NRL Report 6150

Analysis of Errors in Bubble Chamber Track Measurements

R. G. GLASSER

Cosmic Ray Branch
Nucleonics Division

August 7, 1964

U.S. NAVAL RESEARCH LABORATORY
Washington, D.C.

CONTENTS

Copies available from Office of Technical Services
Department of Commerce - $.50

ABSTRACT

The general procedures of propagation of errors have been applied to derive error estimates for the quantities derived in the geometrical reconstruction of tracks and points in space from overdetermined stereoscopic measurements. Formulas are given which are suitable for use in computer programs such as PACKAG. The sources of error explicitly considered are measurement errors and multiple scattering errors. All computations use a Gaussian approximation to the distribution function.

PROBLEM STATUS

This is an interim report on a continuing problem.

AUTHORIZATION

NRL Problem H01-07
Project RR 002-06-41-5004

Manuscript submitted July 17, 1964.

ANALYSIS OF ERRORS IN BUBBLE CHAMBER TRACK MEASUREMENTS

I. INTRODUCTION AND GENERAL REMARKS

In the analysis of bubble chamber data, events are frequently subjected to tests of consistency with alternative interpretations. In order for such tests on alternative hypotheses to be meaningfully carried out, it is necessary to use reasonably accurate values for the relative precision of various measurements. If one is to compare these tests with standard measures of significance, it is also desirable to know the overall precision of these measurements.

The measurements of points on the film are reduced to fitted values of the relevant track parameters by various reconstruction programs. In this report the propagation of errors will be based on the reconstruction program now in use at NRL and the University of Maryland. This program is based on the Berkeley PACKAG program but has been modified to use three views and allow for designation of corresponding points.

In the process of reconstruction, the points measured on the film with respect to standard fiducials are transformed to an origin centered at the optic axis. The coordinates of the measured point are then corrected for various optical effects. The effect of this part of the reconstruction on the errors will be ignored here.

We use a somewhat simplified model of the reconstruction process, which serves to allow the computation of the essential elements of the error propagation.* We neglect all effects of the index of refraction. This principally means that all the distances used normal to the fiducial plane have to be considered as optical path lengths rather than true distances. All cameras will be assumed to lie in one plane.

In discussing measurements based on two cameras, we will assume that camera 1 is at $(-L, 0, -Z)$ and camera 2 at $(+L, 0, -Z)$ (see Fig. 1). We will assume, as given, coordinates (ξ_i, η_i) on the X-Y plane. These will be assumed to be independent variables with a common standard deviation Δ. The coordinates (ξ_i, η_i) are measured with respect to the intersection of the optic axis of camera i with the X-Y plane.

II. POINTS IN SPACE

Neglecting effects of indices of refraction, we can write for the reconstruction formulas:

$$\frac{x + L}{\xi_1} = \frac{L - x}{-\xi_2} = \frac{y}{\eta_1} = \frac{y}{\eta_2} = \frac{Z + z}{Z} . \tag{1}$$

We first consider corresponding points with two cameras. If they truly correspond, $\eta_1 = \eta_2$. The difference $\eta_1 - \eta_2$ has variance $2\Delta^2$. We have used this to estimate Δ.† We use the average of η_1 and η_2 for their common value. Equation (1) for corresponding points leads to

---

*See Appendix A for a brief discussion of the techniques of error propagation.

†All discussion of the values of the constants is in Appendix B.

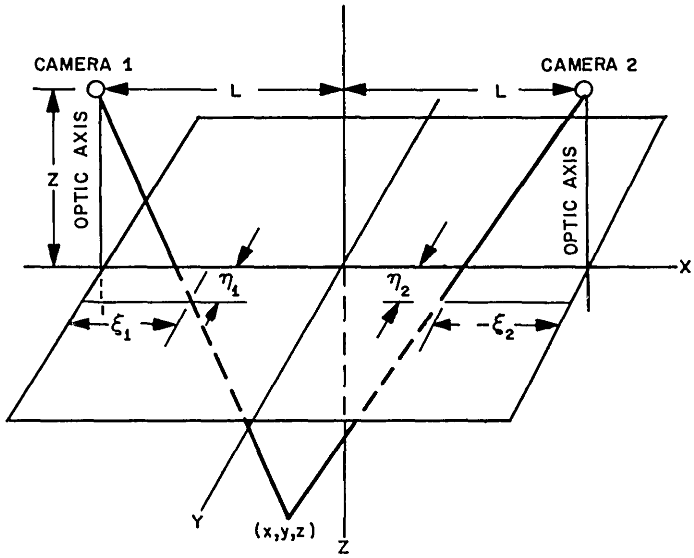

Fig. 1 - Two camera coordinates

$$x = \frac{L(\xi_1 + \xi_2)}{\xi_1 - \xi_2} \tag{2a}$$

$$y = \frac{L(\eta_1 + \eta_2)}{\xi_1 - \xi_2} \tag{2b}$$

$$z = Z\left(\frac{2L}{\xi_1 - \xi_2} - 1\right) . \tag{2c}$$

This leads to a variance matrix for x, y, z:

$$V = A^2 \begin{pmatrix} 1 + \frac{x^2}{L^2} & \frac{xy}{L^2} & \frac{xQ}{L} \\ \frac{xy}{L^2} & 1 + \frac{y^2}{L^2} & \frac{yQ}{L} \\ \frac{xQ}{L} & \frac{yQ}{L} & Q^2 \end{pmatrix} = A^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A^2 \begin{pmatrix} \frac{x}{L} \\ \frac{y}{L} \\ Q \end{pmatrix} \left(\frac{x}{L}, \frac{y}{L}, Q\right) \tag{3}$$

where $A = [(Z + z)/\sqrt{2}\ Z]\Delta$ and $Q = (Z + z)/L$. The second form in Eq. (3) exhibits explicitly the fact that all the difficult terms stem from changing from a conical projection to a Cartesian projection.

To get an idea of the order of magnitude of the terms in this equation, some values appropriate to the Brookhaven National Laboratory 30-inch hydrogen chamber may be cited:

$\Delta \approx 50\ \mu$

$A = 37\ \mu$ near the front of the chamber

$$A = 45\ \mu \text{ near the back of the chamber}$$

$$Q = 5 \text{ to } 6$$

$$\max\ (x^2/L^2) = 1.$$

If corresponding points are not measured, then the reconstruction program constructs an artificial corresponding point by interpolation in measured points, say (ξ_A, η_A) and (ξ_B, η_B) of view 2, using the requirement that $\eta_2 = \eta_1$. The interpolation is not strictly linear, but the calculation of errors as though it were is adequate. From (1) and the interpolation, the values of x, y, z are

$$x = \frac{L(\xi_1 + \xi_2)}{\xi_1 - \xi_2} \tag{4a}$$

$$y = \frac{2L\eta_1}{\xi_1 - \xi_2} \tag{4b}$$

$$z = \frac{2LZ}{\xi_1 - \xi_2} - Z \tag{4c}$$

where

$$\xi_2 = \xi_A \frac{\eta_B - \eta_1}{\eta_B - \eta_A} + \xi_B \frac{\eta_1 - \eta_A}{\eta_B - \eta_A}.$$

If we let

$$\epsilon = \frac{\eta_1 - \eta_A}{\eta_B - \eta_A}; \qquad \theta = \tan^{-1} \frac{\eta_B - \eta_A}{\xi_B - \xi_A}; \qquad \lambda = 1 - \epsilon + \epsilon^2$$

then the variance matrix for x, y, z is

$$V = A^2 \begin{pmatrix} \lambda \csc^2\theta \left(1 + \frac{x^2}{L^2}\right) - \epsilon \frac{x}{L}(1 - \lambda \csc^2\theta) & \text{------}^* & \text{------}^* \\ \left(1 - \frac{x}{L}\right)\operatorname{ctn}\theta - \left(1 + \frac{x}{L}\right)\frac{y}{2L} + \left(1 - \frac{x}{L}\right)\frac{y}{2L}(\lambda\csc^2\theta - 1) & 2 + \frac{y}{L}\operatorname{ctn}\theta + \lambda\frac{y^2}{L^2}\csc^2\theta & \text{------}^* \\ \frac{Q}{2}\left[\left(1 + \frac{x}{L}\right) - \left(1 - \frac{x}{L}\right)(\lambda\csc^2\theta - 1)\right] & Q\left(\operatorname{ctn}\theta + \frac{y}{2L}\lambda\csc^2\theta\right) & \lambda Q^2 \csc^2\theta \end{pmatrix}$$

We also construct a space point by constructing an artificial corresponding point in view 1 corresponding to the measured point in view 2.

Then for tracks with not too large a dip we get for the average variance of these two points,

*Since V is a symmetric matrix, these values have been omitted.

$$V = A^2 \begin{pmatrix} \lambda \csc^2\theta \left(1 + \frac{x^2}{L^2}\right) & \operatorname{ctn}\theta + \lambda \frac{xy}{L^2}\csc^2\theta & \lambda Q \frac{x}{L}\csc^2\theta \\ \operatorname{ctn}\theta + \lambda \frac{xy}{L^2}\csc^2\theta & 2 + \lambda \frac{y^2}{L^2}\csc^2\theta & \lambda Q \frac{y}{L}\csc^2\theta \\ \lambda Q \frac{x}{L}\csc^2\theta & \lambda Q \frac{y}{L}\csc^2\theta & \lambda Q^2 \csc^2\theta \end{pmatrix} . \tag{5}$$

The origin of the correlation terms is, as before, simply due to the conical projection except for the extra X-Y correlation. This is due to the interpolation. If we rotate the variance matrix, but not X and Y, to a coordinate system parallel to the track projection, then we have

$$V = A^2 \begin{pmatrix} 1 + \csc^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A^2(\lambda - 1)\cos^2\theta \begin{pmatrix} 1 & \operatorname{ctn}\theta & 0 \\ \operatorname{ctn}\theta & \operatorname{ctn}^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \csc^2\theta DD^+ \tag{6}$$

where

$$D = \begin{pmatrix} \frac{x}{L}\sin\theta - \frac{y}{L}\cos\theta \\ \frac{y}{L}\sin\theta + \frac{x}{L}\cos\theta \\ Q \end{pmatrix} .$$

The main effect of using noncorresponding points as compared to the previously discussed case of corresponding points is then to make the error in position along the track larger by the factor $(1 + \csc^2\theta)$ since $\lambda - 1$ is small. Considering the restriction normally used that $\csc\theta < 2$, the average factor is $1 + (3\sqrt{3}/\pi) = 2.65$, and the range of values of this factor is 1 to 5.

If more than two views are used, then the problem of errors becomes more complicated. We begin again by considering the case of corresponding points. Let camera i be located at $(C_i, D_i, -Z)$. (We keep all cameras at a common Z just to keep the formulas from becoming even more cumbersome.) The coordinate origin in the X-Y plane is chosen as the center of gravity of the camera projections. Then the formulas corresponding to (1) are

$$\frac{x - C_i}{\xi_i} = \frac{y - D_i}{\eta_i} = \frac{z + Z}{Z} . \tag{7}$$

It is possible to consider several approaches to the solution of (7). We may solve all $2N$ equations for the three unknowns x, y, z as a single overdetermined problem or solve them view pair by view pair as just indicated. If we adopt the first approach, the solution is

$$x = \frac{Z + z}{Z}\,\frac{1}{N}\Sigma\xi_i ; \qquad y = \frac{Z + z}{Z}\,\frac{1}{N}\Sigma\eta_i ; \qquad z = -Z\left[\frac{\Sigma C_i^2 + \Sigma D_i^2}{\Sigma C_i\,\xi_i + \Sigma D_i\,\eta_i} + 1\right] . \tag{8}$$

We get for the variance matrix

$$V = A^2 \begin{pmatrix} \frac{2}{N} + 2\frac{x^2}{R^2} & 2\frac{xy}{R^2} & 2\frac{x(Z+z)}{R^2} \\ 2\frac{xy}{R^2} & \frac{2}{N} + 2\frac{y^2}{R^2} & 2\frac{y(Z+z)}{R^2} \\ 2\frac{x(Z+z)}{R^2} & 2\frac{y(Z+z)}{R^2} & 2\left(\frac{Z+z}{R}\right)^2 \end{pmatrix} \tag{9}$$

where

$$A = \left(\frac{Z+z}{\sqrt{2Z}}\right)\Delta \quad \text{and} \quad R^2 = \Sigma(C_i^2 + D_i^2).$$

In particular for the case $N = 3$ with the cameras at the vertices of an equilateral triangle, we can set $R = 2\sqrt{3L}$ and $Q = (Z+z)/L$, and we get

$$V = A^2 \begin{pmatrix} \frac{2}{3} + \frac{1}{2}\left(\frac{x}{L}\right)^2 & \frac{1}{2}\frac{xy}{L^2} & \frac{1}{2}\frac{x}{L}Q \\ \frac{1}{2}\frac{xy}{L^2} & \frac{2}{3} + \frac{1}{2}\left(\frac{y}{L}\right)^2 & \frac{1}{2}\frac{y}{L}Q \\ \frac{1}{2}\frac{x}{L}Q & \frac{1}{2}\frac{y}{L}Q & \frac{1}{2}Q^2 \end{pmatrix} \tag{9'}$$

If we adopt the alternative approach we take the camera pairs and construct as done before. For the three-camera case this gives three pairs which can be constructed. The formulas are similar to (1) through (3). If we define x, y, z to be the arithmetic average of the three sets thus generated, we get for the variance of these:

$$V = A^2 \begin{pmatrix} \frac{17}{24} + \frac{1}{2}\frac{x^2}{L^2} & \frac{1}{2}\frac{xy}{L^2} & \frac{1}{2}\frac{xQ}{L} \\ \frac{1}{2}\frac{xy}{L^2} & \frac{17}{24} + \frac{1}{2}\frac{y^2}{L^2} & \frac{1}{2}\frac{y}{L}Q \\ \frac{1}{2}\frac{xQ}{L} & \frac{1}{2}\frac{yQ}{L} & \frac{1}{2}Q^2 \end{pmatrix} \tag{10}$$

The difference between (9') and (10) is small enough to justify choosing the reconstruction procedure on other grounds than the fact that (9') is somewhat smaller than (10). In particular the procedure for treating noncorresponding points is much more straightforward if done by view pairs, and the program structure is then simpler if all points are constructed that way.

It would also be possible to use a weighted average of the results from different view pairs, since the spatial dependence certainly gives different errors and therefore implies different weights if the point is not at the center of the chamber. However, while this would improve on (10) it cannot be better than (9) and hence cannot be enough of an improvement to warrant the additional programming complication.

For some purposes it will be satisfactory to further simplify the formulas by replacing the spatial dependence by averages over positions in the chamber. Assuming that the volume is defined by $x^2 + y^2 \leqq L^2$ and $0 \leqq z \leqq L$ we get from (10):

$$V = A^2 \begin{pmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{6} & 0 \\ 0 & 0 & \frac{Q^2}{2} \end{pmatrix} . \tag{11}$$

For two-view corresponding points the values are about

$$V = A^2 \begin{pmatrix} \frac{5}{4} & 0 & 0 \\ 0 & \frac{5}{4} & 0 \\ 0 & 0 & Q^2 \end{pmatrix} . \tag{12}$$

The addition of a third view has improved the precision in x and y by about 50 percent and has doubled the precision in z.

For noncorresponding points we can approximate the error matrix by

$$V = A^2 \begin{pmatrix} \frac{5}{4} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & Q^2 \end{pmatrix} \tag{13}$$

where in this case x is meant to be perpendicular to the track.

In Eqs. (11) through (13) reasonable values for the constants (for the BNL 30-inch chamber) are $A = 40\,\mu$ and $Q = 5.5$.

An additional remark must be made about correlations between different points constructed by the procedure outlined above when corresponding points are not measured. In this case when a space point is constructed from a measurement in view 1 using two points in view 2 to make the artificial corresponding point, then there will also be a point created using one of these same points in view 2 and creating an artificial point in view 1. The average correlation is 1/3 between such points. From the 30-degree stereo restriction each point so constructed will on the average be correlated to 1-1/2 points from other views. Therefore the average effective correlation will be 1/2. This means that if we measure N points for some measurement, we should count them as equivalent to $2N/3$ independent points.

Corresponding points will be created without such correlation. Also the error is smaller (since all three views are used; see Eq. (11)). If we weight points inversely with their variance, then the weight of a corresponding point relative to the perpendicular component of a noncorresponding point is 6/5 x 3/2 x 5/4 = 9/4. We approximately achieve this result if we use each corresponding point twice in all solutions.

If one made an effort to measure corresponding points without so labeling them, then the computation of the average correlation coefficient above, which assumed random spacing of points in one view with respect to the other, would be invalid. The average correlation would be 1/2 instead of 1/3, the effective correlation would be 3/4 instead of 1/2, the number of equivalent independent points for N measured points would be $4N/7$, and the relative weight of corresponding points would be 21/8 instead of 9/4.

III. TRACKS (MEASUREMENT ERRORS)

We are primarily interested not in the location of points in the bubble chamber but in determination of the track parameters used in the kinematics routine. These are, for PACKAG, k (the projected curvature), s (the slope, which is the tangent of the dip angle), and φ (the angle with respect to the X axis of the track projection on the X-Y plane). We also sometimes use the length of the track. It is the errors on these quantities which will be discussed here. In this discussion we will in general use the simplified form (13) of the variance matrix from measurement, assuming that the effect of corresponding points has been adequately taken into account by the procedure described in the last paragraph of Sec. II. In addition to measurement errors, there are several other sources of error for track parameters. The essentially systematic errors will be discussed briefly in Sec. V. The multiple scattering errors will be discussed in Sec. IV. Here we discuss only the effect of the point measurement errors as derived in Sec. II.

The reconstruction we use is based on a parabolic approximation to the track projection and a linear fit to the depth as a function of track length. Corrections are made to the fit for the change in curvature due to energy loss and nonuniform magnetic field and for the change in slope due to the radial field, but these corrections do not involve additional fitted parameters and will be omitted here. We start from the fit to a curve of the form

$$z_i = a + sx_i \tag{14a}$$

$$y_i = \frac{k}{2} x_i^2 + \varphi x_i + b \tag{14b}$$

where it is assumed that the track has been rotated so that its average direction is along the X axis. The parameters φ and s are respectively the azimuth and the slope of the point $x = 0$. Corrections in Eq. (14a) for the difference between x and the track length and in Eq. (14b) for the difference between a circle and parabola are made but will again be neglected here. The k of Eqs. (14) differs from the k of PACKAG only in the factor 0.3B (B in kilogauss) used to convert from units of cm^{-1} to Mev/c.

We fit the data to Eqs. (14) by least squares. The solution to Eqs. (14) is

$$s = \frac{\overline{xz} - \bar{x}\,\bar{z}}{\overline{x^2} - \bar{x}^2} \tag{15a}$$

$$k = \frac{2}{D} \begin{vmatrix} \bar{y} & \bar{x} & 1 \\ \overline{xy} & \overline{x^2} & \bar{x} \\ \overline{x^2 y} & \overline{x^3} & \overline{x^2} \end{vmatrix} \tag{15b}$$

$$\varphi = \frac{1}{D}\begin{vmatrix} \overline{x^2} & \bar{y} & 1 \\ \overline{x^3} & \overline{xy} & \bar{x} \\ \overline{x^4} & \overline{x^2y} & \overline{x^2} \end{vmatrix} \tag{15c}$$

where

$$D = \begin{vmatrix} \overline{x^2} & \bar{x} & 1 \\ \overline{x^3} & \overline{x^2} & \bar{x} \\ \overline{x^4} & \overline{x^3} & \overline{x^2} \end{vmatrix}$$

The dependence of the s, φ, k error matrix on the point errors is, of course, dependent on the exact pattern of points measured for a given track. We can see the main features by assuming that the points are measured at equally spaced intervals. Let

$$x_i = (2i - N - 1)\frac{L}{2N}, \qquad i = 1, 2, \ldots, N,$$

where L is the projected length of the track. We get then

$$x = \overline{x^3} = 0$$

$$\overline{x^2} = \frac{L^2}{12}\,\frac{N+1}{N-1}$$

$$\overline{x^4} = \frac{L^4}{80}\,\frac{N+1}{N-1}\,\frac{N^2 - \frac{7}{3}}{(N-1)^2}.$$

The effects of errors in x on the fit is small except for very steep tracks or tracks which curve through a very large angle. We will neglect this contribution except in $V(s)$. Then,

$$V(k) = \frac{4V(y)}{N\left(\overline{x^4} - \overline{x^2}^2\right)} = \frac{720A^2}{NL^4}\,\frac{(N-1)^3}{(N-2)(N+1)(N+2)} \tag{16a}$$

$$V(\varphi) = \frac{V(y)}{N\,\overline{x^2}} = \frac{12}{NL^2}\,\frac{N-1}{N+1}\,V(y) \tag{16b}$$

$$V(s) = \frac{V(z)}{Nx^2} = \frac{12}{NL^2}\,\frac{N-1}{N+1}\,(Q^2 + s^2)\,V(y)\,. \tag{16c}$$

The correlation terms all vanish. This conclusion depends on the approximation of neglecting the contribution of the x dependence and on the equal spacing of points, but even without these the correlations will be small.

The quantities involved in the above computation are all evaluated at the midpoint of the track. We deal generally in the fitting process with quantities evaluated at the endpoints:

$$k_e = k \pm \int_0^{L/2} \frac{dk}{dR} \approx k \pm \frac{L}{2} \left\langle \frac{dk}{dR} \right\rangle$$

$$\varphi_e = \varphi \pm \int_0^{L/2} k \, dR \approx \varphi \pm \frac{kL}{2}$$

$$s_e = s \pm \int_0^{L/2} k \frac{Hxy}{Hz} \, dR \approx s \, .$$

From these we obtain, assuming that dk/dR is approximately constant and that the error in L is negligible,

$$V(k_e) = V(k) \tag{16a'}$$

$$V(\varphi_e) = V(\varphi) + \frac{L^2}{4} V(k) = \frac{12}{NL^2} \frac{N-1}{N+1} \left[1 + \frac{15\,(N-1)^2}{N^2 - 4} \right] V(y) \tag{16b'}$$

$$V(s_e) = V(s) \tag{16c'}$$

$$V(\varphi_e, k_e) = \pm \frac{L}{2} V(k) = \pm \frac{360}{NL^3} \frac{N-1}{N+1} \frac{(N-1)^2}{N^2-4} V(y) \, . \tag{16d'}$$

The sign of the variance term is the same as the sign of $(\varphi_e - \varphi)$.

More generally we can replace Eqs. (14) by an expression of the form

$$y_i = \sum_{j=1}^{R} \alpha_j \, g_j(x_i) \tag{17}$$

where the α_j are the R parameters of the fit. In Eq. (14b), we had $\alpha_1 = b$, $\alpha_2 = \varphi$, $\alpha_3 = k$, $g_1 = 1$, $g_2 = x_i$, $g_3 = (1/2)\,x_i^2$. The least squares equation then appears as

$$G_{ij}\,\alpha_j = M_i$$

where

$$G_{ij} = \sum_k g_i(x_k) \, g_j(x_k)$$

$$M_i = \sum_k g_i(x_k) \, y_k \, .$$

Again neglecting the small contribution from errors in x,

$$V(\alpha_i, \alpha_j) = (G^{-1} S G^{-1})_{ij} \tag{18}$$

where

$$S_{ij} = \sum_k g_i(x_k) \, g_j(x_k) \, V(y_k) \, .$$

If all of the y_k have the same variance, then we get

$$S = V(y)\,G$$

$$V(\alpha_i, \alpha_j) = V(y)\,G^{-1}_{ij} \,. \tag{19}$$

If we use the assumption of equal spacing, then (19) is equivalent to Eqs. (14). We use (19) to get the errors, since G^{-1}_{ij} is available from the fit. $V(y)$ is estimated by averaging the values over the track from Eq. (6). Since only the $V(y)$ enters in (19), averaging over θ gives

$$V(y) = A^2\left[1 + (\bar{\lambda} - 1)\,\frac{1}{2} + \frac{1}{2}\left(\frac{x}{L}\right)^2\right].$$

We may also be interested in the error in the length of a track. In this case there is little contribution to the error from any source other than the measurement of the two endpoints, since straggling and rectification will produce negligible errors. Thus (using R for length to avoid confusion with L above for projected length),

$$R = \sqrt{(x_N - x_1)^2 + (z_N - z_1)^2}$$

$$V(R) = \frac{2}{1 + s^2}\,(1 + Q^2 s^2)\,V(y)$$

$$V(R, s) = \frac{12\,(N - 1)}{N\,(N + 1)}\,s\,\frac{A^2}{R}\,(Q^2 - 1)\,.$$

IV. MULTIPLE SCATTERING ERRORS

Assume that we have a particle scattering while passing through a set of thin foils (numbered for reference from 1 to n). The initial direction of the particle is 0 and its displacement (we consider only the one-dimensional problem) also zero. The figure illustrates the problem for five plates:

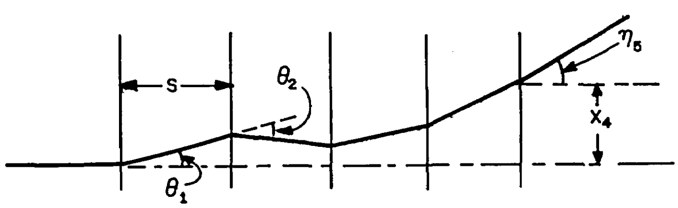

Figure 2

Denote the lateral displacement at the $i+1$th plate produced by the first i plates by x_i, the total angle of scatter after the ith plate by η_i, and the angle of scatter in the ith plate by θ_i (the small angle approximation will be used throughout). We will use the symbol $E(A)$ for the expected value of a statistical variable A. $E(\theta_i) = 0$ by symmetry. $E(\theta_i^2)$ will be the same for all foils, since the thickness is the same. Also,

$$\eta_n = \sum_{i=1}^{n} \theta_i$$

$$x_n = \sum_{i=1}^{n} s\eta_i = \sum_{i=1}^{n} \sum_{j=1}^{i} \theta_j s = s \sum_{j=1}^{n} (n-j+1)\,\theta_j \,.$$

Therefore,

$$E(\eta_n^2) = \sum_{j=1}^{n} E(\theta_j^2) = nE(\theta^2) \tag{20a}$$

$$E(\eta_n x_n) = \sum_{j=1}^{n} (n-j+1)\; sE(\theta_j^2) = sE(\theta^2)\,\frac{n(n+1)}{2} \tag{20b}$$

$$E(x_n^2) = \sum_{j=1}^{n} (n-j+1)^2\; s^2 E(\theta_j^2) = s^2 E(\theta^2)\,\frac{n(2n+1)\;(n+1)}{6}\,. \tag{20c}$$

Equation (18a) confirms the intuitively obvious fact that $E(\eta_n^2)$ is proportional to n and independent of the spacing; that is, in more physical terms, the angle of scatter depends on the amount of material traversed and is independent of the density. We are then justified in setting the expected square of the angle of scatter, $E(\eta_n^2)$, equal to

$$fd = \frac{K^2 d}{(p\beta c)^2}\,,$$

where d is the thickness of the foil and the constant $(K/p\beta c)^2$ has been chosen to give the conventional result. We have not, of course, derived here the p and β dependence.

Now let the numbers of plates in the above example tend to infinity, keeping the total amount of matter the same and the distance t constant and also the total thickness of scatterer D constant:

$$t = ns$$

$$D = nd$$

$$E(\eta^2) = nfd \to fD$$

$$E(\eta x) = \frac{n(n+1)}{2}\, sfd \to \frac{ftD}{2}$$

$$E(x^2) = \frac{n(2n+1)(n+1)}{6}\, s^2 fd \to \frac{ft^2 D}{3}\,.$$

We might as well now make t and D equivalent, taking the density into the constant f, and we have finally

$$E(\eta^2) = ft \tag{21a}$$

$$E(\eta x) = \frac{1}{2}\, ft^2 \tag{21b}$$

$$E(x^2) = \frac{1}{3}\, ft^3\,. \tag{21c}$$

Now consider a particle measured at three points along its trajectory. The measurements are assumed equally spaced. The figure illustrates the situation:

$$x_2 = x_1 + \eta_1 t + y$$

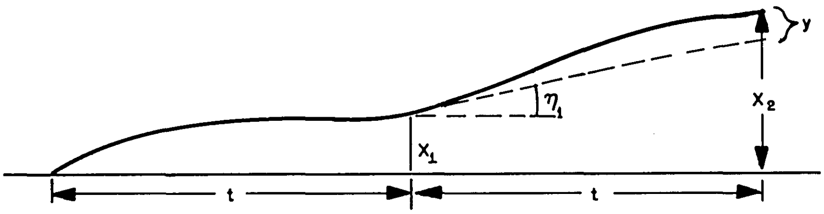

Figure 3

Since y is the additional deflection due to scattering in the second part of the trajectory, it will be independent of the scatter in the first part. We have then from Eq. (2):

$$E(x_1^2) = \frac{1}{3} f t^3 \tag{22a}$$

$$\begin{aligned} E(x_1 x_2) &= E(x_1^2) + t\,E(\eta_1 x) + E(x_1 y) \\ &= \frac{1}{3} f t^3 + \frac{1}{2} f t^3 + 0 \\ &= \frac{5}{6} f t^3 \end{aligned} \tag{22b}$$

$$\begin{aligned} E(x_2^2) &= E(x_1^2) + 2t\,E(x_1 \eta_1) + t^2 E(\eta_1^2) + E(y^2) \\ &= \frac{1}{3} f t^3 + f t^3 + f t^3 + \frac{1}{3} f t^3 \\ &= \frac{8}{3} f t^3 = \frac{1}{3} f t\,(2t)^3 . \end{aligned} \tag{22c}$$

It is informative to calculate the correlation coefficient of x_1 and x_2:

$$\rho = \frac{5/6}{\sqrt{(8/3)(1/3)}} = 0.88 .$$

We can now use these points to measure the apparent position of the track. Let us measure the three parameters x_0, φ', k', where x_0 is the starting point of the track, φ' is the initial direction, and $k' = 1/\rho$ is the curvature. In terms of the coordinates,

$$x_0 = x_0$$

$$k' = \frac{x_2 - 2x_1 + x_0}{t^2}$$

$$\varphi' = \frac{4x_1 - x_2 - 3x_0}{2t} .$$

Straight propagation of errors leads to the variance matrix for k' and φ' (x_0, by assumption, has no error associated with it):

$$G = \begin{array}{c|cc} & k' & \varphi' \\ \hline k' & \frac{2}{3}\, f/t & -\frac{1}{6}\, f \\ \varphi' & -\frac{1}{6}\, f & \frac{1}{3}\, ft \end{array} = \begin{array}{c|cc} & k' & \varphi' \\ \hline k' & \frac{4}{3}\, f/L & -f/6 \\ \varphi' & -f/6 & \frac{fL}{6} \end{array}$$

where $L = 2t$ is the track length. The above considerations were in the plane of the track. If the track has a dip λ with respect to the plane normal to the magnetic field, we are interested in the azimuth angle φ and the curvature k projected on this plane. The transformation is

$$k = \frac{k'}{\cos^2\lambda}$$

$$\varphi = \frac{\varphi'}{\cos\lambda}$$

and the resulting error matrix is

$$G = \begin{array}{c|cc} & k & \varphi \\ \hline k & \frac{4}{3}\, \frac{f}{L\cos^4\lambda} & -f\, \frac{1}{6\cos^3\lambda} \\ \varphi & -f\, \frac{1}{\cos^3\lambda} & \frac{fL}{6\cos^2\lambda} \end{array} .$$

There is also an error induced in the dip determination by the multiple scattering, but this is uncorrelated, in the first approximation, with the results of scattering in the plane and the magnitude of the errors is twice that for the azimuthal angle. After including this error the final result for the error matrix is

$$G = \begin{array}{c|ccc} & k & \lambda & \varphi \\ \hline k & \frac{4}{3}\, \frac{f}{L\cos^4\lambda} & 0 & -\frac{f}{6\cos^3\lambda} \\ \lambda & 0 & \frac{fL}{3} & 0 \\ \varphi & -\frac{f}{6\cos^3\lambda} & 0 & \frac{fL}{6\cos^2\lambda} \end{array} .$$

The correlation coefficient between k and φ is $1/2\sqrt{2} = 0.353$.

The error matrix was derived here from the assumption that the measurement was based on three points only. The measurement of more points certainly makes the error matrix smaller, but not by much. For example, if one calculates the correlation of an intermediate point halfway between two of the points just considered and the points already taken, it is 90%. Thus, only a minor decrease is expected and will be neglected here.

For tracks on which energy loss is significant, we can modify the derivation above. In Eqs. (20) let f depend on the cell (that is, let $E(\theta_j^2) = f_j d$); then we have

$$E(\eta_n^2) = \sum_{j=1}^{n} E(\theta_j^2) = \sum_{j=1}^{n} f_j d \tag{20a'}$$

$$E(\eta_n x_n) = \sum_{j=1}^{n} (n-j+1)\, s\, E(\theta_j^2) = \sum_{j=1}^{n} (n-j+1)\, s f_j d \tag{20b'}$$

$$E(x_n^2) = \sum_{j=1}^{n} (n-j+1)^2 s^2 E(\theta_j^2) = \sum_{j=1}^{n} (n-j+1)^2 s^2 f_j d. \tag{20c'}$$

Going to the continuous limit as before,

$$E(\eta^2) = \int_0^t f(s)\, ds \tag{21a'}$$

$$E(\eta x) = \int_0^t (t-s)\, f(s)\, ds \tag{21b'}$$

$$E(x^2) = \int_0^t (t-s)^2 f(s)\, ds\, . \tag{21c'}$$

In the limit $f(s) = \text{constant}$, Eqs. (21') reduce to Eqs. (21). Again considering a three-point measurement,

$$E(x_1^2) = \int_0^t (t-s)^2 f(s)\, ds \tag{22a'}$$

$$E(x_1 s_2) = \int_0^t \left[(t-s)^2 + t(t-s)\right] f(s)\, ds \tag{22b'}$$

$$E(x_2^2) = \int_0^t (2t-s)^2 f(s)\, ds\, . \tag{22c'}$$

Transforming to k' and φ',

$$E(k'^2) = \frac{1}{t^4}\left[\int_0^t s^2 f(s)\, ds + \int_t^{2t} (2t-s)^2 f(s)\, ds\right]$$

$$E(k'\varphi') = \frac{1}{2t^3}\left[\int_0^t (2t-3s)^2 f(s)\, ds + \int_0^{2t} (2t-s)^2 f(s)\, ds\right]$$

$$E(\varphi'^2) = \frac{1}{4t^2}\left[\int_0^t (2t-3s)^2 f(s)\, ds + \int_0^{2t} (2t-s)^2 f(s)\, ds\right].$$

For a track where f can be considered constant these agree with the relations derived previously for f constant. Now let f be changing and suppose $f(s) = f_0(2t-s)^{-\gamma}$, which might be appropriate, with $\gamma = 0.6$, for a stopping track. Then we can determine a position s_0 such that when k is evaluated at that point, the formula for constant t gives the same value of the scattering. This number will, of course, be different for the three moments. We find s_0 for $E(k^2)$ is 0.6L, for $E(\varphi^2)$ is 0.4L, and for $E(\varphi k)$ is essentially indeterminate since the correlation is negligible (due to an accidental cancellation). Since this is an extreme case, it will never be very bad to evaluate the multiple scattering errors at the midpoint.

V. REMARKS ON OTHER CONTRIBUTIONS TO THE ERRORS

We have assumed in Sec. II that we can neglect errors in optical constants, etc., in the propagation of errors. This is because such errors will have a systematic effect on the measurements and will not give rise to the kind of errors which can be removed by averaging. We should pay a little more attention to the measurement of fiducials, however. For each view we measure the position of three standard fiducials and refer all our measurements to these as a reference. If one of these is off, then as a result (for example, in (2), assuming all ξ_i are off by the same amount) the Z coordinates of all points will be changed systematically within this picture. The effect on the X and Y coordinates is mainly through the conical projection. When only two views are used, all points are systematically affected by about the same amount, and except for steep tracks the effect will be negligible on all computed track parameters. Steep tracks will be systematically made still steeper, however, and this may be desirable to watch out for as a systematic effect. A mistake in the value of the lens-to-chamber distance would have similar effects, but in this case the effects would be constant over the entire experiment rather than changing from one frame to the next.

In computing the errors we have used a simplified model of the track fitting process because it is analytically solvable. This should not strongly affect the discussion of the errors, since we are concerned with the random perturbations from the true curve and the fourth-order fit in PACKAG is adequate. We have not explicitly assumed the errors are Gaussian, since we dealt only with moments, but if the errors are not approximately Gaussian, we cannot expect the quantities computed and called χ^2 to have χ^2 distribution. There is probably no solution to this problem while we depend on human measurers, other than the standard practice of remeasuring events and then looking carefully at the second-time rejects for nonhuman induced sources, such as turbulence, kinks, and odd track structures.

ACKNOWLEDGMENTS

Throughout this report I have omitted references. As far as I am aware, there is no result herein which is totally new. The bibliographic research required to give adequate sources for the formulas would be enormous. Except for multiple scattering, most of the results only exist in the form of informal reports of groups involved. I have certainly benefited over the past few years from discussions with numerous members of the staffs of the Alvarez group at Berkeley, the DD and TC divisions at CERN, and the NRL and University of Maryland bubble chamber groups, but naming all the individuals who have contributed would be impossible.

APPENDIX A

PROPAGATION OF ERRORS

Throughout this report we have used the standard formula for propagation of errors. The general form of this formula can be sketched as follows. Let $z_i = f_i(x_j)$, where x_j are random variables such that $E(x_i) = \mu_i$ and $V(x_i, x_j) = G_{ij}$. Then,

$$E(z_i) = f_i(\mu_j)$$

$$V(z_i, z_j) = \sum_{k,\ell} \frac{\partial f_i}{\partial x_k} \frac{\partial f_j}{\partial x_\ell} G_{k\ell} \tag{A1}$$

where $E(A)$ is the expected value of A and $V(A, B)$ is the variance of A and B. If the f_i are linear functions of x_j, then (A1) is exact. However, this is almost never the case. In a more general case (A1) can be considered an approximation based on an asymptotic series. Consider for simplicity the case of a single function of a single variable. Then $z = f(x)$. Assume that f is analytic in a neighborhood of $x = \mu$, and write the power series

$$z = \mu + \sum_{j=1}^{\infty} a_j (x - \mu)^j . \tag{A2}$$

Taking the expected values in (A2) we get

$$E(z) = \mu + \sum_{j=1}^{\infty} a_j E(x - \mu)^j \tag{A3}$$

and subtracting (A3) from (A2), squaring, and taking the expected value, we get

$$V(z) = \sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n-1} a_j a_{n-j} \left[E\left((x-\mu)^n\right) - E\left((x-\mu)^j\right) E\left((x-\mu)^{n-j}\right)\right] \right\} . \tag{A4}$$

Both (A3) and (A4) are usually only asymptotic series, unless $f(x)$ is a polynomial.

A few specific examples may clarify the situation. Let x have a Gaussian distribution about 0 with standard deviation σ. Then if $z_1 = ax + bx^2$, we can write

$$E(z_1) = b\sigma^2$$

$$V(z_1) = a^2\sigma^2 + 2b^2\sigma^4 .$$

If $b\sigma << a^2$, it will be adequate to use $E(z) = 0$ and $v(z) = a^2\sigma^2$. Any time $b\sigma$ is comparable to a, so that the pair of formulas given above are necessary, the distribution of z will be significantly non-Gaussian. A rough criterion is then: the linear propagation will be inadequate whenever the derivative of f changes significantly within the region of variation of x.

Whatever the values of a and b, however, the bias of z is less than its standard deviation, so it would not normally be considered significant. However, since the bias is always in the same direction for all measurements, if one were to average many measurements of z, so that the variance of the mean value became much smaller, then the value of the bias might become significant compared to this. This situation is met with, for instance, in the case of missing mass computations where significant shifts in mass values can occur, small compared to the error of an individual measurement, but large compared to the error in the central value of a mass distribution.

As a second example let $z = a^2/(x^2 + a^2)$, where x is as before. Then

$$E(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^n\, n!} \left(\frac{\sigma^2}{a^2}\right)^n = 1 - \frac{\sigma^2}{a^2} + 3\frac{\sigma^4}{a^4} - \cdots$$

$$V(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\sigma^{2n}(2n)!}{2^n\, n!} K_n = 2\frac{\sigma^4}{a^4} - 6\frac{\sigma^6}{a^6} + \cdots$$

where

$$K_n = \left| (n+1) - \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{2n}{2j}} \right|$$

are the asymptotic series. Comparison with the exact solutions, which exist for all values of a, shows that the asymptotic solutions are indeed good when they exist; for example, for $a = 2\sigma$ the first three terms (the third is the smallest) gives $E(z)$ to 1/2%. For smaller a the series is useless and the distribution of z is extremely skewed. By the time $a = 2\sigma$, however, the expansion is a reliable guide.

APPENDIX B

VALUES OF THE CONSTANTS

The size of measuring errors, the numbers represented by A (see Eq. (3)) will depend, of course, on many factors. Using film plane digitizing machines with stage accuracy of $2\,\mu$ and digitized to $1\,\mu$, the actual errors induced by the stage and digitization have always been negligible with any of the film we have used. The finite size of the bubble image and the setting inaccuracy on it are the limiting factors. Examining the distribution of differences between the values of the coordinate perpendicular to the stereo axis for points measured as corresponding (see remarks after Eq. (1)), we find an approximate Gaussian with a long tail. The tail is presumably due to misidentification of the point, digitizer failure, etc. We throw out the large values in computing points. From the central peak we estimate Δ to be of the order of $5\,\mu$ on the film, or about $60\,\mu$ in space for the BNL 30-inch chamber. This is similar to the figure obtained for both the 30-cm and 80-cm CERN chambers.

For the value of K we have (Rossi, "High Energy Particles," p. 70)

$$K = \frac{E_s}{\sqrt{2X_0}} = \frac{15\ \text{Mev}}{\sqrt{990\ \text{cm}}} = 0.48\ \text{Mev-cm}^{-1/2}.$$

This value is valid only if the Gaussian approximation to the scattering is valid; where plural scattering is important, as it usually is in hydrogen, this value will underestimate the effective coulomb scattering.

* * *