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Electrodynamics in a Magneto-Ionic Environment

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CONTENTS

ABSTRACT

The equations of electromagnetic wave propagation in a magneto-ionic environment are first derived via the hydrodynamic equations of motion. This subject is presented in order to exhibit explicitly the approximations made in obtaining the equations of electrodynamics as applied to a plasma. An application of the above results are then made to derive a generalized equation of reciprocity for a plasma medium. Last a theoretical discussion of the plasma sheath surrounding a satellite shows that the satellite is charged to a potential of about 0.75 volt and that the sheath extends out about 2.5 cm. Such a sheath may affect the characteristics of an antenna.

PROBLEM STATUS

This is a report on one phase of the problem; work continues on other portions.

AUTHORIZATION

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ELECTRODYNAMICS IN A MAGNETO-IONIC ENVIRONMENT

INTRODUCTION

This report summarizes some of the theoretical work directed toward an understanding of electromagnetic propagation effects in the ionosphere posed by the Lofti experiments. Some of the theory described herein covers much of what may be familiar to a plasma physicist but may not be readily available to others who have just encountered the problem of electromagnetic wave propagation in a magneto-ionic medium.

The first section derives the equations of electrodynamics for a magneto-ionic medium based upon a hydrodynamic approximation to the motion of the constituent particles. While the electrodynamic equations ultimately obtained are identical in form to the Maxwell equations, the resemblance is superficial. To derive these equations, the equations of motion of the plasma are first linearized (a perturbation technique), which involves neglecting products of perturbations. If the electric and magnetic fields were not small, then such approximations could not be made, and the hydrodynamic Maxwell-Lorentz equations would have to be retained in their full generality to properly describe the phenomena.

The second section makes use of the equations derived previously and obtains a generalization of the Rayleigh-Carson theorem for a plasma. The Rayleigh-Carson theorem is a statement of the reciprocity that exists between two radiating elements in isotropic space. The generalization is made here to include the case of a magnetoactive medium.

In the last section of the report application of the work of Jastrow and Pearse is made to estimate the order of magnitude of potential and charges collected by a satellite moving through the ionosphere.

ELECTRODYNAMICS OF PLASMA

The most logical approach to the equations of plasma physics is via the kinetic theory. However, because of the rather lengthy arguments involved this technique will not be utilized. Instead the hydrodynamic equations, as given in Refs. 1-3, will be postulated initially, so that a vast amount of mathematical justification can be circumvented. References 4 and 5 confirm that the kinetic theory ulimately leads to the hydrodynamic equations.

For a plasma containing electrons of number density N_e and mass m_e; ions, N_i, m_i; and neutral molecules N_m, M, the hydrodynamic equations are as follows (1-3):

$$N_e m_e \left[\frac{\partial \mathbf{V}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla)\mathbf{V}_e\right] = -eN_e\left(\mathbf{E} + \frac{\mathbf{V}_e \times \mathbf{H}}{c}\right) - \nabla p_e + m_e N_e \nu_{e_i}(\mathbf{V}_i - \mathbf{V}_e) + m_e N_e \nu_{e_m}(\mathbf{V}_m - \mathbf{V}_e) \tag{1a}$$

$$N_i m_i \left[\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \cdot \nabla)\mathbf{V}_i\right] = eN_i\left(\mathbf{E} + \frac{\mathbf{V}_i \times \mathbf{H}}{c}\right) - \nabla p_i + m_e N_e \nu_{e_i}(\mathbf{V}_e - \mathbf{V}_i) + m_i N_i \nu_{im}(\mathbf{V}_m - \mathbf{V}_i) \tag{1b}$$

$$N_m M\left[\frac{\partial \mathbf{V}_m}{\partial t} + (\mathbf{V}_m\cdot\nabla)\mathbf{V}_m\right] = -\nabla p_m - m_e N_e \nu_{e_m}(\mathbf{V}_m - \mathbf{V}_e) - m_i N_i \nu_{im}(\mathbf{V}_m - \mathbf{V}_i)\ . \tag{1c}$$

Here $\mathbf{V}_e$, $\mathbf{V}_i$, and $\mathbf{V}_m$ are respectively the velocities of the species designated by the subscript and p is the pressure, which is assumed to be isotropic. The frictional forces between various species are, as usual, assumed to be proportional to the relative velocity and collision rates, ν_{e_i}, ν_{e_m}, and ν_{im}. In addition to the equations of motion for each species there are also the equations of continuity:

$$\frac{\partial N_e}{\partial t} + \nabla\cdot(N_e\mathbf{V}_e) = 0 \tag{2a}$$

$$\frac{\partial N_i}{\partial t} + \nabla\cdot(N_i\mathbf{V}_i) = 0 \tag{2b}$$

$$\frac{\partial N_m}{\partial t} + \nabla\cdot(N_m\mathbf{V}_m) = 0\ . \tag{2c}$$

The equations of state are taken to be

$$p_e = kN_eT_e \tag{3a}$$

$$p_i = kN_iT_i \tag{3b}$$

$$p_m = kN_mT_m \tag{3c}$$

where k is Boltzmann's constant. Equations (1a)-(3c) must, of course, be supplemented by the field equations:

$$\nabla\times\mathbf{E} = \frac{-1}{c}\frac{\partial \mathbf{H}}{\partial t} \tag{4a}$$

$$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{4b}$$

$$\nabla\cdot\mathbf{E} = 4\pi\rho \tag{4c}$$

$$\nabla\cdot\mathbf{H} = 0 \tag{4d}$$

$$\mathbf{J} = e(N_i\mathbf{V}_i - N_e\mathbf{V}_e) \tag{4e}$$

$$\rho = e(N_i - N_e)\ . \tag{4f}$$

In all the above equations e represents the absolute value of the electronic charge and each ion has been assumed to be singly ionized.

In the above set of equations, if $\mathbf{E}$ is assumed to be an alternating field, while the other quantities, $\mathbf{V}_i$, N_i, $\mathbf{H}$, etc., are assumed to consist of a constant plus a small perturbation (1), then

$$N_e = n_0 + n_e\ ,\qquad N_i = n_0 + n_i\ ,\qquad N_m = N_0 + n \tag{5}$$

$$\mathbf{V}_e = \mathbf{v}_e^0 + \mathbf{v}_e,\qquad \mathbf{V}_i = \mathbf{v}_i^0 + \mathbf{v}_i \tag{6}$$

$$\mathbf{V}_m = \mathbf{v}_m^0 + \mathbf{v}_m \tag{7}$$

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}\ . \tag{8}$$

Substitution of the above relations in Eqs. (1)-(4) and neglect of products of perturbation terms leads to the linearized equations of motion. In order to study the effects of the motion of the plasma as a whole, the constant parts of the velocities will be retained, so that possible effects of plasma motion may be studied (e.g., the motion of a satellite through a plasma may indeed act as though the plasma had a constant velocity $\mathbf{v}_i^0 = \mathbf{v}_e^0 = \mathbf{v}_m^0$). In any case, then, linearization leads to the equation of motion:

$$\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e^0 \cdot \nabla)\mathbf{v}_e = \frac{-e}{m_e}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_e^0 \times \mathbf{H}_0) + \frac{1}{c}(\mathbf{v}_e^0 \times \mathbf{h}) + \frac{1}{c}(\mathbf{v}_e \times \mathbf{H}_0)\right] - \frac{-\nabla p_e}{m_e n_o}$$

$$+ \nu_{e_i}\left[(\mathbf{v}_i^0 - \mathbf{v}_e^0) + (\mathbf{v}_i - \mathbf{v}_e)\right] + \nu_{em}\left[(\mathbf{v}_m^0 - \mathbf{v}_e^0) + (\mathbf{v}_m - \mathbf{v}_e)\right] \quad (9a)$$

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i^0 \cdot \nabla)\mathbf{v}_i = \frac{e}{m}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_i^0 \times \mathbf{H}_0) + \frac{1}{c}(\mathbf{v}_i^0 + \mathbf{h}) + \frac{1}{c}(\mathbf{v}_i \times \mathbf{H}_0)\right] - \frac{\nabla p_i}{m_i n_0}$$

$$+ \nu_{e_i}\left[(\mathbf{v}_e^0 - \mathbf{v}_i^0) + (\mathbf{v}_e - \mathbf{v}_i)\right] + \nu_{im}\left[(\mathbf{v}_m^0 - \mathbf{v}_i^0) + (\mathbf{v}_m - \mathbf{v}_i)\right] \quad (9b)$$

$$\frac{\partial \mathbf{v}_m}{\partial t} + (\mathbf{v}_m^0 \cdot \nabla)\mathbf{v}_m = \frac{-\nabla p_m}{MN_0} - \frac{m_e}{M}\frac{n_0}{N_0}\nu_{em}\left[(\mathbf{v}_m^0 - \mathbf{v}_e^0) + (\mathbf{v}_m - \mathbf{v}_e)\right]$$

$$- \frac{m_i}{M}\frac{n_0}{N_0}\nu_{im}\left[(\mathbf{v}_m^0 - \mathbf{v}_i^0) + (\mathbf{v}_n - \mathbf{v}_i)\right] \quad (9c)$$

In addition the equations of continuity become .

$$n_0 \nabla \cdot \mathbf{v}_e + (\mathbf{v}_e^0 \cdot \nabla n_e) + \frac{\partial n_e}{\partial t} = 0 \quad (10a)$$

$$n_0 \nabla \cdot \mathbf{v}_i + (\mathbf{v}_i^0 \cdot \nabla n_i) + \frac{\partial n_i}{\partial t} = 0 \quad (10b)$$

$$N_0 \nabla \cdot \mathbf{v}_m + (\mathbf{v}_m^0 \cdot \nabla n) + \frac{\partial n}{\partial t} = 0 \quad (10c)$$

while the charge and current densities are

$$\rho = e(n_i - n_e) \quad (11a)$$

$$J = en_0(\mathbf{v}_i - \mathbf{v}_e) + e(\mathbf{v}_i^0 n_i - \mathbf{v}_e^0 n_e) + eN_0(\mathbf{v}_i^0 - \mathbf{v}_e^0)\,. \quad (11b)$$

The last term in Eq. (11b) represents a gross convection of charge. Now if the fluid is at rest in the Laboratory frame of reference ($\mathbf{v}_e^0 = \mathbf{v}_i^0 = \mathbf{v}_m^0 = 0$), then

$$\frac{\partial \mathbf{v}_e}{\partial t} + \frac{e}{m_e}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_e \times \mathbf{H}_0)\right] + \frac{\nabla p_e}{m_e n_0} = \nu_{e_i}(\mathbf{v}_i - \mathbf{v}_e) + \nu_{e_m}(\mathbf{v}_m - \mathbf{v}_e) \quad (12a)$$

$$\frac{\partial \mathbf{v}_i}{\partial t} - \frac{e}{m_i}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_i \times \mathbf{H}_0)\right] + \frac{\nabla p_i}{m_i n_0} = \nu_{e_i}(\mathbf{v}_e - \mathbf{v}_i) + \nu_{i_m}(\mathbf{v}_m - \mathbf{v}_i) \quad (12b)$$

$$\frac{\partial \mathbf{v}_m}{\partial t} + \frac{\nabla p_m}{MN_0} = \frac{-m_e}{M}\frac{n_0}{N}\,\nu_{e_m}(\mathbf{v}_m - \mathbf{v}_e) - \frac{m_i}{M}\frac{n_0}{N_0}\,(\mathbf{v}_m - \mathbf{v}_i) \tag{12c}$$

$$n_0 \nabla\cdot\mathbf{v}_e + \frac{\partial n_e}{\partial t} = 0 \tag{13a}$$

$$n_0 \nabla\cdot\mathbf{v}_i + \frac{\partial n_i}{\partial t} = 0 \tag{13b}$$

$$N_0 \nabla\cdot\mathbf{v}_m + \frac{\partial n}{\partial t} = 0 \tag{13c}$$

$$\rho = e(n_i - n_e) \tag{14a}$$

$$\mathbf{J} = en_0(\mathbf{v}_i - \mathbf{v}_e) \tag{14b}$$

Substitution of the above relations in Eqs. (1)-(4) and neglect of products of perturbation terms leads to the linearized equations of motion. In order to study the effects of the motion of the plasma as a whole, the constant parts of the velocities will be retained, so that possible effects of plasma motion may be studied (e.g., the motion of a satellite through a plasma may indeed act as though the plasma had a constant velocity $\mathbf{v}_i^0 = \mathbf{v}_e^0 = \mathbf{v}_m^0$). In any case, then, linearization leads to the equation of motion:

$$\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e^0\cdot\nabla)\mathbf{v}_e = \frac{-e}{m_e}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_e^0\times\mathbf{H}_0) + \frac{1}{c}(\mathbf{v}_e^0\times\mathbf{h}) + \frac{1}{c}(\mathbf{v}_e\times\mathbf{H}_0)\right] - \frac{-\nabla p_e}{m_e n_o}$$

$$+ \nu_{e_i}\left[(\mathbf{v}_i^0 - \mathbf{v}_e^0) + (\mathbf{v}_i - \mathbf{v}_e)\right] + \nu_{em}\left[(\mathbf{v}_m^0 - \mathbf{v}_e^0) + (\mathbf{v}_m - \mathbf{v}_e)\right] \tag{9a}$$

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i^0\cdot\nabla)\mathbf{v}_i = \frac{e}{m}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_i^0\times\mathbf{H}_0) + \frac{1}{c}(\mathbf{v}_i^0 + \mathbf{h}) + \frac{1}{c}(\mathbf{v}_i\times\mathbf{H}_0)\right] - \frac{\nabla p_i}{m_i n_0}$$

$$+ \nu_{e_i}\left[(\mathbf{v}_e^0 - \mathbf{v}_i^0) + (\mathbf{v}_e - \mathbf{v}_i)\right] + \nu_{im}\left[(\mathbf{v}_m^0 - \mathbf{v}_i^0) + (\mathbf{v}_m - \mathbf{v}_i)\right] \tag{9b}$$

$$\frac{\partial \mathbf{v}_m}{\partial t} + (\mathbf{v}_m^0\cdot\nabla)\mathbf{v}_m = \frac{-\nabla p_m}{MN_0} - \frac{m_e}{M}\frac{n_0}{N_0}\,\nu_{em}\left[(\mathbf{v}_m^0 - \mathbf{v}_e^0) + (\mathbf{v}_m - \mathbf{v}_e)\right]$$

$$- \frac{m_i}{M}\frac{n_0}{N_0}\,\nu_{im}\left[(\mathbf{v}_m^0 - \mathbf{v}_i^0) + (\mathbf{v}_n - \mathbf{v}_i)\right] \tag{9c}$$

In addition the equations of continuity become .

$$n_0\nabla\cdot\mathbf{v}_e + (\mathbf{v}_e^0\cdot\nabla n_e) + \frac{\partial n_e}{\partial t} = 0 \tag{10a}$$

$$n_0\nabla\cdot\mathbf{v}_i + (\mathbf{v}_i^0\cdot\nabla n_i) + \frac{\partial n_i}{\partial t} = 0 \tag{10b}$$

$$\nabla\cdot\mathbf{D} = \frac{1}{i\omega}\left(4\pi\nabla\cdot\mathbf{J} + i\omega\nabla\cdot\mathbf{E}\right) = \frac{4\pi}{i\omega}\left(\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t}\right) = 0 \tag{19}$$

so $\nabla\cdot\mathbf{D} = 0$ is an expression for the conservation of charge.

In order to gain some insight into the solutions of Eqs. (12) et seq. consider the case where the molecular and ionic motions are negligible; i.e., $m_i = M \to \infty$, so $\mathbf{v}_i = \mathbf{v}_m \to 0$. If it is further assumed that all quantities are harmonic in time (except for H_0 which is a constant) and that the pressure gradients ∇_{pi}, etc., are also negligible relative to the electrodynamic forces, then Eq. (12a) becomes

$$i\omega\mathbf{v}_e + \frac{e}{m_e}\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}_e\times\mathbf{H}_0)\right] = -(\nu_{e_i} + \nu_{e_m})\mathbf{v}_e = -\nu_{eff}\mathbf{v}_e \tag{20}$$

where ν_{eff} is the effective number of collisions. Equation (20) may be solved (6) for $\mathbf{v}_e$; thus

$$i\omega\left[\frac{(1+i\nu_{eff})}{\omega}\mathbf{v}_e - \frac{e\mathbf{H}_0\times\mathbf{v}_e}{im_e\omega c}\right] = \frac{-e}{m_e}\mathbf{E}.$$

Now let

$$\Omega = \frac{eH_0}{m_e\omega}\begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} = \frac{\omega_g}{\omega}\begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

where H_0 is directed along the z axis, and $\omega_g = eH_0/m_ec$ is the gyro or cyclotron frequency. For convenience let $\beta = 1 + i\nu/\omega$; then

$$i\omega(\beta\mathbf{I} + i\Omega)\cdot\mathbf{v}_e = -\frac{e}{m_e}\mathbf{E} \tag{21a}$$

where $\mathbf{I}$ is the unit tensor. Equation (21a) can be rewritten as

$$\mathbf{v}_e = -\frac{e}{i\omega m_e}(\beta\mathbf{I} + i\Omega)^{-1}\cdot\mathbf{E}. \tag{21b}$$

From the definition of $\mathbf{D}$ (Eq. (17)) one obtains

$$\mathbf{D} = \left(\frac{-4\pi e^2 n_0\mathbf{v}_e}{i\omega m_e} + \mathbf{E}\right) = \left[\mathbf{I} - \frac{\omega_0^2}{\omega^2}(\beta\mathbf{I} + i\Omega)^{-1}\right]\cdot\mathbf{E} \tag{22}$$

where $\omega_0 = \sqrt{4\pi e^2 n_0/m_e}$ is the plasma frequency. The dielectric tensor ϵ, by definition (Eq. (18)), is

$$\epsilon = \left[\mathbf{I} - \frac{\omega_0^2}{\omega^2}(\beta\mathbf{I} + i\Omega)^{-1}\right]. \tag{23a}$$

By writing down the components of the tensors as indicated in Eqs. (18) and (20) and performing the indicated inverse operation it is easy to show that the components of ϵ are

$$\epsilon_{xx} = \epsilon_{yy} = \left\{ 1 - \frac{\omega_0^2 (\omega^2 - i\nu_{eff})}{\omega\left[(\omega - i\nu_e)^2 - \omega_g^2\right]} \right\} \tag{23b}$$

$$\epsilon_{xy} = -\epsilon_{yx} = \frac{-i\omega_g \omega_0^2}{\omega\left[\omega - i\nu_{eff})^2 - \omega_g^2\right]} \tag{23c}$$

$$\epsilon_{zz} = \left[1 - \frac{\omega_0^2}{(\omega^2 - i\nu_{eff}\omega)} \right]. \tag{23d}$$

All other components are zero.

If the approximations leading to Eqs. (22) are not made, i.e., Eqs. (12a), (12b), and (12c) are retained while the pressure term is still assumed to be negligible and $m_i = M$, then one is led by the same sort of analysis to the following components for ϵ:

$$\epsilon_{xx} = \epsilon_{yy}, \quad \epsilon_{xy} = \epsilon_{zx} = \epsilon_{yz} = \epsilon_{zy} = 0, \quad \epsilon_{xy} = -\epsilon_{yx}$$

$$\epsilon_{zz} = 1 - \frac{\omega_0^2}{\omega\left[\omega - i(\nu_{e_i} + \nu_{e_m})\right]} \tag{24a}$$

$$\epsilon_{xx} \mp i\epsilon_{xy} = 1 - \frac{\omega_0^2 \left(1 + \dfrac{\nu_{i_m}}{i\omega + \nu_{i_m} \dfrac{n_0}{N_0}} \right)}{A - iB} \tag{24b}$$

$$A = \left[\omega \mp \omega_g - i(\nu_{e_i} + \nu_{e_m})\right] \left(\omega + \Omega g + \frac{\omega\nu_{i_m}}{i\omega + \nu_{i_m} \dfrac{n_0}{N_0}} \right)$$

$$B = \left(\frac{\omega\nu_{e_m} \dfrac{m_e}{M}}{i\omega + \nu_{i_m} \dfrac{n_0}{N_0}} \mp \Omega g \right) \left(\nu_{e_i} + \frac{\nu_{e_m}\nu_{i_m} \dfrac{n_0}{N_0}}{i + i_m \dfrac{n_0}{N_0}} \right)$$

where $\Omega g = eH_0/m_i c$ is the ion cyclotron frequency.

As a result of the previous analysis the equations describing the propagation of waves through a plasma were shown to be

$$\nabla \times \mathbf{E} = \frac{-i\omega}{c} \mathbf{h} \tag{25a}$$

$$\nabla \times \mathbf{h} = \frac{4\pi}{c} (i\omega \mathbf{D} + \mathbf{J}_{ex}) \tag{25b}$$

$$\nabla \cdot \mathrm{D} = 0 \tag{25c}$$

$$\nabla \cdot \mathbf{h} = 0 \tag{25d}$$

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E} \tag{25e}$$

where $\mathbf{J}_{ex}$ is the current density external to the region containing the plasma. In addition one has the relation

$$\nabla \cdot \mathbf{E} = 4\pi\rho = 4\pi n_0 e(n_i - n_e) \tag{25f}$$

which specifies the oscillating electric charge; i.e., $\nabla \cdot \mathbf{E}$ represents the charge due to the polarization of the medium. It should be noted that, by definition, for an incompressible plasma all temporal and spatial gradients vanish; i.e., the quantities n_i, n_e, and n_m are all equal to zero, since they represent a variation in density from the constant value n_0. This implies that $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{v}_i = \nabla \cdot \mathbf{v}_e = 0$. Of course an incompressible plasma does not exist in nature; however, there may be some interest in studying its properties, as the flow of an incompressible fluid is one of the simplest possible.

THE RAYLEIGH-CARSON THEOREM

As pointed out previously (Eq. (23a)) the dielectric tensor is antisymmetric and in the particular case where the magnetic field is directed along the z axis is of the form

$$\boldsymbol{\epsilon} = \begin{Vmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ -\epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{Vmatrix} . \tag{26}$$

The ϵ_{ij} are complex numbers if there are collisions in the gas. In general the dielectric tensor may be written as the sum of a symmetric and antisymmetric part, i.e.,

$$\epsilon_{ij} = a_{ij} + b_{ij} \tag{27}$$

where $a_{ij} = a_{ji}$ and $b_{ij} = -b_{ji}$.

Suppose now that there are two different distributions of fields which will be represented by $\mathbf{E}^1$, $\mathbf{H}^1$, $\mathbf{J}^1$, $\hat{\mathbf{E}}^2$, $\hat{\mathbf{H}}^2$, and $\hat{\mathbf{J}}^2$. The $\hat{\mathbf{E}}^2$, etc., are presumed to satisfy the same set of equations as the $\mathbf{E}^1$, i.e.,

$$\nabla \times \hat{\mathbf{E}}^2 = \frac{-i\omega}{c} \hat{\mathbf{H}}^2 \tag{28a}$$

$$\nabla \times \hat{\mathbf{H}}^2 = \frac{4\pi}{c} (i\omega\hat{\boldsymbol{\epsilon}} \cdot \hat{\mathbf{E}}^2 + \hat{\mathbf{J}}^2_{ex}) \tag{28b}$$

$$\nabla \cdot \hat{\boldsymbol{\epsilon}} \cdot \hat{\mathbf{E}}^2 = 0 \tag{28c}$$

$$\nabla \cdot \hat{\mathbf{H}}^2 = 0 . \tag{28d}$$

The tensor $\hat{\boldsymbol{\epsilon}}$ has not been defined precisely as yet, but its relationship to $\boldsymbol{\epsilon}$ will be demonstrated later.

By use of the field equations for the $\mathbf{E}$ and $\hat{\mathbf{E}}$ it can be shown that

$$(\hat{H}^2\cdot\nabla\times E^1 - E^1\cdot\nabla\times\hat{H}^2) + (\hat{E}^2\cdot\nabla\times H^1 - H^1\cdot\nabla\times\hat{E}^2) = \nabla\cdot(E^1\times\hat{H}^2 - \hat{E}^2\times H^1)$$

$$= \frac{-i\omega}{c}(\hat{H}^2\cdot H^1 - H^1\cdot\hat{H}^2) + \frac{i\omega 4\pi}{c}(\hat{E}^2\cdot\epsilon\cdot E^1 - E^1\cdot\epsilon\cdot\hat{E}^2). \quad (29)$$

In Eq. (29), $(\hat{H}^2\cdot H^1 - H^1\cdot\hat{H}^2)$ vanishes immediately, while $(\hat{E}^2\cdot\epsilon\cdot E^1 - E^1\cdot\epsilon\cdot\hat{E}^2)$ can also be shown to vanish provided $\hat{\epsilon}$ satisfies certain conditions. To establish these conditions, consider

$$\hat{E}^2\cdot\epsilon\cdot E^1 = \hat{E}_i^2\,\epsilon_{ij}\,E_j^1 = (a_{ij} + b_{ij})\,E_j^1\,\hat{E}_i^2$$

and also

$$E^1\cdot\hat{\epsilon}\cdot\hat{E}^2 = E_i^1\,\hat{\epsilon}_{ij}\,\hat{E}_j^2 = (\widehat{a_{ij} + b_{ij}})\,E_i^1\,\hat{E}_j^2.$$

If $\hat{\epsilon}$ is defined so that

$$(\widehat{a_{ij} + b_{ij}}) = (a_{ij} - b_{ij}) \quad (30)$$

then

$$E^1\cdot\hat{\epsilon}\cdot\hat{E}^2 = (\widehat{a_{ij} + b_{ij}})\,E_i^1\,\hat{E}_j^2 = (a_{ij} - b_{ij})\,E_i^1\,\hat{E}_j^2 = (a_{ij} + b_{ij})\,\hat{E}_i^2\,E_j^1 = \hat{E}^2\cdot\epsilon\cdot E^1$$

where the summation indices have been interchanged in the next to the last term and the properties of a_{ij} and b_{ij} have been utilized. It follows that $(\hat{E}^2\cdot\epsilon\cdot E^1 - E^1\cdot\epsilon\cdot\hat{E}^2)$ vanishes if Eq. (30) is satisfied. With these conditions established Eq. (29) may be integrated over all space with the result that

$$\int \nabla\cdot\left[E^1\times\hat{H}^2 - \hat{E}^2\times H^1\right] d\tau = \int_S (E^1\times\hat{H}^2 - \hat{E}^2\times H^1)\cdot dS = \int(\hat{E}^2\cdot J_{ex}^1 - E^1\cdot\hat{J}_{ex}^2)\,d\tau. \quad (31)$$

If the bounding surface S is placed at infinity, then the surface integral vanishes (7); thus the volume integral also vanishes.

The fields $\hat{E}^2$ and $\hat{H}^2$ are simply the electromagnetic fields obtained by replacing the actual magnetoactive plasma in which the dielectric tensor is ϵ by a fictitious medium in which it is ϵ. This is equivalent to reversing the direction of the static magnetic field.

It follows from Eq. (31) that

$$\int \hat{E}^2\cdot J_{ex}^1\,d\tau = \int E^1\cdot\hat{J}_{ex}^2\,d\tau . \quad (32)$$

By interchanging the circumflex in Eq. (32) it immediately follows that

$$\int E^2\cdot\hat{J}_{ex}^1\,d\tau = \int \hat{E}^1\cdot J_{ex}^2\,d\tau . \quad (33)$$

Either equation expresses the same physical content. As a matter of convenience in some applications the integral on the right side of Eq. (32) may be represented by the product $v^1 I^2$. For example suppose that when a voltage v^1 is applied to antenna 1 a current I^2 flows in antenna 2 oriented in the same direction as a result of the voltage applied to the first antenna. Since the applied voltage v^1 only has a value in the region where the voltage is applied (assuming perfect conductors), the integration gives $v^1 I^2$, where I^2 is the current in antenna 2.

CHARGE DISTRIBUTION AROUND A SATELLITE

A satellite moves through a partially ionized medium composed of positive ions, electrons, and neutral molecules. If it is assumed that the ions in the plasma are near thermal equilibrium, then the thermal velocities of the electrons will be approximately 100 times that of the ions. Thus the incident flux of electrons on the satellite will be larger than the ion flux and tend to charge the satellite to a negative potential. The satellite potential can be determined from the requirement that the ion current equal the electron current, for electrical equilibrium. That is, the satellite acquires a negative charge of such a value such that electrons having an energy less than the satellite potential are repelled, thus limiting the electron current to the ion current.

In order to calculate the satellite potential the method of Jastrow and Pearse (8) will be followed. The following parameters of the ionosphere and satellite will be assumed:

Satellite Velocity $V_s = 10^6$ cm/sec

Satellite Radius = R cm

Satellite Potential = φ_0

Electron and ion temperatures, $T_e = T_i = T = 1500°K$

k = Boltzmann's constant = 1.4×10^{-16} erg/°K

Ambient value of electron and ion density, $n_0 = 10^5/cm^3$.

The most probable ion velocity is $\sqrt{2kT/m_i} = 10^5$, so that $V_i << V_s << V_e$. Since the ion velocity is much less than the satellite velocity, the satellite plows into a stationary distribution of ions, so that the flux of positive ions striking the satellite becomes

$$\pi R^2 n_i V_s \,. \tag{34}$$

Since the electron velocity is much greater than the velocity of the satellite, electrons strike the vehicle from all directions. However, only those electrons with velocities V_e such that

$$V_e > \sqrt{2|e\varphi_0|/m_e} \tag{35}$$

will reach the satellite (φ_0 is the satellite potential). In the following the subscript on m_e will be dropped for convenience. The flux of electrons reaching the satellite is

$$4\pi R^2 n_e \int_{\sqrt{|e\varphi_0|/kT}}^{\infty} \frac{\epsilon^{-mc^2/2kT_e}}{(2\pi kT_e/m)^{3/2}} c^2 dc \int_0^{\pi/2} c \sin\theta \cos\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{4\pi R^2 n_e}{\sqrt{2\pi}} \left(\frac{kT}{m}\right)^{1/2} \left(\frac{|e\varphi_0|}{kT} + 1\right) \epsilon^{-|\varphi_0 e|/kT} \tag{36}$$

By equating expression (34) to the right side of Eq. (36) there is obtained

$$\frac{2}{\sqrt{\pi}}\left(\frac{2kT_e}{m}\right)^{1/2}\left(\frac{|e\varphi_0|}{kT}+1\right)\epsilon^{-|e\varphi_0|/kT} = V_s\,. \tag{37}$$

If one assumes a temperature of 1.5×10^3 °K for the electrons, then it is found that $|e\varphi_0|/kT = 5$, from which it follows that $\varphi_0 = 0.75$ volt.

This then is the potential assumed by the satellite. If an electron temperature of 1.5 ev is assumed rather than 0.15 ev, then the satellite potential becomes about 7.5 volts. Approximate experimental data give a value of about 3 to 5 volts.

Now the potential distribution satisfies the equation

$$\nabla^2\varphi = e(n_i - n_e) \tag{38}$$

where for the stationary case

$$n_i,\quad n_e \propto n_0\epsilon^{-|e\varphi|/kT} \tag{39}$$

However, since the ion velocity $v_i << v_s$, the ion distribution cannot move in such a manner as to adjust to the requirement of Eq. (39), so that the ion density cannot depend upon φ. In other words the satellite is moving through a cloud of stationary ions and the satellite potential cannot produce appreciable changes in the motion of the massive ions. On the other hand, the velocities of the electrons are greater than the satellite velocity, so that the charge density for the electrons is as indicated in Eq. (39). The Poisson equation becomes, in this case,

$$\nabla^2\varphi = -4\pi\rho = -4\pi e n_0\left(1 - \epsilon^{-|e\varphi|/kT}\right) \tag{40}$$

which is a modified form of the Debye-Hückle equation.

In order to obtain an approximation to Eq. (40), we note that the charge density is approximately equal to n_0e near the satellite and decreases to zero as the field point moves away from the satellite, as will be verified. As a first approximation, then, it will be assumed that

$$\rho = \begin{cases} n_0 e & R < r < R + \ell \\ 0 & r > \ell + R \end{cases} \tag{41}$$

Here n_0e is the ambient positive charge due to the positive ions (all the electrons are assumed to have been repelled by the negative potential on the satellite) and R is the satellite radius. The quantity ℓ is the screening distance, at which distance from the satellite the negative charge on the satellite is completely screened out.

Integration of Eq. (40) with a constant charge density yields

$$\varphi = \frac{-2\pi n_0 e}{3}r^2 - \frac{a}{r} + b \tag{42}$$

where a and b are constants of integration. According to the assumptions made above

$$\varphi(R) = \varphi_0 = \frac{-2\pi}{3}n_0 eR - \frac{a}{R} + b \tag{43}$$

and

$$\varphi(R+\ell) = \frac{-2\pi}{3}n_0 e(R+\ell)^2 - \frac{a}{R+\ell} + b = 0\,. \tag{44}$$

The surface charge density on the satellite is

$$4\pi\sigma = E_0 = -\left(\frac{\partial \varphi}{\partial r}\right)_R = \frac{-4\pi n_0 e}{3} R + \frac{a}{R^2} .$$

So the total charge is

$$4\pi R^2 \sigma = \frac{-4\pi n_0 e}{3} R^3 + a . \tag{45}$$

In order to screen out the electric field due to the charge on the satellite the total positive charge surrounding the satellite must equal that on the device itself; thus

$$\frac{-4\pi n_0 e}{3} R^3 + a = \frac{4\pi}{3} n_0 e \left[(R+\ell)^3 - R^3\right] .$$

Then

$$a = \frac{4\pi}{3} n_0 e (R+\ell)^3 . \tag{46}$$

From Eqs. (44) and (45) it follows that

$$\ell^2 = \frac{|\varphi_0|}{2\pi n_0 e} = \frac{2|\varphi_0 e|}{kT} \frac{kT}{4\pi n_0 e^2} \tag{47}$$

or

$$\ell = \sqrt{\frac{2\varphi e}{kT}} \lambda_D \tag{48}$$

where $\lambda_D = \sqrt{kT/4\pi n_0 e}$ is the Debye length.

In the case under consideration, $\lambda_D = 0.8$ cm and $\ell = 2.5$ cm. According to Eq. (48), ℓ will be larger the smaller n_0 is for a given value of φ_0. The satellite potential φ_0 does not depend upon the electron density in this approximation but only upon the mean electron energy and satellite velocity.

The question now arises as to whether the plasma sheath surrounding a satellite antenna can affect the reception of electromagnetic signals. In the discussion of the characteristics of the sheath it was found that the electron density near the satellite was e^{-5} or about two orders of magnitude below the normal plasma electron density and that the sheath extended outward from the satellite a distance of 2.5 cm or so. However, the ion density was near its normal value in this region. Since it is only the electrons that affect the propagation of electromagnetic signals (for frequencies above the ion plasma and cyclotron frequencies), it is reasonable to approximate the sheath by free space. That is, in such a model the normal plasma extends to a distance ℓ from the satellite and the spherical shell of radii R and $(R+\ell)$ is presumed to be free space.

It now appears unlikely that the plasma sheath can affect the propagation of vlf signals, since $k\ell \ll 1$; i.e., the thickness of the sheath is a minute fraction of the wavelength (k = propagation constant).

This argument will be placed on a more rigorous basis by considering the interaction of a plane wave with a plasma void. Consider the situation in which a plane wave of wavelength λ_p propagating in a magnetoactive plasma is normally incident upon an infinite slab of thickness 2ℓ containing no plasma where $\ell/\lambda_p \ll 1$. In general the plane wave has three components of electric intensity between which there is a linear relationship (3), viz., $\alpha E_x + \beta E_y + \gamma E_z = 0$. Now since there is no charge in the slab, the normal component E_z of the vector **E**, produces a charge $\epsilon_{zz} E_z$ on the boundary of the plasma upon which the electromagnetic signal is incident. Furthermore since the component E_z must vanish inside the slab (the component of the electric field along the direction of propagation cannot exist in an uncharged medium), a charge identical to that on the incident boundary must exist on the exit boundary so that the normal component E_z of the transmitted field is identical to the incident. The tangential components E_x and E_y are transmitted across the slab such that the signal incident on the exit boundary is proportional to $\epsilon^{i\omega 2\ell/c} \approx 1$; thus the tangential components of the transmitted signal can differ from the incident only by quantities of the order of $\omega 2\ell/c \ll 1$. Furthermore all transmitted components must satisfy the same linear relationship $\alpha E_x + \beta E_y + \gamma E_z = 0$ as for the incident signal. Thus the signal on both slabs are the same within quantities of the order $\omega 2\ell/c$, and the antenna inside the slab is subjected to the same tangential fields as exist without. (This conclusion would not be correct if the thickness of the slab were an appreciable fraction of a wavelength.) Since the antenna is subjected to the same fields inside as exist outside, the voltage induced in the antenna should not be influenced by the plasma sheath, at least to first approximation.

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The equations of electromagnetic wave propagation in a magneto-ionic environment are first derived via the hydrodynamic equations of motion. This subject is presented in order to exhibit explicitly the approximations made in obtaining the equations of electrodynamics as applied to a plasma. An application of the above results are then made to derive a generalized equation of reciprocity for a plasma medium. Last a theoretical discussion of the plasma sheath surrounding a satellite shows that the satellite is charged to a potential of about 0.75 volt and that the sheath extends out about 2.5 cm. Such a sheath may affect the characteristics of an antenna.

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