NRL REPORT NO. N-3265

A STUDY OF MULTIPLE COMPTON SCATTERING

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Problem No. 31N03-08 March 26, 1948

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ABSTRACT

An experimental and theoretical investigation has been made of the γ-radiation in water produced by a uniform distribution of radioactive cobalt. The intensity and spectral distribution of the radiations are determined essentially by the multiple Compton scattering that takes place in the liquid. An approximate quantitative treatment is presented in which the multiple scattering is considered as a succession of steps which are the same for every emitted γ-ray. Although the method is quite crude, the numerical results both for the absolute counting rate of a submerged counter and for the effect of shielding around the counter are in close agreement with experiment.

PROBLEM STATUS

This is an interim report on this problem; work is continuing.

A STUDY OF MULTIPLE COMPTON SCATTERING

INTRODUCTION

Much experimental work has been done on the scattering of X-rays in liquids*, which has been concerned primarily with the diffraction pattern produced by a beam striking a liquid target. Some scattering studies have also been made by measuring the intensity of an X-ray beam as it passes through a liquid. However, in none of this work has the multiple character of the scattering been emphasized. To study the latter point, it was decided to use a gamma-emitting radio-isotope uniformly distributed throughout a large volume of water. Measurements of the quanta existing in the solution can then be made by means of a Geiger counter and estimates of the energy distribution can be determined by means of the absorption of thin metallic sheets surrounding the counter. The use of a uniform distribution of radioactive material simplifies the experimental procedure since this eliminates the geometrical factors introduced by a beam.

THEORY

Consider an infinitely large volume of water in which there are η radioactive atoms per unit volume. A quantum emitted by a radioactive atom experiences multiple Compton scattering as it travels through the liquid so that the path taken by an individual quantum is quite complicated. If the scattering were uniform in angle, the quantum's path would correspond to the trajectory of the problem of random flights. Actually it is more complex because the cross section for scattering depends upon both energy and angle. The probability that the quantum will scatter in a small cone $d\Omega$ in the forward direction is greater for quanta of high energy while for low energy quanta it is almost the same in the forward and the backward direction.

Now, according to the Compton formula, the wavelength change experienced by a quantum upon being scattered by an electron, is:

$$\delta\lambda = \frac{h}{mc}(1 - \cos\theta) \tag{1}$$

Here θ is the scattering angle and h/mc is the Compton wavelength. If a quantum suffers several such scatterings as it travels through the liquid, its energy E_k after the k-th scattering is:

$$E_k = \frac{hc}{\lambda_o + \sum_k \delta\lambda_k} \tag{2}$$

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* A. H. Compton and S. K. Allison, X-rays in theory and experiment, D. Van Nostrand (1936), Chapter II

Here $\lambda_0 = hc/E_0$ and is the wavelength corresponding to the energy E_0 which the quantum had on emission.

Since the scattering cross section depends on the energy and since the energy is different for each scattering, the general characteristics of the quantum's path change from one end to the other. Initially, the quantum is scattered more in the forward direction, and the distance, l, between successive scatterings (mean free path) is relatively long. As the energy is reduced by successive scatterings, the forward direction is no longer predominant. Further, the mean free path decreases as the energy decreases* (Figure 1) so that successive scatterings occur more frequently at the lower energies. The quantum energy thus eventually reaches a sufficiently low value so that it disappears by photoelectric absorption.

To study the scattering analytically, the quanta will be divided into groups which are determined by the total number of scatterings, i.e., the k-th group has experienced k-scatterings. All quanta of a particular group will not have the same energy but will have energies distributed about a mean value E_k. Each group has also a mean free path l_k, which is a function of the E_k.

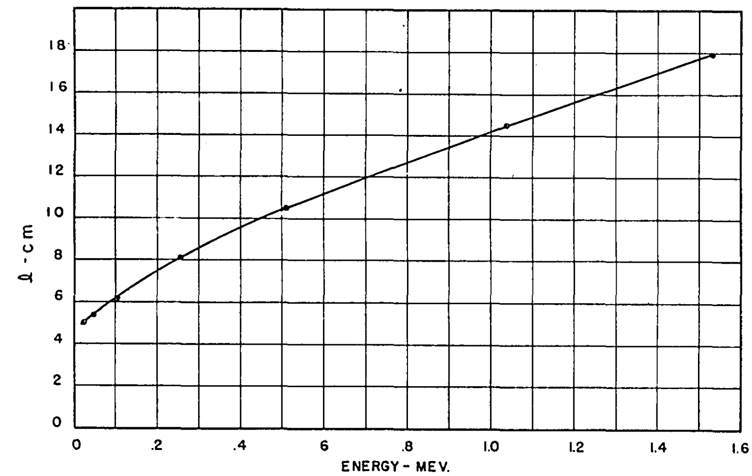

Fig. 1 - Mean Free Path of -Rays in Water

The total cross section, σ, for scattering is obtained by integrating the Klein-Nishina formula over all angles.† It may be decomposed into the sum $\sigma_a + \sigma_s$ where σ_a is the absorption due to scattering and σ_s is the cross section for true scattering.‡ The ratio σ_s/σ is the fraction of incident energy that is lost in the form of a scattered quantum and is shown as a function of energy in Figure 2. Values of this ratio are obtained by integrating the scattered intensity over all angles and dividing the result by the total cross section. The energy after several scatterings can be computed by successive application of this ratio. Table I indicates the various mean energy values taken by a quantum for successive scatterings, starting with an initial energy of 1.2 MEV.

In order to observe the effects of multiple scattering in the water, some means of detecting the quanta must be used such as a Geiger counter submerged in the liquid. The results of measurement made by the counter are to be related to the radioactive source strength.

* Handbook of Chemistry and Physics, 30 edition, Chemical Rubber Publishing Co. (1947), p. 2005. Mean free paths were computed from the linear absorption coefficients which were computed from the mass absorption coefficients given in this handbook.

† H. Heitler, The quantum theory of radiation, Oxford (1944), Chapter III ¶ 16

‡ Compton and Allison, l.c., p. 260

According to the usual definition of counter efficiency (Appendix I), a Geiger counter registers counts of the k-th group, at a rate R_k, given by the integral,

$$R_k = \int_{\Delta E_k} \int_A \epsilon(E_k)\ \overline{F_k} \cdot \overline{d\sigma}\ dE_k\ , \tag{3}$$

where $\overline{F_k}$ is the incident flux of quanta of group k and $\epsilon(E_k)$ is the counter efficiency. Both of these quantities are functions of the energy. The surface integral is extended over the effective surface of the counter, and E_k is extended over the range ΔE_k of the energies of this group of quanta. The total flux of quanta is the sum of $\overline{F_k}$ over all possible values of k.

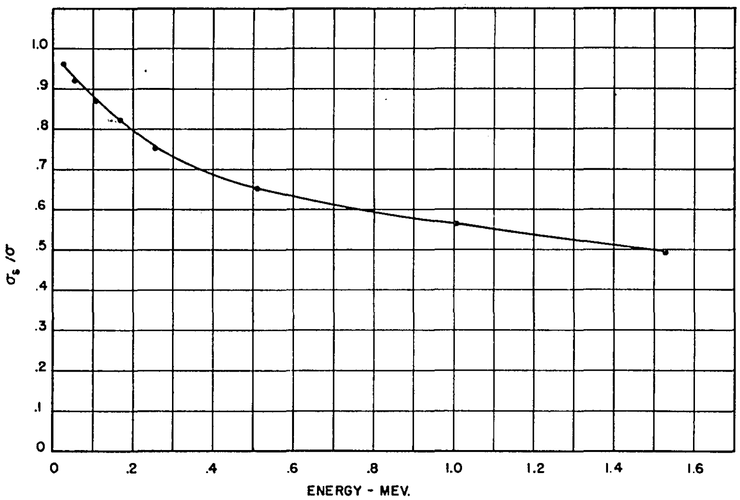

Fig. 2 - Fraction of the Incident Energy Scattered in the Form of γ-Rays According to the Klein-Nishina Formula

The integral of the flux F over the surface of the counter can be evaluated by choosing a coordinate system as shown in Figure 3. Place the counter along the Z axis and let its projected area be A. Because of the cylindrical symmetry of the counter, its projected area is A when viewed from any point in the XY plane. Now the probability that a quantum emitted in an element of volume dV will pass through the counter is

$$\frac{A \sin\theta}{4\pi\rho^2}$$

If there are N_k quanta of group k emitted per unit volume per second, the rate of emission of

TABLE I

Energy and Mean Free Path of Successively Scattered Quanta

Energy (MEV)	σ_S/σ	l_k (cm)	$\epsilon_k\, l_k \times 10^3$	Filter Cutoff
1.20	.545	15.6	112	
.654	.63	11.6	41	
.412	.69	9.8	18	
.280	.73	8.5	12.3	
.202	.79	7.5	11.3	
.161	.825	7.0	11.2	Pb + Cu 0.190 MEV
.133	.845	6.6	12.2	
.112	.86	6.3	12.6	
.096	.875	6.1	13.4	
.084	.89	5.9	13.0	
.075	.90	5.8	5.8	
.067	- -	5.8	5.6	Cu .066 MEV
		$\Sigma l_k = 96$	$\Sigma \epsilon_k\, l_k = .268$	

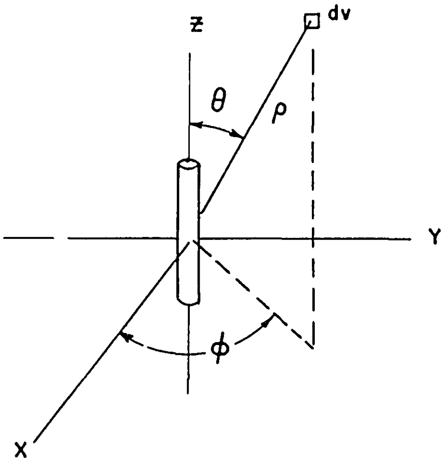

Fig. 3 - Coordinate System for Obtaining the Total Flux through a Cylindrical Counter

this group of quanta from dV is $N_k dV$. As these quanta travel away from dV they experience scatterings and so pass out of the k-th group. However, the fraction remaining in the k-th group after they have traveled a distance ρ is $e^{-\rho/l_k}$ where l_k is the mean free path. The number of quanta of the k-th group passing through the projected area, A, per second is

$$\frac{AN_k\, e^{-\rho/l_k}}{4\pi\rho^2} \sin\theta \, dV.$$

This is then integrated over the whole sphere, and the total number of quanta incident on the counter per second is found to be:

$$\int_A \overline{F}_k \cdot d\overline{\sigma} = 2 \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi/2} \frac{AN_k \operatorname{Sin}^2\theta \, e^{-\rho/l_k}}{4\pi} \, d\rho \, d\phi \, d\theta$$
$$= \frac{\pi}{4} N_k l_k A. \tag{4}$$

By substituting equation (4) in (3) and summing over all possible values of "k", the counting rate is:

$$R = \sum_0^{\infty} R_k = \frac{\pi A}{4} \int_{\Delta E} \Sigma\, N_k l_k \epsilon_k \, dE_k = \frac{\pi A}{4} \sum_0^{\infty} N_k l_k \underline{\epsilon_k} \, ,$$

where $\underline{\epsilon_k}$ represents a mean counter efficiency for the k-th group of quanta.

The sum $\sum^{\infty} N_k l_k \underline{\epsilon_k}$ can be approximated by assuming that photoelectric absorption in the water is negligible until the quantum is degraded in energy to a point that is near K-absorption limit of oxygen. This means that the rate of emission of all groups of quanta from unit volume is the same, i.e. $N_k = N$. Therefore N must also be equal to the rate of emission of quanta from unit volume by the radioactive material. That is, $N = \alpha\eta\lambda$ where $\eta\lambda$ is the disintegration rate per unit volume and α is the number of quanta emitted per disintegration. If it is assumed that all quanta are scattered only a finite number of times, m, between emission and absorption in the water, then the upper limit of the sum becomes m. The first assumption is certainly not correct as there is some photoelectric absorption at all energy values, but it is quite small for the high energies. The second assumption is also incorrect, as some quanta can be scattered an unlimited number of times. However m can be taken as the mean number of times that the quanta are scattered before absorption. This is finite, as the number of quanta experiencing an infinite number of scattering is vanishingly small. By the use of these approximations, the counting rate can be written as

$$R = \frac{\pi A}{4} \alpha\eta\lambda \sum_0^m \underline{\epsilon_k} l_k \, . \tag{5}$$

DESCRIPTION OF EXPERIMENT

The experiment consisted of measuring the counting rate of a Geiger counter suspended in an aqueous solution of a radioactive isotope. The tank containing the solution was cylindrical in shape, with a radius of 3 feet and a height of 6 feet and was constructed from $\frac{1}{4}$-inch boiler plate. A layer of beeswax was placed on the inside of the tank to prevent rusting and contamination of the tank. A plate of $\frac{1}{4}$-inch sheet iron, large enough to cover approximately half the area of the tank was laid across the top of the tank. As an aid in positioning the counter, a slot $3\frac{1}{2}$ inches wide colinear with a diameter of the tank was cut into the iron sheet (Figure 4).

The counter was placed in a water-tight aluminum tube (0.4 g/cm^2), two inches in diameter and five feet long. This tube was immersed in the solution through the slot in the sheet and placed in position as shown in Figure 4. The Geiger tube was mounted on a light wooden frame and lowered to the bottom of the aluminum tube. The Geiger tube itself was made of 1/32-inch copper tubing, one inch in diameter, with an effective length of 8 inches. It was filled with neon and ether to a total pressure of 30 cm, of which 5 cm was ether. A tungsten wire .003 inch in diameter was used as an anode. The Geiger counter was operated at 1100 volts.

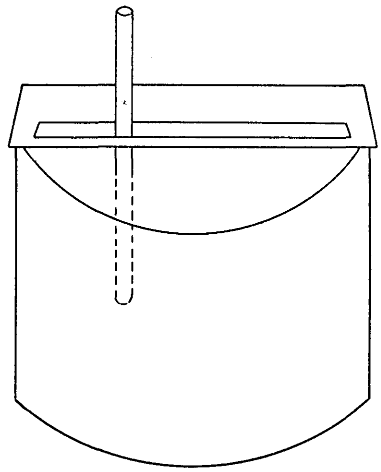

Fig. 4 - Experimental Arrangement of Water Tank

Background measurements were made in the tank with the counter placed in various positions. Results of these measurements are indicated in Figure 5. The unsymmetrical shape of the curve is due to contamination of the room in which the experiment was conducted. After the background measurements were completed, a 0.0395-g sample of Co^{60} (in the form of $CoSO_4$) which had an activity of 0.0191 millicuries was dissolved in the tank water. The specific activity in the tank was then 4.00×10^{-9} millicuries per cubic centimeter (mc/cm^3). The count was then observed at the various positions in the tank and plotted as shown in the lower curve of Figure 6. These curves represent the true counting rate, i.e., total count less background. After these measurements were completed, a 0.0368 millicure sample of Co^{60} was added to the solution, which made the total specific activity in the tank 11.9×10^{-9} mc/cm^3.* Results of these measurements are also shown in Figure 6. The ratio of the ordinates of these two curves is precisely that of the measured activities of the Co samples before being put in solution.

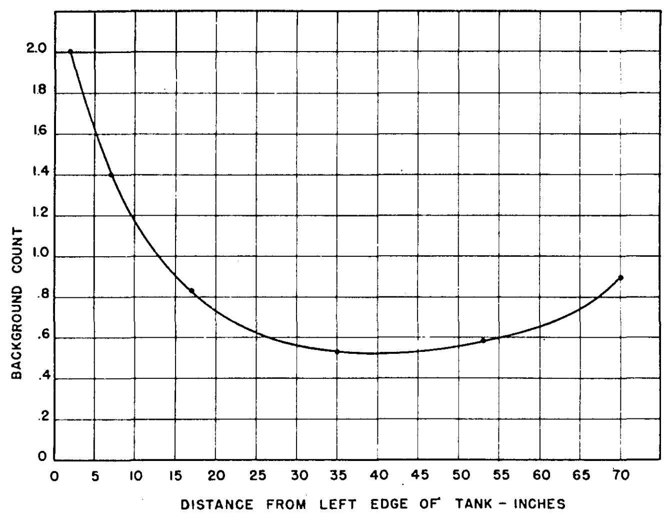

Fig. 5 - Background Counting Rate in the Water

The last part of the experiment consisted of taking measurements with various types of shields around the counter. This was done to obtain information on the spectral distribution of

* Co^{60} emits a β-particle (.3 MEV) and two γ-rays of 1.1 and 1.3 MEV energy per disintegration. The number of quanta emitted per unit volume of solution per second is $2 \times 11.9 \times 10^{-9} \times 3.71 \times 10^{7} = 0.884$.

the quanta striking the counter. All of these measurements were taken in the center of the tank with the counter submerged 3 feet. Observations were made with an aluminum shield of 0.81 g/cm^2 placed around the counter. This shield was replaced by a lead shield of 0.81 g/cm^2 and measurements taken again. Further measurements were also taken with a 1.60 g/cm^2 shield. Results of these measurements are shown in Table II.

TABLE II

Effect of Shielding on a Submerged Counter

Shield	Counts/Sec	Transmission	Attenuation
Lead - 0.81 g/cm^2	6.52	0.73	0.27
Lead - 1.60 g/cm^2	6.17	0.69	0.31
Al. - 0.81 g/cm^2	8.52	0.95	0.045
No Shield	8.92	1	0.0

DISCUSSION OF RESULTS

The curves of Figure 6 show the counting rate due to the γ-rays emitted from the radioactive material as a function of the distance from the edge of the tank. Because the curves are quite flat near the center, the effect of the discontinuity at the edge of the tank is negligible. It is further seen that the "edge effect" decreases rapidly as the counter is moved towards the center and is quite small at distances of 20 inches from the edge of the tank.

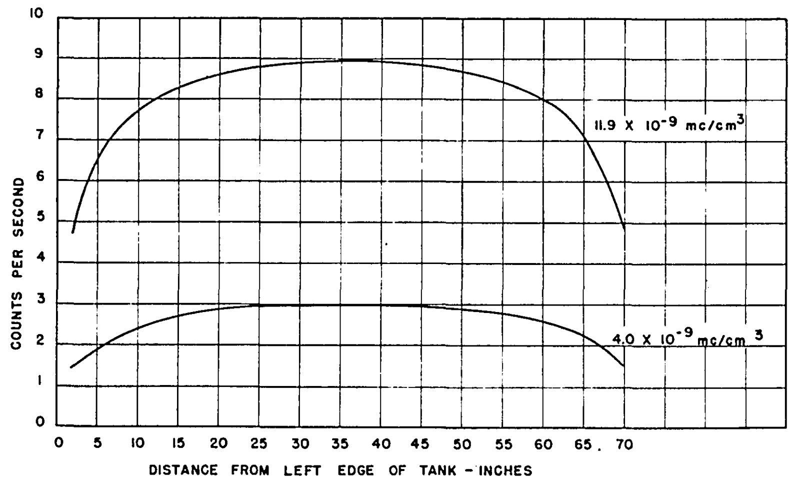

Fig. 6 - Counting Rate from Co^{60} Uniformly Distributed in the Water

It was shown in the theory that the counting rate is approximately:

$$R = \frac{\pi A}{4} \alpha \eta \lambda \sum_{0}^{m} \epsilon_k l_k \,. \qquad (5)$$

In order to evaluate this expression, the product ϵl was calculated from the data of Figures 1 and 7* and was plotted against the energy as shown in Figure 8. The value of $\epsilon_k l_k$ for the scattered quantum energies given in Table I are represented by the vertical lines in Figure 8. $\Sigma\, \epsilon_k l_k$ is represented by the sum of the length of these lines which is 0.266. By using $\alpha\eta\lambda = 0.884$ (preceding section) and A = 50 cm^2, the counting rate was calculated to be 9.24. This is within 3 percent of the observed counting rate.

The total path length for a quantum is $\sum^{m} l_k$. Values of l_k corresponding to the successively scattered quanta are given in Table I, the sum of which is 96 cm.

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* Figure 7 is taken from H. Bradt et al. Helv Phys Acta 19: 77-90, 1946.

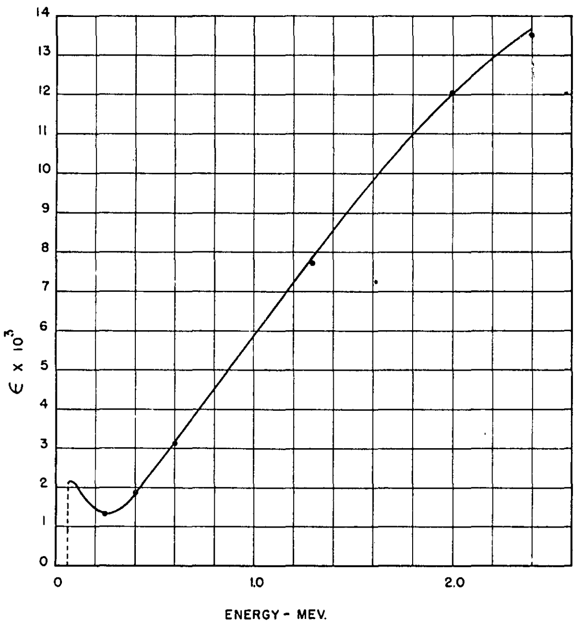

Fig. 7 - Efficiency of Copper Geiger-Mueller Counter

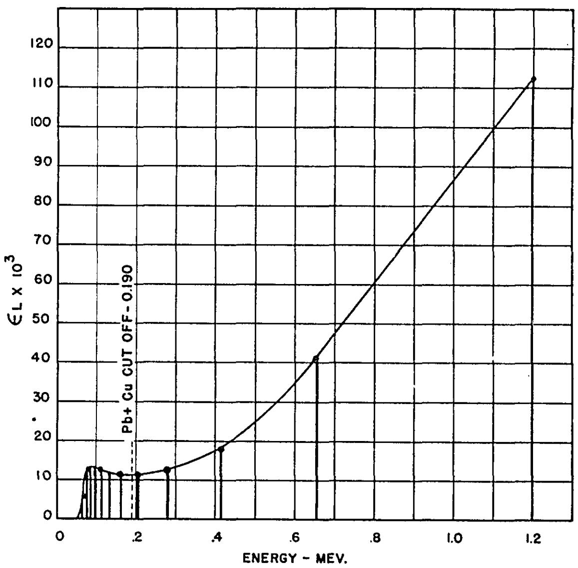

Fig. 8 - Product of Efficiency and Mean Free Path for a Cu Counter Submerged in Water

Results of the shielding experiment given in Table II indicate that an aluminum filter of 0.81 g/cm^2 reduces the counting rate by about 5 percent, while a lead filter of the same mass reduces the counting rate by 27 percent. Further lead filtering has little effect on the counting rate.

The effect of the aluminum filter can be understood by reference to the transmission curves of Figure 9. These curves show that the aluminum filter introduces an attenuation of approximately 5 percent for all energy values which lie above the copper cut-off. This value is the observed result. Cut-off is taken to correspond to an attenuation of 0.50.

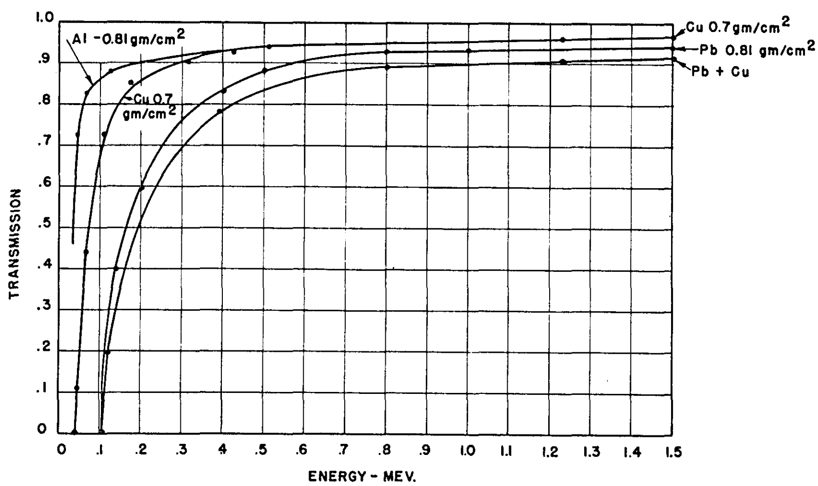

Fig. 9 - Attenuation Introduced by Shielding

Figure 9 shows that the introduction of the lead filter changes the cut-off energy from 0.06 MEV (Cu) to 0.190 MEV (Cu + Pb). The values of $\epsilon_k l_k$ (Figure 8) contributing to the counting rate are those that lie above the new cut-off energy. By taking the sum of the length of these lines, it is found that $\Sigma\ \epsilon_k l_k = 0.192$. This corresponds to a counting rate of 6.36^0, so that the ratio of shielded to unshielded count is 6.68/9.24 = .72, which is almost exactly the observed value.

The approximations made in the theory of neglecting photoelectric absorption in the water and the use of a finite sum in place of the infinite series are rather crude. It is therefore remarkable that such close agreement between the experimental and theortical results were obtained, particularly with regard to the absolute value of the counting rate. It further indicates that this approximate description will probably be useful in analyzing other problems which involve multiple Compton scattering.

* * *

APPENDIX I

CALIBRATION OF COUNTER

Measurements of the counter efficiency were made by several different methods, the results of which agree generally with data published elsewhere.* The efficiency, ϵ, of a counter is defined as:

$$\epsilon = R/S, \tag{6}$$

where R is the counting rate due to the number of quanta, S, incident on the counter per second. The quanta incident on a counter of length b and effective width, a, at a distance 1 cm from a source is:

$$S = \alpha\, \eta\, \lambda \times 3.71 \times 10^7 \frac{\Omega}{4\pi} \tag{7}$$

where:

α = number of quanta emitted per disintegration,
$\eta\lambda$ = activity of source in millicuries, and
Ω = solid angle subtended at the source by the counter.

If the distance 1 is comparable with the length b of the counter, the solid angle is:

$$\Omega = \frac{a\,.b}{1\sqrt{1^2 + b^2/4}} \qquad (b \gg a)$$

Results of such measurements with Co^{60} sources are summarized in Table III.

TABLE III

Counter Efficiency for Co^{60} γ-Radiation

SOURCE	1 (inches)	Count (less B.G.)	ϵ
0.67 μc	9.41	2.45 /sec	7.93×10^{-3}
0.67 μc	7.25	4.06	8.39
..0399 Mc	24.00	26.6	8.15
	Mean ϵ = $(8.16 \pm 0.12) \times 10^{-3}$		

* H. Bradt et al. Helv Phys Acta 19: 77-90, 1946

Similar measurements were also carried out with a radium source which yielded a result of $\epsilon = 4.25 \times 10^{-3}$. This represents a mean efficiency since radium emits on the average 2.7 quanta per disintegration of energy ranging from .2 to 2.3 MEV with a mean energy of 0.76 MEV.*

Efficiency measurements were also carried out with an X-ray machine at quantum energies of 0.135 MEV. The counter was placed one meter from the center of focus of the X-ray tube and was completely shielded in lead. A small aperture of variable dimensions was placed in the lead shield so that the number of quanta striking the tube could be varied. Measurements of the intensity of the X-ray beam were made by a 0.001 Roentgen Victoreen meter.

In order to calculate the efficiency, the flux of quanta in the X-ray beam had to be related to the number of Roentgens indicated by the Victoreen meter. Since one r is equivalent to one e.s.u. of charge per cc, the number of ion pairs formed per second is:

$$\text{Number of ion pair/sec} = \frac{r/s}{4.8 \times 10^{-10}},$$

where r/s is the number of Roentgens delivered per second. It requires 35 volts to form one ion pair in air under standard conditions of pressure and temperature so that the energy absorbed per second is:

$$\text{Energy/sec} = \frac{35\, r/s}{h\nu \times 4.8 \times 10^{-10}} \quad . \tag{8}$$

If the quanta producing the ionization has an energy of $h\nu$ (volts), the number of such quanta is:

$$\Delta n = \frac{35\, r/s}{h\nu \times 4.8 \times 10^{-10}} \tag{9}$$

Now as described in the theory, the total scattering cross-section (photoelectric absorption is neglected) is $\sigma = \sigma_a + \sigma_s$, from which the ratio of the absorption due to scattering to the total scattering is:

$$\sigma_a/\sigma = 1 - \sigma_s/\sigma \quad . \tag{10}$$

Since σ_a is proportional to the energy given the recoil electrons, the ratio σ_a/σ is the fraction of the total scattering that is effective in producing local ionization. If the X-ray beam has a flux f, the quanta absorbed per second in passing through a unit distance is μf where μ is the linear absorption coefficient. Of this, the fraction σ_a/σ is effective in producing local ionization, i.e., $\mu f \sigma_a/\sigma$.

Substituting this in equation (9) for Δn, it is found that the flux is:

$$f = \frac{35\, r/s}{h\nu\mu(\sigma_a/\sigma) \times 4.8 \times 10^{-10}} \quad . \tag{11}$$

---

*Latyshev, G. D., Revs Modern Phys, 19: 132-145, April 1947

By using the efficiency as defined in equation (6) and equation (11), it is found that:

$$\epsilon = \frac{R h \nu (\sigma_a/\sigma) \mu \times 4.8 \times 10^{-10}}{35 A r/s}, \qquad (12)$$

where A is the area of the counter tube exposed to the X-ray beam. From Figure 2 for $h\nu = 0.130$ MEV, σ_a/σ was found to be 0.17. Results of the measurements for the X-ray machine are tabulated below:

COUNTER EFFICIENCY
FOR X-RAYS

Voltage	130 kv
r/s	1.67×10^{-6}
Counting rate	183 sec
Area of tube	3.22 sq cm
ϵ	1.76×10^{-3}

The curve of Figure 7 shows the efficiency of a copper counter as given by H. Bradt et al.* The counter tube used in the NRL experiment had an efficiency just slightly higher than the curve indicates.

* * *

* H. Bradt et al. Helv. Phys acta 19: 77-90, 1946

PRNC-5292-4-13-48-150