest that some of the statistics of the surface have been ignored in the self-consistent formulation of the scattering problem. On the other hand, S (or N) is the only arbitrary quantity in the statistics of the surface, and any average or statistical description of the scattering properties of the surface should be essentially in terms of S.

The parallel plates of Reference 1 are regularly spaced with a common spacing which we may call S_0. The scattering properties of the 'surface' of Reference 1 depend on S_0 in somewhat the same way that the present properties depend on S; differences in the functional dependencies on S_0 and on S would shed light on the extent to which the randomness of the present surface actually affects the deduced scattering. Unfortunately this comparison is not easy, owing to the fact that the mathematically convenient case in Reference 1 is $S_0 = 0(1)$, whereas the presently convenient case is $S >> 1$ and the grazing incidence ($\sqrt{k^2 - p^2} = k \sin\mu << k$ in Equation (2.1)).

We now obtain an approximate expression for R, using Equations (2.5) and (2.16). With $\beta = \frac{1}{2}\pi$, we have

$$R = \frac{4\pi^2 NKf(-k\sin\mu)}{k\sin\mu}. \tag{3.1}$$

We bypass most of the material of Section II, resorting to an approximation valid for the case $Q = N/k \ll 1$ and $\mu \ll 1$, i.e., for large spacings between the plates and for nearly grazing incidence. Here in Figure 3b the poles and zeros in the interior of the strip U_0, L_0 lie close to the origin, and those in the interior of the adjacent strips lie near $\pm\pi$, well away from the contour of integration. The outlying poles and zeros are important only in their effect on the behavior of various functions in the strip U_0, L_0 ($-\frac{1}{2}\pi \le \mathrm{Re}(\theta) \le \frac{1}{2}\pi$); under the present assumptions, the close grouping of poles and zeros outside of the strip means that their effects cancel in the strip. Thus, to some approximation, R depends only on the poles and zeros in the strip L_0, U_0. Instead of evaluating R through the formal procedure of Section II, we now reapproach the factorization problem from the point of view just outlined.

With $\beta = \frac{1}{2}\pi$, $p = k \cos\mu$, $w = k \sin\theta$, $Q = N/k$ in Equation (2.8), we may write down the result of expanding the various trigonometric functions in power series, obtaining, to a first approximation

$$\frac{2\pi K f(k \sin\theta)\,(\theta - \sqrt{\mu^2 + 2iQ})\,(\theta + \sqrt{\mu^2 + 2iQ})}{k(\theta - \mu)\,(\theta + \mu)} = \frac{i}{2\pi k(\theta - \mu)} + \psi\,. \tag{3.2}$$

Here φ has been expressed as a simple pole, so that the general procedure of Equation (5) is immediately suggested. We obtain

$$\frac{2\pi K f(k \sin\theta)\,(\theta - \sqrt{\mu^2 + 2iQ})}{(\theta - \mu)} = \frac{i2\mu}{2\pi(\theta - \mu)(\mu + \sqrt{\mu^2 + 2iQ})}$$

or

$$2\pi^2 K f(-k \sin\mu) = \frac{-i\mu}{[\mu + \sqrt{\mu^2 + 2iQ}\,]^2} = \frac{i\mu}{4Q^2}\,[\sqrt{\mu^2 + 2iQ} - \mu]^2\,. \tag{3.3}$$

Hence

$$R = \frac{-2iQ}{[\mu + \sqrt{\mu^2 + 2iQ}\,]^2} = \frac{[\sqrt{\mu^2 + 2iQ} - \mu]^2}{-2iQ}\,. \tag{3.4}$$

This clearly gives the correct $R \to -1$ when $\mu \to 0$.

The radiation from each plate is symmetric about the plane of the plate, and each plate is perpendicular to the plane $x = 0$. Thus the power scattered back along the direction of the incident wave is readily calculated from $f(-k \sin\mu)$. We use (2.3) and the subsequent expression for the current $I(u)$ ($u \sim x$ with $\beta = \frac{1}{2}\pi$) to determine the field radiated by a plate with edge at $x = 0$, $y = 0$, to a point $x = r \sin\mu$, $y = -r \cos\mu$. Taking the absolute square of the result and multiplying by N, we conclude that the power back-scattered in this direction, per unit length of surface, is, at range r,

$$N \frac{8\pi^3 K^2}{kr}\,|f(-k \sin\mu)|^2\,.$$

Substituting the f of (3.3) we obtain

$$\text{`back-scatter'} = \frac{2Q}{\pi r} \frac{\mu^2}{|\mu + \sqrt{\mu^2 + 2iQ}\,|^4} . \tag{3.5}$$

The constant factor on the right is not guaranteed, and the whole calculation would have to be repeated in any event to obtain any correlation with the practical radar case with point source, and with appreciable scattering in the z-direction. Nevertheless, the formula may be compared with other two-dimensional back-scattering formulas as they may occur. Finally, the dependence of the back-scatter on Q and μ is of primary interest; for fixed Q, the back-scatter varies as μ^2 for small μ.

Now we wish to discuss the physical significance of the approximate results (3.4) and (3.5), both obtained in the mathematical approximation that all but the most important features of the material of Section II have been neglected. Although the problem ostensibly contains two parameters, Q and μ, one can set $z = \frac{1}{2}\mu Q^{-\frac{1}{2}}$ and obtain, from (3.4)

$$R = -[z - iz + \sqrt{1 - 2iz^2}\,]^{-2} = -[\sqrt{1 - 2iz^2} - z + iz]^2 . \tag{3.6}$$

Similarly (3.5) yields

$$\text{`back-scatter'} = \frac{2}{\pi r} z^2 |R|^2 . \tag{3.7}$$

In Figures 4 and 5, these quantities have been plotted against the single parameter z. The physical significance of z is as follows. If one calculates by physical optics the field illuminating the edge of one plate in the partial shadow of a plate at distance S, one finds that z^2 is a measure of the number of complete Fresnel zones entering the calculation. Thus in the limit $z >> 1$, there is very little shadowing effect and R and 'back-scatter' should be computable as superpositions of fields scattered by the single, isolated plates discussed in Section I. The asymptotic forms

$$R \simeq \simeq i/8z^2, \quad \text{`back-scatter'} = C/z^2, \quad z >> 1 \tag{3.8}$$

are also those found in the manner just suggested; here the simplest verification is to see that the $f(-k \sin \theta)$ of (3.3) is asymptotically equal to that of (1.13).

We have already discussed the case $z << 1$, which led to $R \simeq -1$ and 'back-scatter' $\alpha \mu^2$ or αz^2. The singularities in the principal strip of the θ-plane migrate with change in z in such a manner that various singularities cancel to give the simple results found in the limiting cases. We may perhaps regard the present results as first terms in expansions obtained by taking into account the groups of poles and zeros found in the successive strips in the θ-plane of Section II. The terms contributed by these poles and zeros in the nth strip are perhaps interpretable as arising from fields diffracted and reflected n times around the edge of a plate.

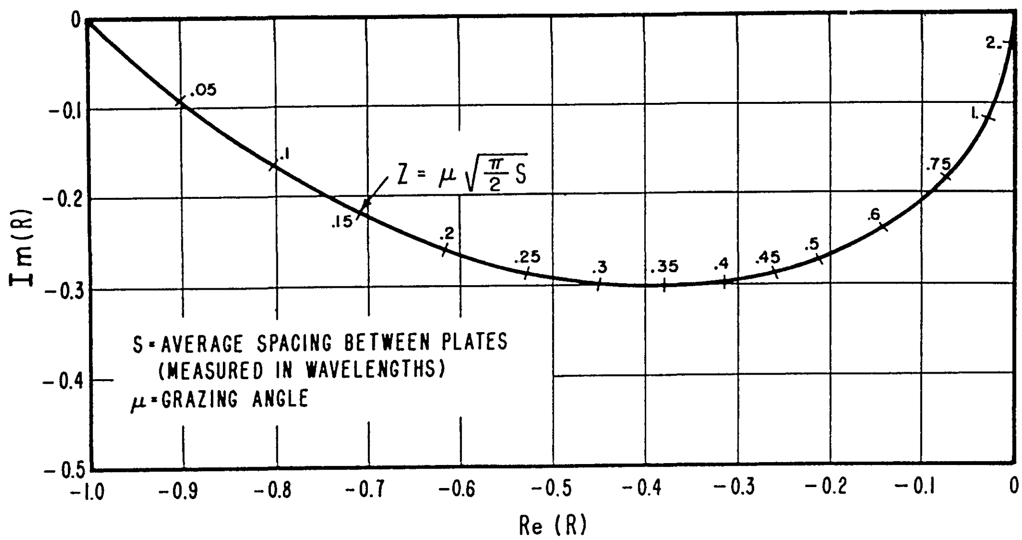

Figure 4 - Complex reflection coefficient, R, of the parallel-plate medium for large average spacing and nearly grazing incidence

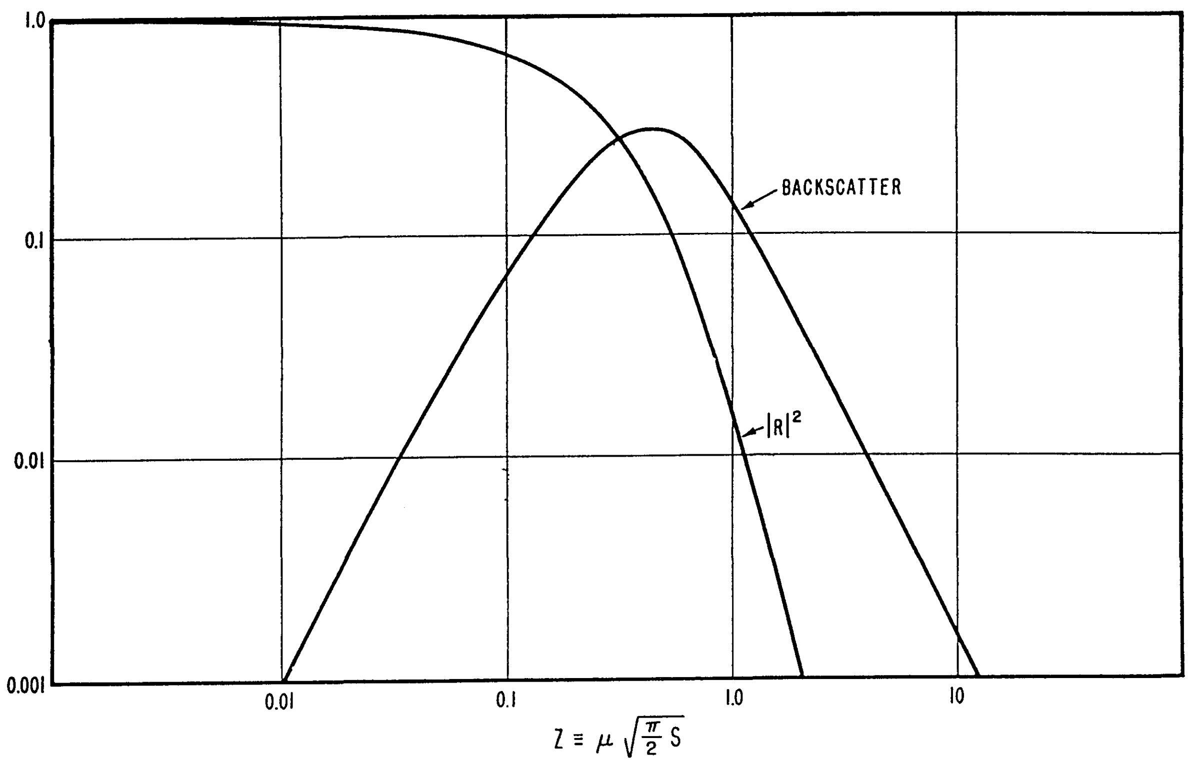

Figure 5 - Specularly reflected and back-scattered power from parallel-plate medium

In connection with the foregoing approximations, it is interesting to consider an analogous problem, the propagation of a plane wave through a large region D, containing, per unit volume, N' large reflecting disks of area A. Suppose all disks are parallel to the yz-plane and that a plane monochromatic wave $U_0 = \exp(ikx)$ is incident, where k is the propagation constant of the air, or of the disk-free space. If the field incident on one disk is uniform over the disk and is represented in phase and amplitude by U, then a unit area of the disk acts as a source of a spherical (scalar) wave given by $U(ik/2\pi r)\exp(ikr)$, where r is distance from the unit area to any other point in space. (Choice of the proportionality constant is determined so as to give complete shadow immediately behind a sufficiently large disk.) Thus, neglecting edge effects, we may express the total field at a point in space p by

$$U(p) = U_0(p) + \int_D N' \int_{\text{disk with center at } q} (ik/2\pi\,|\sigma, p|)\exp(ik\,|\sigma, p|)\, U(\sigma)\, d\,\text{area}(\sigma)\, d\,\text{vol}(q) \qquad (3.9)$$

where U_0 is the incident field radiated from some source.

Now we are going to take a time average of the total field, under the assumption that the disks are swirling randomly about in D, remaining always parallel to the yz-plane. Toward this end, we further assume that the field incident on a particular disk is the average field at the location of the disk which would be found in the absence of that disk. (This is a "self-consistent field" assumption similar to that made in atomic physics, and made in the Section II.) With these assumptions and the well-known property of the Green's function $\exp(ikr)/4\pi r$, we apply the operator $\nabla^2 + k^2$ to both sides of (3.9) and get, in D,

$$(\nabla^2 + k^2)\, U = (-2ikN'A)\, U,$$

or

$$\nabla^2 U + [k^2(1 + iN'A/k)^2 + N'^2A^2]\, U = [\nabla^2 + k'^2]\, U = 0. \qquad (3.10)$$

Thus, in D, for small N', the effective propagation constant k' is given by $k' \simeq k(1 + iN'A/k)$. Hence, the average plane wave behaves like $e^{ikx}\, e^{-N'Ax}$; its power falls off like $e^{-2N'Ax}$, so that, in agreement with well-known theory, one large disk of area A effectively removes from the average transmitted field <u>twice</u> the power incident on it. (The general mathematical approach used here is that of L. L. Foldy, Phys. Rev., <u>67</u>, p. 109, 1945.)

Now we consider that D fills the space $y < 0$, the disks remaining normal to the x-axis. With the plane wave $\exp(ikx\cos\mu - iky\sin\mu)$ incident on D from above, we have some wave with amplitude T transmitted into D, and some wave $R\exp(ikx\cos\mu + iky\sin\mu)$ reflected into $y > 0$ at the $y = 0$ interface. We calculate R and T on the assumptions that the total wave and its y-derivative are continuous across $y = 0$, use $k' = k(1 + iN'A/k)$ as the propagation constant applying in $y < 0$, and obtain

$$R = \frac{-2iQ'}{\left[\sin\mu + \sqrt{2iQ' + \sin^2\mu}\right]^2} \qquad (3.11)$$

where $Q' = N'A/k$. With the obvious identification $N'A = N$ (the N of Section II), so that $Q' = Q$ (the Q of (3.4)), we have an apparent generalization of the reflection coefficient formula (3.4), for which validity was claimed only in the case $\mu << 1$, $Q = N/k << 1$. The two formulas become identical only at grazing incidence ($\mu << 1$). Here the most important currents must lie in the exposed edges of the disks near $y = 0$, but the propagation constant k' was derived on the premise of a negligible effect of the disk's edge on the currents induced in it. Thus whether the agreement of (3.11) with (3.4) lends more or less credence to the latter equation is not clear.

SECTION IV - DIFFRACTION BY A HALF-PLANE BETWEEN TWO DIFFERENT MEDIA

We now consider the factorization encountered in the problem of the diffraction of a plane wave by a perfectly conducting half-plane which lies in the plane interface between two electromagnetically different media. When either the electric or magnetic vector of the plane incident wave is parallel to the half-plane's edge, the equivalent of h(w) in (2) or (2.8) turns out to be of the form

$$h(w) = 1/(C \cdot \sqrt{k_1^2 - w^2} + \sqrt{k_2^2 - w^2})$$

where k_1 and k_2 are the propagation constants in the two media and C is a constant depending on the four complex electromagnetic constants effective at angular frequency ω. We shall now take $k_2 > k_1 > 0$, on the grounds that the case of lossy propagation constants may eventually be obtained by analytic continuation. Writing $k_1/k_2 = k$, we have the problem of factoring $h' = 1/h = C'\sqrt{1 - w'^2} + \sqrt{1 - k^2 w'^2}$; $w' = w/k_1$ where C' is again a constant. As function of w', h' has branch points at $w' = 1$, $w' = 1/k$, as shown together with the integration contour in Figure 6. To remove the branch points we make the transformation $w' = \mathrm{sn}(u, k)$. (The function sn(u, k) is a Jacobian elliptic function; for this and subsequent elliptic functions, the reader is referred to References 3 and 4.) The structure of sn(u, k) in the u-plane is shown in Figure 6b, where poles, zeros, and even points (points about which sn is an even function) are marked with the symbols ∞, 0, and e, respectively. The S-shaped integration contour into which the real w-axis maps is indicated by Γ, and, over this contour, the Fourier inversion integral $J(x) = \int_{-\infty}^{\infty} g(w)\exp(iwx)dw$ transforms essentially in $J(x) = \int_{\Gamma} g(k_1 \mathrm{sn}(u,k))\exp(ix\,\mathrm{sn}(u,k))\mathrm{cn}(u,k)\mathrm{dn}(u,k)du$.

The structure of cn is indicated in Figure 6c and that of dn in Figure 6e. Then if $g(k_1 \mathrm{sn})$ has the even properties of sn on $]$ (say), and $x > 0$, the contour may be transformed into $]$ and thence into a diagonal connecting iK' with $2K - iK'$ plus a horizontal connecting $2K - iK'$ with $-iK'$. On the diagonal, the integrand is an odd fuction of position with respect to the midpoint K owing to the fact that cn is odd about K; similarly, on the horizontal, the integrand is odd about the midpoint $K - iK'$ owing to dn. Hence, under the assumptions, $J(x) = 0$.

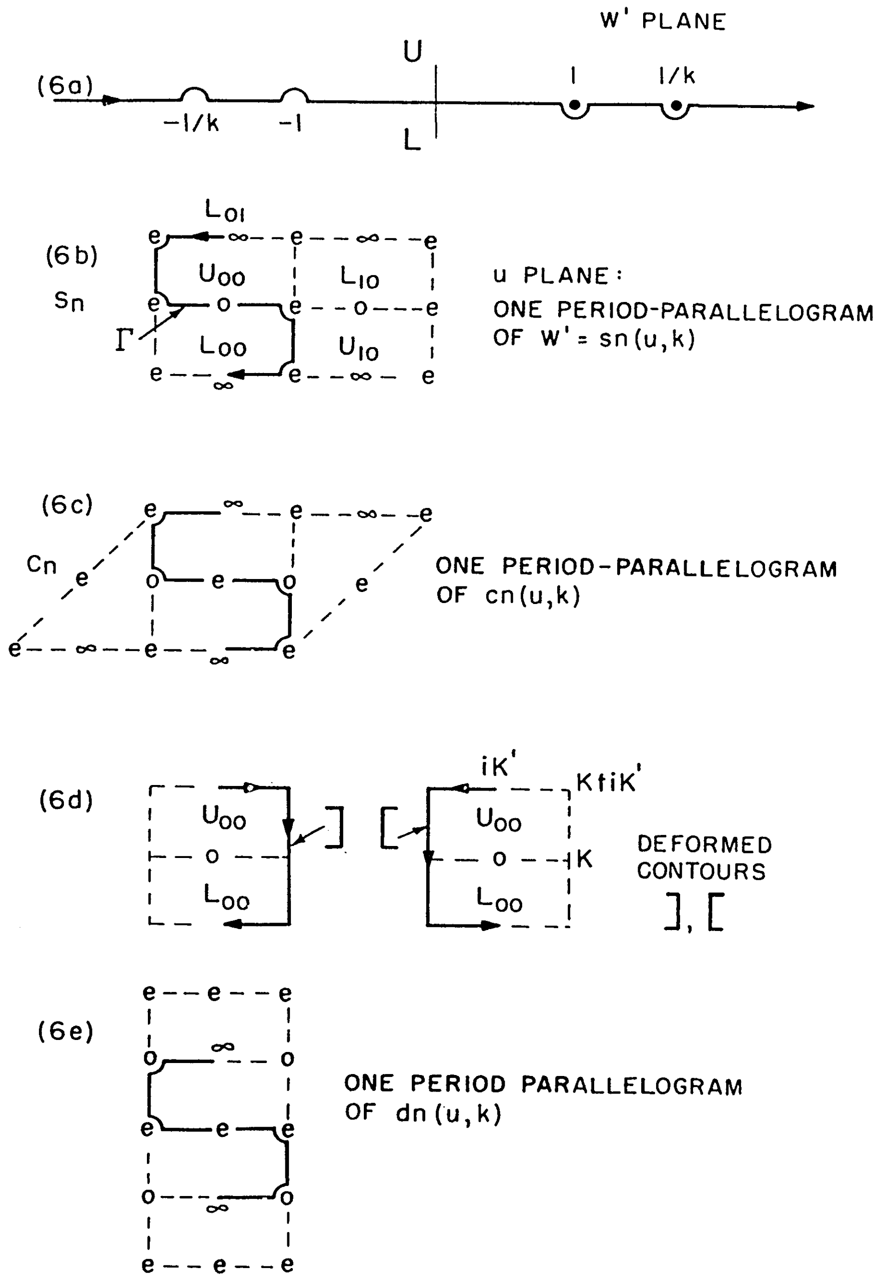

Figure 6 - Contours and singularities for the two-medium diffraction problem.

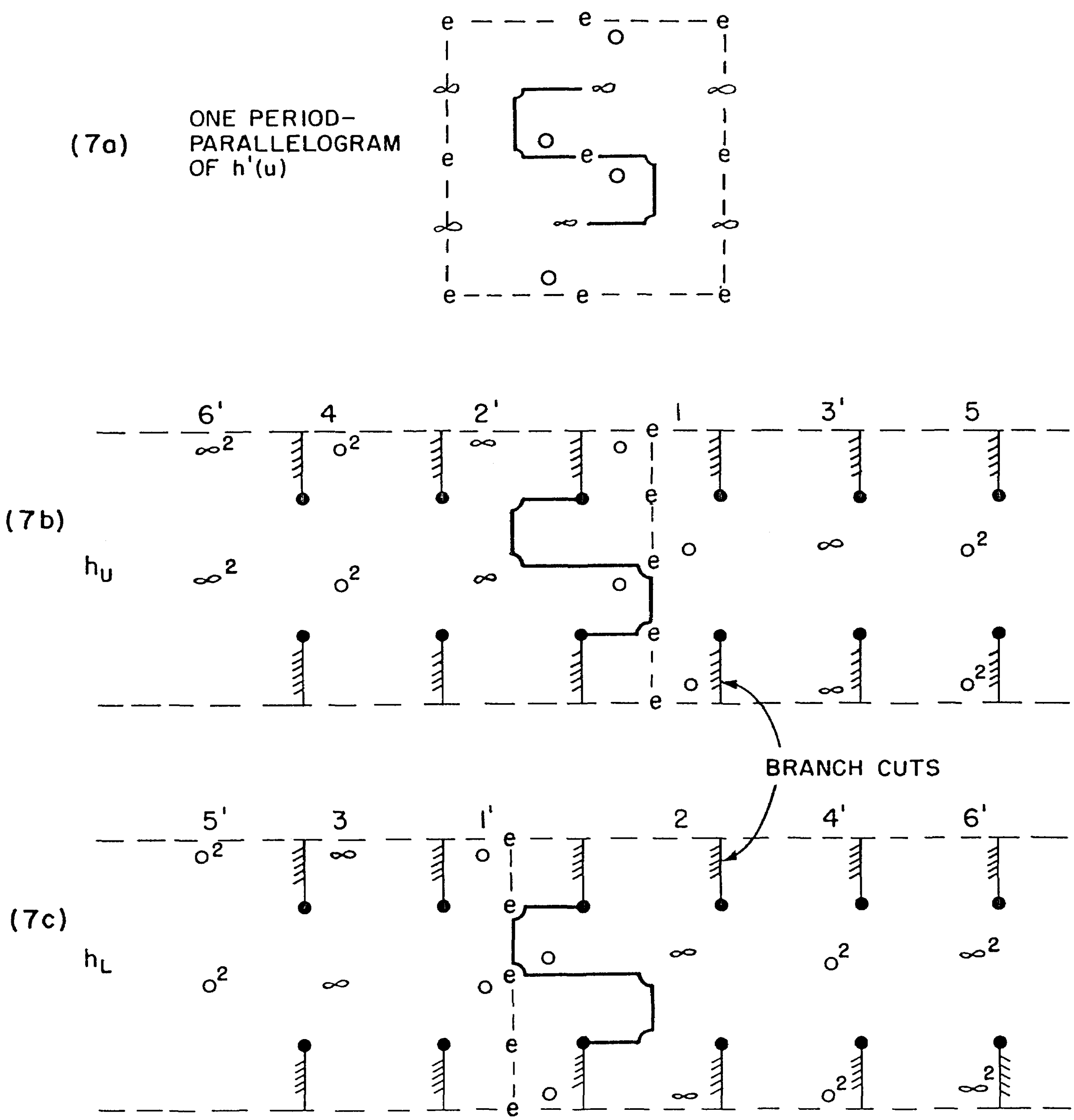

Figure 7 - Structure of h, h_U, and h_L

marked 0 and ∞. The fact that zeros of h' are missing in the regions $U_{2n-1,n}$ and $L_{2n-1,n}$ (as seen by comparing Figure 7a with Figure 6b) means that 1/h(w) has no zeros in two of the four sheets of the Riemann surface into which the w-plane must be developed because of the four branch points of Figure 6a. The periods of h' are 4K and 4iK' (see Figure 6d for K, K').

The factor h_U is even about the even points K, K + iK, K - iK', of sn(u, k). It might be assumed that h_U is therefore periodic in u with period 2iK', but branch cuts are required and the period is 4iK', the 'vertical' period of h'. The same conclusion holds for h_L, and because h' conveniently has an even point at u = 0, we may take $h_L(u) = h_U(-u)$. The zig-zag argument must be invoked because of the lack of zeros of h' in the odd-numbered vertical strips; for clarity we show the structure of one period of the factor h_L in Figure 7b, that of h_U in Figure 7c.

The zero of h' in $U_{0,0}$ belongs to h_L, and is the start of a zig-zag argument as indicated by the (1) at the top of the figure on the vertical through the zero in question. The even property of h_L implies zeros in the next strip to the left; this deduction forms step two in the argument, as indicated by (2) on the appropriate vertical at the top of Figure 7b. These zeros were not found in h', so that cancelling poles must be found in h_U. This third step in the argument is indicated by (3) on the appropriate vertical at the top of Figure 4h. Corresponding poles must be found symmetrically disposed with respect to the even points of h_U, along the vertical marked (4) in Figure 7c, etc. The zero of h' in $L_{0,0}$ starts a similar zig-zag argument which can be followed through the numbers (1'), (2'),....

The poles of h' are symmetrically disposed with respect to the even points of the factors; the resulting singularities in the factors are marked with heavy dots. In the neighborhood of the dot at u = iK' (say) both factors are asymptotically proportional to $(u - iK')^{-1/2}$; this is the behavior requiring branch cuts in each factor, cuts which may be taken to avoid the contour and which cause the periods of the factors to be 4iK'. It seems simplest to regard each of the total structures in Figures 7b and 7c as a product of two factors, one having the structure given by the dots, the other containing no branch points and having the structure arrived at by the zig-zag arguments. The two structures shown in Figures 7b and 7c clearly have the requisite even properties, and their product has the structure of h' as indicated in Figure 7a.

Under variation of the physical parameters, the zero of h' in $U_{0,0}$ may migrate into $L_{1,0}$ and then, perhaps into $U_{1,0}$. It is seen that this zero must remain in h_L, so that factors and subsequent results will vary analytically throughout the migration. Corresponding to a plane incident wave, the present analogue of the φ of (2) or (2.8) will have a single pole in the w-plane (about which the contour in Figure 6a is properly deformed). In the u-plane, φ will then have the even properties of sn so that the separation of φ/h_U into a term R in $U_{0,0}$ and a term in $L_{0,0}$ should be essentially as easy as in the previous cases.

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