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A Nonlinear Filtering Technique for Digitized Images Degraded by Film-Grain Noise

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A class of nonlinear recursive filtering and smoothing algorithms is investigated for use in digital processing of images degraded by film-grain noise. These algorithms are developed by using a Bayesian perturbation analysis and a class of "almost Gaussian" conditional probability distributions to obtain results that are formally accurate to first order in the perturbation parameters. The nonlinear features of the resulting algorithms are used to account for the "signal dependence" of film-grain noise and the nonlinear relation between optical density and intensity. These algorithms are tested on a photographic film negative. Some improvement over conventional Wiener filtering is found.

CONTENTS

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A NONLINEAR FILTERING TECHNIQUE FOR DIGITIZED IMAGES DEGRADED BY FILM-GRAIN NOISE

I. INTRODUCTION AND NOTATION

One kind of image degradation that often occurs is film-grain noise, which is produced by a photographic emulsion during the process of recording an image on film. An extensive account of the mechanisms and behavior of this type of noise can be found in Naderi [1], James [2], and included references. This report investigates the use of digital image-processing techniques to suppress this grain noise by implementing a particular class of nonlinear filters. The emphasis here is on processing an image in terms of the "optical density" variable, which is a logarithmic (point-by-point) function of the light intensity transmitted by a film negative, because the grain noise is of the convenient additive form in this variable.

One troublesome feature of film-grain noise is the fact that it is signal dependent, i.e., its root-mean-square (rms) value varies with the local average optical density of the film. The conventional "optimal" Wiener filter for suppressing this noise ignores its signal dependence [1]. The type of nonlinear filter applied here, however, takes the first-order effects of this signal dependence into account, and experimentally shows some improvement over the conventional Wiener filter. This nonlinear filter is based on Bayesian probability concepts, and it results from a perturbation analysis of a linear filtering problem. Hence, considerable background of relevant linear filtering is also presented.

Authorities disagree over the relative importance of grain-noise signal dependence and the difficulties inherent in the nonlinear relation between image intensity and film density [3,4]. It happens that the class of nonlinear filters developed here can also be used to account for such nonlinearities in images of sufficiently low contrast, so this application is also explored to some extent. This particular nonlinearity turns out to have rather severe effects in practice, however, and the validity of the perturbation analysis used here is somewhat dubious in this context.

Unless otherwise indicated, lower-case letters in the following material denote (real) column vectors or scalars. Matrices are denoted by capital Roman letters. A^T denotes the transpose of a matrix A, and tr(A) its trace. It will also be convenient to manipulate three-way matrices, which are denoted by capital Greek letters. For continuity of notation, the following definitions are adopted for such a matrix Γ, with vector **x** and matrices A and B of compatible dimensions, and with repeated indices denoting summation:

$$(\Gamma x)_{ij} = \Gamma_{ij\sigma} x_\sigma \qquad \text{(matrix)}$$

$$(A\Gamma)_{ijk} = A_{i\sigma}\Gamma_{\sigma jk} \qquad \text{(three-way matrix)}$$

$$(\Gamma B)_{ijk} = \Gamma_{ij\sigma} B_{\sigma k} \qquad \text{(three-way matrix)}$$

$$(Ax^T)_{ijk} = A_{ij} x_k \qquad \text{(three-way matrix)}$$

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$$(\Gamma')_{ijk} = \Gamma_{jki} \qquad \text{(three-way matrix)}$$

$$(\Gamma^T)_{ijk} = \Gamma_{kji} \qquad \text{(three-way matrix)}$$

$$[\mathrm{Tr}(\Gamma)]_i = \Gamma_{\sigma i \sigma} \qquad \text{(column vector)}$$

$$[\mathrm{Tr}(\Gamma\Gamma)]_{ij} = \Gamma_{\lambda i \sigma}\Gamma_{\sigma j \lambda} \qquad \text{(matrix).}$$

Also, Γ is called symmetric if $\Gamma = \Gamma' = \Gamma'' = \Gamma_T$.

II. BACKGROUND

A. Linear Filtering

Because of its virtually complete lack of spatial correlation, grain noise can always be reduced by local spatial averaging, provided of course that the averaging is done over an area whose dimensions are large compared to that of a typical grain ($\sim 1\ \mu$m). Perhaps the simplest procedure of this sort is to assign the optical density

$$\frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} z(r, \theta)\, d\theta\, dr$$

to each point of the processed image, where (r, θ) denotes a polar coordinate system centered at the point of interest, z is the recorded optical density, and R is some small radius. In other words, the density at each point in the processed image is merely the average of the observed density in a small circle of radius R centered at that point. In digital image processing, of course, the integrals are approximated by discrete sums.

The averaging nature of this procedure means that it achieves grain noise suppression only at the cost of a loss of resolution. Some penalty of this sort is always unavoidable, but more sophisticated "filtering" methods can achieve greater grain-noise suppression for a given reduction in resolution. A common type of procedure is linear filtering, which corresponds to *weighted* spatial averaging. In its most general form, this means (in rectangular coordinates) that the density b at a point (x, y) is computed as

$$b(x, y) = \iint W(x, y, u, v)\, z(u, v)\, du\, dv,$$

where the region of integration is the entire image. In practice, the weighting function $W(x, y, u, v)$ is almost always chosen so that its values depend only on $x - u$ and $y - v$ (except near the edge of the image). The computed density in this case can be interpreted as the two-dimensional convolution of the weighting function with the observed image; that is,

$$b(x, y) = \iint W(0, 0, u - x, v - y)\; z(u, v)\, du\, dv.$$

This can be expressed more concisely as

$$b = \iint h(x, y)\, z(x, y)\, dx\, dy,$$

where x and y are now local coordinates centered at the point whose filtered density is being computed, and where $h(x, y) = W(0, 0, x, y)$.

Clearly, the local spatial averaging method described above is a particular kind of linear filter, with

$$h(x, y) = \begin{cases} \dfrac{1}{\pi R^2} & \text{if } \sqrt{x^2 + y^2} \leqslant R \\ 0 & \text{for greater values of } \sqrt{x^2 + y^2}. \end{cases}$$

Another type of weighting function commonly employed is the "Gaussian window," with

$$h(x, y) = k \, \frac{e^{-\frac{x^2+y^2}{2\sigma^2}}}{2\pi\sigma^2},$$

where σ and k are given constants. This function has the appealing property of being isotropic, meaning that the weight given to the observed density at an offset (x, y) from the point whose density is being computed depends only on the magnitude $\sqrt{x^2 + y^2}$ of the offset and not its orientation. This is also true of the preceding weighting function. A third popular type of weighting function

$$h(x, y) = \frac{k}{4a^2} \, e^{-\frac{|x|+|y|}{a}} \qquad \text{("exponential window")}$$

is not isotropic, but has attractive computational properties and has at least 90° rotational symmetry. Sketches of these weighting functions are shown in Fig. 1.

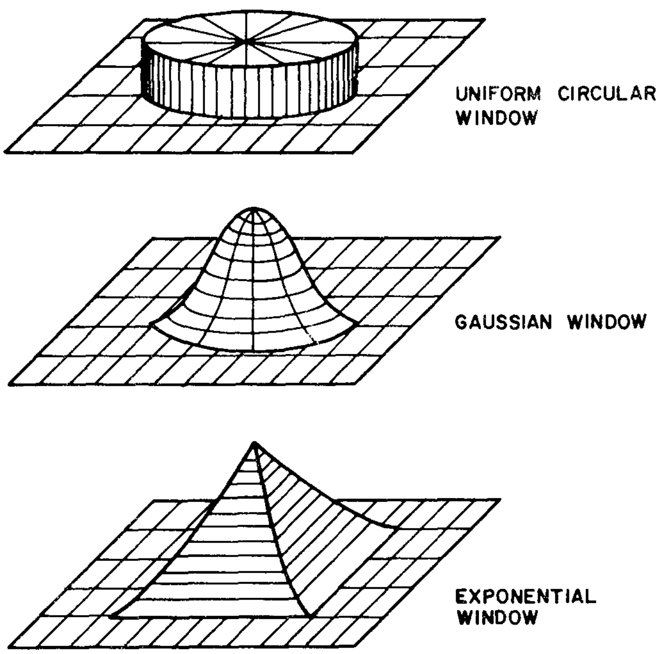


Fig. 1 — Weighting functions corresponding to three kinds of linear filters

It is generally regarded as reasonable to approximate grain noise, within a uniformly exposed region at least, as spatially uniform additive white noise because of the absence of spatial correlation. Without going into details, this means that the relative grain-noise suppression for a weighting function $h(x, y)$ can be expressed as

$$\frac{\sigma_1}{\sigma_0} = \frac{1}{k} \sqrt{\int\int h^2(x, y)\,dx\,dy}$$

where

$k \quad = h(x, y)\,dx\,dy$ (a normalization constant)

$\sigma_i \quad =$ rms noise-induced density fluctuation in processed image (using h)

$\sigma_0 \quad =$ corresponding rms fluctuation with a uniform window of unit area.

In digital image processing, the pixel size is generally used as the unit area. This result enables us to find, for example, the relative sizes of R, σ, and a in the preceding three filters that yield equal grain-noise suppression, namely

$$\sigma = \frac{R}{2} = \frac{2a}{\sqrt{\pi}}.$$

The form of these three parameters is such that increasing them increases the degree of grain-noise suppression, and decreases, at the same time, the resolution. The differing tradeoffs between resolution and noise suppression can now be compared by plotting the responses of these three filters to a semi-infinite step for equal grain-noise suppression. The results are summarized in Fig. 2, and are independent of the step size and degree of noise suppression except for scale changes. The differences are not dramatic, but this example does illustrate how this tradeoff can be affected by the type of filter used. For instance, if one adopts as a criterion of resolution the distance required for the response to rise from one-quarter to three-quarters of its maximum value, then for the orientation shown, the exponential window is about 20% better than the uniform circular window with the same degree of grain-noise suppression. It is even better for other orientations.

B. Bayesian Filtering

Another approach to this problem is to postulate or otherwise determine an a priori probability distribution for the set of possible original images and then compute its conditional probability distribution, given the observed noise-corrupted image. In terms of a spatially discrete image, whose pixels are regarded for convenience as continuously variable in optical density, this typically means specifying the probability densities $p(x)$ and $p(z_i/\mathbf{x}_i)$, with $i = 1, \ldots, N$; where $\mathbf{x}$ is an N-dimensional real vector whose components correspond to the optical densities of the individual pixels, and where z_i is the observed optical density of the ith pixel. The desired posterior probability density $p(x/z_1, \cdots, z_N)$ can then be determined in principle by using the Bayes rule. The number N of pixels in an image is typically in the hundreds of thousands, however, and such a procedure is not computationally feasible at present except in special cases. Another drawback to this approach is that, whereas $p(z_i/x_i)$ is often definitely known, the prior probability density $p(x)$ usually is not, and is fairly arbitrary.

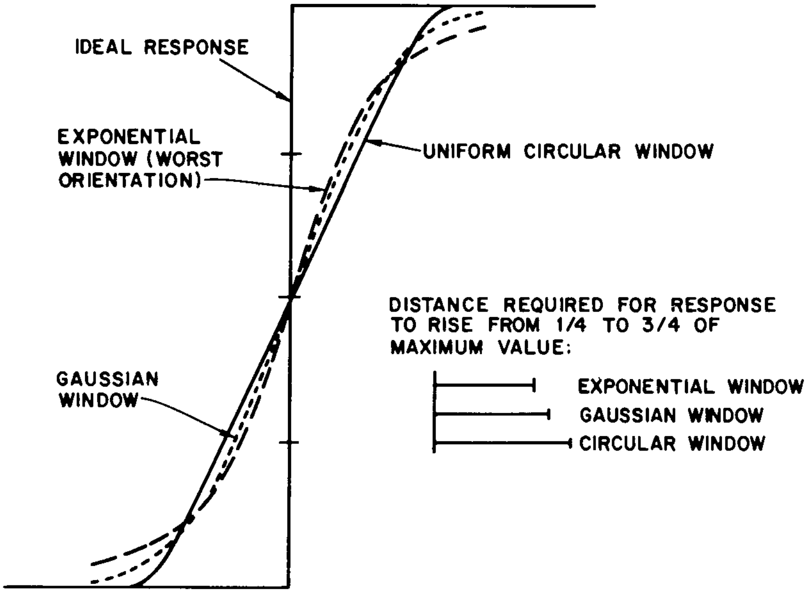


Fig. 2 — Responses to semi-infinite step for three two-dimensional filters with equal grain-noise suppression

With the Bayesian approach, one is usually content with finding either the mean or mode of the posterior probability density as a good estimate of the original image, given the observed image. An important property in this regard is that if

$$p(z_i/x_i) = x_i + w_i \tag{1}$$

and if $p(x)$ and $p(w_1, \cdots, w_N)$ are both multivariate Gaussian probability densities with a priori known parameters (mean and covariance), then the posterior probability density $p(x/z)$ is also multivariate Gaussian. This means that its mean and mode are identical. Furthermore, the value of this conditional mean for each pixel is a linear combination of the observed optical densities of the pixels. Hence, the process of computing the conditional mean, which we will denote as Bayesian filtering here, falls into the category of linear filters described in the preceding section under these particular conditions of additive Gaussian measurement noise and a Gaussian prior. Incidentally, the Bayes filter in this context is also the same as the Wiener filter, which is optimal in the least squares sense.

Because of this property, one way of constructing linear filters for grain-noise suppression is to treat the noise as additive and Gaussian and to use the corresponding Bayes filter for some Gaussian prior distribution that is roughly consistent with the situation at hand. The rationale for this construction is that the resulting linear filter is optimal in at least some sense for a similar situation, and therefore hopefully good for the image actually being processed. However, this procedure requires that the grain-noise magnitude be approximated as independent of the optical density of the underlying image, which is considerably at variance with experimental evidence [5]. Much of the emphasis in this report is on modifying the Bayes filter by perturbation methods to account more accurately for dependence of the grain noise on optical density. The resulting filters are no longer linear, however, and cannot be interpreted as weighted spatial averaging.

As a concession to computational tractability, this Bayesian filtering approach can be applied to the pixel rows and columns of a discrete image individually, resulting in a one-dimensional filter for each row or column [1]. Each such filter is based on an a priori joint probability distribution for the optical density values of only the pixels in the row or column in question. This prior distribution is chosen to be consistent with the main features of the image, such as resolution and rms optical density fluctuation. The filtering then amounts to computing the posterior mean of the pixel densities in that row or column only, given the noise-corrupted observations thereof. The grain noise, being statistically independent for each pixel, is the same as in the two-dimensional analysis. Because it treats each row and column individually, this one-dimensional analysis does not address the possibility of correlations between adjacent rows or columns in the original image, which typically exist to a high degree, nor the fact that rows and columns intersect. Realistically, then, the use of such methods in two-dimensional image processing can be only partly justified on Bayesian grounds. Also, the further simplification is often made of using the same prior distribution, and hence the same filter, for each row and column. This is a reasonable approximation in view of the other inaccuracies already incurred.

If the entire image is processed, these one-dimensional filters are then combined by some other rationale. One procedure, described by Naderi [1], is to filter each row and column of the observed image separately, and then assign to each pixel in the processed image the optical density that is the average of the filtered values for that pixel in the row and column to which it belongs. Another method is to filter each row separately and then filter the columns of the *resulting* image individually. If the row and column filters are linear and if their results are combined linearly in processing the two-dimensional image, then the composite filtering operation is a linear filter of the type described in the preceding section, and the weighting function of the corresponding weighted spatial average can be examined directly as a guide in combining the constituent one-dimensional filters. For linear row and column filters, both procedures just mentioned are linear combinations of this sort. A special case of particular interest occurs when the row and column filters all correspond to a single one-dimensional weighting function of the spatially invariant Gaussian form

$$h(x) = k\, e^{-\frac{x^2}{2\sigma^2}}; \qquad k,\ \sigma \text{ are known constants}$$

except near the edges. This case is noteworthy because if these one-dimensional filters are combined in the second way described above (i.e., by cascading the row and column filtering operations), then, except for edge effects, the resulting two-dimensional linear filter has the "Gaussian window" type of weighting function described in the preceding section. This weighting function is desirable because of its isotropic properties.

Most of this report is further restricted to filters composed of one-dimensional Bayes filters for rows and columns of image pixels. Although these constituent filters are generally nonlinear, the types we examine are derived from a perturbation analysis of linear filters, so the weighted spatial averaging interpretation of the corresponding unperturbed linear filters is used as a guide in combining the nonlinear filters into two-dimensional filters.

C. Recursive Implementations

It is sometimes possible to reduce the computational burden of Bayesian filtering by describing the optical density fluctuations of the original image as a Markov process and exploiting this Markov property to construct a recursive, or sequential, implementation of the corresponding Bayesian filter. This approach is emphasized in this report, and is especially suited to the type of one-dimensional line filters just described.

As an example, consider the case of the density-independent grain noise of Eq. (1) with a Gaussian prior distribution for the original image fluctuations having the autocorrelation function

$$E(u_{i+n,\, j+m} u_{i,\, j}) = \sigma^2 e^{-k(|n|+|m|)}, \tag{2}$$

where $u_{i,\, j}$ denotes the optical density of the i, j th pixel. For simplicity, regard all random variables as having zero mean, which introduces no loss of generality. According to the theory described in the preceding section, one could aways compute the (Gaussian) conditional probability distribution of the original image, given the observed image in a single "step" by defining a vector $\mathbf{x}$ composed of all the image pixels. The conditional mean $\hat{\mathbf{x}}$ and covariance matrix P of this vector (which completely determine this posterior distribution) would then be given by the well-known equations [6]

$$P = M - \frac{1}{r} M(I + M)^{-1}M \tag{3}$$

and

$$\hat{\mathbf{x}} = \frac{1}{r} P\mathbf{z}, \quad \text{(filtered image vector)} \tag{4}$$

where

$\mathbf{z}$ = vector of observed pixel values

M = covariance matrix of a priori distribution of $\mathbf{x}$

r = grain noise variance (a scalar constant).

If the image is an N × N array of pixels, however, $\mathbf{x}$, $\hat{\mathbf{x}}$, and $\mathbf{z}$ here are N^2 vectors, and P and M are $N^2 \times N^2$ matrices. For realistic values of N (often 512), it is impossible even to store M or P in current computers.

According to Habibi [7], the image statistics of Eq. (2) can be obtained as the output of the two-dimensional Markov process with

$$u_{i,\, j} = \rho(u_{i-1,\, j} + u_{i,\, j-1}) - \rho^2 u_{i-1,\, j-1} + w_{i,\, j}, \tag{5}$$

where $\{w_{i,\, j}\colon\ i, j = 0, \cdots, N\}$ is a set of independent zero-mean Gaussian random variables, each with variance $\sigma^2(1 - \rho^2)$, in which

$$\rho = e^{-k}. \tag{6}$$

It is apparent from the form of Eq. (5) that the sequence of diagonals in the pixel array can be realized as a one-dimensional-vector Markov process by constructing a state vector consisting of pairs of diagonals. For convenience, we assume a square N × N image and make all the state vectors in the sequence of the same dimension by adding fictitious components to the diagonals beyond the edge of the actual image. If the ith augmented diagonal is denoted by $\mathbf{y}_i$ (an N-vector), then this is equivalent to a Markov process with the linear "dynamics" of

$$\begin{bmatrix} y_{i+1} \\ \hline y_{i+2} \end{bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline -\rho^2 I & \rho K_{i+2} \end{array}\right] \begin{bmatrix} y_i \\ \hline y_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \hline \omega_{i+2} \end{bmatrix}; \quad i = 1, \cdots, N-2, \tag{7}$$

where the ω_i are a sequence of independent vector zero-mean Gaussian random variables, each with a diagonal covariance $\underline{Q}$ having all diagonal elements equal to $\sigma^2(1 - \rho^2)$. The quantity K_i is a matrix of the form

$$\begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & 0 & \cdots & 0 & \vdots \\ \vdots & 1 & 0 & \ddots & & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 & 1 & \vdots \\ \vdots & 0 & \cdots & 0 & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & \underbrace{\cdots}_{i} & \cdots & \cdots & 0 \end{bmatrix}$$

where the nonzero diagonal "block" shown in detail is in the center of the matrix. The observed pixel densities can also be expressed in terms of this sequence of state vectors as

$$\zeta_i = [0 \mid \underline{H}_i] \begin{bmatrix} y_{i-1} \\ \hline y_i \end{bmatrix} + n_i, \tag{8}$$

where ζ_i is now an N-vector of observations and the n_i are independent, identically distributed zero-mean vector Gaussian random variables with diagonal covariance matrix R, all of whose diagonal terms are equal to r. The matrices $\underline{H}_i$ are of the diagonal form

$$\underline{H}_i = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \underbrace{1 \;\; \ddots \;\; 1}_{i} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

where the i nonzero terms are in the center of the matrix. With this construction, the standard theory of Kalman filtering and "linear smoothing" can be applied [6] to the realization of Eqs. (7) and (8) to obtain the posterior probability distribution of the optical densities of the image pixels according to the following formulas:

$$\hat{x}_{i+1} = F_i\hat{x}_i + \frac{1}{r} \mathrm{P}_{i+1}\mathrm{H}_{i+1}^T(\zeta_{i+2} - \mathrm{H}_{i+1}\mathrm{F}_i\hat{x}_i) \tag{9}$$

$$\mathrm{P}_i = \mathrm{M}_i - \mathrm{M}_i\mathrm{H}_i^T(\mathrm{H}_i\mathrm{M}_i\mathrm{H}_i^T + r\mathrm{I})^{-1}\mathrm{H}_i\mathrm{M}_i \tag{10}$$

$$\mathbf{M}_{i+1} = \mathbf{F}_i \mathbf{P}_i \mathbf{F}_i^T + \mathbf{Q} \tag{11}$$

$$\bar{x}_i = \hat{x}_i + \mathbf{P}_i \mathbf{F}_i^T \mathbf{M}_{i+1}^{-1} (\bar{x}_{i+1} - \mathbf{F}_i \hat{x}_i); \quad \bar{x}_N = \hat{x}_N \tag{12}$$

$$\mathbf{B}_i = \mathbf{P}_i - \mathbf{P}_i \mathbf{F}_i^T \mathbf{M}_{i+1}^{-1} (\mathbf{M}_{i+1} - \mathbf{B}_{i+1}) \mathbf{M}_{i+1}^{-1} \mathbf{F}_i \mathbf{P}_i; \quad \mathbf{B}_N = \mathbf{P}_N, \tag{13}$$

where

$$\mathbf{H}_i = [0 \mid \underline{\mathbf{H}}_{i+1}], \quad \mathbf{Q} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underline{\mathbf{Q}} \end{array}\right], \quad \text{and } \mathbf{F}_i = \left[\begin{array}{c|c} 0 & \mathbf{I} \\ \hline -\rho^2 \mathbf{I} & \rho \mathbf{K}_{i+1} \end{array}\right].$$

This equation system has the form of a forward recursion, consisting of Eqs. (9) – (11), followed by a reverse recursion (Eqs. (12) and (13)), which uses the results of the forward recursion. The initial values $\hat{\mathbf{x}}_1$ and $\mathbf{M}_1$ are specified a priori. The interpretation here is that the posterior distribution of the composite vector $\left[\begin{array}{c} \mathbf{y}_i \\ \hline \mathbf{y}_{i+1} \end{array}\right]$ is Gaussian, with mean $\bar{\mathbf{x}}_i$ and covariance matrix $\mathbf{B}_i$.

Although the description of this recursive implementation is more complicated, the dimension of the vectors and matrices is 2N instead of N^2. This reduces the actual computational burden enormously, typically by a factor of 100,000, and the storage requirements to an even greater extent. However, the above recursions do not determine the "cross-correlations" $E(y_i y_j)$ for $|i-j| > 1$. These features of the posterior distribution are not usually of interest, and in any event can be computed from other known formulas [8]. The computational requirements of this recursive procedure are still formidable, though probably within the realm of possibility. Hence, it will not be pursued further in this report.

Another particular recursive implementation, which will be pursued in greater detail later, is for a class of one-dimensional Bayesian line flters. The one-dimensional version of the autocorrelation function (2) for the original image is

$$E(u_{i+n} u_i) = \sigma^2 e^{-k|n|}. \tag{14}$$

This can be obtained, for the zero-mean Gaussian case, as the output of the scalar Markov process

$$u_{i-1} = f u_i + w_i; \quad i = 1, \cdots, N-1 \tag{15}$$

in which the w_i are independent, identically distributed, scalar, Gaussian random variables with variance q, and where

$$f = e^{-k} \tag{16}$$

and

$$q = \sigma^2 (1 - e^{-2k}). \tag{17}$$

With the same grain-noise model as before for the observed image, the observed optical density z_i of the ith pixel in such a line is clearly

$$z_i = u_i + \mathrm{v}_i; \quad i = 0, \cdots, N \tag{18}$$

where the v_i are random variables with variance r. The posterior mean and variance of the optical densities of the individual pixels are then given by the scalar version of the recursions (9) – (13), which is

$$\left.\begin{aligned} \hat{u}_{i+1} &= f\hat{u}_i + \frac{p_{i+1}}{r}(z_{i+1} - f\hat{u}_i); \quad \hat{u}_o = z_o && (19) \\ p_i &= \frac{m_i r}{m_i + r}; \quad p_o = r && (20) \\ m_{i+1} &= f^2 p_i + q && (21) \end{aligned}\right\} \text{Kalman filter}$$

and

$$\left.\begin{aligned} \bar{u}_i &= \hat{u}_i + \left(\frac{fp_i}{m_{i+1}}\right)(\bar{u}_{i+1} - f\hat{u}_i); \quad \bar{u}_N = \hat{u}_N && (22) \\ b_i &= p_i - \left(\frac{fp_i}{m_{i+1}}\right)^2 (m_{i+1} - b_{i+1}); \quad b_N = p_N && (23) \end{aligned}\right\} \text{linear smoother}$$

To reiterate, the conditional probability density of u_i, given the observed image (which is only a line here), is Gaussian with mean $\bar{u}_i$ and variance b_i.

Except near the edges, the p, m and b variables are essentially constant as functions of i. This permits us to reduce the Bayes filter of Eqs. (19) and (22) to a one-dimensional convolution filter of the form

$$u_i = \sum_{j=0}^{N} h(i-j)z(j). \tag{24}$$

Working out the details, we note that h in Eq. (24) is given by

$$h(n) = \left(\frac{m + r - fr}{m + r + fr}\right) e^{-k|n|}; \quad k = \ell n \frac{fr}{m + r} \tag{25}$$

if edge effects are neglected. This is a one-dimensional linear Bayes filter of the type described in the preceding section. When applied successively to the rows and then the columns (or vice versa) of a two-dimensional pixel array, it results in a two-dimensional linear filter of the exponential window type described in Section II-A, with

$$h(i, j) = \text{constant} \times e^{-\frac{|i|+|j|}{a}}$$

Hence, this type of two-dimensional filter can be implemented recursively for an N × N image by 2N successive applications of the one-dimensional filter of Eqs. (19) and (22). Such an implementation has the property of at least being a Bayes filter for the corresponding one-dimensional image described by (14) and (18), and is extremely simple computationally.

Unfortunately, as noted earlier, this particular two-dimensional filter is not isotropic. Ideally, one would like to construct a one-dimensional Bayes filter like (19) and (22) that would give a Gaussian one-dimensional weighting function away from the end points. Such a filter would result in an isotropic "Gaussian window" in two dimensions when applied successively to the rows and columns. This goal does not seem quite attainable, but arbitrarily close approximations to it can be achieved at the expense of resorting to ever higher-order Markov processes in place of that of Eq. (15). For example, the next-higher (second) order approximation can be constructed by defining a Markov process with state vector **x** such that

$$\mathbf{x}_i = \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}$$

and by giving $\mathbf{x}$ the dynamics of a "critically damped second-order system" driven by white noise. The recursions analogous to (19) and (22) in the resulting Bayes filter for the $\mathbf{x}$ sequence now involve 2×2 matrix operations. The form of the equivalent one-dimensional convolution function for the u's is shown in Fig. 3 for the limiting case of very small pixel spacing. For comparison, this figure also shows the Gaussian window and exponential window (the first-order approximation), that give the same degree of grain-noise suppression. It is apparent that this second-order approximation is indeed closer in form to the Gaussian window.

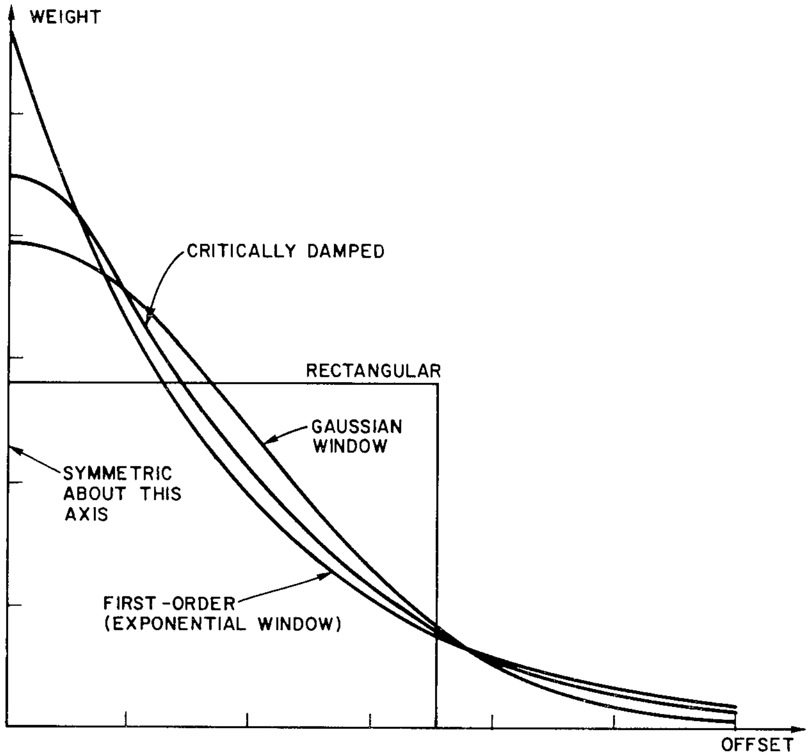


Fig. 3 — Convolution functions for one-dimensional filters with equal grain-noise suppression

III. A CLASS OF NONLINEAR BAYESIAN FILTERS

A. Formulation

All Bayesian filters described in detail so far have been for the case of density-independent grain noise. Although the additive, Gaussian, and spatially uncorrelated character of photographic grain noise is fairly well confirmed by experiment, its rms magnitude is known [1] to increase with the optical density of the original image as approximately

$$\sigma(\mathrm{D}) = \sigma(\mathrm{D}_0)\left(\frac{\mathrm{D}}{\mathrm{D}_0}\right)^{1/3} \tag{26}$$

where

D = film density at the point in question

D_0 = reference density (usually 1)

σ = rms grain noise value.

In this section we seek to construct Bayesian filters that take this density dependence into account. They will be developed according to a perturbation analysis for small D/D_0, however, and will be only first-order approximations to exact Bayesian filters. The exact form of the a priori probability distribution that should be used for the original image is a more imponderable issue, and there is less reason to depart from the convenient Gaussian distribution described earlier. However, there is some motivation for dealing with spatial image fluctuations that are Gaussian in the intensity variable, rather than density, because many optical processes result in image transformations that are linear in terms of intensity, and the class of Gaussian probability distributions is closed under linear transformations. Dealing with such cases leads to the inclusion of either measurement nonlinearities or nonlinear recursion relationships, depending on whether the analysis is performed for the optical density or intensity variable.

These considerations lead to the formulation of a general problem in which a state vector **x** evolves sequentially according to the dynamics

$$x_{i+1} = \mathrm{F}_i x_i + w_i + \Gamma_i'' x_i x_i + \Omega_i'' w_i w_i; \quad i = 0, \quad \cdots, \mathrm{N}-1 \tag{27}$$

and generates the noisy observations

$$z_i = \mathrm{H}_i x_i + \Phi_i'' x_i x_i + \mathrm{v}_i; \quad i = 0, \ \cdots, \mathrm{N} \tag{28}$$

where the following conditions hold.

- The components of the three-way matrices Γ_i, Ω_i, and Φ_i are approximately infinitesimal, say of order h. All other quantities are of order unity, including F_i^{-1}.

- Given $x_0, \ \cdots, x_N$, the w_i and v_i are statistically independent zero-mean Gaussian random variables such that

  $$\mathrm{cov}(\mathrm{w}_i) = \mathrm{Q}_i + 2\Psi_i' x_i$$

  and

  $$\mathrm{cov}(\mathrm{v}_i) = \mathrm{R}_i + 2\Upsilon_i' x_i$$

  to first order in h. Ψ_i and Υ_i are of order h, and Q_i, R_i, and R_i^{-1} are of order unity.

- Q_i and R_i are symmetric and positive-semidefinite, R_i is positive-definite. $\Gamma_i = \Gamma_i^T$, $\Omega_i = \Omega_i^T$, $\Phi_i = \Phi_i^T$, $\Psi_i = \Psi_i^T$, and $\Upsilon_i = \Upsilon_i^T$. $\mathrm{N} << \frac{1}{h}$ and $||\mathrm{F}_i|| \leqslant 1$.

The objective here is to determine, at least to first order in h, the conditional probability distribution of x_i (for each i individually) given the observations $z_0, \cdots, z_N$ when x_0 has an a priori probability density of the general form given by the following expression to first order in h :

$$p_{x_0}(x) = \left[1 + \frac{1}{3}(x - \bar{x})^T(\mathrm{V}^{-1}\Lambda\mathrm{V}^{-1})'\mathrm{V}^{-1}(x - \bar{x})(x - \bar{x})\right]\left[\frac{e^{-\frac{1}{2}(x-\bar{x})^T\mathrm{V}^{-1}(x-\bar{x})}}{(2\pi)^{n/2}|\mathrm{V}|^{1/2}}\right]. \tag{29}$$

In the above, V and Λ are symmetric, n is the dimension of x, Λ is of order h, and $\bar{x}$, V, and V^{-1} are of order unity. This problem is treated in multivariate form to accommodate the kinds of recursive realizations of image fluctuations described in the preceding section.

The problem will be treated sequentially by first determining $p(x_i/\mathrm{Z}_i)$ for successively greater values of i, where $\mathrm{Z}_i = \{z_0, \cdots, z_i\}$, and then, starting with $p(x_N/\mathrm{Z}_N)$, finding $p(x_i/\mathrm{Z}_N)$ for successively smaller values of i. It turns out that, to first order in h, all of these probability densities are of the form of (29). In general, Eq. (29) can assume negative values for sufficiently large magnitudes of $(x - \bar{x})$ and must be modified slightly to be a proper probability density. Because of the rapid decay of the exponential factor and the orders of magnitude assumed for the parameters, however, such modifications can be confined to a region whose probability mass is negligible to arbitrary order for sufficiently small h. Hence, for brevity, Eq. (29) itself will be treated as a proper density. Since it has the form of a Gaussian density multiplied by a polynomial, it follows from standard results for Gaussian moments that its integral over R^n is unity and its first three central moments are given by

$$\mathrm{E}(x) = \bar{x} + \mathrm{Tr}(\mathrm{V}^{-1}\Lambda), \tag{30}$$

$$\mathrm{cov}(x) = \mathrm{V}, \tag{31}$$

and

$$\mathrm{E}[(x - \mathrm{E}(x))(x - \mathrm{E}(x))^T(x - \mathrm{E}(x))^T] = 2\Lambda \tag{32}$$

to first order in h. Furthermore, if $y = \mathrm{A}x + b$, where A and b are constants and A^{-1} exists and is of order unity, then $p(y)$ is also a probability density of form (29), with the parameters $\bar{x}$, V, and Λ replaced respectively by $\mathrm{A}\bar{x} + b$, AVA^T and $(\mathrm{A}\Lambda\mathrm{A}^T)'\mathrm{A}^T$. It is also useful to note that the apparently more general probability "density"

$$p(x) = \left[1 + \lambda^T(x - a) + \frac{1}{2}\,\mathrm{tr}\{\mathrm{L}[(x-a)(x-a)^T - \bar{\mathrm{V}}]\} + \frac{1}{3}(x-a)^T\bar{\Lambda}(x-a)(x-a)\right]$$
$$\times \left[\frac{e^{-\frac{1}{2}(x-a)^T\bar{\mathrm{V}}^{-1}(x-a)}}{(2\pi)^{n/2}|\bar{\mathrm{V}}|^{1/2}}\right], \tag{33}$$

in which L is symmetric and λ and L are of order h, can be reduced to the form of (29) to first order in h. This is done by making the substitutions

$$\bar{x} = a + \bar{\mathrm{V}}\lambda, \tag{34}$$

$$\mathrm{V} = \bar{\mathrm{V}} + \bar{\mathrm{V}}\mathrm{L}\bar{\mathrm{V}}, \tag{35}$$

and

$$\Lambda = (\bar{\mathrm{V}}\bar{\Lambda}\bar{\mathrm{V}})'\bar{\mathrm{V}}, \tag{36}$$

and expanding the exponential and determinant functions in Eq. (33) about the values $\bar{x}$ and V. Incidentally, if either form is expanded about its mean and covariance to first order, the result is a standard first-order Edgeworth expansion of $p(x)$. Thus, an earlier proposal of Sorenson and Stubberud [9] to approximate posterior density functions by Edgeworth expansions in Bayesian filtering is accurate to first order in this context.

Now, if $y = x + \Gamma''xx$, with $\Gamma = \Gamma^T$ of order h, then either $x = y - \Gamma''yy$ to first order or $|x| >> 1$ for any y of order unity. The Jacobian matrix $[\partial x_i/\partial y_j]$ can be expressed as $I + 2\Gamma''x$, whose determinant is $1 + 2\mathrm{tr}(\Gamma x)$ to first order. If x has the density (29), then the standard formula for the transformation of probability densities gives

$$p(y) = \left[1 + \frac{1}{3}(y - \bar{x})^T(V^{-1}\Lambda V^{-1})'V^{-1}(y - \bar{x})(y - \bar{x}) - 2\,\mathrm{tr}(\Gamma y)\right]$$

$$\times \left[\frac{e^{-\frac{1}{2}(y-\Gamma''yy-\bar{x})^T V^{-1}(y-\Gamma''yy-\bar{x})}}{(2\pi)^{n/2}|V|^{1/2}}\right] \tag{37}$$

to first order for y of order unity. This follows because large values of x have negligible probability, and $y - x$ is otherwise of order h. Equation (37) is also an adequate approximation for larger y because x cannot then be of order unity. Expanding the exponential factor of Eq. (37) about $(y - \bar{x})$ and using Eqs. (33)–(36), we note that $p(y)$ is again of form Eq. (29) to first order, with the parameters $\bar{x}$, V, and Λ from $p(x)$ changed respectively to

$$\bar{x} + \Gamma''\bar{x}\bar{x} - 2V\,\mathrm{Tr}(\Gamma'), \tag{38}$$

$$V + 2(V\Gamma' + \Gamma''V)'\bar{x}, \tag{39}$$

$$\Lambda + V\Gamma V + (V\Gamma V)' + (V\Gamma V)''. \tag{40}$$

We proceed from this point by considering two arbitrary adjacent indices i and $i + 1$, dropping for brevity the subscripts from x_i, F_i , Q_i, w_i, and Ω_i, and denoting x_{i+1} by y. First, $p(y/Z_{i+1})$ is derived to first order as a probability density of form (29) for $p(x/Z_i)$ also of this form, giving a forward recursion on i. Then, for $p(y/Z_N)$ of form (29), the density $p(x/Z_N)$ is derived as another density of this form, which specifies a reverse recursion. Since $p(x_0)$ is of form (29), these two results together establish that all $p(x_i/Z_N)$ have this form, and they provide formulas for computing the parameters of these densities. It is possible that the random variables z_i can by chance assume such excessively large values that the assumed approximations are not valid, but this only happens with negligible probability for sufficiently small h.

B. Forward Recursion

Suppose that the probability density $p(x/Z_i)$ is of form (29) with parameters $\bar{x}_i$, V_i, and Λ_i, and with corresponding mean denoted by $\hat{x}_i$. Letting t denote Fx implies that t has a probability density of the form of Eq. (29) with

$$\bar{x} = F\bar{x}_i,$$

$$V = FV_iF^T,$$

$$\Lambda = (F\Lambda_iF^T)'F^T.$$

Letting $s = Fx + \Gamma''xx$ with $\Gamma = F^{-T}\Gamma_iF^{-1}$ implies that $s = t + \Gamma''tt$. Hence, the density of s is also of the form (29) to first order, with corresponding parameters $\bar{s}$, $\bar{V}$ and $\bar{\Lambda}$ given by (38)–(40) as

$$\bar{s} = \bar{x} + \Gamma''\overline{xx} - 2\mathrm{V}\,\mathrm{Tr}(\Gamma'),$$
$$\bar{\mathrm{V}} = \mathrm{V} + 2(\mathrm{V}\Gamma' + \Gamma''\mathrm{V})'\bar{x},$$

and

$$\bar{\Lambda} = \Lambda + \mathrm{V}\Gamma\mathrm{V} + (\mathrm{V}\Gamma\mathrm{V})' + (\mathrm{V}\Gamma\mathrm{V})''.$$

If Q^{-1} exists, it follows from a first-order expansion of the exponential and determinant in $p(w/x)$ that

$$p(w/s) = p_{w/x}(w, \mathrm{F}^{-1}s) = [1 - \mathrm{tr}\{\mathrm{Q}^{-1}\Psi' s(\mathrm{I} - \mathrm{Q}^{-1}ww^T)\}]\left[\frac{e^{-\frac{1}{2}w^TQ^{-1}w}}{(2\pi)^{n/2}|\mathrm{Q}|^{1/2}}\right] \tag{41}$$

to first order in h, where $\Psi' = \Psi_i' \mathrm{F}^{-1}$. For $r = w + \Omega'' ww$, Eqs. (37)–(40) imply that

$$p(r/s) = \Big[1 - \mathrm{Tr}^T(\Omega' + \Omega'')r - \mathrm{tr}\{\mathrm{Q}^{-1}\Psi' s(\mathrm{I} - \mathrm{Q}^{-1}rr^T)\}$$
$$+ \frac{1}{3}\, r^T\{\mathrm{Q}^{-1}\Omega'' + \Omega'\mathrm{Q}^{-1} + (\mathrm{Q}^{-1}\Omega'')'\}rr\Big]\left[\frac{e^{-\frac{1}{2}r^TQ^{-1}r}}{(2\pi)^{n/2}|\mathrm{Q}|^{1/2}}\right] \tag{42}$$

to first order. Since $y = r + s$ by definition,

$$p(y/\mathrm{Z}_i) = \int_{R^n} p_s(y - r)\,p_{r/s}(r, y - r)\,dr$$
$$= \int_{R^n} \frac{k(r)}{(2\pi)^n|\bar{\mathrm{V}}\mathrm{Q}|^{1/2}}\, e^{-\frac{1}{2}[r^TQ^{-1}r + (y-r-\bar{s})^T\bar{V}^{-1}(y-r-\bar{s})]}\,dr \tag{43}$$

to first order in h, where the polynomial factor $k(r)$ is given by

$$k(r) = 1 + \frac{1}{3}\,(y - r - \bar{s})^T(\bar{\mathrm{V}}^{-1}\bar{\Lambda}\bar{\mathrm{V}}^{-1})'\bar{\mathrm{V}}^{-1}(y - r - \bar{s})(y - r - \bar{s}) - \mathrm{Tr}^T(\Omega' + \Omega'')r$$
$$- \mathrm{tr}\,[\mathrm{Q}^{-1}\Psi'(y - r)(\mathrm{I} - \mathrm{Q}^{-1}rr^T)] + \frac{1}{3}\, r^T[\mathrm{Q}^{-1}\Omega'' + \Omega'\mathrm{Q}^{-1} + (\mathrm{Q}^{-1}\Omega'')']rr.$$

Completing the square in the exponent of the integrand gives

$$p(y/\mathrm{Z}_i) = \frac{e^{-\frac{1}{2}[(y-\bar{s})^TM^{-1}(y-\bar{s})]}}{(2\pi)^{n/2}|\mathrm{M}|^{1/2}} \int_{R^n} k(r)\, \frac{e^{-\frac{1}{2}[r-QM^{-1}(y-\bar{s})]^T(V-VM^{-1}V)^{-1}[r-QM^{-1}(y-\bar{s})]}}{(2\pi)^{n/2}|\mathrm{V} - \mathrm{V}\mathrm{M}^{-1}\mathrm{V}|^{1/2}}\,dr \tag{44}$$

where $\mathrm{M} = \mathrm{V} + \mathrm{Q}$. This result does not involve Q^{-1}, and in fact holds for singular Q as well. In the latter case, however, a more complicated derivation is required, in which the integration of Eq. (43) and the expansion corresponding to (42) are performed over the subspace of R^n for which $p(r/s)$ is nonsingular. The integral of (44) has the form of the expected value of a third-degree polynomial in r and $(y - \bar{s})$ with respect to a Gaussian distribution for r whose mean is proportional to $(y - \bar{s})$. Therefore, it can be expressed in the form

$$\text{constant} + \lambda^T(y - \bar{s}) + \frac{1}{2}\,\mathrm{tr}\,\{\mathrm{L}[(y - \bar{s})(y - \bar{s})^T - \mathrm{M}]\} + \frac{1}{3}\,(y - \bar{s})^T(\mathrm{M}\Lambda^*\mathrm{M})'\mathrm{M}(y - \bar{s})(y - \bar{s}),$$

where the constant is independent of $y - \bar{s}$. It can be shown that L and Λ^* can be taken as symmetric here because M is, and that λ, L, and Λ^* are of order h because $\bar{\Lambda}$ and Ψ are. Since

$p(y/Z_i)$ is a probability density, the constant term must be unity. Thus, $p(y/Z_i)$ is of the form (33), and therefore reducible to the form of (29) to first order. The first three central moments, which determine this density, can be found by performing the integration of Eq. (44) and substituting the result into the earlier definitions. However, it is simpler just to evaluate them by using (27) and decomposing the expectations into marginal expectations of conditional expectations given x. This gives the result

$$E(y/Z_i) = F\hat{x}_i + \mathrm{Tr}[\Gamma_i(\hat{x}_i\hat{x}_i^T + V_i) + \Omega Q] \triangleq \bar{x}_{i+1} \tag{45}$$

$$\mathrm{cov}(y/Z_i) = FV_iF^T + Q + 2(\Psi_i + FV_i\Gamma_i' + \Gamma_i''V_iF^T)'\hat{x}_i \tag{46}$$

$$\frac{1}{2}\,E[(y-\bar{x}_{i+1})(y-\bar{x}_{i+1})^T(y-x_{i+1})^T/Z_i] = (F\Lambda_iF^T)'F^T + \Theta + \Theta' + \Theta'';$$

where

$$\Theta = FV_i\Gamma_iV_iF^T + \Psi_i'V_iF^T + Q\Omega Q. \tag{47}$$

Turning now to the updating of this conditional density with the next measurement z_{i+1}, we delete for brevity the subscripts of v_{i+1}, H_{i+1}, Φ_{i+1}, R_{i+1}, and Υ_{i+1}. Since, to first order $p(v/y)$ is mean-zero Gaussian with covariance matrix $R + 2\Upsilon'y$, $p(z/y)$ is Gaussian with the same covariance matrix and mean $Hy + \Phi''yy$. Expanding the exponential and determinant of this density function to first order about the values indicated below, we obtain

$$p(z/y) = [1 + (z-Hy)^TR^{-1}\Phi''yy - \mathrm{tr}\{R^{-1}\Upsilon'y[I - R^{-1}(z-Hy)(z-Hy)^T]\}] \times \left[\frac{e^{-\frac{1}{2}(z-Hy)^TR^{-1}(z-Hy)}}{(2\pi)^{m/2}|R|^{1/2}}\right],$$

where m is the dimension of z. Since v is independent of the preceding noise values, the updated conditional state density is proportional to the product $p(z/y)p(y/Z_i)$ as a function of y, which follows from the Bayes rule. Recalling that $p(y/Z_i)$ is the probability density specified by (45)–(47), completing the square in the exponent of this product, and using the "matrix inversion lemma" [6], we obtain

$$p(y/Z_{i+1}) = g\Big[1 + \frac{1}{3}(y-\bar{y})(V^{-1}\Lambda V^{-1})'V^{-1}(y-\bar{y})(y-\bar{y}) - \mathrm{tr}\{R^{-1}\Upsilon'y[I - R^{-1}(z-Hy)(z-Hy)^T]\}\Big]\left[\frac{e^{-\frac{1}{2}(y-a)^T\bar{V}^{-1}(y-a)}}{(2\pi)^{n/2}|\bar{V}|^{1/2}}\right], \tag{48}$$

where g is a constant of proportionality,

$$\bar{V} = V - VH^T(R + HVH^T)^{-1}HV,$$

$$a = \bar{y} + \bar{V}H^TR^{-1}(z - H\bar{y}),$$

and $\bar{y}$, V, and Λ are now used to denote the parameters of $p(y/Z_i)$ as defined by (29). The polynomial factor of (48) can be expressed to first order as

$$1 + \lambda^T(y-a) + \frac{1}{2}\,\mathrm{tr}\{L[(y-a)(y-a)^T - \bar{V}]\} + \frac{1}{3}(y-a)^T\bar{\Lambda}(y-a)(y-a), \tag{49}$$

in which

$$\begin{aligned}\lambda = {} & [2\Phi - \mathrm{H}^T\mathrm{R}^{-1}(\mathrm{Y} + 2\mathrm{HV}\Phi')']\bar{y}(\mathrm{R} + \mathrm{HVH}^T)^{-1}(z - \mathrm{H}\bar{y}) - \mathrm{H}^T\mathrm{R}^{-1}\Phi''\bar{y}\bar{y} \\ & + \mathrm{Tr}\{[\mathrm{H}(\mathrm{V}^{-1}\Lambda)'\mathrm{H}^T + 2\mathrm{HV}\Phi'' + \mathrm{Y}](\mathrm{R}+\mathrm{HVH}^T)^{-1}(z-\mathrm{H}\bar{y})(z-\mathrm{H}\bar{y})^T \\ & \times (\mathrm{R}+\mathrm{HVH}^T)^{-1}\} - \mathrm{H}^T\mathrm{R}^{-1}\mathrm{Tr}[\mathrm{HV}\Phi\mathrm{VH}^T(\mathrm{R}+\mathrm{HVH}^T)^{-1}(z-\mathrm{H}\bar{y})(z-\mathrm{H}\bar{y})^T \\ & \times (\mathrm{R}+\mathrm{HVH}^T)^{-1}\} - \mathrm{Tr}(\mathrm{YR}^{-1}),\end{aligned}$$

$$\begin{aligned}\mathrm{L} = {} & 2[(\mathrm{V}^{-1}\Lambda\mathrm{V}^{-1})'\mathrm{H}^T + \Phi' - \mathrm{H}^T\mathrm{R}^{-1}(\mathrm{Y} + \Phi''\mathrm{VH}^T) + (\mathrm{H}^T\mathrm{R}^{-1}\mathrm{YR}^{-1}\mathrm{H} - \mathrm{H}^T\mathrm{R}^{-1}\Phi'')'\mathrm{VH}^T \\ & - (\mathrm{YR}^{-1}\mathrm{H})''](\mathrm{R}+\mathrm{HVH}^T)^{-1}(z-\mathrm{H}\bar{y}) \\ & + 2(\mathrm{H}^T\mathrm{R}^{-1}\mathrm{YR}^{-1}\mathrm{H} - \mathrm{H}^T\mathrm{R}^{-1}\Phi'' - \Phi'\mathrm{R}^{-1}\mathrm{H})'\bar{y},\end{aligned}$$

and

$$\begin{aligned}\bar{\Lambda} = {} & (\mathrm{V}^{-1}\Lambda\mathrm{V}^{-1})'\mathrm{V}^{-1} + (\mathrm{H}^T\mathrm{R}^{-1}\mathrm{Y} - \Phi')\mathrm{R}^{-1}\mathrm{H} + [(\mathrm{H}^T\mathrm{R}^{-1}\mathrm{Y} - \Phi')\mathrm{R}^{-1}\mathrm{H}]' \\ & + [(\mathrm{H}^T\mathrm{R}^{-1}\mathrm{Y} - \Phi')\mathrm{R}^{-1}\mathrm{H}]''.\end{aligned}$$

Consequently, according to (33)—(36), $p(y/\mathrm{Z}_{i+1})$ can be reduced to the form of (29). If we follow the suggestion of Sorenson and Stubberud [10], we can express the first three central moments of this density, denoted here as $\hat{x}_{i+1}$, V_{i+1} and $2\Lambda_{i+1}$, to first order as

$$\begin{aligned}\hat{x}_{i+1} = {} & \bar{x}_{i+1} + \mathrm{V}(\mathrm{H}+2\Phi''\bar{x}_{i+1})^T[\mathrm{R}+2\mathrm{Y}'\bar{x}_{i+1} + (\mathrm{H}+2\Phi''\bar{x}_{i+1})\mathrm{V}(\mathrm{H}+2\Phi''\bar{x}_{i+1})^T]^{-1} \\ & \times [z - \mathrm{H}\bar{x}_{i+1} - \Phi''\bar{x}_{i+1}\bar{x}_{i+1} - \mathrm{Tr}(\mathrm{V}\Phi)] + \bar{\mathrm{V}}\mathrm{Tr}\{(\mathrm{R}+\mathrm{HVH}^T)^{-1}[\mathrm{Y} + 2\mathrm{HV}(\Phi \\ & - \mathrm{H}^T\mathrm{R}^{-1}\mathrm{Y}') + \mathrm{H}(\mathrm{V}^{-1}\Lambda)'\mathrm{H}^T](\mathrm{R}+\mathrm{HVH}^T)^{-1}(z-\mathrm{H}\bar{x}_{i+1})(z-\mathrm{H}\bar{x}_{i+1})^T \\ & - (\mathrm{R}+\mathrm{HVH}^T)]\} - \bar{\mathrm{V}}\mathrm{H}^T\mathrm{R}^{-1} \times \mathrm{Tr}\{(\mathrm{R}+\mathrm{HVH}^T)^{-1}\mathrm{HV}\Phi\mathrm{VH}^T(\mathrm{R}+\mathrm{HVH}^T)^{-1} \\ & \times [(z-\mathrm{H}\bar{x}_{i+1})(z-\mathrm{H}\bar{x}_{i+1})^T - (\mathrm{R}+\mathrm{HVH}^T)]\},\end{aligned} \tag{50}$$

$$\begin{aligned}\mathrm{V}_{i+1} = {} & \mathrm{V} - \mathrm{V}(\mathrm{H}+2\Phi''\bar{x}_{i+1})^T[\mathrm{R}+2\mathrm{Y}'\bar{x}_{i+1} \\ & + (\mathrm{H}+2\Phi''\bar{x}_{i+1})\mathrm{V}(\mathrm{H}+2\Phi''\bar{x}_{i+1})^T]^{-1}(\mathrm{H}+2\Phi''\bar{x}_{i+1})\mathrm{V} \\ & + 2\{(\bar{\mathrm{V}}\mathrm{V}^{-1}\Lambda\mathrm{V}^{-1}\bar{\mathrm{V}})'\mathrm{H}^T + [\bar{\mathrm{V}}(\mathrm{H}^T\mathrm{R}^{-1}\mathrm{YR}^{-1}\mathrm{H} - \Phi'\mathrm{R}^{-1}\mathrm{H} - \mathrm{H}^T\mathrm{R}^{-1}\Phi'')\bar{\mathrm{V}}]'\mathrm{VH}^T \\ & + [\bar{\mathrm{V}}(\Phi - \mathrm{H}^T\mathrm{R}^{-1}\mathrm{Y}' - \mathrm{Y}''\mathrm{R}^{-1}\mathrm{H})\bar{\mathrm{V}}]'\}(\mathrm{R}+\mathrm{HVH}^T)^{-1}(z-\mathrm{H}\bar{x}_{i+1}),\end{aligned} \tag{51}$$

and

$$\Lambda_{i+1} = (\bar{\mathrm{V}}\bar{\Lambda}\bar{\mathrm{V}})'\bar{\mathrm{V}}. \tag{52}$$

This result, together with those of Eqs. (45)—(47), is the same to first order as that proposed on more ad hoc grounds by Sorenson and Stubberud [10] for obtaining $\mathrm{E}(x_{i+1}/\mathrm{Z}_{i+1})$ and $\mathrm{cov}(x_{i+1}/\mathrm{Z}_{i+1})$ from $\mathrm{E}(x_i/\mathrm{Z}_i)$ and $\mathrm{cov}(x_i/\mathrm{Z}_i)$, except for the inclusion of the measurement-driven term in (51) and the last two terms of (50) containing the scatter matrix of $(z - \mathrm{H}\bar{x}_{i+1})$. The use of such scatter matrix terms was proposed by Lee and Tung [11] in a closely related context.

C. Reverse Recursion

We now seek to determine $p(x_i/Z_N)$ as a probability density of the form of (29) to first order, given that $p(x_i/Z_i)$ and $p(x_{i+1}/Z_N)$ are both of this form. For brevity, the latter two densities are denoted as

$$p(x/Z_i) = \left[1 + \frac{1}{3}(x-\bar{x})^T\Delta(x-\bar{x})(x-\bar{x})\right]\left[\frac{e^{-\frac{1}{2}(x-\bar{x})^T V^{-1}(x-\bar{x})}}{(2\pi)^{n/2}|V|^{1/2}}\right] \tag{53}$$

and

$$p(y/Z_N) = \left[1 + \frac{1}{3}(y-\bar{y})^T\Theta(y-\bar{y})(y-\bar{y})\right]\left[\frac{e^{-\frac{1}{2}(y-\bar{y})B^{-1}(y-\bar{y})}}{(2\pi)^{n/2}|B|^{1/2}}\right]. \tag{54}$$

The results for the general case, although obtainable by the methods used below, are very cumbersome, and their interpretation is still not well understood. Hence, the development is limited from this point to the case of zero Γ_i, Ω_i, and Ψ_i, which simplifies the dynamics of (27) to

$$y = Fx + w; \quad w \text{ is Gaussian } (0, Q) \tag{55}$$

with the omission, as before, of subscripts. From the Markov property of the x's, $p(x/y, Z_N) = p(x/y, Z_i)$, so that

$$p(x/Z_N) = \int p(x/y, Z_i)\, p(y/Z_N)\, dy, \tag{56}$$

where the integration is performed over the subspace of R^n for which $p(x/y, Z_i)$ is nonsingular. For brevity, we again treat only the case of nonsingular Q, in which case this subspace is R^n itself. However, the result does not involve Q^{-1}, and can be verified for singular Q as well by resorting to a more complicated derivation. From the Bayes rule, $p(x/y, Z_i)$ is proportional as a function of x to the product $p(y/x, Z_i)\ p(x/Z_i)$. Furthermore, the Markov property of the x's can be exploited again to replace the first factor in this product by $p(y/x)$. From Eq. (55),

$$p(y/x) = \frac{e^{-\frac{1}{2}(y-Fx)^T Q^{-1}(y-Fx)}}{(2\pi)^{1/2n}|Q|^{1/2}}. \tag{57}$$

The product of functions (53) and (57) can be normalized so that its integral is unity by completing the square in the exponent, which leads to

$$\begin{aligned} p(x/y, Z_i) = \Big\{ & 1 + (\lambda + Au + \Sigma'' uu)(x - \tilde{x} - VF^T M^{-1}u) \\ & + \operatorname{tr}\left[\frac{1}{2}L + \Pi' u\right][(x - \tilde{x} - VF^T M^{-1}u)(x - \tilde{x} - VF^T M^{-1}u)^T - S] \\ & + \frac{1}{3}(x - \tilde{x} - VF^T M^{-1}u)^T \Delta (x - \tilde{x} - VF^T M^{-1}u)(x - \tilde{x} - VF^T M^{-1}u)\Big\} \\ & \times \left\{\frac{e^{-\frac{1}{2}(x-\tilde{x}-VF^T M^{-1}u)^T S^{-1}(x-\tilde{x}-VF^T M^{-1}u)}}{(2\pi)^{n/2}|S|^{1/2}}\right\} \end{aligned} \tag{58}$$

where

$$
\begin{aligned}
u &= y - \bar{y}, \\
\tilde{x} &= \bar{x} + \mathrm{VF}^T\mathrm{M}^{-1}(\bar{y} - \mathrm{F}\bar{x}), \\
\mathrm{M} &= \mathrm{FVF}^T + \mathrm{Q}, \\
\mathrm{S} &= \mathrm{V} - \mathrm{VF}^T\mathrm{M}^{-1}\mathrm{FV}, \\
\pi &= (\mathrm{M}^{-1}\mathrm{FV}\Delta)', \\
\Sigma &= \mathrm{M}^{-1}\mathrm{FV}\Delta\mathrm{VF}^T\mathrm{M}^{-1}, \\
\lambda &= \mathrm{Tr}[\Sigma(\bar{y} - \mathrm{F}\bar{x})(\bar{y} - \mathrm{F}\bar{x})^T], \\
\mathrm{L} &= 2\Delta\mathrm{VF}^T\mathrm{M}^{-1}(\bar{y} - \mathrm{F}\bar{x}), \text{ and} \\
\mathrm{A} &= 2\Sigma''(\bar{y} - \mathrm{F}\bar{x}).
\end{aligned}
$$

Using (54) and (58) in (56), completing the square in the resulting exponential factor, and deleting second-order terms leads eventually to the equation

$$
\begin{aligned}
p_{x-\tilde{x}}(t/\mathrm{Z}_N) = \left[\frac{e^{-\frac{1}{2}t^TG^{-1}t}}{(2\pi)^{n/2}|\mathrm{G}|^{1/2}}\right] \int_{R^n} \Bigg\{ & 1 + \lambda^T t + \frac{1}{2}\mathrm{tr}[\mathrm{L}(tt^T - \mathrm{S})] \\
& + \frac{1}{3}\, t^T\Delta tt - [\mathrm{M}^{-1}\mathrm{FV}\lambda + \mathrm{Tr}(\mathrm{S}\Pi)]^T u - \frac{1}{2}\,\mathrm{tr}(\mathrm{M}^{-1}\mathrm{FVA}uu^T) \\
& + \frac{1}{3}\, u^T(\Theta - \Sigma'\mathrm{VF}^T\mathrm{M}^{-1})uu \Bigg\} \left\{\frac{e^{-\frac{1}{2}(u-KQ^{-1}Ft)^TK^{-1}(u-KQ^{-1}Ft)}}{(2\pi)^{n/2}|\mathrm{K}|^{1/2}}\right\} du, \qquad (59)
\end{aligned}
$$

where

$$\mathrm{G} = \mathrm{V} - \mathrm{VF}^T\mathrm{M}^{-1}(\mathrm{M} - \mathrm{B})\mathrm{M}^{-1}\mathrm{FV} \qquad (60)$$

and

$$
\begin{aligned}
\mathrm{K} &= \mathrm{B} - \mathrm{BM}^{-1}\mathrm{FVG}^{-1}\mathrm{VF}^T\mathrm{M}^{-1}\mathrm{B} \qquad (61) \\
&= \mathrm{Q}(\mathrm{MB}^{-1}\mathrm{Q} + \mathrm{FVF}^T)^{-1}\mathrm{M} \quad \text{(by the matrix-inversion lemma).}
\end{aligned}
$$

The integrand of (59), as a function of u, has the form of a third degree polynomial multiplied by a Gaussian density. Therefore, the integral can be evaluated by using standard results for Gaussian moments. This procedure gives

$$p(x/Z_N) = \left[1 + \bar{\lambda}^T(x - \bar{x}) + \frac{1}{2}\,\mathrm{tr}\{\bar{L}\{(x - \bar{x})(x - \bar{x})^T - G\}\}\right.$$
$$\left. + \frac{1}{3}\,(x - \bar{x})^T\bar{\Delta}(x - \bar{x})(x - \bar{x})\right]\left[\frac{e^{-\frac{1}{2}(x-\bar{x})^T G^{-1}(x-\bar{x})}}{(2\pi 9^{n/2}|G|^{1/2}}\right], \tag{62}$$

where

$$\bar{\lambda} = \lambda + F^TQ^{-1}K[\mathrm{Tr}(K\Theta - K\Sigma'VF^TM^{-1}) - M^{-1}FV\lambda - \mathrm{Tr}(S\Pi)], \tag{63}$$

$$\bar{L} = L - F^TQ^{-1}KM^{-1}FVLVF^TM^{-1}KQ^{-1}F \tag{64}$$

$$\bar{\Delta} = \Delta + F^TQ^{-1}K(\Theta - \Sigma'VF^TM^{-1})KQ^{-1}F]'KQ^{-1}F. \tag{65}$$

Since $p(x/Z_N)$ is of the form (33), it is reducible to the form of (29) to first order. It is convenient at this point to define the more meaningful variables.

$$\Lambda = (V\Delta V)'V \quad (1/2 \text{ third central moment of } p(x/Z_i)),$$

$$\Xi = (B\Theta B)'B \quad (1/2 \text{ third central moment of } p(y/Z_N)),$$

$$\hat{x} = \bar{x} + \mathrm{Tr}(V^{-1}\Lambda) \quad (\text{mean of } p(x/Z_i)),$$

$$\tilde{y} = \bar{y} + \mathrm{Tr}(B^{-1}\Xi) \quad (\text{mean of } p(y/Z_N)).$$

The application of (33)—(36) to (62) then shows, after much manipulation, that the first three central moments of $p(x/Z_N)$ can be expressed to first order as

$$\hat{x} + VF^TM^{-1}(\tilde{y} - F\hat{x}) + G[V^{-1} - F^T(MB^{-1}Q + FVF^T)^{-1}F]$$
$$\times \mathrm{Tr}\{M^{-1}F\Lambda F^TM^{-1}[(\tilde{y} - F\hat{x})(\tilde{y} - F\hat{x})^T - (M + B)]\}, \tag{66}$$

$$G + 2[GV^{-1}\Lambda V^{-1}G - GF^T(MB^{-1}Q + FVF^T)^{-1}F\Lambda F^T(QB^{-1}M + FVF^T)^{-1}FG]'$$
$$\times F^TM^{-1}(\tilde{y} - F\hat{x}), \tag{67}$$

and

$$2[(KV^{-1}\Lambda V^{-1}K)'V^{-1}K + \{KF^TMB^{-1}Q + FVF^T)^{-1}[(MB^{-1}\Xi B^{-1}M)'B^{-1}M$$
$$- (F\Lambda F^T)'F^T](QB^{-1}M + FVF^T)^{-1}FK\}'FK]. \tag{68}$$

D. Overall Result

It is convenient for further perturbation analyses to express the preceding results in terms of the "nominal" covariance matrices of $p(x_i/Z_i)$ and $p(x_i/Z_N)$ for the case in which the perturbation parameters are all zero. If these matrices are denoted respectively by P_i and B_i, it follows from standard Kalman filtering and Bayesian smoothing results [6] that they are generated to first order by the recursions

$$\mathrm{M}_{i+1} = \mathrm{F}_i\mathrm{P}_i\mathrm{F}_i^T + \mathrm{Q}_i; \quad \mathrm{M}_0 \text{ specified a priori,} \tag{69}$$

$$\mathrm{P}_i = \mathrm{M}_i - \mathrm{M}_i\mathrm{H}_i^T(\mathrm{H}_i\mathrm{M}_i\mathrm{H}_i^T + \mathrm{R}_i)^{-1}\mathrm{H}_i\mathrm{M}_i - 2\mathrm{M}_i\mathrm{Tr}(\mathrm{R}_i^{-1}\Upsilon_i\mathrm{R}_i^{-1}\Upsilon_i)\mathrm{M}_i, \tag{70}$$

and

$$\mathrm{B}_i = \mathrm{P}_i - \mathrm{P}_i\mathrm{F}_i^T\mathrm{M}_{i+1}^{-1}(\mathrm{M}_{i+1} - \mathrm{B}_{i+1})\mathrm{M}_{i+1}^{-1}\mathrm{F}_i\mathrm{P}_i; \quad \mathrm{B}_N = \mathrm{P}_N. \tag{71}$$

Let us define the following quantities:

$$\mathrm{D}_i = \frac{1}{2}\,[\mathrm{cov}(x_i/\mathrm{Z}_i) - \mathrm{P}_i] \tag{72}$$

$$\mathrm{E}_i = \frac{1}{2}\,[\mathrm{cov}(x_i/\mathrm{Z}_N) - \mathrm{B}_i] \tag{73}$$

$$\Lambda_i = \frac{1}{2} \text{ third central moment of } p(x_i/\mathrm{Z}_i) \tag{74}$$

$$\Theta_i = \frac{1}{2} \text{ third central moment of } p(x_i/\mathrm{Z}_N) \tag{75}$$

$$\mathrm{K}_{i+1} = \mathrm{B}_{i+1} - \mathrm{B}_{i+1}\mathrm{M}_{i+1}^{-1}\mathrm{F}_i\mathrm{P}_i\mathrm{B}_i^{-1}\mathrm{P}_i\mathrm{F}_i^T\mathrm{M}_{i+1}^{-1}\mathrm{B}_{i+1} \tag{76}$$

$$\hat{x}_i = \mathrm{E}(x_i/\mathrm{Z}_i) \tag{77}$$

$$\tilde{x}_i = \mathrm{E}(x_i/\mathrm{Z}_N) \tag{78}$$

and suppress for brevity in the following notation the subscript i for the variables $\hat{x}$, F, Γ, P, Q, D, Ψ, Ω, and Λ, and the subscript $i+1$ for the variables M, H, R, Φ, Υ, Θ, E, B, K, $\tilde{x}$, and z. Then, for the general case with nonzero perturbation parameters, the preceding recursions can be expressed to first order as follows:

<u>Forward Recursion</u> $(i = -1, 0, \cdots, \mathrm{N})$

$$\bar{x} = \mathrm{F}\hat{x} + \mathrm{Tr}[\Gamma(\hat{x}\hat{x}^T + \mathrm{P}) + \Omega\mathrm{Q}]; \quad i \geqslant 0 \tag{79}$$

$$\overline{\mathrm{D}} = \mathrm{FDF}^T + (\Psi + \mathrm{FP}\Gamma' + \Gamma''\mathrm{PF}^T)'\hat{x}; \quad i \geqslant 0 \tag{80}$$

$$\begin{aligned}\overline{\Lambda} = {} & (\mathrm{F}\Lambda\mathrm{F}^T)'\mathrm{F}^T + \mathrm{FP}\Gamma\mathrm{PF}^T + \Psi'\mathrm{PF}^T + \mathrm{Q}\Omega\mathrm{Q} + (\mathrm{FP}\Gamma\mathrm{PF}^T + \Psi'\mathrm{PF}^T + \mathrm{Q}\Omega\mathrm{Q})' \\ & + (\mathrm{FP}\Gamma\mathrm{PF}^T + \Psi'\mathrm{PF}^T + \mathrm{Q}\Omega\mathrm{Q})''; \quad i \geqslant 0\end{aligned} \tag{81}$$

$$\begin{aligned}\hat{x}_{i+1} = {} & \bar{x} + \mathrm{P}_{i+1}\mathrm{H}^T\mathrm{R}^{-1}[z - \mathrm{H}\bar{x} - \Phi''\overline{xx} - \mathrm{Tr}(\mathrm{M}\Phi)] + 2\mathrm{P}_{i+1}[\mathrm{M}^{-1}\overline{\mathrm{D}}\mathrm{H}^T + \Phi\bar{x} \\ & - \mathrm{H}^T\mathrm{R}^{-1}(\mathrm{HM}\Phi' + \Upsilon)'\bar{x}](\mathrm{R} + \mathrm{HMH}^T)^{-1}(z - \mathrm{H}\bar{x}) + \mathrm{P}_{i+1}\mathrm{Tr}\{(\mathrm{R} + \mathrm{HMH}^T)^{-1} \\ & \times [[\mathrm{I} + (\Upsilon\mathrm{R}^{-1} + \mathrm{R}^{-1}\Upsilon)'\bar{x}]\ \Upsilon + 2\mathrm{HM}(\Phi - \mathrm{H}^T\mathrm{R}^{-1}\Upsilon') - (\mathrm{H}^T\mathrm{R}^{-1}[\mathrm{HM}\Phi\mathrm{MH}^T]'')' \\ & + \mathrm{H}(\mathrm{M}^{-1}\overline{\Lambda})'\mathrm{H}^T](\mathrm{R} + \mathrm{HMH}^T)^{-1}[(z - \mathrm{H}\bar{x})(z - \mathrm{H}\bar{x})^T - (\mathrm{R} + 2\Upsilon'\bar{x} + \mathrm{HMH}^T)]]\}\end{aligned} \tag{82}$$

$$\begin{aligned}\mathrm{D}_{i+1} = {} & \mathrm{P}_{i+1}\mathrm{M}^{-1}\overline{\mathrm{D}}\mathrm{M}^{-1}\mathrm{P}_{i+1} + [\mathrm{P}_{i+1}(\mathrm{H}^T\mathrm{R}^{-1}\Upsilon\mathrm{R}^{-1}\mathrm{H} - \Phi'\mathrm{R}^{-1}\mathrm{H} - \mathrm{H}^T\mathrm{R}^{-1}\Phi'')\mathrm{P}_{i+1}]'\bar{x} \\ & + \{(\mathrm{P}_{i+1}\mathrm{M}^{-1}\overline{\Lambda}\mathrm{M}^{-1}\mathrm{P}_{i+1})'\mathrm{H}^T \\ & + [\mathrm{P}_{i+1}(\mathrm{H}^T\mathrm{R}^{-1}\Upsilon\mathrm{R}^{-1}\mathrm{H} - \Phi'\mathrm{R}^{-1}\mathrm{H} - \mathrm{H}^T\mathrm{R}^{-1}\Phi'')\mathrm{P}_{i+1}]'\mathrm{MH}^T \\ & + [\mathrm{P}_{i+1}(\Phi - \mathrm{H}^T\mathrm{R}^{-1}\Upsilon' + \Upsilon''\mathrm{R}^{-1}\mathrm{H})\mathrm{P}_{i+1}]'\}(\mathrm{R} + \mathrm{HMH}^T)^{-1}(z - \mathrm{H}\bar{x})\end{aligned} \tag{83}$$

$$\Lambda_{i+1} = (P_{i+1}M^{-1}\overline{\Lambda}M^{-1}P_{i+1})'M^{-1}P_{i+1} + (P_{i+1}H^TR^{-1}YR^{-1}HP_{i+1})'P_{i+1}$$
$$- (P_{i+1}\Phi P_{i+1})'R^{-1}HP_{i+1}$$
$$+ [(P_{i+1}H^TR^{-1}YR^{-1}HP_{i+1})'P_{i+1} - (P_{i+1}\Phi P_{i+1})'R^{-1}HP_{i+1}]'$$
$$+ [(P_{i+1}H^TR^{-1}YR^{-1}HP_{i+1})'P_{i+1} - (P_{i+1}\Phi P_{i+1})'R^{-1}HP_{i+1}]''. \tag{84}$$

Reverse Recursion for $\Gamma, \Psi, \Omega = 0$ $\quad (i = N-1, \cdots, 0)$

$$\tilde{x}_i = \hat{x} + PF^TM^{-1}(\bar{x} - F\hat{x}) + 2(I - PF^TM^{-1}F)DF^TM^{-1}(\tilde{x} - F\dot{\hat{x}})$$
$$+ B_i[P^{-1} - F^T(MB^{-1}Q + FPF^T)^{-1}F]\mathrm{Tr}\{M^{-1}F\Lambda F^TM^{-1}[(\tilde{x} - F\hat{x})(\tilde{x} - F\hat{x})^T$$
$$- (M+B)]\}; \quad \tilde{x}_N = \hat{x}_N \tag{85}$$

$$E_i = D - (I - PF^TM^{-1}F)DF^TM^{-1}(M-B)M^{-1}FP - PF^TM^{-1}(M-B)M^{-1}FD(I - F^TM^{-1}FP)$$
$$+ PF^TM^{-1}(FDF^T - E)M^{-1}FP + [B_iP^{-1}\Lambda P^{-1}B_i - B_iF^T(MB^{-1}Q + FPF^T)^{-1}$$
$$\times F\Lambda F^T(QB^{-1}M + FPF^T)^{-1}FB_i]F^TM^{-1}(\tilde{x} - F\hat{x}); \quad E_N = D_N \tag{86}$$

$$\Theta_i = (KP^{-1}\Lambda P^{-1}K)'P^{-1}K + \{KF^T(MB^{-1}Q + FPF^T)^{-1}[(MB^{-1}\Theta B^{-1}M)'B^{-1}M$$
$$- (F\Lambda F^T)'F^T](QB^{-1}M - FPF^T)^{-1}FK\}'FK; \quad \Theta_N = \Lambda_N. \tag{87}$$

For the index $i = -1$, the quantities $\bar{x}$, $\overline{D}$ and $\overline{\Lambda}$ are specified by the a priori distribution of x_0, not by (79)—(81). For the common limiting case of a "flat prior" (i.e., an a priori probability density for x_0 which has an approximately infinite diagonal covariance matrix), a reasonable procedure for initiating the recursions here would be to make M_0 a diagonal matrix with very large diagonal entries, and use zero for $\bar{x}$, $\overline{D}$ and $\overline{\Lambda}$ in the initial iteration of (82)—(84). For the special case of $H_0 = I$, this procedure could be circumvented entirely by using

$$P_0 = R_0,$$
$$\hat{x}_0 = z_0 - \Phi_0''z_0z_0 - R_0\mathrm{Tr}(Y_0R_0^{-1}),$$
$$D_0 = (Y_0 - R_0\Phi_0' - \Phi_0''R_0)z_0, \quad \text{and}$$
$$\Lambda_0 = Y_0'R_0 - R_0\Phi_0R_0 + (Y_0'R_0 - R_0\Phi_0R_0)' + (Y_0'R_0 - R_0\Phi_0R_0)''.$$

Since all posterior probability densities $p(x_i/Z_N)$ are of the form of (29), they are completely specified to first order by their first three central moments, which in turn are specified to first order as $\tilde{x}_i$, $B_i + 2E_i$, and $2\Theta_i$ by the system of recursions (69)—(71) and (79)—(87).

Asymptotic Results

In practice, the pixel spacing is often so small that the number of pixels is no longer small compared to $1/h$ for the nonlinearities that are present. In such cases, one is typically faced with a situation in which the problem parameters assume the form

$$F_i = I + \epsilon F(t)$$
$$\Gamma_i = \epsilon\Gamma(t)$$
$$\Omega_i = \Omega(t)$$

$$H_i = H(t)$$
$$\Phi_i = \Phi(t)$$
$$R_i = \frac{1}{\epsilon} R(t)$$
$$Q_i = \epsilon Q(t)$$
$$\Upsilon_i = \frac{1}{\epsilon} \Upsilon(t)$$
$$\Psi_i = \epsilon \, \Psi(t)$$

where ϵ denotes the small quantity 1/N and t denotes the quantity i/N, a variable ranging from 0 to 1. As long as $\epsilon >> h^2$, a consistent derivation of the above results can still be developed, in this case by folllowing the sort of procedure used in Refs. 12 and 13, except that all terms of order $h\epsilon$ or larger are now retained. This analysis becomes extremely tedious, however, and is not exhibited here.

IV. EXPERIMENTAL RESULTS

A. Filtering in Density Variable

To evaluate the performance of these filters experimentally, we chose a test image as similar as possible to one used by Naderi [1] in work with some related filters. Our image was generated by scanning a photographic negative on a comparatively grainy film (Kodak Tri-X) with a microdensitometer, using a 5-μm × 5-μm aperture to produce the 128-pixel by 128-pixel digital image shown in Fig. 4. The optical densities of the light, medium, and dark areas of this image average 2.1, 1.2, and 0.3, respectively. The appearance and statistical parameters of this example agree fairly closely with those of Naderi, which were entirely computer-generated. In addition to the photographic grain noise, our image undoubtedly contains some shot noise contributed by the microdensitometer, but this is believed to be relatively minor and is not considered further.

Further following Naderi's approach, we constructed our filter by combining one-dimensional filters for the pixel rows and columns, each of which is treated as a noisily observed Markov process vector with first-order correlation of 0.85. This means that the original one-dimensional "images" can all be regarded as generated by a simple scalar process of the form of (27),

$$x_{i+1} = fx_i + w_i; \quad w_i \text{ is Gaussian } (0,q) \tag{88}$$

provided that the Markov process is assumed to be stationary. In the above,

$x_i = u_i - \bar{u}$,

u_i = optical density of the ith pixel,

$\bar{u}$ = average pixel density of the entire (two-dimensional) image,

$f = 0.85$,

$$\left.\begin{array}{l} q = \sigma^2(1 - f^2) \text{ and} \\ \sigma^2 = \text{variance of optical density fluctuations in} \\ \quad\ \text{the two-dimensional original image.} \end{array}\right\} \text{from Eq. (17)}$$

In addition to grain noise, optical and chemical blurring are present in the actual photographically recorded image. Naderi estimated the magnitude of this blurring and took it into account, but the resulting contribution to the final filter design turned out to be relatively minor in this case because the correlation coefficient f between adjacent pixels is so close to unity that it overwhelms the compensation in the design for the expected optical and chemical blurring. This blurring still affects the filter's output appreciably, but mostly because it affects the input. Thus, although for simplicity these blurring phenomena are not considered in the present filter design, the situation in this respect is still comparable to that of Naderi.

The grain-noise model adopted here is such that the variance is a linear fit to that of (26) in the range of the optical density levels of interest. Hence, the observed image corresponding to (88), with the average optical density $\bar{u}$ subtracted for convenience, can be treated as a sequence $\{z_i\}$ such that

$$z_i = x_i + \mathrm{v}_i \tag{89}$$

where, given the x_i sequence, the v_i are independent zero-mean Gaussian random variables with

$$\mathrm{E}(\mathrm{v}_i^2/x_i) = r + 2\eta x_i = r + 2\eta(u_i - \bar{u}) \tag{90}$$

in the x_i range of interest. For the image examined here, r and η were actually determined by matching the values given by (90) and those observed at $u_i = \bar{u} \pm \sigma$, for which

$$\bar{u} = 1.2$$

and

$$\sigma = 0.6.$$

The resulting parameters are approximately

$$r = 0.1$$

and

$$\eta = 0.04.$$

Since η is so small, it can be regarded as an approximately infinitesimal parameter, so (88) and (89) represent a simple scalar instance of the class of observation systems analyzed in the preceding section. Hence, the Bayes filter for this system can be found to first order as a special case of the recursions (79) — (87), for which the desired output is the $\hat{x}_i$ sequence of posterior mean values. Once these are found, the average optical density $\bar{u}$ has to be added back to the result for each pixel. Neglecting edge effects allows the M_i, P_i, B_i, Λ_i, and Θ_i variables to be treated as constants, denoted here by m, p, b, λ, and θ. This reduces the pertinent equations to the following:

$$\begin{aligned}\hat{x}_{i+1} = f\hat{x}_i &+ \left[\frac{p}{r} + 2\left(\frac{fp}{m}\right)^2 \frac{d_i}{r} - 2\eta\left(\frac{p}{r}\right)^2\left(\frac{f\hat{x}_i}{m}\right)\right](z_{i+1} - f\hat{x}_i) \\ &+ \left[\frac{\lambda}{r^2} + \frac{\eta p}{r^2}\left(\frac{r-3m}{m+r+2\eta\hat{x}_i}\right)\right][(z_{i+1} - f\hat{x}_i)^2 - (m + r + 2\eta\hat{x}_i)]\end{aligned} \tag{91}$$

$$d_{i+1} = \left[\frac{fp}{m}\right]^2 d_i + \eta \left[\frac{p}{r}\right]^2 f\hat{x}_i + \left[\frac{\lambda}{r} - 2\eta\left[\frac{p}{r}\right]^2\right](z_{i+1} - f\hat{x}_i) \tag{92}$$

$$\tilde{x}_i = \hat{x}_i + \frac{f}{m}\left[p + 2\frac{qd_i}{m}\right](\tilde{x}_{i+1} - f\hat{x}_i) + \frac{f^2 q\lambda}{m^3}\left[(\tilde{x}_{i+1} - f\hat{x}_i)^2 - (m+b)\right], \tag{93}$$

where (91) and (92) are evaluated successively for $i = 0, 1, \cdots, N-1$ (where N = 127 in the present case), beginning with

$$\hat{x}_0 = z_0$$

and

$$d_0 = \eta z_0,$$

and (93) is evaluated in reverse order for the $\tilde{x}_i$, beginning with $\tilde{x}_N = \hat{x}_N$. The constants p, m, b, and λ are determined from (69)—(71) and (84) by requiring that the variables in these equations don't change between adjacent indices. This yields

$$p = \frac{1}{2f^2}\left[f^2 R + q - R + \sqrt{(f^2 R + q - R) + 4f^2 qR}\right]; \quad R = \frac{r^2}{r + 2\eta^2}, \tag{94}$$

$$m = f^2 p + q, \tag{95}$$

$$b = \frac{p - (fp/m)^2 m}{1 - (fp/m)^2}, \tag{96}$$

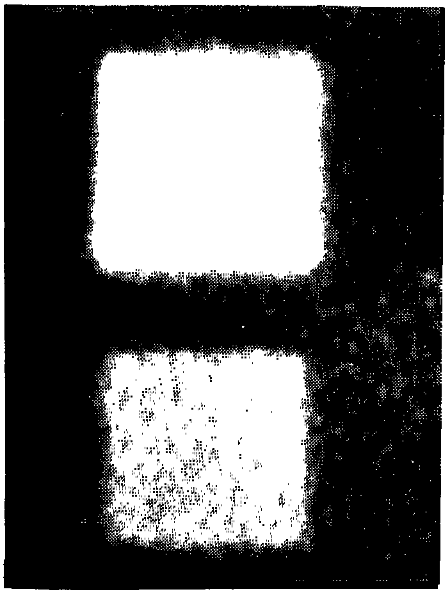
and

$$\lambda = \frac{3p^3\eta}{r^2[1 - (fp/m)^3]}. \tag{97}$$

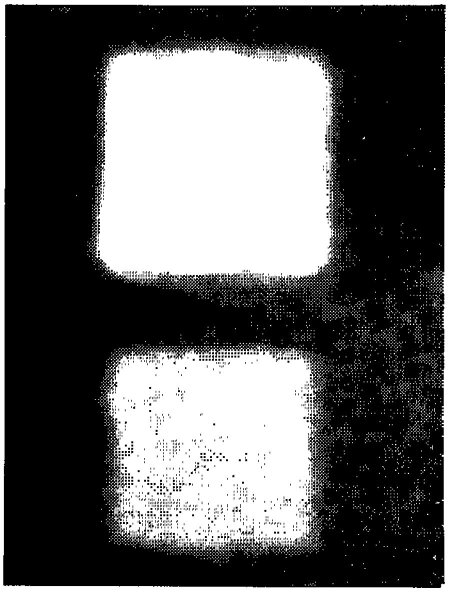
Note that the parameter θ is not needed for the limited purpose of computing only the $\tilde{x}_i$ values.

The one-dimensional Bayesian filter determined by recursions (91)—(93) was used to construct a two-dimensional filter for the image at hand by applying it successively to the pixel rows and columns in the manner described in Section IIC. The rationale for this construction is that this filter is a perturbation of the linear one-dimensional filter of Eqs. (19)—(23). It has already been shown that when this unperturbed filter is used on a two-dimensional image in this way, the result can, except for edge effects, be interpreted as a two-dimensional weighting function of the reasonable "exponential window" form. Hence, when the perturbations are added to the one-dimensional filters to account for the grain-noise density-dependence, it is reasonable to combine them in the same way. It should be noted that this is not the procedure used by Naderi [1], so the treatments become distinct at this point.

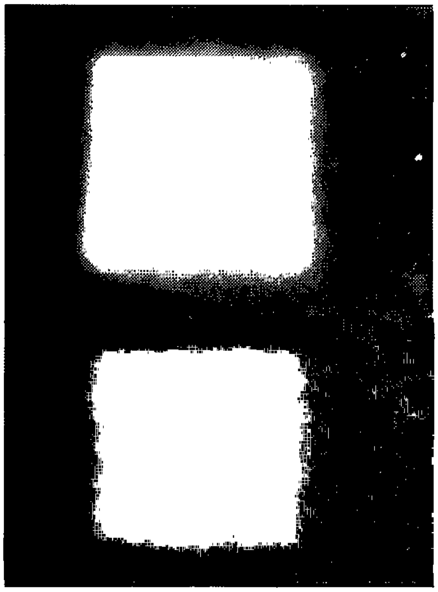
Figure 4 shows the result of applying this composite filter to the test image, which is shown in the same figure. For comparison, the output of the corresponding unperturbed (linear) filter is also displayed. The constituent one-dimensional linear filters used here can be obtained simply by setting $\eta = d_0 = 0$ in (91)—(97); therefore, by (90), these are the Bayes filters resulting from approximating the grain noise as independent of the optical density.



a. Test image with photographic grain noise



b. Nonlinearly filtered image



c. Linearly filtered image

Figure 4

Hence, the difference between Figs. 4b and 4c is that which results from taking the density-dependence of the grain noise into account, at least to within the accuracy of the linear variance fit used here. This difference is not dramatic, but it is noticeable. As might be expected on intuitive grounds, the more accurate representation of the grain noise leads to relatively more filtering in regions of high optical density, where the noise magnitude is larger. The range of optical densities is quite high in this example, which indicates a good degree of robustness in the approximations and perturbation procedures used in this context.

Direct grain-noise filtering results appear unimpressive partly because the same visual perception tends to be created automatically by the observer. For this reason, we investigated the value of such filtering as a preliminary operation used to improve the subsequent results from more intuitive nonlinear image processing procedures. The so-called noise-cheating algorithm described in Naderi [1] was used for this purpose. This algorithm is essentially a variant of gray-level slicing with an extra step added to clean up boundaries between regions of distinct gray levels. Also, the gray levels to which the pixels of the input image are quantized are spaced 4σ apart, where σ is the rms noise value (possibly a slowly varying function of gray level) of the input image.

Figure 5 shows the results of applying this noise-cheating algorithm to (i) the unfiltered test image, (ii) the linearly filtered image, and (iii) the nonlinearly filtered image. In these cases, the 4σ spacing for the quantization levels were determined as follows:

(i) By direct use of (26), with $\bar{u} = 1.2$ and $\sigma(u) = \sqrt{r} = 0.31$ (result varies with gray level u)

(ii) In this case, the result of (i) was multiplied by the noise-suppression factor described in Section IIA for the exponential window weighting function corresponding to this filter, which is linear.

(iii) In this case, the action of the nonlinear filter at any given gray level u was approximated as that of the linear filter of (ii) when the grain-noise variance parameter r is replaced by $r + 2\eta u$. The computation then proceeded as in (ii).

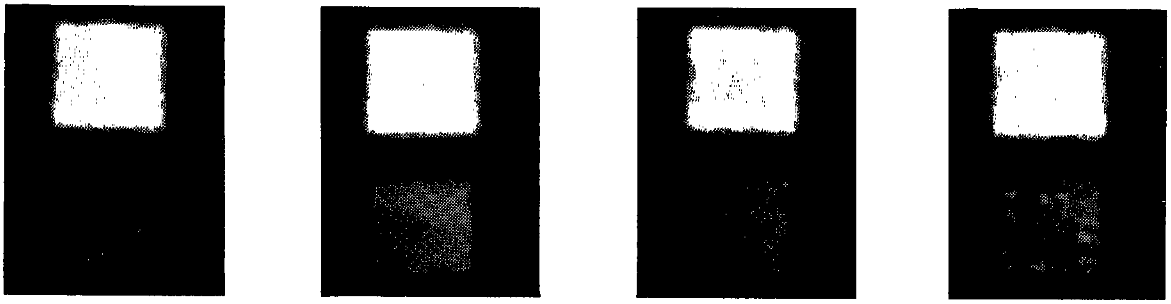
The appearance of the end result is affected by shifts of the quantization levels in each case, so several different choices are shown for each. Although no general improvement in appearance is apparent from the filtering, it should be remembered that the quantized gray levels in the output are more closely spaced in the case of the filtered images, and more closely spaced in the bright areas (and less so for the darker areas) in the nonlinearly filtered image than in the linearly filtered one. For example, part (iv) of Fig. 5 shows the effect of applying this algorithm to the unfiltered test image with the more closely spaced quantization levels that were used for the nonlinearly filtered image in part (iii). It is apparent that the filtering allows a finer degree of gray scale resolution for roughly the same degree of geometric degradation of the image.

B. Filtering in Intensity Variable

For reasons cited earlier, the estimated image is often desired in terms of the intensity variable, which is related nonlinearly to the optical density by the relation

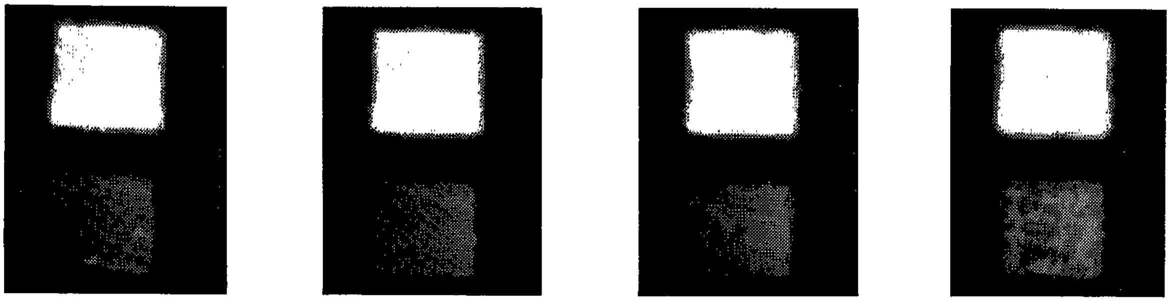
a b c d

i. a-d Unfiltered test image — four initial levels



a b c d

ii. a-d Linearly filtered image — four initial levels



a b c d

iii. a-d Nonlinearly filtered image — four initial levels

a b c d

iv. Unfiltered test image — quantization levels of part iii

Fig. 5 — Applications of noise-cheating algorithm

$$y = -\log_{10}(e)\ \ell n\ x = -0.434\ \ell n\ x \tag{98}$$

where

y = optical density

x = transmitted intensity/incident intensity (= 1 here).

Two basic approaches to this problem are examined here, each of which is based on one-dimensional line filters constructed according to the above perturbation analysis. In the first approach, the observed optical density is regarded as a nonlinear measurement of the intensity, and the intensity variable is filtered directly. In the second, the density variable is filtered after the dynamics of the intensity fluctuations, which are presumed linear and Gaussian, are converted into equivalent nonlinear dynamics for the density fluctuations. In both cases, the optical density is used as the measurement variable. It is the variable that behaves experimentally as one corrupted by noise that is approximately additive and Gaussian, as required by the perturbation theory used in the filter construction. The resulting one-dimensional filters are "cascaded" as before to produce the final two-dimensional filter. The work here is fairly preliminary and is limited to the case of intensity fluctuations of the simple form described by Eq. (88).

1. Nonlinear Measurement Approach

In this approach, the dynamics of the image fluctuations are still of the form (88), except that the u_i-variable now denotes (relative) intensity instead of optical density, and $\bar{u}$ denotes average intensity. Subtracting the average value of the observed optical density given by (98) as

$$-0.434\ \ell n\ \bar{u},$$

and rescaling the difference (and the associated noise variable) leads to a measurement variable of the form

$$z_i = x_i + \phi x_i^2 + v_i, \tag{99}$$

where ϕ is chosen to approximate the logarithmic nonlinearity in the range of interest. This is again an instance of the class of observation systems treated in the last section, and is the same as the preceding case of optical density filtering except for the added complication of a measurement nonlinearity. The filtering equations corresponding to (91)—(97) become the following in the present case:

$$\begin{aligned}\hat{x}_{i+1} = f\hat{x}_i &+ \frac{p}{r}\left[z_{i+1} - f\hat{x}_i - \phi(f^2\hat{x}_i^2 + m)\right] + 2\Big[(fp/m)^2(d_i/r) \\ &- \frac{fp}{m+r}\left(\eta + (m-r)\phi\right)\hat{x}_i\Big](z_{i+1} - f\hat{x}_i) \\ &+ \left[\frac{\lambda}{r^2} + \frac{\eta p}{r^2}\left(\frac{r-3m}{m+r+2\eta\hat{x}_i}\right) + 2\phi(p/r)^2\right]\left[(z_{i+1} - f\hat{x}_i)^2 - (m + r + 2\eta\hat{x}_i)\right],\end{aligned} \tag{100}$$

$$\begin{aligned}d_{i+1} = (fp/m)^2 d_i &+ (\eta - 2r\phi)(p/r)^2 f\hat{x}_i \\ &+ \left[\frac{\lambda}{r} - 2(\eta - r\phi)(p/r)^2 - \frac{\phi p^2}{m+r}\right](z_{i+1} - fx_i),\end{aligned} \tag{101}$$

and

$$\lambda = \frac{3p^3(\eta - r\phi)}{r^2[1 - (fp/m)^3]}. \tag{102}$$

The equations for $\tilde{x}_i$, p, m, and b remain unchanged as (93) and (94)–(96).

2. Nonlinear Dynamics Approach

In this approach, the results of the intensity variable filtering are expressed in terms of the optical density variable. The final result can, of course, be transformed back to an intensity representation by using the inverse of (98). With intensity fluctuations of the form (88), now expressed as

$$u_{i+1} - \bar{u} = f(u_i - \bar{u}) + w_i, \tag{103}$$

in which

$\bar{u}$ = average intensity

u_i = intensity of i th pixel

w_i = constant-variance Gaussian noise sequence,

a careful expansion to second order in the image fluctuations shows that the corresponding optical density has dynamics given by

$$x_{i+1} = fx_i + \omega_i - \frac{f(1-f)}{2\alpha} x_i^2 + \frac{f}{\alpha} x_i\omega_i + \frac{1}{2\alpha}\omega_i^2, \tag{104}$$

where

$\alpha = \log_{10} e \cong 0.434$

$\omega = -\dfrac{\alpha}{\bar{u}} w_i$ (another Gaussian noise sequence, variance denoted by q)

$x_i = i$ th optical density $- \bar{x}$, $\bar{x} = -\alpha \, \ell n \, \bar{u}$.

Again, the zeroth-order parameters f and q were chosen here to match the observed image characteristics (rms *density* fluctuations and degree of correlation between adjacent pixels). To first order (104) is equivalent to

$$x_{i+1} = fx_i + \omega_i - \frac{f(1-f)}{2\alpha} x_i^2 + \frac{1}{2\alpha}\omega_i^2 \tag{105}$$

with

$$E(\omega_i^2/x_i) = q + 2\frac{fq}{\alpha} x_i. \tag{106}$$

The forward recursion of (79)–(84) includes the present case, with the dynamics of (105) and (106), and the measurements of (89) and (90). Neglecting edge effects by using the steady-state values of p, m, and q from (94)–(96), and

$$\lambda = \left[\frac{3p^3}{1 - (fp/m)^3}\right]\left[\frac{m^2 - f^3p^2}{2m^3\alpha} + \frac{\eta}{r^2}\right] \tag{107}$$

leads to the following first-order forward recursion:

$$\hat{x}_{i+1} = f\hat{x}_i - \frac{f(1-f)}{2\alpha}(\hat{x}_i^2 + p) + \frac{q}{2\alpha} + \frac{p}{r}\left[z_{i+1} - f\hat{x}_i + \frac{f(1-f)}{2\alpha}(\hat{x}_i^2 + p) - \frac{q}{2\alpha}\right]$$

$$+ 2\frac{fp}{m+r}\left[\frac{f}{m}d_i + \left(\frac{m-fp}{m\alpha} - \frac{\eta}{r}\right)\hat{x}_i\right](z_{i+1} - f\hat{x}_i)$$

$$+ \frac{r}{(m+r)^3}\left[f^3\lambda + \frac{3}{2\alpha}(m^2 - f^3p^2) + \frac{\eta m^2}{p(r+2\eta\hat{x}_i)}(r-3m)\right]$$

$$\times\left[(z_{i+1} - f\hat{x}_i)^2 - (m + r + 2\eta\hat{x}_i)\right] \tag{108}$$

$$d_{i+1} = \left(\frac{fp}{m}\right)^2 d_i + \left[\eta\left(\frac{p}{r}\right)^2 + \left(\frac{m-fp}{\alpha}\right)\left(\frac{p}{m}\right)^2\right]f\hat{x}_i + \left[\frac{\lambda}{r} - 2\eta\left(\frac{p}{r}\right)^2\right](z_{i+1} - f\hat{x}_i). \tag{109}$$

The general reverse recursion described by (85)—(87) must be extended to generate the $\tilde{x}_i$'s for the present case. There is no conceptual difficulty in simply including the extra terms in the analysis there, but the general result becomes extremely complicated. Currently we are unable to simplify it due to insufficient insight. For the simple special case at hand, however, these details have been worked out for the limited purpose of computing only the posterior means $\tilde{x}_i$. The result is the following reverse recursion:

$$\tilde{x}_i = \hat{x}_i + \frac{fp}{m}\left[\tilde{x}_{i+1} - f\hat{x}_i + \frac{f(1-f)}{2\alpha}(\hat{x}_i^2 + p) - \frac{q}{2\alpha}\right] + \left[\frac{fp}{m^2\alpha}(fm + 2fp - 3m)\hat{x}_i\right.$$

$$\left. + 2\left(\frac{fq}{m^2}\right)d_i\right](\tilde{x}_{i+1} - f\hat{x}_i) + \frac{f}{m^3}\left[fq\lambda + \frac{p}{2\alpha}(fpm - m^2 - 3fpq)\right]$$

$$\times\left[(\tilde{x}_{i+1} - f\hat{x}_i)^2 - (m + \mathrm{b})\right] + \frac{fpb}{\alpha m^3}(fpm - m^2 - 3fpq) + \frac{3f^3pb}{m\alpha}; \quad \tilde{x}_N = \hat{x}_N. \tag{110}$$

3. Numerical Results

Both of these approaches were applied to the test pattern used previously. Unfortunately, the logarithmic nonlinearity here is so severe that, with the rather extreme contrasts present in this particular image, the first procedure would not work without some serious compromises being made.

In the first procedure, the observed optical density is treated as a noisy nonlinear measurement of the intensity. It can be shown from (98) and (89) after much manipulation that (suppressing the i subscripts)

$$z = \epsilon + \frac{1}{0.868}\epsilon^2 + \mathrm{v}, \tag{111}$$

where

x = pixel intensity

$\bar{x}$ = average intensity over image (= 0.063 here)

$z = (y - \bar{y}) + v$

y = pixel optical density

$\bar{y} = -0.434 \; \ell n \; \bar{x}$,

to second order in $x - \bar{x}$, the departure of the intensity from its average value. By definition, the intensity value can be recovered from the ϵ variable as

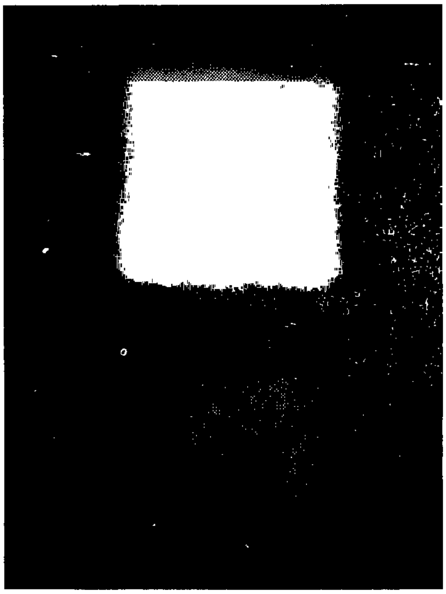
$$x = \bar{x} \left[1 - \frac{\epsilon}{0.434} \right]. \tag{112}$$

Ideally, one would wish to use $\phi = 1/0.868$ in the perturbation equations (100) – (102) for filtering the ϵ variable in this case, so that (99) and (111) agree for small fluctuations of ϵ. However, these fluctuations reach values of ± 0.8, which reduces (112) to an absurdity, giving possibly negative intensity values. Experimentally, the use of this "low contrast" value for ϕ led to numerical instability when used on the test image. In order to achieve reasonable experimental results, we found it necessary to reduce ϕ from this low-contrast value by a factor of about ten. This in effect sacrificed low-contrast accuracy for the sake of a better overall fit of the large-scale intensity fluctuations encountered in this image. So much had to be sacrificed, however, that the use of this ϕ parameter to represent the nonlinearities here is of doubtful validity and should probably be confined to images of much lower contrast. The output of the filter, modified as just described, is shown in Fig. 6. Rather than displaying the actual intensity variable specified by (112), Fig. 6a shows the filtered values of ϵ, the variable on which the filter actually operates. Since (112) is linear and $\tilde{\epsilon}$ is the conditional mean of ϵ, the conditional mean of the intensity, $\tilde{x}$, is given by

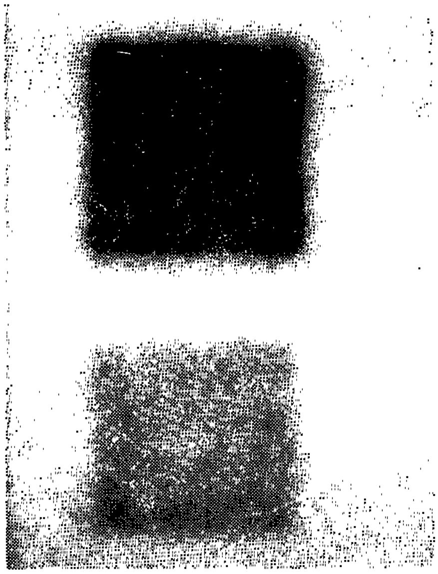
$$\tilde{x} = \bar{x} \left[1 - \frac{\tilde{\epsilon}}{0.434} \right].$$

Hence, this result is the same as the negative of the filtered intensity, at least to within an arbitrary bias and scale factor. Figure 6b shows the result of applying the appropriate linear transformation to the $\tilde{\epsilon}$ estimate for obtaining the filtered image in the original intensity variable. The exact details of this transformation depend on the dynamic range of the particular display device, etc. Also, an optical enlargement of the original photographic negative (i.e., the unfiltered intensity) is shown in Fig. 6c for comparison.

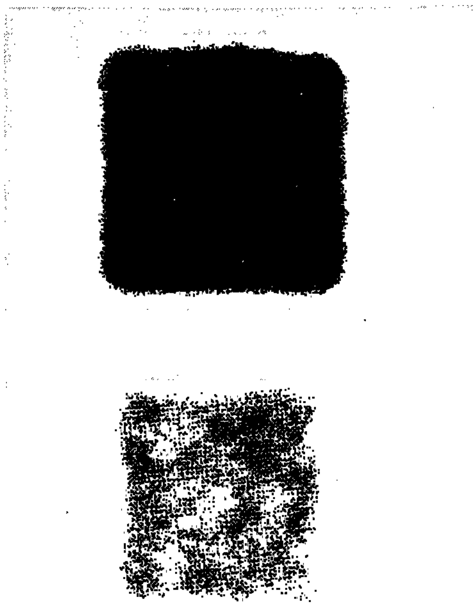
The nonlinearities here did not have such a catastrophic effect on the second method, in which the filtering is performed on the optical density variable with nonlinear (one-dimensional) fluctuation dynamics. Here, acceptable performance could be achieved with the nonlinearity coefficients at roughly their low-contrast values, although this appeared to be near the borderline of feasibility for this image. Thus it appears that this nonlinear dynamics approach, while more complicated, is basically more robust than the nonlinear measurements approach. Figure 7a shows the direct result of using this filter on the test image, which appears in terms of the optical density variable. Figure 7b shows the same result after having been transformed to the intensity representation by applying the inverse of (98). The optical enlargement of the original photograph is repeated for convenience in Fig. 7c.



a. Result of nonlinear measurements filter, ϵ variable

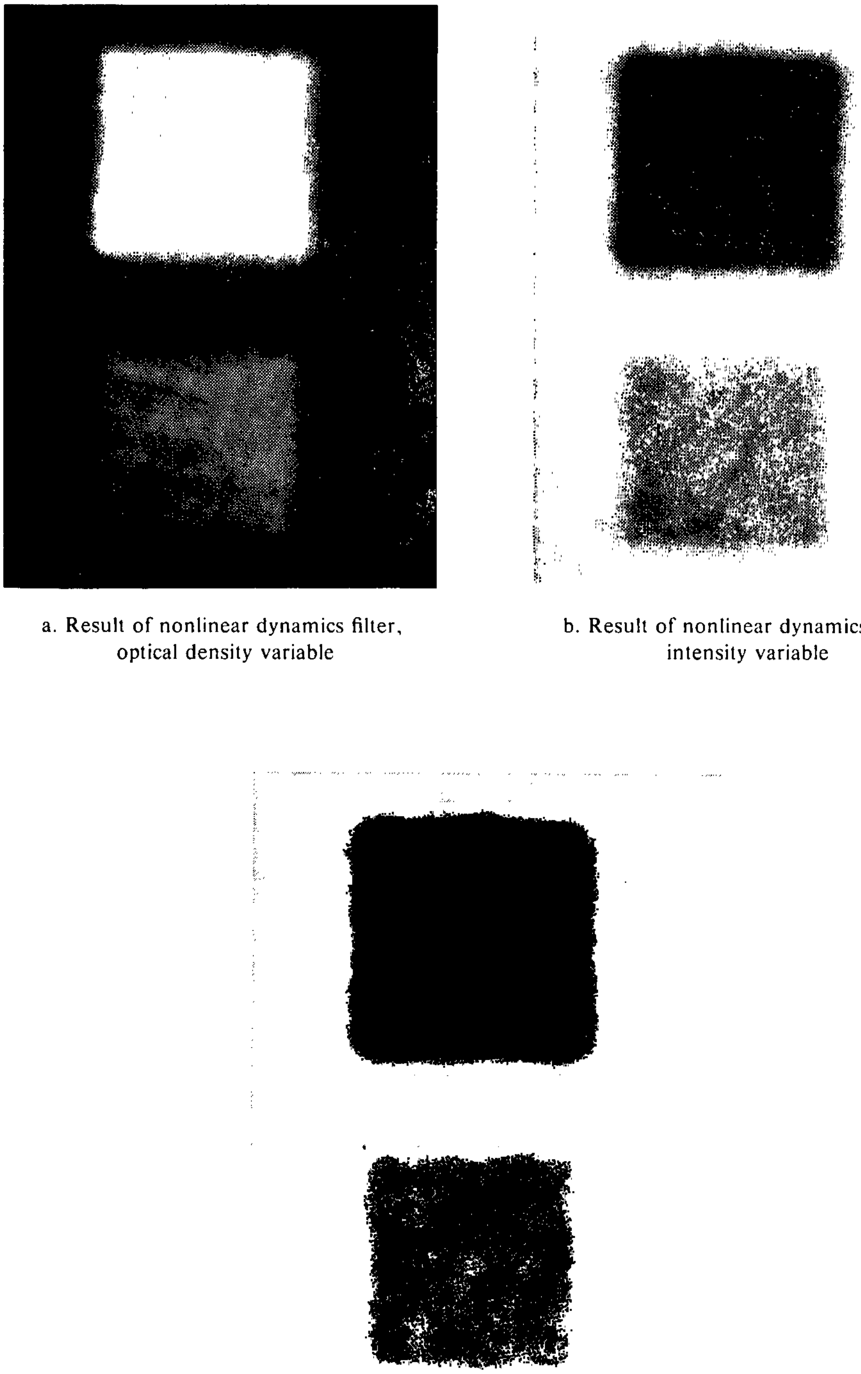


b. Result of nonlinear measurements filter, intensity variable



c. Optically enlarged photographic image (intensity variable)

Figure 6



a. Result of nonlinear dynamics filter, optical density variable

b. Result of nonlinear dynamics filter, intensity variable

c. Optically enlarged photographic image

Figure 7

In transforming these last results to the intensity variable, it would have been more satisfying to compute the conditional mean of the intensity itself instead of the nonlinear (exponential) transformation of the estimated optical density. To do this, however, requires a knowledge of the conditional variance of the optical density in the two-dimensional context, which is accurate to first order in the perturbations. The two-dimensional image's conditional mean has been inferred here by the cascading of one-dimensional conditional means, which was justified on the basis of the relation between these two types of conditional means in the corresponding linear case without perturbations. Yet it is difficult to see how this reasoning could be extended to the conditional variance perturbations, because they form part of the output of the nonlinear filter but not of the corresponding linear one. This information could be obtained, however, through a full two-dimensional formulation of the problem, such as that described in Section II-C, thus enabling the conditional mean of the pixel intensities to eventually be determined. Such a formulation would be much more formidable, however, and is beyond the scope of this report.

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