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SIGNAL-TO-NOISE RATIO IN ENVELOPE MATCHING SYSTEMS

Samuel F. George

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Approved by:

Dr. R. M. Page, Superintendent, Radio Division III

NAVAL RESEARCH LABORATORY
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ABSTRACT

Proposals of methods enabling predetection pulse integration by using envelope matching or the cross-correlation principle have created recently considerable interest and speculation as to the relative merits of the several systems. In this report is a discussion of the pulse requirements permitting integration, an examination of two of the methods for obtaining these conditions, and an analysis of the improvement in sensitivity resulting from integration before detection. It is established that envelope matching accomplishes r-f gating and enables subsequent pulse-to-pulse integration before detection, but does not in itself otherwise increase the sensitivity. It is further shown that the improvement in signal-to-noise ratio is the same for both systems analyzed and is a function of the total integration time. Specifically, if n pulses are integrated, the improvement in power S/N relative to a single pulse is of order n.

PROBLEM STATUS

This is one of a series of reports on this problem. Work is continuing.

AUTHORIZATION

NRL Problem R14-01R
NR 514-010

SIGNAL-TO-NOISE RATIO IN ENVELOPE MATCHING SYSTEMS

INTRODUCTION

The advantage of predetection integration on the signal-to-noise ratio in radar tracking systems has long been appreciated, but recently proposed schemes for performing "phase coherent" integration have revived interest in the problem. Although none of these methods has yet been realized in practice, sufficient emphasis is now being given to storage devices to make the subject of predetection integration worthy of renewed theoretical consideration. It is the purpose of this work to discuss the conditions enabling such integration, to examine methods of meeting these exacting conditions, and to analyze some of the processes thus made available to accomplish the desired increase in sensitivity.

THE PHASE REQUIREMENTS

As soon as one begins to study the problem, it is at once evidentthat, in order to benefit from pulse-to-pulse integration before detection, a knowledge of r-f or i-f carrier phase is required. To take full advantage of such integration, one must depend upon the voltage addition of "phase coherent" carriers from pulse to pulse and the power addition of noncoherent random noise from pulse to pulse. Hence, before integration of echo pulses, the inherent phase characteristic of the original transmitted pulses must be known or determinable. This is possible, and several systems have been proposed employing various methods. Pulsed CW systems can employ either storage or regeneration* techniques, but such systems have serious power limitations. Magnetron systems in general do not generate phase coherent pulses and are subject to frequency modulation during the pulse duration; consequently they must rely on storage techniques to preserve phase.

This condition introduces the additional requirement of envelope matching and exact reproduction of transmitter modulation. It will be seen in the theory to follow that predetector integration techniques offer the greatest improvement in sensitivity and hence utilize the cross-modulation process to fullest advantage when the echo pulses and stored pulses have identical modulation waveforms including both amplitude and frequency modulation. Hence, it becomes almost imperative in magnetron systems to use storage devices to preserve exact knowledge of pulse characteristics. The theory of operation then depends upon delaying by storage the exact transmitted pulses until such time as the echo pulses from a target return, at which time the two pulses are compared. Ideally, comparison should show identical modulation waveforms, identical phase (the delay being correctly

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* This means the generation of pulses at a later time and not regeneration in the ordinary electronic engineering sense.

adjusted which is a form of range gating), but different amplitudes. However, in practice it may not be quite so simple.

It is a well-known fact that pulses reflecting from a target exhibit a certain degree of distortion in shape and phase. However, existing experimental evidence would tend to indicate that these distortions are not so serious as to make predetection integration impossible, although there will result a deterioration over the ideal sensitivity improvement. A considerable amount of additional experimental data must be taken before it will be possible to determine whether and under what conditions the average pulse shape and phase distortions are sufficient to wipe out the increase in sensitivity resulting from predetector integration.

USE OF CROSS-CORRELATION PRINCIPLE

Once a method has been established for remembering the shape, modulation, and inherent phase characteristic of individual transmitted pulses, the next step is to determine how this information can be used to improve sensitivity. In the first place, any frequency modulation implanted on the original transmitted pulses either unintentionally or for purposes of range resolution is not required for angle tracking and hence may be eliminated from angle circuits. Secondly, there is a considerable amount of noise on the echo pulses which cannot be considered a part of the desired tracking modulation, and it is important to remove as much of this noise as possible. Thirdly, receiver noise is the factor limiting the ultimate range of the system, and hence the effect of this random noise must be reduced as much as possible.

A number of ways are perhaps at once suggested for performing some or all of the above requirements. A most obvious one might be an attempt to use the stored reference pulses to produce a very stable intermediate frequency carrier which could then be narrowly filtered down to the tracking sidebands. This can be done. However, a more elegant and instructive method of thinking is as follows. It is known that the cross-correlation process has inherent properties of enabling an increase in signal-to-noise ratio, so let the predetection integration process be compared to see if it is analogous. The first step in cross-correlation is multiplication of the two functions displaced in time by τ. The second step is integration over the fundamental period T which is later allowed to become large. The third step is to average the resultant sum over the period. Now this same sequence can be performed in a radar system. The multiplication step can be obtained in a mixer, using the returning echo pulses for one function and the delayed transmitter pulses for the other function—a variable delay $\tau = \tau_1 - \tau_2$ is thus provided. It is the purpose of range tracking to make $\tau = 0$. The integration and averaging can be done separately or both at once by a number of different methods, but unlike true cross-correlation, the integration time is limited by the ultimate tracking modulation bandwidth.

Now that it has been shown than an envelope matching system operates on the cross-correlation principle, the next step will be to develop the mathematical theory of operation.

THE THEORY OF ENVELOPE MATCHING

Before developing specific theory, let us examine a typical pulse as it is treated in an envelope matching system. For complete generality, let the nth transmitted pulse have the form

$$\text{nth pulse} = A_n f(t) \cos\left[\omega_{rf} t + mF(t) + \phi_n\right] \tag{1}$$

where f(t) is amplitude modulation, F(t) represents phase or frequency modulation, and ϕ_n represents an initial inherent phase characteristic of the pulse. Now this pulse is split into two parts and sent along different paths; one part is used for echo tracking, and the other is stored and delayed. Let the returning echo pulse be given by

$$\text{nth echo} = A_s f(t - \tau_s) \cos \left[\omega_{rf}(t - \tau_s) + mF(t - \tau_s) + \phi_n + \phi_s\right] \tag{2}$$

where τ_s represents the transit time of the pulse from the transmitter to the target and back to the receiver, ϕ_s is an arbitrary phase shift due to target aspect and is also a function of τ_s, and A_s is the reduced pulse amplitude. Let the stored pulse after a delay of τ_d and a definite attenuation be given by

$$\text{nth stored} = A_d f(t - \tau_d) \cos \left[\omega'_{rf}(t - \tau_d) + mF(t - \tau_d) + \phi_n\right] \tag{3}$$

where $\omega'_{rf} = \omega_{rf} - \omega_{if}$ has been accomplished by control of record and playback rates. These two pulses are now combined in a product mixer such that the output is proportional to the product of Equations (2) and (3) yielding

$$e_0 \propto A_s A_d f(t - \tau_s) f(t - \tau_d) \cos \Big[\omega_{if} t - (\omega_{rf}\tau_s - \omega'_{rf}\tau_d) + \phi_s + m\left\{F(t - \tau_s) - F(t - \tau_d)\right\}\Big] \tag{4}$$

where only terms around ω_{if} are retained. An examination of Equation (4) discloses that if $\tau_s \neq \tau_d$, the amplitude modulation components will yield somewhat less than maximum e_0, or perhaps even zero (when the pulses do not overlap at all). Thus, we see that the maximum amplitude of e_0 is for $\tau_s = \tau_d$, the case of perfect register or envelope matching. Frequency modulation is present in the output unless we have perfect register. The phase angle depends upon both r-f and i-f unless $\tau_s = \tau_d$. Finally, the phase characteristic of the new pulse is arbitrary unless $\phi_s \rightarrow 0$. It will be valuable to examine Equation (4) for the case of perfect envelope matching:

$$e_0 \propto A_s A_d f^2(t - \tau_s) \cos \left[\omega_{if}(t - \tau_s) + \phi_s\right]. \tag{5}$$

Here we see that the frequency modulation has vanished, irrespective of its form, and hence the receiver bandwidth required to pass the new pulse is dictated only by the amplitude modulation factor $f^2(t - \tau_s)$. It is interesting to note that, for $\phi_s = 0$, Equation (5) represents a pulse delayed in time by τ_s and having a zero initial phase characteristic. This means that each resultant pulse will have the same ultimate phase characteristic and hence a sequence of pulses will be "phase identical." It is therefore possible to add pulses either sequentially or simultaneously.

In order to determine how the signal-to-noise ratio is affected by this envelope matching process, let us modify Equation (2) by receiver noise; omitting the frequency modulation:

$$f(t) = A_s f(t - \tau_s) \cos \left[\omega_{rf}(t - \tau_s) + \phi_n + \phi_s\right] + \mathcal{E} \sum_{i=-l}^{l} \cos \left[(\omega_{rf} + i\Delta\omega)\, t + \phi_i\right] \tag{6}$$

where this noise is defined over one pulse period. If $2l\Delta\omega = B_{rf}$ is the r-f bandwidth, then the mean square noise $E_n^2 = \mathcal{E}^2 l$. The stored pulses are considered noise free and so can be represented by Equation (3) without the f.m. The product mixer output for the case of perfect envelope matching, and $\phi_S = 0$, is then

$$e_0 \propto A_S A_d f^2(t - \tau_S) \cos \left[\omega_{if}(t - \tau_S)\right]$$

$$+ A \mathcal{E} f(t - \tau_S) \sum_{i=-l}^{l} \cos \left[(\omega_{if} + i \Delta\omega) t + \omega'_{rf} \tau_S - \phi_n + \phi_i\right]. \qquad (7)$$

Examination of Equation (7) shows at once that one major effect of the operation is that of gating the noise, since $f(t - \tau_S)$ is zero outside of the gate length δ. If $f(t - \tau_S)$ represents a band limited spectrum, then the output signal-to-noise ratio will be altered slightly in addition to the gating factor δ/τ due to the presence of $f^2(t - \tau_S)$ on the pulse itself.

The conclusion may then be drawn that the envelope matching process serves (1) for frequency conversion, (2) for obtaining proper pulse-to-pulse phase relations, and (3) for r-f gating. These factors enable i-f integration by either of two methods as discussed in NRL report 3699 (Reference (3)). In either case the improvement in sensitivity, over a single pulse and based on power S/N, is the order of n, the number of pulses integrated.

PHASE COHERENCE AND I-F INTEGRATION

If we can assume a pulsed radar system in which there exists pulse-to-pulse phase coherence, then the envelope matching process provides the basis for integration by extreme bandwidth narrowing at intermediate frequencies. For short square pulses of r-f carrier ω_{rf} and prf represented by ω_p the transmitter output may be written

$$f(t) = E F\left(1 + \sum_{n=1}^{\infty} C_n \cos n\omega_p t\right) \cos (\omega_{rf} t + \phi_0) \qquad (8)$$

where E is peak pulse amplitude, F is the duty factor, and $C_n = 2 (\sin n\pi F)/n\pi F$. By simple trigonometric manipulation Equation (8) may be rewritten as

$$f(t) = \frac{E F}{2} \sum_{n=-\infty}^{\infty} C_n \cos \left[(\omega_{rf} + n\omega_p) t + \phi_0\right]. \qquad (9)$$

Actually these terms cannot extend indefinitely as indicated because of finite bandwidth limitations. Let them extend to $\pm N$, where $N f_p = 1/\delta$ as shown in Figure 1.

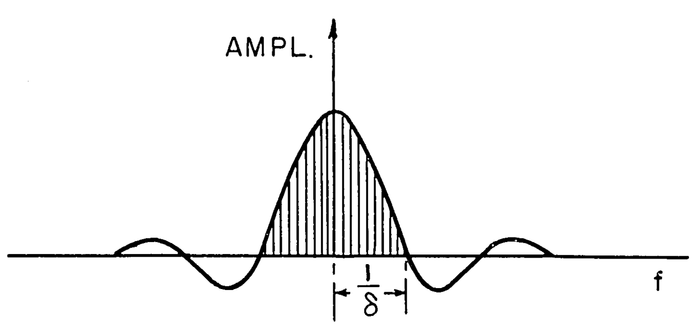

Figure 1 - Original pulses - before envelope matching

Now we are going to store the pulses as given in Equation (9), thus delaying them. Then they are played back later. Let the phase delay be τ_d and then the stored or reference pulses can be represented by

$$f(t) = A_1 \sum_{n=-N}^{N} C_n \cos\left[(\omega_{rf} + n\omega_p)(t - \tau_d) + \phi_0\right] \tag{10}$$

where $A_1 \propto EF$ and may be nearly equal to $EF/2$. For this analysis we are considering these reference pulses to be noise free, and hence Equation (10) represents the entire picture.

Let us now look at the pulses as they return from a fixed target. They will have a greatly reduced amplitude and a phase delay depending upon range, hence

$$g(t) = A_2 \sum_{n=-N}^{N} C_n \cos\left[(\omega_{rf} + n\omega_p)(t - \tau_s) + \phi_0\right]. \tag{11}$$

There may also be noise present which we may assume to be random in nature and follow a normal distribution. We may represent this noise by

$$n(t) = \mathcal{E} \sum_{i=-l}^{l} \cos\left[(\omega_{rf} + i\Delta\omega)t + \phi_i\right] \tag{12}$$

where ϕ_i is a random phase angle and $l\Delta\omega = N\omega_p$. If we call E_n the rms value of n(t) over the entire bandwidth, then

$$E_n = \mathcal{E}\sqrt{l} \quad \text{and} \quad E_n^2 \Delta\omega = \mathcal{E}^2 N\omega_p. \tag{13}$$

Thus, the signal and noise at r-f level may be written as

$$g(t) = A_2 \sum_{n=-N}^{N} C_n \cos\left[(\omega_{rf} + n\omega_p)(t - \tau_s) + \phi_0\right] + \mathcal{E} \sum_{i=-l}^{l} \cos\left[(\omega_{rf} + i\Delta\omega)t + \phi_i\right]. \tag{14}$$

Here, the r-f bandwidth is $2\,Nf_p$.

We now play back the delayed reference pulses as given by Equation (10) and by a local oscillator produce a different frequency, ω'_{rf}, such that $\omega'_{rf} - \omega_{rf} = \omega_{if}$. These pulses now are put through a product mixer with those of Equation (14) such that the mixer output is

$$M_0(t) = f(t) \cdot g(t) = A_1 \sum_{n=-N}^{N} C_n \cos\left[(\omega'_{rf} + n\omega_p)(t - \tau_d) + \phi_0\right] \cdot$$
$$\left\{A_2 \sum_{n=-N}^{N} C_n \cos\left[(\omega_{rf} + n\omega_p)(t - \tau_s) + \phi_0\right] + \mathcal{E} \sum_{i=-l}^{l} \cos\left[(\omega_{rf} + i\Delta\omega)t + \phi_i\right]\right\}. \tag{15}$$

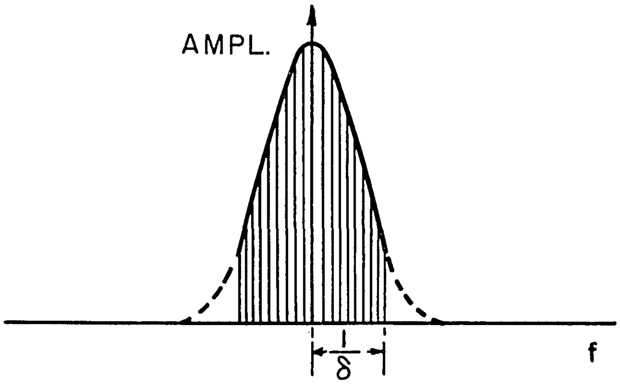

Figure 2 - New pulse spectrum - after envelope matching

We can separate the signal from the noise here and hence get some idea of signal-to-noise ratios. Let, then,

$$M_0(t) = S_0(t) + N_0(t). \tag{16}$$

Thus, expanding (15) and keeping only terms of order ω_{if}, we have

$$S_0(t) = \frac{A_1 A_2}{2} \sum\sum_{n,m=-N}^{N} C_n C_m \cos\left[\omega_{if} t + (n-m)\omega_p t + \omega_p(m\tau_s - n\tau_d) + \omega_{rf}\tau_s - \omega'_{rf}\tau_d\right], \tag{17}$$

$$N_0(t) = \frac{A_1 \mathcal{E}}{2} \sum_{n=-N}^{N} \sum_{i=-l}^{l} C_n \cos\left[\omega_{if} t + (n\omega_p - i\Delta\omega)t + \phi_0 - \phi_i\right]. \tag{18}$$

We will consider only the cases in which the i-f bandwidth is equal to or less than the r-f bandwidth as shown in Figure 2. Hence, all of the terms in Equations (17) and (18) do not contribute. Also, at this point, we will consider only the case of perfect register, or $\tau_s = \tau_d = \tau$, say. Hence Equation (17) becomes

$$S_0(t) = \frac{A_1 A_2}{2} \sum\sum_{n,m=-N}^{N} C_n C_m \cos\left[\omega_{if} + (n-m)\omega_p\right](t-\tau),$$

$$= \frac{A_1 A_2}{2} \sum_{j=-N}^{N} A_j \cos(\omega_{if} + j\omega_p)(t-\tau) \tag{19}$$

where the coefficients A_j are given by the expression

$$A_j = \sum_{n=-N}^{N-|j|} C_n C_{n+|j|}. \tag{20}$$

To determine the rms value of noise, Equation (18) can best be handled in parts. First, let us determine the noise contribution at any frequency $\omega_{if} + \omega$, where ω lies between 0 and ω_p. From Equation (18), $\omega = n\omega_p - i\Delta\omega$. It is best to look at n first, and then for any given value of n selected over the range $-N \leq n \leq +N$, determine how many values of $i(-l \leq i \leq +l)$ can be found to yield $\omega(0 \leq \omega \leq \omega_p)$. It is easily seen that there can be no values of i for n = -N and one value each for i where n = -N + 1, -N + 2, . ., N - 1, N. Hence we may write

$$N_0(t)_0^{\omega_p} = \frac{A_1\mathcal{E}}{2} \sum_{n=-N+1}^{N} C_n \cos\left[(\omega_{if} + \omega)t + \bar{\phi}_n\right]. \tag{21}$$

By similar arguments we may write an expression for the noise contributions in the interval $(k - 1)\omega_p \leq \omega \leq k\omega_p$, giving

$$N_0(t)_{(k-1)\omega_p}^{k\omega_p} = \frac{A_1\mathcal{E}}{2} \sum_{n=-N}^{N-k} C_n \cos\left[(\omega_{if} + \omega)t + \bar{\phi}_n\right]. \tag{22}$$

From Equation (22) we may write the incremental noise power as

$$P_k(\omega)_{(k-1)\omega_p}^{\omega_p} \Delta\omega = \frac{A_1^2 E_n^2}{8N\omega_p} \sum_{n=-N}^{N-k} C_n^2 \Delta\omega. \tag{23}$$

The total noise power in the interval $(k - 1)\omega_p \leq \omega \leq k\omega_p$ is the integral

$$P_k = \int_{(k-1)\omega_p}^{k\omega_p} P_k(\omega)d\omega = \frac{A_1^2 E_n^2}{8N} \sum_{n=-N}^{N-k} C_n^2. \tag{24}$$

The total noise power for $0 \leq \omega \leq N\omega_p$ is thus

$$P_T = 2 \sum_{k=1}^{N} P_k = \frac{A_1^2 E_n^2}{4N} \sum_{k=1}^{N} \sum_{n=-N}^{N-k} C_n^2. \tag{25}$$

Before proceeding to determine the output signal-to-noise ratio, a few comments about the theory are in order. First, the process of envelope matching insofar as the signal-to-noise ratio is concerned is the equivalent of range gating at r-f and hence reduces the noise power by the duty factor. Hence, aside from detector losses, no improvement should be expected over a conventional video range gated radar system. Secondly, the type of noise interference considered is random or white noise whose power varies as the bandwidth. A different analysis is needed to handle interference of a type which can be discriminated against by increasing the transmitted spectrum bandwidth. And thirdly, pulse-to-pulse phase coherence is necessary to permit this type of analysis and to allow i-f integration by narrow-banding directly.

DEVELOPMENT OF FORMULAS

Considerable attention was given to the problem of determining just what type of signal-to-noise ratio would most nearly approach the one actually used in a radar system.

It was finally decided to use the ratio of mean signal power to mean noise power. In all cases we are considering perfect envelope matching or perfect register ($\tau_s = \tau_d = \tau$) during mixing.

Considering the pulses and noise within a frequency bandwidth at r-f or i-f of $2/\delta$, we have the following sensitivities:

(1) Before mixing (ungated noise)
Mean Signal Power - [from Equation (14)]

$$S = \frac{A_2^2}{2} \sum_{-N}^{N} C_n^2 ; \tag{26}$$

Mean Noise Power - [from Equation (13)]

$$\text{Noise} = E_n^2 ; \tag{27}$$

Signal-to-Noise Ratio

$$S/N = \frac{A_2^2 \sum_{-N}^{N} C_n^2}{2E_n^2} . \tag{28}$$

(2) Before mixing (gated noise)
Mean Signal Power - same as Equation (26);

Mean Noise Power

$$\text{Noise} = E_n^2/N ; \tag{29}$$

Signal-to-Noise Ratio

$$S/N = \frac{A_2^2 N \sum_{-N}^{N} C_n^2}{2E_n^2} . \tag{30}$$

(3) After mixing (perfect register)
Mean Signal Power - [from Equation (19)]

$$S = \frac{A_1^2 A_2^2}{8} \sum_{-N}^{N} A_j^2 ; \tag{31}$$

Mean Noise Power - [from Equation (25)]

$$\text{Noise} = \frac{A_1^2 E_n^2}{4N} \sum_{k=1}^{N} \sum_{n=-N}^{N-k} C_n^2 ; \tag{32}$$

Signal-to-Noise Ratio

$$S/N = \frac{A_2^2 N \sum_{-N}^{N} A_j^2}{2E_n^2 \sum_{k=1}^{N} \sum_{n=-N}^{N-k} C_n^2}. \tag{33}$$

SUMMATION APPROXIMATIONS

The straightforward evaluation of the summations found in Equations (26) through (33) becomes very laborious for $N > 10$ and thus it was highly desirable to develop approximations. Fortunately, very close and simple approximations were obtained for all summations required. These will be presented below.

(1) Approximation for $\sum_{-N}^{N} C_n^2$.

We can prove by partial integration that

$$\int_0^x \frac{\sin^2 t}{t^2}\, dt = \mathrm{Si}(2x) - \frac{\sin^2 x}{x},$$

for let $\quad u = \sin^2 t \qquad dv = (1/t^2)\, dt$

$\quad du = \sin 2t\,dt \qquad v = -(1/t)$

$$\int_0^x \frac{\sin^2 t}{t^2}\, dt = -\frac{\sin^2 t}{t}\Big|_0^x + \int_0^x \frac{\sin 2t}{t}\, dt.$$

Let $\quad y = 2t, \quad ydt = tdy,$ then

$$\int_0^x \frac{\sin^2 t}{t^2}\, dt = -\frac{\sin^2 x}{x} + \int_0^{2x} \frac{\sin y}{y}\, dy.$$

Therefore

$$\int_0^x \frac{\sin^2 t}{t^2}\, dt = \mathrm{Si}(2x) - \frac{\sin^2 x}{x}. \tag{34}$$

Since tables exist for Si(x), we may approximate the required sum by use of the well-known Euler-Maclaurin formula:

$$\frac{1}{\omega}\int_a^{a+r\omega} f(x)dx = \frac{1}{2}f(a) + \sum_{n=1}^{r-1} f(a+n\omega) + \frac{1}{2}f(a+r\omega) + T, \tag{35}$$

where T denotes certain correction terms. Since $T = 0\ [B_1\omega f'(a)/2\,!]$ and ω is small, T here can be ignored. Hence, for $a = 0$,

$$\sum_{1}^{r} f(n\omega) = \frac{1}{\omega}\int_0^{r\omega} f(x)\,dx - \frac{1}{2}f(0). \tag{36}$$

If we let $x = \pi$ in Equations (34) and (36) we have

$$\sum_{1}^{N} \frac{\sin^2 n\pi/N}{(n\pi/N)^2} = \frac{N\,Si(2\pi)}{\pi} - \frac{1}{2},$$

$$\sum_{0}^{N} C_n^2 = 1.805N + 2. \tag{37}$$

Therefore:

$$\sum_{-N}^{N} C_n^2 = 3.61N. \tag{38}$$

Check Values

N	Computed	By (37)	Error
3	7.4	7.4	0%
10	20	20	0%
100	181.6	182	less than 1/4%

(2) Approximation for $\sum_{k=1}^{N}\sum_{n=-N}^{N-k} C_n^2$.

This double summation can be expressed as follows:

$$\sum_{k=1}^{N}\sum_{n=-N}^{N-k} C_n^2 = N\sum_{0}^{N} C_n^2 + \left\{\sum_{1}^{N-1} C_n^2 + \sum_{1}^{N-2} C_n^2 + \ldots + \sum_{1}^{2} C_n^2 + \sum_{1}^{1} C_n^2\right\}. \tag{39}$$

By examining a plot of C_n^2 it can be seen that the curve may be very closely approximated by a straight line, where

$$y = y_0 + mx, \quad y_0 = 4 \text{ and } m = -y_0/x_0,$$

$$y_k = y_0 + \frac{kmx_0}{N} \text{ where increments } 0 < x < \pi \text{ are } x_0/N;$$

$$\sum_{1}^{P} y_k = \sum_{1}^{P} y_0 - y_0/N \sum_{1}^{P} k = Py_0 - \frac{y_0 P(P+1)}{2N}$$

$$= \frac{2P(2N - P - 1)}{N};$$

$$\sum_{P=1}^{N-1} \sum_{k=1}^{P} y_k = 4 \sum_{1}^{N-1} P - \frac{2}{N} \sum_{1}^{N-1} P - \frac{2}{N} \sum_{1}^{N-1} P^2$$

$$= \left(4 - \frac{2}{N}\right) \frac{N(N-1)}{2} - \frac{2}{N} \frac{N(N-1)(2N-1)}{6}$$

$$= \frac{4N^2}{3} - 2N + \frac{2}{3}.$$

Combining (37) and (39) we have

$$\sum_{k=1}^{N} \sum_{n=-N}^{N-k} C_n^2 = 3.14N^2 + 0.67. \tag{40}$$

For large N, we may approximate the square root of Equation (40) by

$$\sqrt{\sum_{k=1}^{N} \sum_{n=-N}^{N-k} C_n^2} \doteq 1.77N. \tag{41}$$

Check Values

N	Computed	By (41)	Error
3	5.32	5.31	less than 1/5%
10	17.7	17.7	0%

(3) Approximation for $\sum_{-N}^{N} A_j^2$.

To evaluate this summation we use the following arguments. If we had a complete pulse ($N \to \infty$) and squared it, the spectrum of the squared pulse would have the same form as that of the original pulse except that the coefficients would be altered. If $A = E/N$ is the d-c value of the original pulse, then $A = E^2/N$ is the d-c value of the squared pulse. Hence, we see that $A' = A^2 N$. Now in squaring the pulse, A' becomes $A^2/2$ and hence each new C_n' must be $2NC_n$ such that

$$A' C_n' = \frac{E^2}{N} C_n' = 2A' NC_n \quad \text{or} \quad A' C_n = \frac{E^2 C_n'}{2N^2}.$$

Hence, we have

$$\sum_{-N}^{N} A_j^2 = 4N^2 \sum_{-N}^{N} C_n^2 \doteq 14.4N^3. \tag{42}$$

RESULTS

We are now in a position to interpret and evaluate Equations (26) through (33). First, it will be interesting to determine whether this method of analysis agrees with the interpretation of Equation (7); i. e., that the envelope matching process acts merely as an r-f gate insofar as the signal-to-noise ratio is concerned. For this purpose the whole pulse after mixing is considered—up to a bandwidth of $2/\delta$. The following expression gives the ratio of output S/N to input S/N where the input noise is gated:

$$R = \frac{\sum_{-N}^{N} A_j^2}{\sum_{-N}^{N} C_n^2 \sum_{k=1}^{N} \sum_{n=-N}^{N-k} C_n^2} = 1.3. \tag{43}$$

The value 1.3 instead of unity as would be expected arises from the use of finite instead of unlimited pulse bandwidth. Again it must be concluded that the envelope matching in itself does not produce an increase in sensitivity. However, envelope matching does give a very stable intermediate frequency and provides also the phase coherence which makes possible integration by direct bandwidth narrowing.

Let us analyze the change in sensitivity by the foregoing method. This can most easily be done in two steps: (1) First determine the ratio of the original gated S/N to the S/N resulting from the i-f carrier, with a bandwidth of noise up to but not including the first spectral line, and (2) determine subsequent improvement due to further narrowing down to the modulation bandwidth. The ratio of output to gated input S/N as defined in (1) above is

$$R = \frac{A_0^2}{\sum_{-N}^{N} C_n^2 \sum_{-N}^{N} C_n^2} = 1. \tag{44}$$

Hence, filtering down through the first spectral line leaves the signal-to-noise ratio unchanged. However, it must be recalled that, by this process, the pulse is lost and only the i-f carrier with its associated tracking modulation remains. If we have pulses of width δ and repetition period τ, then the noise bandwidth here is $1/\tau$. If we assume the desired modulation bandwidth to be B_m, then the i-f bandwidth can be further reduced to this value. The improvement in so doing will be $1/\tau B_m$, and will be the maximum improvement possible by this type of integration.

Now let us compare the two methods of integration assuming a total integration time of $1/B_m$ second:

I. Pulse shape retained
   a. Number of pulses integrated $n = 1/$ [illegible] n
   b. Improvement in sensitivity $1/\tau B_m$

II. Pulse shape lost
   a. Final bandwidth B_m
   b. Improvement in sensitivity $1/\tau B_m$

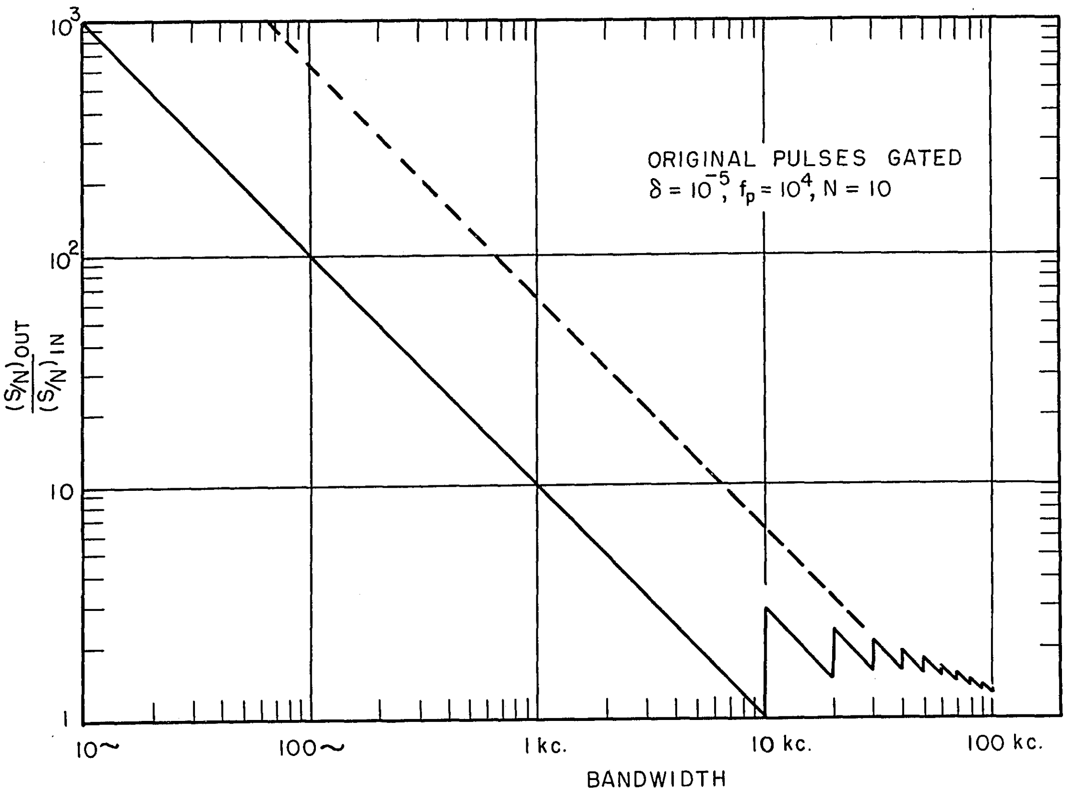

Figure 3 - Ratio $(S/N)_{out}/(S/N)_{in}$ vs. bandwidth

It can be concluded, then, that the two methods of integration give essentially the same improvement in sensitivity. A plot showing the manner in which the S/N changes with bandwidth is given in Figure 3.

SUMMARY

The conditions enabling predetection integration have been discussed and two rather promising methods have been considered in some detail. In both cases envelope matching, requiring storage or regeneration of the original identical transmitted pulses, was used as the first step of a cross-correlation process. It was demonstrated that the envelope matching process produced:

(1) "Phase identity" or "phase coherence" and stable intermediate frequency, hence permitting integration

(2) Range gating at radio frequencies

(3) No improvement in S/N over a gated input

The analysis showed that both methods of integration yielded the same improvement in power signal-to-noise ratio for a given integration time. This improvement was equal to the number of pulses integrated or to $n = 1/\tau B_m$ where τ is recurrence period and $1/B_m$ is total integration time.

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