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An Analytical Study of Radar Returns in the Presence of a Rough Sea Surface

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20. Abstract (Continued)

surface. Terms beyond first order (second and third) are included in order to explain cross-polarization effects, which have been observed experimentally. Limited results on fourth-order terms are also obtained. However, when the analysis is carried only to fourth order and these terms are averaged, they do not contribute to the cross-polarization and their effect is seen only if specular returns are present.

CONTENTS

AN ANALYTICAL STUDY OF RADAR RETURNS IN THE PRESENCE OF A ROUGH SEA SURFACE

INTRODUCTION

The analytical study described in this report was originally undertaken in an attempt at a theoretical explanation of certain experimental results on radar return from ship targets corrupted by sea return. In the process of carrying out the study, the author first reviewed the literature with a view toward applying the existing theoretical results directly to the problem at hand. Previous theoretical analyses by S. O. Rice at the Bell Telephone Laboratories (BTL) [1], a group of investigators at the Naval Research Laboratory (NRL) [2 through 21], other investigators in the Soviet Union [22 through 30], and still others in the U.S. and elsewhere [31 through 45] had produced results sufficient to provide good estimates of equivalent radar cross sections of the sea surface that explained a number of experimental observations [3,46,47].

The perturbation approach originally introduced in the radiowave context by S. O. Rice in a 1951 paper [1] was used in many of the subsequent analytical studies. It was found that the first-order perturbation results did not explain cross-polarization effects in the return.* These effects are present in second-order results, which were obtained formally by Rice [1] and later by Valenzuela [6,20,21].

In evaluating the problem at hand, the author concluded that (if the perturbation approach were to be used) second-order fields at the very least would be needed to explain the experimental observations because cross polarization effects are significant in this problem. It was also concluded that, in the presence of specular returns and ship target returns, it might be important to have third- and fourth-order fields in some cases. The reasons for these conclusions are indicated below.

The quantities calculated in this study are ensemble averages of: (a) twofold products of field or received signal voltages (e.g., mean radar cross section, cross correlation function of two received signal voltages corresponding to different polarizations or different delays, etc.) and (b) fourfold products of fields or received signal voltages (e.g., cross correlation of two received signal *powers* corresponding to different polarizations or delays, etc., or the power fluctuation). We designate terms involving products of fields of orders j, k, ℓ, m etc. by $(jk\ell m)$ in fourfold products and (j,k) in twofold products. We also note that under the assumption of Gaussian statistics of the surface fluctuations, odd-order averages are zero and specular and target returns are considered as deterministic and of zero-order in the perturbation parameter. Based on all of this, the twofold product averages carried

*There are some exceptions to this statement; e.g., Wright [2,5] using first order theory, considers cross-polarization due to surface tilt. That effect, which is also covered in the present analysis, is a type of cross polarization from a different mechanism than the "quasi-crosspolarization" discussed later in this present report.

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to fourth order will contain terms (0000), (0011), (0002), (0013), (0112), (0022), (0004), and (1111). In the absence of zero-order terms (i.e., return from regions with no specular or target return) the twofold averages are confined to terms (11), (13) and (22) and the fourfold averages to the single term (1111). In this case, if we were to carry out the twofold product averages to second-order and the fourfold product averages to fourth order, then the first-order fields would suffice. However, we could not expect cross polarization effects from such an analysis. These effects would appear in the twofold product averages only if we were to carry these out to fourth order (i.e., if we include the (22) and (13) terms) and would be entirely absent from the fourfold product averages. Their appearance in the latter type of averages would require at least inclusion of terms (1122) and (1113) which are sixth order and obtainable from first, second, and third order fields.

If zero-order fields *are* present, then an evaluation of twofold product averages to second order would include (00), (11), and (02). The (02) term, requiring a second-order field, may be comparable in magnitude to the (11) term in the part of the return that is parallel-polarized (i.e., the VV or HH return) and should be included. The HV and VH return is eliminated in the averaging process for the (02) term; hence, no cross polarization will appear if we truncate the series beyond second order. Again the VH and HV returns average to zero in these terms, and hence, cross polarization does not appear. In the case of fourfold product averages, all three of the terms (0000), (0002), and (0011) must be included for second-order accuracy in the VV and HH returns.

Since the second-order fields *must* be included in order to capture the cross-polarization effects, it is highly desirable to use them to calculate the fourth-order terms (22) and (0022) in twofold and fourfold product averages respectively. If this were done, however, then the calculation would not be complete without the inclusion of terms (13) and (04) in the twofold product averages and the terms (0013) and (0004) in the fourfold product averages. These terms, also of fourth order, should be comparable in magnitude to the (22) and (0022) terms already included. Their absence would give an incomplete indication of fourth-order effects in the VV and HH returns and some of them would also contain cross-polarization components (i.e., VH and HV).

Now let us proceed to a discussion of the analysis, which begins with an electromagnetic wave in free-space incident on a sea surface with two scales of roughness. There is small-scale roughness ("ripple" or "capillary waves") and large-scale roughness ("swell" or "gravity waves"). The true picture of a sea surface, of course, is a continuum of scales of surface roughness. Examination of previous work [6,7,39] demonstrates that precise analytical treatment of this entire continuum is extremely difficult. Hence a model accounting for the two limiting scales is the most general model for which numerical solutions can be obtained without great difficulty.

The large-scale roughness, where the local radius of curvature of the surface is assumed to be large in terms of wavelength, is assumed to obey the "Kirchoff approximation," i.e., in the absence of ripple fluctuations, return signals would only be seen from regions of normal incidence, just as if the mean surface were an infinite flat plane.* Such an approach

*As already indicated, many previous theoretical studies have made use of this perturbation approach or its equivalent.

depends for its validity on the supposition that the mean surface is nearly flat over many wavelengths.

The incident and scattered waves in both media are expressed as superposition of plane waves. The boundary conditions are then written and the perturbation theory is formulated. The electric and magnetic fields are expressed as superpositions of plane wave fields at an arbitrary point in the free space medium and are then determined to fourth order in the perturbation parameter. The electric field vector at the radar location resulting from illumination of, and consequently scattering from, a small region on the surface is thereby determined as a superposition of plane wave fields. It is recognized that the radar is well into the far zone of the illuminated region (denoted by S) and, hence, the backscattered radiation looks like a plane wave at the radar. Consequently, a single plane wave is extracted from the solution, i.e., that wave propagating from the region S (subtending a sufficiently small angle at the radar to be equivalent to a point scatterer). The field vectors from all such regions illuminated by the antenna pattern are then superposed, i.e., the solution is integrated over a set of spherical angular coordinates (θ, ϕ), the integrals being weighted by the complex antenna pattern function and the product of spherical wave Green's functions e^{2jkr}/r^2.

This approach accounts for low orders of *local* multiple scattering due to ripple fluctuations. This is inherent in the perturbation theory. It does not account for *large-scale* multiple scattering, i.e., the return from the region S is assumed to be entirely independent of the illumination of other regions. The latter requirement of the theory would be useful for treating a sea surface with a great deal of large scale swell. If the swell is small, it should not be a very significant effect. If the slope of the swell surface were very small, then illumination of any region by secondary radiation from other regions would result from wide angle scattering, which should be small relative to direct illumination from the radar transmitter. A large swell surface, however, might result in situations where a given region S sees specular reflection from a number of adjacent waves. In this case, the secondary illumination could conceivably approach the same order of magnitude as the direct illumination. Such situations should be rare except with an unusually rough sea surface and hence the neglect of this effect is probably not a serious weakness of the theory in most cases.

The author investigated an alternative approach used by some workers [39] using the "Stratton-Chu" or "Kirchoff-Huygens" integral equation form expressing the fields at a point in space in terms of the fields on the surface. It was found that this approach gives answers equivalent to those obtained by the direct approach outlined above once the standard approximations are invoked. These integral equation forms are merely different ways of expressing the field equations and *not* solutions of these equations. It is still necessary to solve a boundary value problem in order to determine the fields on the surface. Hence it is more efficient to solve the boundary value problem directly to find the field at a point in space due to illumination of the surface by the incident wave. This was the approach used by Rice in his classic 1951 paper [1], and the use of the integral equation form does not really improve on that approach under the assumptions made in our model.

The approach used here is essentially that of Rice [1] which has been used by subsequent workers (e.g. Valenzuela [4,6,20,21]). The differences between the present development and that of Rice are as follows:

(1) The Fourier integral is used directly instead of Rice's Fourier series with a later approach to the continuous limit. In terms of results these approaches are identical, although there are questions of existence of Fourier integrals of random processes. As long as one understands and accounts for the specialized meaning of the Fourier transform of the random process while carrying out the development, this is not an important distinction between the development in this report and that originally used by Rice.

(2) The wave incident on the region S is assumed to approximate a plane wave with arbitrary polarization. H and V solutions from Rice's work could be weighted and superposed to treat the case of arbitrary incident wave polarization. This could have been done in the present work. The principal reason for not doing it this way is that it was desired to express the final results in a coordinate system with origin at the radar and z-axis along the antenna beam. Accounting for the large-scale swell necessitated the use of a complicated transformation between the coordinate system used in the basic boundary value problem and that in which the final results were to be expressed. In view of this coordinate transformation, there was not significant labor-saving in treating horizontally and vertically polarized incident fields separately.

(3) The fields are calculated to fourth order in the perturbation parameter. Most previous work confined itself to first-order fields. Rice [1] and Valenzuela [4,6] calculated the second-order fields and used them to evaluate the (2,2) term in the average power (i.e., the average of the square of the second-order fields. Valenzuela calculated second-order doppler spectra in later papers [20,21]. The reasons for our inclusion of higher order fields were delineated earlier in this section.

Further extensions from most previous theoretical studies are as follows:

(1) The present work includes the effect of pulsing or other forms of modulation of the transmitted RF wave. Previous treatments are generally confined to pure CW transmission.

(2) The present work includes integration over all incidence angles weighted by the antenna pattern shaping function.

(3) The present work includes calculation of crosscovariance functions of two (prerectification) received signal voltages and crosscovariance functions of deviations of two received signal powers from their mean values (where the two returns might correspond, for example, to two different polarizations or two different radar-target geometries). As degenerate cases, the two voltages or powers may be the same but with different delays, in which case these expressions degenerate into autocovariance functions of received signal voltage or power. With zero delay difference, the latter expressions degenerate into mean power and "power fluctuation" respectively. All of these possibilities are contained in the averages evaluated in this work.

(4) The present work includes effects of target signals in the ocean environment, i.e., it considers the entire "system" problem and not merely the CW return at a single angle as do most of the previous theoretical studies of sea scatter. The target return and sea return can be considered as statistically independent; hence, in calculating return signal power using only first-order fields, sea return and target return add incoherently and there is no coupling between the two. In accounting for higher order terms or in calculating fourfold product averages, terms arise that are products of sea return and target return. This sort of coupling is accounted for in the present work.

To summarize these points of difference between this work and previous work, the latter has stressed the calculation of the scattering cross section of a patch of slightly rough sea surface. The present work should be considered as a "systems study" in which the traditional theory with small modifications is applied to a system consisting of (a) a radar whose beam illuminates a region of the sea surface, possibly a large region and (b) a target which the radar may be trying to detect or locate. The analysis accounts for possible pulsing or other modulation of the radar signal, the shape of the antenna pattern, and the polarization of the radar signal. A certain form of correlation processing is done on the received signal. The analysis accounts for this processing and focuses on calculation of certain statistical averages both with and without the processing and both with and without the presence of a target.

Previous theoretical analyses of scattering from a rough surface have been examined. The author has concluded that much of the work reported in the literature has as its basis the classical paper written by S. O. Rice in 1951 [1]. If one were going to use the perturbation theory approach to the problem, it would seem to be very difficult to improve in a major way on Rice's basic work. Subsequent workers have usually made use of methods of approach equivalent to that used by Rice. The present author has also followed Rice's approach, with a few small extensions and variations which were described earlier. Instead of using Rice's results directly or those of Valenzuela [4,6], the author has chosen to rederive the results. This was a very easy (although exceedingly tedious) task and was for the purpose of adapting the analysis to easy extension and casting it in a form that could be readily prepared for digital computation. It would have been difficult to use Rice's or Valenzuela's published results directly for the calculations to be made in this study.

Unfortunately, it is true that the third- and fourth-order field perturbations are very cumbersome and may devour computer time at a prohibitively high rate. They are too cumbersome for hand calculation within a reasonable time, although such calculation is simple in principle. If these computations are prohibitively expensive, then a great deal of significant information can still be obtained from the first- and second-order fields alone. In the author's opinion, it is still worthwhile to have worked out and reported these higher order terms because it may become important and economically feasible to compute them at some later time. This would be particularly true in cases where cross-polarization effects are very important. In such cases, the second-order terms are the lowest order terms that capture these effects, and third-order terms would be required for any refinement on the simplest results. All of this was discussed early in this section.

An effect not accounted for in the analysis presented in this report is that of shadowing. This effect assumes increasing importance as the grazing angle decreases and the surface roughness increases [25]. Since numerical studies at near grazing incidence were not contemplated by the author, no analysis was done to account for shadowing. However, if such cases are to be studied based on the theory presented here, it will be necessary to supplement the model with further analysis that includes this effect.

THE BOUNDARY VALUE PROBLEM; CALCULATION OF THE FIELDS

Consider a rough boundary between homogeneous medium #1 (free space) and an arbitrary homogeneous medium called medium #2 (e.g., the sea). The mean boundary surface can be considered as an infinite flat plane. The coordinate system to be used is a right-handed rectangular system whose origin is on the mean surface, whose (x,y) plane is the mean surface itself, and whose z-axis is directed into free-space. The geometry is illustrated in Fig. 1, where the "mean" incident, reflected, and transmitted waves are shown in the standard way for a case where the incident wave is a plane wave and the fluctuations of the surface are designated by a function $z = z(x,y,t)$ which is a random function of the three variables x,y and the time t and has zero mean. The incident wave, although shown as a plane wave in Fig. 1, will be arbitrary and possibly pulsed in the analysis to follow, but will be specified as plane at a certain point in the analysis. The constitutive parameters of medium #1 (free-space) are denoted in the usual way by (ϵ_o, μ_o) and those of medium #2 are (ϵ_c, μ_o).

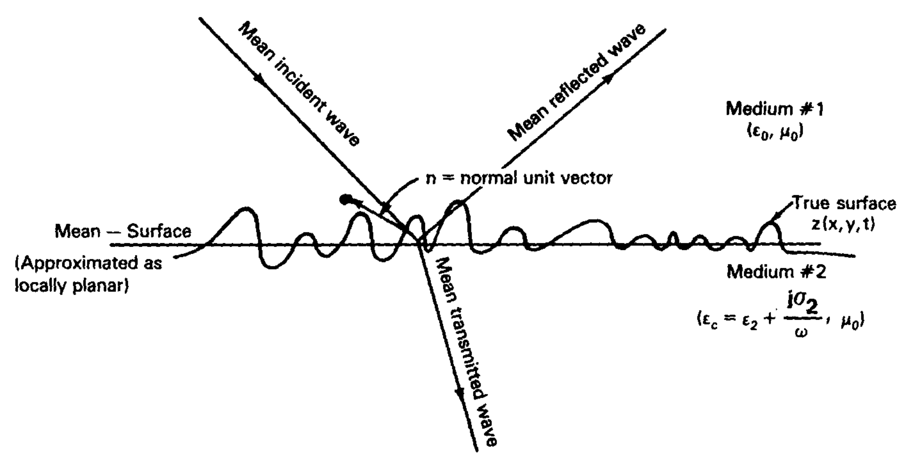

Fig. 1 — Geometry of problem

Denoting incident, reflected, and transmitted wave fields by subscripts i, r, and t respectively, we will represent all field vectors by column vectors and use matrix notation throughout, having found that this leads to a much more efficient and compact analysis than longhand methods. The electric and magnetic field vectors are as follows:*

$$\underset{\to}{E}(\underset{\to}{\rho},z,t) = \begin{bmatrix} E_{ix}(\underset{\to}{\rho},z,t) \\ r \\ t \\ E_{iy}(\underset{\to}{\rho},z,t) \\ r \\ t \\ E_{iz}(\underset{\to}{\rho},z,t) \\ r \\ t \end{bmatrix} = \int_{-\infty}^{\infty} d\omega e^{-j\omega t} \overline{\underset{\to}{E}}_i(\underset{\to}{\rho},z,\omega) \tag{2.1.a}$$

$$\underset{\to}{H}(\underset{\to}{\rho},z,t) \quad \begin{bmatrix} H_{ix}(\underset{\to}{\rho},z,t) \\ r \\ t \\ H_{iy}(\underset{\to}{\rho},z,t) \\ r \\ t \\ H_{iz}(\underset{\to}{\rho},z,t) \\ r \\ t \end{bmatrix} = \int_{-\infty}^{\infty} d\omega e^{-j\omega t} \overline{\underset{\to}{H}}_i(\underset{\to}{\rho},z,\omega) \tag{2.1.b}$$

where $\underset{\to}{\rho} = (x,y)$ and

$$\overline{\underset{\to}{E}}(\underset{\to}{\rho},z,\omega) = \int \cdots \int d\underset{\to}{\beta} e^{j\frac{\omega}{c}\left[(\underset{\to}{\beta}\cdot\underset{\to}{\rho}) \mp \begin{bmatrix} \beta_z & (\underset{\to}{\beta}) \\ \beta_z & (\underset{\to}{\beta}) \\ \gamma_z & (\underset{\to}{\beta}) \end{bmatrix} z\right]} \widetilde{\underset{\to}{E}}_{\substack{i\\r\\t}}(\underset{\to}{\beta},\omega) \tag{2.1.c}$$

where

$$\underset{\to}{\beta} = \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix}, \quad \begin{aligned} \beta_z &= \beta_z(\underset{\to}{\beta}) = +\sqrt{1-\beta_x{}^2-\beta_y{}^2} \\ \gamma_z &= \gamma_z(\underset{\to}{\beta}) = +\sqrt{\nu^2-\beta_x{}^2-\beta_y{}^2} \end{aligned}$$

$$\overline{\underset{\to}{H}}(\underset{\to}{\rho},z,\omega) = \int \cdots \int d\underset{\to}{\beta} e^{j\frac{\omega}{c}\left[(\underset{\to}{\beta}\cdot\underset{\to}{\rho}) \mp \begin{bmatrix} \beta_z & (\underset{\to}{\beta}) \\ \beta_z & (\underset{\to}{\beta}) \\ \gamma_z & (\underset{\to}{\beta}) \end{bmatrix} z\right]} P_{\substack{i\\r\\t}}(\underset{\to}{\beta}) \widetilde{\underset{\to}{E}}_{\substack{i\\r\\t}}(\underset{\to}{\beta},\omega) \tag{2.1.d}$$

*Note that some of our notation is patterned after that of Rice's classical paper [1].

$$P_{\substack{i\\r}}(\underset{\rightarrow}{\beta}) = \begin{bmatrix} 0 & \pm\beta_z & \beta_y \\ \mp\beta_z & 0 & -\beta_x \\ -\beta_y & \beta_x & 0 \end{bmatrix} \tag{2.1.e}$$

$$P_t(\underset{\rightarrow}{\beta}) = \begin{bmatrix} 0 & \gamma_z & \beta_y \\ -\gamma_z & 0 & -\beta_x \\ -\beta_y & \beta_x & 0 \end{bmatrix} \tag{2.1.f}$$

ν = complex refractive index of medium #2

The local normal to the rough surface at a point $(x,y,z(x,y,t))$ and at time t is denoted by $\underset{\rightarrow}{n}(\underset{\rightarrow}{\rho},z,t)$. or as a column vector by

$$n(\underset{\rightarrow}{\rho},z,t) = \begin{bmatrix} n_x(\underset{\rightarrow}{\rho},z,t) \\ n_y(\underset{\rightarrow}{\rho},z,t) \\ n_z(\underset{\rightarrow}{\rho},z,t) \end{bmatrix} = \frac{1}{\sqrt{1+|\nabla_z|^2}} \cdot \begin{bmatrix} -z_x \\ -z_y \\ 1 \end{bmatrix} \tag{2.2}$$

where $z_x = \dfrac{\partial z}{\partial x}$, $z_y = \dfrac{\partial z}{\partial y}$, $\nabla_z = \underset{\rightarrow}{i}_x \dfrac{\partial z}{\partial x} + \underset{\rightarrow}{i}_y \dfrac{\partial z}{\partial y}$; $(\underset{\rightarrow}{i}_x, \underset{\rightarrow}{i}_y, \underset{\rightarrow}{i}_z)$ = unit basis vectors.

In standard vector notation, the tangential boundary conditions are:

$$\left\{ \underset{\rightarrow}{n} x [\underset{\rightarrow}{E}_1 + \underset{\rightarrow}{E}_r - E_t] \right\}_{z=z(\underset{\rightarrow}{\rho},t)} = 0 \tag{2.3.a}$$

$$\left\{ \underset{\rightarrow}{n} x [\underset{\rightarrow}{H}_i + \underset{\rightarrow}{H}_r - \underset{\rightarrow}{H}_t] \right\}_{z=z(\underset{\rightarrow}{\rho},t)} = 0 \tag{2.3.b}$$

In matrix notation with the aid of (2.1.a,b), Eqs. (2.3.a,b) take the form

$$\frac{1}{\sqrt{1+|\nabla_z|^2}} \iint d\underset{\rightarrow}{\Omega}\, N(\underset{\rightarrow}{\rho},z(\underset{\rightarrow}{\rho},t)) \Big\{ [\underset{\rightarrow}{\widetilde{E}}_1(\underset{\rightarrow}{\Omega}) e^{-j\frac{\omega}{c}\beta_z(\underset{\rightarrow}{\beta})z(\underset{\rightarrow}{\rho},t)}$$

$$+ \underset{\rightarrow}{\widetilde{E}}_r(\Omega) e^{j\frac{\omega}{c}\beta_z(\underset{\rightarrow}{\beta})z(\underset{\rightarrow}{\rho},t)}] - [\underset{\rightarrow}{\widetilde{E}}_t(\underset{\rightarrow}{\Omega}) e^{-j\frac{\omega}{c}\gamma_z(\underset{\rightarrow}{\beta})z(\underset{\rightarrow}{\rho},t)}] \Big\} e^{j\frac{\omega}{c}[\underset{\rightarrow}{\beta}\cdot\underset{\rightarrow}{\rho} - ct]} = 0 \tag{2.4.a}$$

(where $\underset{\rightarrow}{\Omega}$ = shorthand notation for $(\beta_x, \beta_y, \omega)$)

$$\frac{1}{\sqrt{1+|\nabla_z|^2}}\iint d\underset{\rightarrow}{\Omega}\, N(\underset{\rightarrow}{\rho}, z(\underset{\rightarrow}{\rho}, t))\Big\{[P_i(\underset{\rightarrow}{\Omega})\underset{\rightarrow i}{\widetilde{E}}(\underset{\rightarrow}{\Omega})e^{-j\frac{\omega}{c}\beta_z(\underset{\rightarrow}{\beta})z(\underset{\rightarrow}{\rho},t)}$$

$$+ P_r(\underset{\rightarrow}{\Omega})\underset{\rightarrow r}{\widetilde{E}}(\underset{\rightarrow}{\Omega})e^{+j\frac{\omega}{c}\beta_z(\underset{\rightarrow}{\beta})z(\underset{\rightarrow}{\rho},t)}] \tag{2.4.b}$$

$$- [P_t(\underset{\rightarrow}{\Omega})\underset{\rightarrow t}{\widetilde{E}}(\underset{\rightarrow}{\Omega})e^{-j\frac{\omega}{c}\gamma_z(\underset{\rightarrow}{\beta})z(\underset{\rightarrow}{\rho},t)}]\Big\}e^{j\frac{\omega}{c}[(\underset{\rightarrow}{\beta}\cdot\underset{\rightarrow}{\rho}) - \omega t]} = 0$$

where

$$N(\underset{\rightarrow}{\rho}, z, t) = \begin{bmatrix} 0 & -1 & -z_y \\ 1 & 0 & z_x \\ z_y & -z_x & 0 \end{bmatrix}$$

At this point (following Rice's treatment) we invoke the divergence equation for the electric field, as follows:

$$\nabla \cdot \underset{\rightarrow}{E} = 0 \tag{2.5}$$

Using the electric field representations (2.1.a,c), Eq. (2.5) takes the form

$$\iint d\underset{\rightarrow}{\Omega}\left[j\Big\{[\beta_x \underset{\substack{\rightarrow ix\\ r\\ t}}{\widetilde{E}}(\underset{\rightarrow}{\Omega}) + \beta_y \underset{\substack{\rightarrow iy\\ r\\ t}}{\widetilde{E}}(\underset{\rightarrow}{\Omega})]\right.$$

$$\left. + \begin{bmatrix} -\beta_z(\underset{\rightarrow}{\beta}) \\ +\beta_z(\underset{\rightarrow}{\beta}) \\ -\gamma_z(\underset{\rightarrow}{\beta}) \end{bmatrix} \widetilde{E}_{\substack{iz\\ r\\ t}}(\underset{\rightarrow}{\Omega})\Big\}e^{-j\omega t}\exp\left(j\frac{\omega}{c}(\underset{\rightarrow}{\beta}\cdot\underset{\rightarrow}{\rho}) + \begin{bmatrix} -\beta_z(\underset{\rightarrow}{\beta}) \\ +\beta_z(\underset{\rightarrow}{\beta}) \\ -\gamma_z(\underset{\rightarrow}{\beta}) \end{bmatrix} z(\underset{\rightarrow}{\rho},t)\right)\right] = 0 \tag{2.6}$$

Multiplying (2.6) by

$$e^{-j\frac{\omega'}{c}(\underset{\rightarrow}{\beta}'\cdot\underset{\rightarrow}{\rho} - ct)},$$

integrating over all $\underset{\rightarrow}{\rho}$ and t and invoking the relation

$$\iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho}\, e^{j\frac{\omega'}{c}(\underset{\rightarrow}{\beta} - \underset{\rightarrow}{\beta}')\cdot\underset{\rightarrow}{\rho}} = (2\pi)^2\left(\frac{c}{\omega'}\right)^2 \delta(\beta_x - \beta_x{}')\delta(\beta_y - \beta_y{}') \tag{2.7.a}$$

$$\iint\limits_{-\infty}^{\infty} dt\; e^{-j(\omega - \omega')t} = 2\pi\;\delta(\omega - \omega') \tag{2.7.b}$$

we obtain

$$\widetilde{E}_{\substack{iz\\r\\t}}\,(\underset{\rightarrow}{\Omega}) = \begin{matrix} +\dfrac{1}{\beta_z}\left\{\beta_x \widetilde{E}_{\substack{ix\\r\\t}} + \beta_y \widetilde{E}_{\substack{iy\\r\\t}}\right\}. \\ -\dfrac{1}{\beta_z} \\ +\dfrac{1}{\gamma_z} \end{matrix} \tag{2.8}$$

We can now construct a new matrix relationship, i.e.,

$$\underset{\rightarrow}{\widetilde{E}}_{\substack{i\\r\\t}} = \begin{bmatrix} \widetilde{E}_{\substack{ix\\r\\t}} \\ \widetilde{E}_{\substack{iy\\r\\t}} \\ \begin{pmatrix} +\dfrac{1}{\beta_z} \\ -\dfrac{1}{\beta_z} \\ +\dfrac{1}{\gamma_z} \end{pmatrix} [\beta_x \widetilde{E}_{\substack{ix\\r\\t}} + \beta_y \widetilde{E}_{\substack{iy\\r\\t}}] \end{bmatrix} = M_{\substack{i\\r\\t}}\, \underset{\rightarrow}{\widetilde{\widetilde{E}}}_{\substack{i\\r\\t}} \tag{2.9}$$

where

$$M_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \left(\dfrac{\beta_x}{\beta_z}\right) & \left(\dfrac{\beta_y}{\beta_z}\right) \end{bmatrix}, M_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \left(\dfrac{-\beta_x}{\beta_z}\right) & \left(\dfrac{-\beta_y}{\beta_z}\right) \end{bmatrix}, M_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \left(\dfrac{\beta_x}{\gamma_z}\right) & \left(\dfrac{\beta_y}{\gamma_z}\right) \end{bmatrix}$$

$$\underset{\rightarrow}{\widetilde{\widetilde{E}}}_{\substack{i\\r\\t}} = \begin{bmatrix} \widetilde{E}_{\substack{ix\\r\\t}} \\ \widetilde{E}_{\substack{iy\\r\\t}} \end{bmatrix}$$

From (2.1.a,b), (2.4.a,b) and (2.9) we construct the following matrices,

$$NM_i = \begin{bmatrix} \dfrac{-\beta_x z_y}{\beta_z} & -\left(1 + \dfrac{\beta_y z_y}{\beta_z}\right) \\ \left(1 + \dfrac{\beta_x z_x}{\beta_z}\right) & \dfrac{\beta_y z_x}{\beta_z} \\ z_y & -z_x \end{bmatrix} \tag{2.10.a}$$

$$NM_r = \begin{bmatrix} \dfrac{\beta_x z_y}{\beta_z} & -\left(1 - \dfrac{\beta_y z_y}{\beta_z}\right) \\ \left(1 - \dfrac{\beta_x z_x}{\beta_z}\right) & \dfrac{-\beta_y z_x}{\beta_z} \\ z_y & -z_x \end{bmatrix} \tag{2.10.b}$$

$$NM_t = \begin{bmatrix} -\dfrac{\beta_x z_y}{\gamma_z} & -\left(1 + \dfrac{\beta_y z_y}{\gamma_z}\right) \\ \left(1 + \dfrac{\beta_x z_x}{\gamma_z}\right) & \dfrac{\beta_y z_x}{\gamma_z} \\ z_y & -z_x \end{bmatrix} \tag{2.10.c}$$

At this point we adopt a notational convention which will be used many times throughout the development to follow, wherein a prime will be used to indicate an interchange of x and y components, e.g., if $a = \beta_x z_y$, then $a' = \beta_y z_x$. Continuing the construction of needed matrices with the aid of this notational device, we have

$$NP_i M_i = I = \begin{bmatrix} I_{11} & I_{12} \\ I_{12}' & I_{11}' \\ I_{31} & I_{31}' \end{bmatrix} \tag{2.10.d}$$

where

$$I_{11} = \left(\frac{1 - \beta_y{}^2}{\beta_z} \right) + \beta_y z_y$$

$$I_{12} = \beta_x \left(\frac{\beta_y}{\beta_z} - z_y \right)$$

$$I_{31} = \left(\frac{\beta_x \beta_y}{-\beta_z} \right) z_y + \left(\frac{1 - \beta_y{}^2}{\beta_z} \right) z_x$$

$$NP_r M_r = II = \begin{bmatrix} II_{11} & II_{12} \\ II_{12}' & II_{11}' \\ II_{31} & II_{31}' \end{bmatrix} \qquad (2.10.e)$$

where

$$II_{11} = -\left(\frac{1 - \beta_y{}^2}{\beta_z} \right) + \beta_y z_y$$

$$II_{12} = -\beta_x \left(\frac{\beta_y}{\beta_z} + z_y \right)$$

$$II_{31} = -\left(\frac{\beta_y \beta_y}{\beta_z} \right) z_y - \left(\frac{1 - \beta_y{}^2}{\beta_z} \right) z_x$$

$$NP_t M_t = III = \begin{bmatrix} III_{11} & III_{12} \\ III_{12}' & III_{11}' \\ III_{31} & III_{31}' \end{bmatrix} \qquad (2.10.f)$$

where

$$III_{11} = \left(\frac{\nu^2 - \beta_y{}^2}{\gamma_z}\right) + \beta_y z_y$$

$$III_{12} = \beta_x\left(\frac{\beta_y}{\gamma_z} - z_y\right)$$

$$III_{31} = \left(\frac{\beta_x\beta_y}{\gamma_z}\right) z_y + \left(\frac{\nu^2 - \beta_y{}^2}{\gamma_z}\right) z_x .$$

We can now use (2.10.a, ..., f) to construct a 4 X 4 matrix equation consisting of the x and y components of Eqs. (2.4.a,b) (with the factor

$$\frac{1}{\sqrt{1 + |\nabla_z|^2}}$$

removed) as follows:

$$\iint d\underset{\rightarrow}{\Omega}\, A(\underset{\rightarrow}{\Omega},\underset{\rightarrow}{\rho},t)\, \hat{\tilde{\underset{\rightarrow}{E}}}(\underset{\rightarrow}{\Omega}) e^{j\frac{\omega}{c}\,[(\underset{\rightarrow}{\beta}\cdot\underset{\rightarrow}{\rho}) - ct]} = \iint d\underset{\rightarrow}{\Omega}\, B(\underset{\rightarrow}{\Omega},\underset{\rightarrow}{\rho},t)\, \approx{\underset{\rightarrow 1}{E}}(\Omega) e^{j\frac{\omega}{c}\,[(\underset{\rightarrow}{\beta}\cdot\underset{\rightarrow}{\rho}) - ct]} ,$$

where $\underset{\rightarrow}{\Omega}$ symbolizes the set of variables $(\beta_x, \beta_y, \omega)$ and where

$$A(\underset{\rightarrow}{\Omega},\underset{\rightarrow}{\rho},t) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ -A_{12}' & -A_{11}' & -A_{14}' & -A_{13}' \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{32}' & A_{31}' & A_{34}' & A_{33}' \end{bmatrix} \qquad (2.11.a)'$$

$$B(\underset{\rightarrow}{\Omega},\underset{\rightarrow}{\rho},t) = \begin{bmatrix} B_{11} & B_{12} \\ -B_{12}' & -B_{11}' \\ B_{31} & B_{32} \\ B_{32}' & B_{31}' \end{bmatrix} \qquad (2.11.b)'$$

$$\approx{\underset{\rightarrow 1}{E}} = \begin{bmatrix} \tilde{E}_{ix} \\ \tilde{E}_{iy} \end{bmatrix} \qquad (2.11.c)'$$

$$\hat{\underset{\rightarrow}{\tilde{E}}} = \begin{bmatrix} \tilde{E}_{rx} \\ \tilde{E}_{ry} \\ \tilde{E}_{tx} \\ \tilde{E}_{ty} \end{bmatrix} \tag{2.11.d)'}$$

and where

$$\begin{aligned} A_{11} &= \frac{\beta_x}{\beta_z} z_y \, e^{j\frac{\omega}{c}\beta_z z} \\ A_{12} &= -\left(1 - \frac{\beta_y}{\beta_z} z_y\right) e^{j\frac{\omega}{c}\beta_z z} \\ A_{13} &= \frac{\beta_x z_y}{\gamma_z} \, e^{-j\frac{\omega}{c}\gamma_z z} \\ A_{14} &= \left(1 + \frac{\beta_y z_y}{\gamma_z}\right) e^{-j\frac{\omega}{c}\gamma_z z} \end{aligned} \tag{2.11.e)'}$$

$$\begin{aligned} A_{31} &= -\left[\frac{(1-\beta_y{}^2)}{\beta_z} - \beta_y z_y\right] e^{j\frac{\omega}{c}\beta_z z} \\ A_{32} &= -\left[\frac{\beta_x\beta_y}{\beta_z} + \beta_x z_y\right] e^{j\frac{\omega}{c}\beta_z z} \\ A_{33} &= -\left[\frac{(\nu^2-\beta_y{}^2)}{\gamma_z} + \beta_y z_y\right] e^{-j\frac{\omega}{c}\gamma_z z} \\ A_{34} &= -\left[\frac{\beta_x\beta_y}{\gamma_z} - \beta_x z_y\right] e^{-j\frac{\omega}{c}\gamma_z z} \end{aligned} \tag{2.11.f)'}$$

$$B_{11} = \frac{\beta_x z_y}{\beta_z} e^{-j\frac{\omega}{c}\beta_z z}$$

$$B_{12} = \left(1 + \frac{\beta_y z_y}{\beta_z}\right) e^{-j\frac{\omega}{c}\beta_z z}$$

$$B_{31} = -\left[\left(\frac{1-\beta_y{}^2}{\beta_z}\right) + \beta_y z_y\right] e^{-j\frac{\omega}{c}\beta_z z}$$

$$B_{32} = -\left[\frac{\beta_x\beta_y}{\beta_z} - \beta_x z_y\right] e^{-j\frac{\omega}{c}\beta_z z}$$

We now invoke our perturbation theory, in which terms of A and B not involving z, z_x or z_y are considered zero order. Thus, using ζ as an ordering parameter, we have

$$A(\underset{\to}{\Omega},\underset{\to}{\rho},t) = A^{(0)}(\underset{\to}{\beta},\omega) + \zeta A^{(1)}(\underset{\to}{\beta},\omega,\underset{\to}{\rho},t) \tag{2.12.a}$$

$$B(\underset{\to}{\Omega},\underset{\to}{\rho},t) = B^{(0)}(\underset{\to}{\beta}) + \zeta B^{(1)}(\underset{\to}{\beta},\omega,\underset{\to}{\rho},t) \tag{2.12.b}$$

$$A^{(0)}(\underset{\to}{\beta},\omega) = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ A^{(0)}_{31} & A^{(0)}_{32} & A^{(0)}_{33} & A^{(0)}_{34} \\ A^{(0)}_{32} & A^{(0)\prime}_{31} & A^{(0)}_{34} & A^{(0)\prime}_{33} \end{bmatrix} \tag{2.12.a)$''$}$$

where

$$A^{(0)}_{31} = -\left(\frac{1-\beta_y{}^2}{\beta_z}\right)$$

$$A^{(0)}_{32} = -\frac{\beta_x\beta_y}{\beta_z}$$

$$A^{(0)}_{33} = -\left(\frac{\nu^2-\beta_y{}^2}{\gamma_z}\right)$$

$$A_{34}^{(0)} = \frac{-\beta_x \beta_y}{\gamma_z}$$

$$B^{(0)}(\underset{\rightarrow}{\beta}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ A_{31}^{(0)} & A_{32}^{(0)} \\ A_{32}^{(0)} & A_{31}^{(0)'} \end{bmatrix} \qquad \text{(2.12.b)}''$$

$$A^{(1)}(\beta,\omega,\rho,t) = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} & A_{14}^{(1)} \\ -A_{12}^{(1)'} & -A_{11}^{(1)'} & -A_{14}^{(1)'} & -A_{13}^{(1)'} \\ A_{31}^{(1)} & A_{32}^{(1)} & A_{33}^{(1)} & A_{34}^{(1)} \\ A_{32}^{(1)'} & A_{31}^{(1)'} & A_{34}^{(1)'} & A_{33}^{(1)'} \end{bmatrix} \qquad \text{(2.12.a)}'''$$

where

$$A_{11}^{(1)} = \frac{\beta_x}{\beta_z} z_y \, e^{j\frac{\omega}{c}\beta_z z}$$

$$A_{12}^{(1)} = \frac{\beta_y}{\beta_z} z_y \, e^{j\frac{\omega}{d}\beta_z z} - \left(e^{j\frac{\omega}{c}\beta_z z} - 1\right)$$

$$A_{13}^{(1)} = \frac{\beta_x}{\gamma_z} z_y \, e^{-j\frac{\omega}{c}\gamma_z z}$$

$$A_{14}^{(1)} = \frac{\beta_y}{\gamma_z} z_y \, e^{-j\frac{\omega}{c}\gamma_z z} + \left(e^{-j\frac{\omega}{c}\gamma_z z} - 1\right)$$

$$A_{31}^{(1)} = \beta_y z_y \, e^{j\frac{\omega}{c}\beta_z z} - \left(\frac{1-\beta_y^2}{\beta_z}\right)\left(e^{j\frac{\omega}{c}\beta_z z} - 1\right)$$

$$A_{32}^{(1)} = -\beta_x \left[z_y \, e^{j\frac{\omega}{c}\beta_z z} + \frac{\beta_y}{\beta_z}\left(e^{j\frac{\omega}{c}\beta_z z} - 1\right)\right]$$

$$A_{33}^{(1)} = -\beta_y z_y e^{-j\frac{\omega}{c}\gamma_z z} - \left(\frac{\nu^2 - \beta_y^2}{\gamma_z}\right)\left(e^{-j\frac{\omega}{c}\gamma_z z} - 1\right)$$

$$A_{34}^{(1)} = \beta_x \left[z_y e^{-j\frac{\omega}{c}\gamma_z z} - \frac{\beta_y}{\gamma_z}\left(e^{-j\frac{\omega}{c}\gamma_z z} - 1\right)\right]$$

$$B^{(1)}(\underset{\rightarrow}{\beta},\omega,\underset{\rightarrow}{\rho},t) = \begin{bmatrix} B_{11}^{(1)} & B_{12}^{(1)} \\ -B_{12}^{(1)\prime} & -B_{11}^{(1)} \\ B_{31}^{(1)} & B_{32}^{(1)} \\ B_{32}^{(1)\prime} & B_{31}^{(1)\prime} \end{bmatrix} \tag{2.12.b)'''}$$

where

$$B_{11}^{(1)} = \frac{\beta_x}{\beta_z} z_y e^{-j\frac{\omega}{c}\beta_z z}$$

$$B_{12}^{(1)} = \frac{\beta_y}{\beta_z} z_y e^{-j\frac{\omega}{c}\beta_z z} + \left(e^{-j\frac{\omega}{c}\beta_z z} - 1\right)$$

$$B_{31}^{(1)} = -\beta_y z_y e^{-j\frac{\omega}{c}\beta_z z} - \left(\frac{1 - \beta_y{}^2}{\beta_z}\right)\left(e^{-j\frac{\omega}{c}\beta_z z} - 1\right)$$

$$B_{32}^{(1)} = \beta_x \left[z_y e^{-j\frac{\omega}{c}\beta_z z} - \frac{\beta_y}{\beta_z}\left(e^{-j\frac{\omega}{c}\beta_z z} - 1\right)\right]$$

Expanding $\hat{\tilde{E}}$ in a perturbation series,

$$\underset{\rightarrow}{\hat{\tilde{E}}} = \underset{\rightarrow}{\hat{\tilde{E}}}^{(0)} + \zeta \underset{\rightarrow}{\hat{\tilde{E}}}^{(1)} + \zeta^2 \underset{\rightarrow}{\hat{\tilde{E}}}^{(2)} + \ldots . \tag{2.13}$$

and employing (2.12.a,b) and (2.13) in 2.11), we can write (2.11) in the form

$$\iint d\underset{\rightarrow}{\beta} \int d\omega \, e^{j\frac{\omega}{c}[\underset{\rightarrow}{\beta}\cdot\underset{\rightarrow}{\rho}) - ct]} \Big\{ [A^{(0)}(\underset{\rightarrow}{\beta},\omega)\hat{\underset{\rightarrow}{E}}^{(0)}(\underset{\rightarrow}{\beta},\omega) - B^{(0)}(\underset{\rightarrow}{\beta})\tilde{\tilde{\underset{\rightarrow}{E}}}_i(\underset{\rightarrow}{\beta},\omega)]$$

$$+ \zeta[A^{(0)}(\underset{\rightarrow}{\beta},\omega)\, \hat{E}^{(1)}(\underset{\rightarrow}{\beta},\omega) - (B^{(1)}(\underset{\rightarrow}{\beta},\omega,\underset{\rightarrow}{\rho},t)\hat{\underset{\rightarrow}{E}}_i(\underset{\rightarrow}{\beta},\omega) - A^{(1)}(\underset{\rightarrow}{\beta},\omega,\underset{\rightarrow}{\rho},t)\hat{\underset{\rightarrow}{E}}^{(0)}(\underset{\rightarrow}{\beta},\omega))]$$

$$+ \zeta^2[A^{(0)}(\underset{\rightarrow}{\beta},\omega)\hat{\underset{\rightarrow}{E}}^{(2)}(\underset{\rightarrow}{\beta},\omega) - (- A^{(1)}(\underset{\rightarrow}{\beta},\omega,\underset{\rightarrow}{\rho},t)\hat{\underset{\rightarrow}{E}}^{(1)}(\underset{\rightarrow}{\beta},\omega))] + \ldots \tag{2.14}$$

$$+ \zeta^n[A^{(0)}(\underset{\rightarrow}{\beta},\omega)\hat{\underset{\rightarrow}{E}}^{(n)}(\underset{\rightarrow}{\beta},\omega) - (- A^{(1)}(\underset{\rightarrow}{\beta},\omega,\underset{\rightarrow}{\rho},t)\hat{\underset{\rightarrow}{E}}^{(n-1)}(\underset{\rightarrow}{\beta},\omega))] + 0(\zeta^{n+1})\Big\}$$

Multiplication of (2.14) by

$$e^{-j\frac{\omega'}{c}[(\underset{\rightarrow}{\beta}'\cdot\underset{\rightarrow}{\rho}) - ct]}$$

and integration of product on $\underset{\rightarrow}{\rho}$ and t from $-\infty$ to $+\infty$, with the aid of (2.7.a,b) yields the following set of perturbation equations (after interchanging $(\underset{\rightarrow}{\beta}',\underset{\rightarrow}{\omega}')$ and (β,ω) and changing order of integration)

$$A^{(0)}(\underset{\rightarrow}{\beta},\omega)\hat{\underset{\rightarrow}{E}}^{(0)}(\underset{\rightarrow}{\beta},\omega) = B^{(0)}(\underset{\rightarrow}{\beta})\tilde{\tilde{\underset{\rightarrow}{E}}}_i(\underset{\rightarrow}{\beta},\omega) \tag{2.15.0}$$

$$A^{(0)}(\underset{\rightarrow}{\beta},\omega)\hat{\underset{\rightarrow}{E}}^{(1)}(\underset{\rightarrow}{\beta},\omega) = \frac{1}{(2\pi)^3}\iint_{-\infty}^{\infty} d\underset{\rightarrow}{\beta}' \int_{-\infty}^{\infty} d\omega' \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho} \int_{-\infty}^{\infty} dt\, e^{\frac{j}{c}[(\omega'\underset{\rightarrow}{\beta}' - \omega\underset{\rightarrow}{\beta})\cdot\underset{\rightarrow}{\rho} - c(\omega' - \omega)t]} \ldots \tag{2.15.1}$$

$$(B^{(1)}(\underset{\rightarrow}{\beta}',\omega',\underset{\rightarrow}{\rho},t)\tilde{\tilde{\underset{\rightarrow}{E}}}_i(\underset{\rightarrow}{\beta}',\omega') - A^{(1)}(\underset{\rightarrow}{\beta}',\omega',\underset{\rightarrow}{\rho},t)\underset{\rightarrow}{E}^{(0)}(\underset{\rightarrow}{\beta}',\omega'))$$

$$A^{(0)}(\underset{\rightarrow}{\beta},\omega)\hat{\underset{\rightarrow}{E}}^{(2)}(\underset{\rightarrow}{\beta},\omega) = \frac{1}{(2\pi)^3}\iint_{\infty}^{\infty} d\underset{\rightarrow}{\beta}' \int_{\infty}^{\infty} d\omega' \iint_{\infty}^{\infty} d\underset{\rightarrow}{\rho} \int_{\infty}^{\infty} dt\, e^{\frac{j}{c}[(\omega'\underset{\rightarrow}{\beta}' - \omega\underset{\rightarrow}{\beta})\cdot\underset{\rightarrow}{\rho} - c(\omega' - \omega)t]} \ldots \tag{2.15.2}$$

$$(- A^{(1)}(\underset{\rightarrow}{\beta}',\omega',\underset{\rightarrow}{\rho},t)\hat{\underset{\rightarrow}{E}}^{(1)}(\underset{\rightarrow}{\beta}',\omega'))$$

$$A^{(0)}(\underset{\to}{\beta},\omega)\underset{\to}{\hat{\tilde{E}}}^{(n)}(\beta,\omega) = \frac{1}{(2\pi)^3}\iint_{-\infty}^{\infty} d\underset{\to}{\beta}' \int_{-\infty}^{\infty} d\omega' \iint_{-\infty}^{\infty} d\underset{\to}{\rho} \int_{-\infty}^{\infty} dt\, e^{\frac{j}{c}[(\omega'\underset{\to}{\beta}' - \omega\underset{\to}{\beta})\cdot\underset{\to}{\rho} - c(\omega' - \omega)t]} \tag{2.15n}$$

$$(- A^{(1)}(\underset{\to}{\beta}',\omega';\underset{\to}{\rho},t)\underset{\to}{\hat{\tilde{E}}}^{(n-1)}(\underset{\to}{\beta}',\omega')); n=3,4, \ldots$$

Solution of (2.15.0) yields

$$\underset{\to}{\hat{\tilde{E}}}^{(0)}(\underset{\to}{\beta},\omega) = \hat{R}^{(0)}(\underset{\to}{\beta},\omega)\underset{\to}{E}_i(\underset{\to}{\beta},\omega), \tag{2.16.0}$$

where

$$\hat{R}^{(0)}(\beta,\omega) = [A^{(0)}(\beta,\omega)]^{-1} B^{(0)}(\beta,\omega),$$

and the inverse of $A^{(0)}$ is given by

$$[A^{(0)}(\underset{\to}{\beta},\omega)]^{-1} = b_0(\underset{\to}{\beta}) = \frac{1}{(\beta_z + \gamma_z)(1 - \beta_z{}^2 + \beta_z\gamma_z)} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ -b_{12}' & -b_{11} & b_{14} & b_{13}' \\ b_{11} & b_{32} & b_{13} & b_{14} \\ -b_{32}' & -b_{11} & b_{14} & b_{13}' \end{bmatrix} \tag{2.16.0$'$}$$

where

$$\begin{aligned} b_{11} &= - \beta_x\beta_y(\beta_z - \gamma_z), \\ b_{12} &= \beta_z\nu^2 - (\beta_z - \gamma_z)\beta_y{}^2, \\ b_{13} &= - (\beta_z\gamma_z + \beta_y{}^2), \\ b_{14} &= \beta_x\beta_y, \text{ and} \\ b_{32} &= - \gamma_z - (\beta_z - \gamma_z)\beta_y{}^2. \end{aligned}$$

Solution of (2.15.1) follows from (2.16.0), i.e.,

$$\underset{\to}{\hat{\tilde{E}}}^{(1)}(\underset{\to}{\beta},\omega) = \iint d\underset{\to}{\beta}_1\, d\omega_1\, d\underset{\to}{\rho}_1\, dt_1\, e^{\frac{j}{c}[(\omega_1\underset{\to}{\beta}_1 - \omega\underset{\to}{\beta})\cdot\underset{\to}{\rho}_1 - c(\omega_1 - \omega)t_1]} \ldots \tag{2.16.1}$$

$$\hat{R}^{(1)}(\underset{\to}{\beta},\omega/\underset{\to}{\beta}_1,\omega_1,\underset{\to}{\rho}_1,t_1)\underset{\to}{\tilde{\tilde{E}}}_i(\underset{\to}{\beta}_1,\omega_1),$$

where

$$\hat{R}^{(1)}(\underset{\rightarrow}{\beta},\omega/\underset{\rightarrow}{\beta}_1,\omega_1;\underset{\rightarrow}{\rho}_1,t_1) = \frac{1}{(2\pi)^3}\,[A^{(0)}(\underset{\rightarrow}{\beta},\omega)]^{-1}\Big\{\underset{\rightarrow}{B}^{(1)}(\underset{\rightarrow}{\beta}_1,\omega_1;\underset{\rightarrow}{\rho}_1,t_1)$$

$$- A^{(1)}(\underset{\rightarrow}{\beta}_1,\omega_1;\underset{\rightarrow}{\rho}_1,t_1)[A^{(0)}(\underset{\rightarrow}{\beta}_1,\omega_1)]^{-1}\,B^{(0)}(\underset{\rightarrow}{\beta}_1,\omega_1)\Big\}.$$

From (2.15.2), (2.16.0) and 2.16.1),

$$\hat{\underset{\rightarrow}{\tilde{E}}}^{(2)}(\underset{\rightarrow}{\beta},\omega) = \iint d\underset{\rightarrow}{\beta}_1\, d\omega_1\, d\underset{\rightarrow}{\rho}_1\, dt_1\, d\underset{\rightarrow}{\beta}_2\, d\omega_2\, d\rho_2\, dt_2 \cdots$$

$$e^{\frac{j}{c}[(\omega_1\underset{\rightarrow}{\beta}_1 - \omega\underset{\rightarrow}{\beta})\cdot\underset{\rightarrow}{\rho}_1 - c(\omega_1 - \omega)t_1 + (\omega_2\underset{\rightarrow}{\beta}_2 - \omega_1\underset{\rightarrow}{\beta}_1)\cdot\underset{\rightarrow}{\rho}_2 - c(\omega_2 - \omega_1)t_2]} \cdots \tag{2.16.2}$$

$$\hat{R}^{(2)}(\underset{\rightarrow}{\beta},\omega/\underset{\rightarrow}{\beta}_1,\omega_1/\underset{\rightarrow}{\beta}_2,\omega_2;\underset{\rightarrow}{\rho}_1,t_1/\underset{\rightarrow}{\rho}_2,t_2)\,\underset{\rightarrow}{\tilde{E}}_i(\beta_2,\omega_2),$$

where

$$\hat{R}^{(2)}(\underset{\rightarrow}{\beta},\omega/\underset{\rightarrow}{\beta}_1,\omega_1/\underset{\rightarrow}{\beta}_2,\omega;\underset{\rightarrow}{\rho}_1,t_1/\underset{\rightarrow}{\rho}_2,t_2) =$$

$$-\frac{1}{(2\pi)^6}\,[A^{(0)}(\underset{\rightarrow}{\beta},\omega)]^{-1}A^{(1)}(\underset{\rightarrow}{\beta}_1,\omega_1;\underset{\rightarrow}{\rho}_1,t_1)[A^{(0)}(\underset{\rightarrow}{\beta}_1,\omega_1)]^{-1}\Big\{B^{(1)}(\underset{\rightarrow}{\beta}_2,\omega_2;\underset{\rightarrow}{\rho}_2,t_2)$$

$$- A^{(1)}(\underset{\rightarrow}{\beta}_2,\omega_2;\underset{\rightarrow}{\rho}_2,t_2([A^{(0)}(\underset{\rightarrow}{\beta}_2,\omega_2)]^{-1}B^{(0)}(\underset{\rightarrow}{\beta}_2,\omega_2)\Big\}.$$

The general (n^{th}) term of $\hat{\underset{\rightarrow}{\tilde{E}}}$ is (from (2.15.n))

$$\hat{\underset{\rightarrow}{\tilde{E}}}^{(n)}(\underset{\rightarrow}{\beta},\omega) = \iint d\underset{\rightarrow}{\beta}_1\, d\omega_1\, d\underset{\rightarrow}{\rho}_1\, dt_1\, d\underset{\rightarrow}{\beta}_2\, d\omega_2\, d\underset{\rightarrow}{\rho}_2\, dt_2 \ldots d\underset{\rightarrow}{\beta}_n\, d\omega_n\, d\underset{\rightarrow}{\rho}_n\, dt_n \cdots$$

$$\begin{aligned} &e^{\frac{j}{c}[(\omega_1\underset{\rightarrow}{\beta}_1 - \omega\underset{\rightarrow}{\beta})\cdot\underset{\rightarrow}{\rho}_1 - c(\omega_1 - \omega)t_1 + (\omega_2\underset{\rightarrow}{\beta}_2 - \omega_1\underset{\rightarrow}{\beta}_1)\cdot\underset{\rightarrow}{\rho}_2 - c(\omega_2 - \omega_1)t_2} \cdots \\ &\quad + (\omega_{n-1}\underset{\rightarrow}{\beta}_{n-1} - \omega_{n-2}\underset{\rightarrow}{\beta}_{n-2})\cdot\underset{\rightarrow}{\rho}_{n-1} - c(\omega_{n-1} - \omega_{n-2})t_{n-1} \\ &\quad + (\omega_n\underset{\rightarrow}{\beta}_n - \omega_{n-1}\underset{\rightarrow}{\beta}_{n-1})\cdot\underset{\rightarrow}{\rho}_n - c(\omega_n - \omega_{n-1})t_n] \cdots \end{aligned} \tag{2.16.n}$$

$$\hat{R}^{(n)}(\underset{\rightarrow}{\beta},\omega/\underset{\rightarrow}{\beta}_1,\omega_1/\underset{\rightarrow}{\beta}_2,\omega_2/\ldots/\underset{\rightarrow}{\beta}_n,\omega_n;\underset{\rightarrow}{\rho}_1 t_1/\underset{\rightarrow}{\rho}_2,t_2/\ldots/\underset{\rightarrow}{\rho}_n,t_n)\underset{\rightarrow}{\tilde{E}}_1(\underset{\rightarrow}{\beta}_n,\omega_n),$$

where

$$\hat{R}^{(n)}(\underset{\to}{\beta},\omega/\underset{\to}{\beta}_1,\omega_1/\underset{\to}{\beta}_2,\omega_2/\cdots/\underset{\to}{\beta}_n,\omega_n;\underset{\to}{\rho}_1,t_1/\underset{\to}{\rho}_2,t_2/\cdots/\underset{\to}{\rho}_n,t_n) =$$

$$-(-1)^n \frac{1}{(2\pi)^{3n}} [A^{(0)}(\underset{\to}{\beta},\omega)]^{-1} A^{(1)}(\underset{\to}{\beta}_1,\omega_1;\underset{\to}{\rho}_1,t_1)$$

$$[A^{(0)}(\underset{\to}{\beta}_1,\omega_1)]^{-1} A^{(1)}(\underset{\to}{\beta}_2,\omega_2;\underset{\to}{\rho}_2,t_2) \cdots [A^{(0)}(\underset{\to}{\beta}_2,\omega_2)]^{-1} A^{(1)}(\underset{\to}{\beta}_3,\omega_3;\underset{\to}{\rho}_3,t_3) \cdots$$

$$[A^{(0)}(\underset{\to}{\beta}_{n-2},\omega_{n-2})]^{-1} A^{(1)}(\underset{\to}{\beta}_{n-1},\omega_{n-1};\underset{\to}{\rho}_{n-1},t_{n-1})$$

$$[A^{(0)}(\underset{\to}{\beta}_{n-1},\omega_{n-1})]^{-1} \Big\{ B^{(1)}(\underset{\to}{\beta}_n,\omega_n;\underset{\to}{\rho}_n,t_n) - A^{(1)}(\underset{\to}{\beta}_n,\omega_n;\underset{\to}{\rho}_n,t_n)$$

$$[A^{(0)}(\underset{\to}{\beta}_n,\omega_n)]^{-1} B^{(0)}(\underset{\to}{\beta}_n,\omega_n) \Big\}.$$

The vector $\hat{\underset{\to}{\tilde{E}}}$ is not of direct use to us. It would be preferable to have the reflected wave vector $\underset{\to}{\tilde{E}}_r$. To obtain $\underset{\to}{\tilde{E}}_r$ we invoke (2.9) and note that

$$\underset{\to}{\tilde{E}}_r = M_r \underset{\to}{\tilde{\tilde{E}}}_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(\frac{\beta_x}{\beta_z}) & -(\frac{\beta_y}{\beta_z}) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{E}_{rx} \\ \tilde{E}_{ry} \\ \tilde{E}_{tx} \\ \tilde{E}_{ty} \end{bmatrix} = Q(\underset{\to}{\beta}) \hat{\underset{\to}{\tilde{E}}}, \tag{2.17}$$

where

$$Q(\underset{\to}{\beta}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -(\frac{\beta_x}{\beta_z}) & -(\frac{\beta_y}{\beta_z}) & 0 & 0 \end{bmatrix}.$$

The unit vectors corresponding to horizontal and vertical polarization are defined in Appendix I (Eqs. (I-24.a,b) and (I-27)) and are denoted by $\underset{\to}{\ell}_V$ and $\underset{\to}{\ell}_H$ respectively. The specialized meaning of horizontal and vertical polarization in this report, different from the standard meaning, are explained in Appendix I.

The vertically and horizontally polarized components of $\tilde{\underset{\rightarrow}{E}}_r$ are denoted by

$$E_{rV} = \underset{\rightarrow}{\ell}{}_V^T \, \tilde{\underset{\rightarrow}{E}}_r, \text{ and} \tag{2.18.a}$$

$$E_{rH} = \underset{\rightarrow}{\ell}{}_H^t \, \tilde{\underset{\rightarrow}{E}}_r. \tag{2.18.b}$$

In all that follows we will adopt the notational convention introduced in Appendix IV wherein all explicit indications of dependence on ω are dropped.

We now relate $\overset{\approx}{E}_i$ to E_{iV} and E_{iH}, the vertically and horizontally polarized components of the incident wave vector $\underset{\rightarrow}{E}_i$. From (2.8) and (2.9)

$$\overset{\approx}{\underset{\rightarrow}{E}}_i \,(\underset{\rightarrow}{\beta}) = L(\underset{\rightarrow}{\beta})\overline{\tilde{\underset{\rightarrow}{E}}}_i \,(\underset{\rightarrow}{\beta}), \tag{2.19}$$

where

$$\overline{\tilde{\underset{\rightarrow}{E}}}_i \,(\underset{\rightarrow}{\beta}) = \begin{bmatrix} E_{iV} & (\underset{\rightarrow}{\beta}) \\ E_{iH} & (\underset{\rightarrow}{\beta}) \end{bmatrix},$$

$$L(\underset{\rightarrow}{\beta}) = \frac{1}{(\ell'_{Vx}\ell'_{Hy} - \ell'_{Hx}\ell'_{Vy})} \begin{bmatrix} \ell'_{Hy} & -\ell'_{Vy} \\ -\ell'_{Hx} & \ell'_{Vx} \end{bmatrix} \text{ and,}$$

$$\ell'_{\substack{Hx\\y\\Vx\\y}} = \ell_{\substack{Hx\\y\\Vx\\y}} + \frac{\ell_{Vz}}{\beta_z}\, \substack{x\\y\\x\\y}\ .$$

The remaining analysis, detailed in Appendix IV, results in expressions for vertically and horizontally polarized electric field components at the receiving point, given both vertically and horizontally polarized transmitted fields.

THE RECEIVED RADAR SIGNAL

We denote the received signal at the radar by the symbol $u_{Aa}(t)$. This represents a quantity proportional to the electric field component along the direction of polarization selected by the receiver. The subscripts A and a denote received and transmitted polarizations respectively. We can use the standard representation for narrow band signals, i.e.,

$$u_{Aa}(t) = Re\left\{U_{Aa}(t)e^{-j\omega_o t}\right\}, \tag{3.1}$$

where $U_{Aa}(t)$ is the complex envelope of $u_{Aa}(t)$, ω_o is the central frequency of the signal spectrum and $U_{Aa}(t)$ varies slowly compared with

$$e^{-j\omega_o t}.$$

The most general quantity that will ultimately be needed involves the product of two signals possibly with relative time displacement Δt and different polarizations, time-averaged over an RF cycle (where an overline indicates the time averaging process, and of course

$$\overline{e^{\mp 2j\omega_o t}} = 0.$$

This is given by

$$\hat{U}_{AaBb}(t,\Delta t) = \frac{1}{2} Re(\hat{U}_{AaBb}(t,\Delta t) e^{j\omega_o \Delta t}), \tag{3.2}$$

where

$$\hat{U}_{AaBb}(t,\Delta t) = U_{Aa}(t) U^*_{Bb}(t + \Delta t),$$

and where B and b represent received and transmitted polarizations that may be different from A and a.

A special case of (3.2) is the signal power, or equivalently (if proportionality constants are set equal to unity) the quadratically rectified signal voltage,

$$\hat{U}_{AaAa}(t,0) = \frac{1}{2} Re(\hat{U}_{AaAa}(t,0) = \frac{1}{2} \hat{U}_{AaAa}(t,0), \tag{3.3}$$

where

$$U_{AaAa}(t,0) = \left|U_{Aa}(t)\right|^2.$$

Another quantity of interest, in its most general form, is the product of two quadratically rectified signals with relative time displacement Δt and possibly different polarizations. This is obtained from (3.3) and is given by

$$\hat{\hat{U}}_{AaBb}(t,\Delta t) = \hat{U}_{AaAa}(t,0) \hat{U}_{BbBb}(t + \Delta t,0) = \left|U_{Aa}(t)\right|^2 \left|U_{Bb}(t + \Delta t)\right|^2, \tag{3.4}$$

neglecting the factor 1/4 which does not affect the results significantly.

We now invoke the simplifying assumption that the incident wave is a CW signal, amplitude modulated by a pulsing waveform of arbitrary shape, but whose bandwidth is small compared with RF. More accurately, we should solve the boundary value problem for each frequency ω contained in the incident waveform and integrate on ω over the entire spectrum. This process can be approximated for practical purposes by treating the signal as if the incident wave were CW with frequency $\omega = \omega_o$ and multiplying the resulting scattered CW field by the pulsing function. This implies that, within the small band covered by the pulse, the scattered field waveform is approximately independent of frequency.

This simplified model is implemented through the introduction of a function $g(r_o)$, which weights different contributions to the return in proportion to the delay, which in turn is proportional to r_o, the distance from the radar to the illuminated point on the scattering surface.

Another weighting function on the signal is the antenna pattern function $f(\theta'_3, \phi'_3)$ which weights contributions from different angular positions (θ'_3, ϕ'_3)*. Still another is the factor

$$\left(e^{2j\frac{\omega}{c} r_o} / r_o^2\right)$$

present in any two-way radar signal return from a point target.

Allowing for the possibility that the pulsing function and antenna pattern function may differ for different polarizations we will place subscripts Aa on them and construct an overall weighting function

$$F_{Aa}(\theta'_3, \phi'_3, t) = K \frac{g_{Aa}(r_o(\theta'_3, \phi'_3, t)) f^2_{Aa}(\theta'_3, \phi'_3) e^{2j\frac{\omega}{c} r_o(\theta'_3, \phi'_3, t)}}{[r_o(\theta'_3, \phi'_3, t)]^2}, \tag{3.5}$$

where the radar-surface geometry fixes the functional dependence of r_o on the angles (θ'_3, ϕ'_3) where $f_{Aa}(\theta'_3, \phi'_3)$ is the one-way field pattern of the antenna for the polarizations denoted by Aa, where k is a constant containing radar parameters, and where the possible time dependence in r_o arises from the expression (Appendix III, Eq. III. 24),

$$r_o \simeq \hat{r}_o + \Delta r_o, \tag{3.6}$$

where r_o is the distance from the radar to a point on the horizontal (calm) sea surface, given (through manipulation of (III.25) in Appendix III) by

$$\hat{r}_o = \frac{h_R}{\cos\theta'_3 \cos\gamma - \sin\theta'_3 \sin\gamma \cos\phi'_3}, \qquad (3.6.a)'$$

*See Appendix I (beginning below Eq. I.14) for definition of the angles (θ'_3, ϕ'_3).

where h_R is the radar altitude and γ the beam angle relative to vertical, and where (from (III.24) or (III.26))

$$\Delta r_o = - H(x',y') \frac{h_R}{\hat{r}_o}. \tag{3.6.b)'}$$

$H(x',y')$ represents the mean sea surface as a function of the horizontal coordinate (x',y'), h_R is radar altitude and $\hat{r}_o$ is given by (3.6.a)$'$.

If $H(x',y') \neq 0$ in (3.6.b)$'$, i.e., if the mean sea surface is not perfectly horizontal everywhere, then, since (through manipulation of (III.15.f) in Appendix III)

$$x' = x'_R + h_R \frac{(\sin\theta'_3 \cos\phi'_3 \cos\gamma + \cos\theta'_3 \sin\gamma)}{(\cos\theta'_3 \cos\gamma - \sin\theta'_3 \sin\gamma \cos\phi'_3)}, \tag{3.6.c)'}$$

and

$$y' = y'_R - \frac{h_R \sin\theta'_3 \sin\phi'_3}{(\cos\theta'_3 \cos\gamma - \sin\theta'_3 \sin\gamma \cos\phi'_3)}, \tag{3.6.d)'}$$

where x'_R and y'_R are the horizontal coordinates of the radar relative to a fixed origin of coordinates.

The time dependence in Δr_o arises from a possible time-dependence in x'_R and y'_R which would appear in x', and y' through (3.6.c,d)$'$ and hence would appear in $H(x', y')$ and in (3.6.b)$'$. This would occur through horizontal motion of the radar. Vertical radar motion would manifest itself in time dependence of h_R, which would render $\hat{r}_o$ time-dependent through (3.6.d)$'$ and would also appear in Δr_o through (3.6.b,c,d)$'$. Another possible source of time dependence is a time variation of the beam angle γ, which would appear in r_o through Eqs. (3.6) and (3.6.a,b,c,d)$'$. If the mean-sea surface were perfectly horizontal everywhere and the beam direction fixed, then r_o and hence F_{Aa} would be independent of time.

We now specify the form of the radar return signal as a superposition of returns from patches at position $(r_o, \theta'_3, \phi'_3)$ relative to the radar, each return weighted by the function $F_{Aa}(\theta'_3, \phi'_3, t)$ as given in (3.5) and a factor S_{Aa} derived from the perturbation solution of the boundary value problem. The results of this line of reasoning are

$$U_{Aao(t)} = \int_o^{\pi} d\theta'_3 \sin\theta'_3 \int_o^{2\pi} d\phi' \, F_{Aa}(\theta'_3,\phi'_3,t) \, S_{Aao}(\underset{\to}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t)), \tag{3.7.0}$$

$$U_{Aa1}(t) = \int_o^{\pi} d\theta_3' \sin\theta_3' \int_o^{2\pi} d\phi' \, F_{Aa}(\theta_3',\phi_3,t) S_{Aa1}(\hat{\underset{\rightarrow}{\alpha}}_o \, (\theta_3',\phi_3',t)) \ldots$$

$$Z(-\frac{2\omega}{c}\hat{\underset{\rightarrow}{\alpha}}_o \, (\theta_3',\phi_3',t)), \tag{3.7.1}$$

$$U_{Aa2}(t) = \int_o^{\pi} d\theta_3' \sin\theta_3' \int_o^{2\pi} d\phi_3' \, F_{Aa}(\theta_3',\phi_3',t) \iint d\underset{\rightarrow}{k} S_{Aa2}(\hat{\underset{\rightarrow}{\alpha}}_o \, (\theta_3',\phi_3',t), \cdot \underset{\rightarrow}{k}) \ldots$$

$$Z(\underset{\rightarrow}{k}) Z(-\frac{2\omega}{c}\hat{\underset{\rightarrow}{\alpha}}_o \, (\theta_3',\phi_3',t) - \underset{\rightarrow}{k}), \tag{3.7.2}$$

$$U_{Aa3}(t) = \int_o^{\pi} d\theta_3' \sin\theta_3' \int_o^{2\pi} d\phi_3' \, F_{Aa}(\theta_3',\phi_3',t) \iint d\underset{\rightarrow}{k}_1 \iint d\underset{\rightarrow}{k}_2 S_{Aa3} \ldots$$

$$(\hat{\underset{\rightarrow}{\alpha}}_o \, (\theta_3',\phi_3',t), \cdot \underset{\rightarrow}{k}_1, \underset{\rightarrow}{k}_2) Z(\underset{\rightarrow}{k}_1) Z(\underset{\rightarrow}{k}_2) \ldots Z(-\frac{2\omega}{c}\hat{\underset{\rightarrow}{\alpha}}_o(\theta_3',\phi_3',t) - \underset{\rightarrow}{k}_1 - \underset{\rightarrow}{k}_2), \text{ and} \tag{3.7.3}$$

$$U_{Aa4}(t) = \int_o^{\pi} d\theta_3' \sin\theta_3' \int_o^{2\pi} d\phi_3' F_{Aa}(\theta_3',\phi_3',t) \iint d\underset{\rightarrow}{k}_1 \iint d\underset{\rightarrow}{k}_2 \iint d\underset{\rightarrow}{k}_3 \; S_{Aa4} \ldots$$

$$(\hat{\underset{\rightarrow}{\alpha}}_o(\theta_3',\phi_3',t), \, \underset{\rightarrow}{k}_1, \underset{\rightarrow}{k}_2, \underset{\rightarrow}{k}_3) Z(\underset{\rightarrow}{k}_1) Z(\underset{\rightarrow}{k}_2) Z(\underset{\rightarrow}{k}_3) \ldots \tag{3.7.4}$$

$$Z(-\frac{2\omega}{c}\hat{\underset{\rightarrow}{\alpha}}_o(\theta_3',\phi_3',t) - \underset{\rightarrow}{k}_1 - \underset{\rightarrow}{k}_2 - \underset{\rightarrow}{k}_3),$$

where $\hat{\underset{\rightarrow}{\alpha}}_o$ is defined in Appendix I, Eq. (I.27.a), as the projection on the x-y plane of the unit vector $\hat{\alpha}_o$ directed from the radar to the illuminated surface point.

Note that the $\hat{\underset{\rightarrow}{\alpha}}_o$ in (3.7.0,1,2,3,4) is indicated as a function of the angles (θ_3',ϕ_3') and a possible function of time. The time-dependence arises again through possible motion of the radar and beam and departures of the mean sea surface from perfectly horizontal. To show this, we invoke (I.27.a) and (I.28), dropping subscripts in (I.28) and also Eqs. (I.8.a,b) and (I.10.a,b) with the results

$$\hat{\underset{\rightarrow}{\alpha}}_o = (\alpha_{ox}, \alpha_{oy}), \tag{3.8}$$

where

$$\alpha_{ox} = \frac{1}{\sqrt{1 + |\nabla' H|^2}} \Big\{ \sin\theta_3' \cos\phi_3' [\cos\Phi \cos\gamma - |\nabla' H| \sin\gamma] \tag{3.8.a)'}$$

$$- \sin\theta_3' \sin\phi_3' \sin\Phi + \cos\theta_3' [\cos\Phi \sin\gamma + |\nabla' H| \cos\gamma] \Big\}, \text{ and}$$

$$\alpha_{oy} = \Big\{ - \sin\theta_3' \cos\phi_3' \sin\Phi \cos\gamma - \sin\theta_3' \sin\phi_3' \cos\Phi$$

$$- \cos\theta_3' \sin\Phi \sin\gamma \Big\}. \tag{3.8.b)'}$$

(Where $|\nabla' H|$ and γ were defined above in connection with Eqs. (3.6) and (3.6.a,b,c,d)′, and where $\cos\Phi = (\partial H/\partial x')/|\nabla' H|$, $\sin\Phi = (\partial H/\partial y')/|\nabla' H|$, and the coordinate systems have been defined such that Φ is equal to zero for a perfectly horizontal surface.)

We have referred to polarization in this section only implicitly. The subscripts Aa and Bb used in (3.1) through (3.5) and (3.7.0,...,4) refer to particular polarization situations. The equations in Appendix IV from which (3.7.0,...,4) are justified contain quantities with subscripts VV, VH, HV, and HH, where the first subscript indicates received polarization and the second subscript indicates transmitted polarization. Thus, e.g., if the transmitted and received polarizations corresponding to u_{Aa} in (3.1) were both vertical, then the subscripts Aa would be indicated as VV. If the transmitted polarization corresponding to $u_{Bb}(t)$ were vertical and the received polarization were horizontal, then subscripts Bb would be indicated as HV, If the transmitted and received polarizations were circular, then the subscripts Aa or Bb might indicate (for example) LR, implying that the transmitted polarization is right-handed and the received polarization is left-handed. In this case (3.1) would be written as follows:

$$u_{LR}(t) = Re\Big\{U_{LR}(t) d^{-j\omega_o t}\Big\}, \tag{3.9}$$

where

$$U_{LR}(t) = U_{VR}(t) + jU_{HR}(t),$$

where the subscripts VR and HR imply respectively vertically and horizontally polarized received radiation with right-handed circular transmitted polarization.

ENSEMBLE AVERAGES OVER SURFACE FLUCTUATIONS

The quantities used in the final results will be ensemble-averaged over small-scale surface fluctuations. This averaging process is discussed in Appendix II and results given there are used as a basis for much of what follows.

Averages of Voltage Products

If we expand $u_{Aa}(t)$ as given in (3.1) through fourth order, we obtain

$$u_{Aa}(t) = u_{Aao}(t) + u_{Aa1}(t) + \ldots + u_{Aa4}(t). \tag{4.1}$$

The averaging of (3.2) with the aid of (3.7.0,...,4), (II.4), (II.13) and (II.16) leads to

$$<\hat{u}_{AaBb}(t,\Delta t)> = \hat{u}_{AaBbo}(t,\Delta t) + <\hat{u}_{AaBb2}(t,\Delta t)> + <\hat{u}_{AaBb4}(t,\Delta t)>, \tag{4.2}$$

where the subscripts 0, 2, and 4 refer to perturbation order and where (it is understood that the argument of all Aa factors is t, that of Bb factors is $t + \Delta t$ and all quantities $\hat{U}_{AaBb}$ have arguments $(t,\Delta t)$)

$$\hat{u}_{AaBBo} = \frac{1}{2} Re\, (\hat{U}_{AaBbo} e^{j\omega_o \Delta t}), \qquad (4.2.a)'$$

where

$$\hat{U}_{AaBbo} = U_{Aao} U^*_{Bbo},$$
$$<u_{AaBb2}> = \frac{1}{2} Re\, (<\hat{U}_{AaBb2}> e^{j\omega_o \Delta t}), \qquad (4.2.b)'$$

where

$$<\hat{U}_{AaBb2}> = U_{Aao} <U^*_{Bb2}> + <U_{Aa2}> U^*_{Bbo} + <U_{Aa1} U^*_{Bb1}>,$$
$$<\hat{u}_{AaBb4}> = \frac{1}{2} Re\, (<\hat{U}_{AaBb4}> e^{j\omega_o \Delta t}), \qquad (4.2.c)'$$

where

$$<\hat{U}_{AaBb4}> = <U_{Aa2} U^*_{Bb2}> + <U_{Aa1} U^*_{Bb3}> + <U_{Aa3} U^*_{Bb1}>$$
$$+ U_{Aao} <U^*_{Bb4}> + <U_{Aa4}> U^*_{Bbo}. \qquad (4.2.d)'$$

Averages of Power (or Quadratically Rectified Voltage) Products

The ensemble average of $\hat{\hat{U}}_{AaBb}(t,\Delta t)$, obtained from (3.4), is:

$$
\begin{aligned}
<\hat{\hat{U}}_{AaBb}(t,\Delta t)> = &<\hat{U}_{AaAa}(t,0)><\hat{U}_{BbBb}(t+\Delta t, 0)> \\
&+<[\hat{U}_{AaAa}(t,0) - <\hat{U}_{AaAa}(t,0)] \ldots \\
&[\hat{U}_{BbBb}(t+\Delta t, 0) - <\hat{U}_{BbBb}(t+\Delta t, 0)>]>,
\end{aligned}
\tag{4.3}
$$

where each of the factors in the first term can be considered as specializations of factors in terms in (4.2.a,b,c)′ and the second term, expanded to fourth order, with the aid of (3.2), (3.7.0,...,4), (II.13) and (II.16) is

$$
\begin{aligned}
<\Delta\hat{\hat{U}}_{AaBb}(t,\Delta t)> = &<[\hat{U}_{AaAa}(t,0) - <\hat{U}_{AaAa}(t,0)>] \ldots \\
&[\hat{U}_{BbBb}(t+\Delta t, 0) - <\hat{U}_{BbBb}(t+\Delta t, 0)>]> = \\
&<\hat{U}_{AaAa1}(t,0)\hat{U}_{BbBb1}(t+\Delta t, 0)> \\
&+<\hat{U}_{AaAa1}(t,0)\hat{U}_{BbBb3}(t+\Delta t, 0)> \\
&+<\hat{U}_{AaAa3}(t,0)\hat{U}_{BbBb1}(t+\Delta t, 0)> \\
&+<\hat{U}_{AaAa2}(t,0)\hat{U}_{BbBb2}(t+\Delta t, 0)> \\
&-<\hat{U}_{AaAa2}(t,0)><\hat{U}_{BbBb2}(t+\Delta t, 0)>,
\end{aligned}
\tag{4.4}
$$

whose individual terms are (where it is understood that arguments of Aa factors are $(t,0)$ and those of Bb factors are $(t+\Delta t, 0)$):

$$
\begin{aligned}
<\hat{U}_{AaAa1}\hat{U}_{BbBb1}> = 2\,Re\,[&U^*_{Aao}U_{Bbo}<U_{Aa1}U^*_{Bb1}> \\
&+ U^*_{Aao}U^*_{Bbo}<U_{Aa1}U_{Bb1}>],
\end{aligned}
\tag{4.4.a)'}
$$

$$
\begin{aligned}
<\hat{U}_{AaAa1}\hat{U}_{BbBb3}> = 2\,Re\,[&U^*_{Aao}U_{Bbo}<U_{Aa1}U^*_{Bb3}> \\
&+ U^*_{Aao}U^*_{Bbo}<U_{Aa1}U_{Bb3}> \\
&+ U^*_{Aao}<U^*_{Aa1}U_{Bb1}U_{Bb2}> \\
&+ U_{Aao}<U^*_{Bb1}U^*_{Aa1}U_{Bb2}>],
\end{aligned}
\tag{4.4.b)'}
$$

$$<\hat{U}_{AaAa3}\hat{U}_{BbBb1}> = 2\,Re\,[U^*_{Bbo}U_{Aao}<U_{Bb1}U^*_{Aa3}> + U^*_{Bbo}U^*_{Aao}<U_{Bb1}U_{Aa3}> + U^*_{Bbo}<U^*_{Bb1}U_{Aa1}U_{Aa2}> + U_{Bbo}<U^*_{Aa1}U^*_{Bb1}U_{Aa2}>], \tag{4.4.c$'$}$$

$$<\hat{U}_{AaAa2}\hat{U}_{BbBb2}> = 2\,Re\,[U_{Aao}U^*_{Bbo}<U^*_{Aa2}U_{Bb2}> + U^*_{Aao}U^*_{Bbo}<U_{Aa2}U_{Bb2}> + U_{Aao}<U_{Bb1}U^*_{Bb1}U^*_{Aa2}> + U_{Bbo}<U_{Aa1}U^*_{Aa1}U^*_{Bb2}>] + <U^*_{Aa1}U_{Aa1}U^*_{Bb1}U_{Bb1}> - <U_{AaAa2}><U_{BbBb2}> \tag{4.4.d$'$}$$

$$-<\hat{U}_{AaAa2}(t,0)><\hat{U}_{BbBb2}(t+\Delta t,0)> = -\left\{2\,Re\,[U_{Aao}<U^*_{Aa2}>] + <\left|U_{Aa1}\right|^2>\right\}\cdots\left\{2\,Re\,[U_{Bbo}<U^*_{Bb2}>] + <\left|U_{Bb1}\right|^2>\right\}. \tag{4.4.e$'$}$$

From (3.7.0,...,4), (4.2), (4.2.a,b,c)$'$, (4.4), (4.4.a,...,e)$'$, and a number of results in Appendix II, we can express $<\hat{U}_{AaBb}(t,\Delta t)>$ and $<\Delta\hat{U}_{AaBb}(t,\Delta t)>$ in more compact approximate forms. First, from (4.2), (4.2.a,b,c)$'$, (3.7.0,...,4), and (II.17, 18, 19, 20, 21),

$$<\hat{U}_{AaBb}(t,\Delta t)> = \frac{1}{2} Re\,(<\hat{U}_{AaBb}(t,\Delta t)> e^{j\omega\Delta t}), \tag{4.5}$$

where

$$<\hat{U}_{AaBb}(t,\Delta t)> = U_{Aao}U^*_{Bbo} + U_{Ao}[<U^*_{Bb2}> + <U^*_{Bb4}>] + U^*_{Bbo}[<U_{Aa2}> + <U_{Aa4}>] + <U_{Aa1}U^*_{Bb1}> + <U_{Aa2}U^*_{Bb2}> + <U_{Aa1}U^*_{Bb3}> + <U_{Aa3}U^*_{Bb1}> \tag{4.5.1$'$}$$

where it is understood that the argument of U_{Aa} is always t and that of U^*_{Bb} is always $t + \Delta t$, and where (from 3.7.0)

$$U_{Aao} = \int_o^{\pi} d\theta'_3 \sin\theta'_3 \int_o^{2\pi} d\phi'_3 \; F_{Aao}(\theta'_3,\phi'_3,t) S_{Aao}(\hat{\alpha}_o(\theta'_3,\phi'_3,t)). \qquad (4.5.2)'$$

From (3.7.2) and (II.18)

$$<U_{Aa2}> = (\frac{c}{2\omega})^2 \sum_{\substack{k \\ (\hat{\underset{\rightarrow}{\alpha}}_o = 0)}}^{1} F_{Aa}(\theta'_{3k}, \phi'_{3k}, t) \iint d\underset{\rightarrow}{k}\, W(\underset{\rightarrow}{k}) S_{Aa2}(\underset{\rightarrow}{0},\underset{\rightarrow}{k}) \qquad (4.5.3)'$$

where $W(\underset{\rightarrow}{k})$ is the spatial spectrum of the fluctuations. From (3.7.4) and (II.21,18,9)

$$<U_{Aa4}> = (\frac{c}{2\omega})^2 \sum_{\substack{k \\ (\hat{\alpha}_o = 0)}}^{1} F_{Aa}(\theta'_{3k} k\, \phi'_{3k}, t) \iint d\underset{\rightarrow}{k}_1 W(\underset{\rightarrow}{k}_1) \iint d\underset{\rightarrow}{k}_2 W(\underset{\rightarrow}{k}_2) \ldots$$

$$[S_{Aa4}(\underset{\rightarrow}{0}, \underset{\rightarrow}{k}_1, -\underset{\rightarrow}{k}_1, \underset{\rightarrow}{k}_2) + S_{Aa4}(\underset{\rightarrow}{0}, \underset{\rightarrow}{k}_1, \underset{\rightarrow}{k}_2, -\underset{\rightarrow}{k}_1) \qquad (4.5.4)'$$

$$+ S_{Aa4}(\underset{\rightarrow}{0}, \underset{\rightarrow}{k}_1, \underset{\rightarrow}{k}_2, -\underset{\rightarrow}{k}_2)].$$

From (3.7.1) and II.19,14),

$$<U_{Aa1} U^*_{Bb1}> = (\frac{c}{2\omega})^2 \int_o^{\pi} d\theta'_3 \sin\theta'_3 \int_o^{2\pi} d\phi'_3 \; W(-\frac{2\omega}{c} \hat{\underset{\rightarrow}{\alpha}}_o(\theta'_3,\phi'_3,t) \ldots$$

$$F_{Aa}(\theta'_3, \phi'_3, t) F^*_{Bb}(\theta'_3, \phi'_3, t + \Delta t) \qquad (4.5.5)'$$

$$S_{Aa1}(\hat{\underset{\rightarrow}{\alpha}}_o(\theta'_3, \phi'_3, t)(S^*_{Bb1}(\hat{\underset{\rightarrow}{\alpha}}_o(\theta'_3, \phi'_3, t + \Delta t)).$$

From (3.7.2) and II.21,19,14),

$$<U_{Aa2} U^*_{Bb2}> = <U_{Aa2}><U^*_{Bb2}> + (\frac{c}{2\omega})^2 \int_o^{\pi} d\theta'_3 \sin\theta'_3 \ldots$$

$$\int_o^{\pi} d\phi'_3 F_{Aa}(\theta'_3, \phi'_3, t) F^*_{Bb}(\theta'_3, \phi'_3, t + \Delta t) \iint d\underset{\rightarrow}{k} W(\underset{\rightarrow}{k}) W \ldots$$

$$(-\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t)-\underset{\rightarrow}{k})S_{Aa2}(\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t),\underset{\rightarrow}{k})\ \ldots \tag{4.5.6$'$}$$

$$[S^*_{Bb2}(\alpha_o(\underset{\rightarrow}{\hat{\theta}'_3},\phi'_3,t+\Delta t),\underset{\rightarrow}{k})$$

$$+S^*_{Bb2}(\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t),-\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t)-\underset{\rightarrow}{k})],$$

where $<U_{Aa2}>$ and $<U^*_{Bb2}>$ are specializations of (4.5.3)$'$.

From (3.7.1,3) and (II.21,18,19,14),

$$<U_{Aa1}U^*_{Bb3}> \ = (\frac{c}{2\omega})^2\int_o^{\pi} d\theta'_3 \sin\theta'_3 \int_o^{2\pi} d\theta'_3\ W(-\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t))\ \ldots$$

$$F_{Aa}(\theta'_3,\phi'_3,t)F^*_{Bb}(\theta'_3,\phi'_3,t+\Delta t)A_{Aa1}$$

$$(\underset{\rightarrow}{\hat{\alpha}}(\theta'_3,\phi'_3,t))\iint d\underset{\rightarrow}{k}W(\underset{\rightarrow}{k})[S^*_{Bb3} \tag{4.5.7$'$}$$

$$(\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t);-\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t);\underset{\rightarrow}{k})$$

$$+S^*_{Bb3}(\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t);\underset{\rightarrow}{k},-\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t))$$

$$+S^*_{Bb3}(\underset{\rightarrow}{\hat{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t);\underset{\rightarrow}{k},-\underset{\rightarrow}{k}))].$$

From (3.7.0,...,4), (4.4), (4.4a,...e)$'$, and (II.17,18,19,20,21,23,24,25,26),

$$\begin{aligned}<\Delta\hat{\hat{U}}_{AaBb}(t,\Delta t)>\simeq 2\,Re\,[&U^*_{Aao}U_{Bbo}<U_{Aa1}U^*_{Bb1}> \\ &+U^*_{Aao}U_{Bbo}<U_{Aa1}U^*_{Bb3}>+U^*_{Bbo}U_{Aao}<U_{Bb1}U^*_{Aa3}> \\ &+U_{Aao}U^*_{Bbo}<U^*_{Aa2}U_{Bb2}>+\frac{1}{2}\left|<U_{Aa1}U^*_{Bb1}>\right|^2 \\ &-U_{Aao}U_{Bbo}<U^*_{Aa2}><U^*_{Bb2}>-U_{Aao}U^*_{Bbo}<U^*_{Aa2}><U_{Bb2}> \\ &+U^*_{Aao}<U_{Bb2}><U^*_{Aa1}U_{Bb1}>+U^*_{Bbo}<U_{Aa2}><U^*_{Bb1}U_{Aa1}>],\end{aligned} \tag{4.6}$$

where all factors in terms of (4.6) are specializations of (4.5.2,...,7)$'$.

In the special case where no zero-order terms exist (i.e., no specularly reflecting surface and no targets), then (4.5) and (4.6) assume much simpler forms. In this case (from (4.5) with all zero-order terms set equal to zero)

$$<\hat{U}_{AaBb}(t,\Delta t)> = \frac{1}{2} Re\ [<U_{Aa1} U^*_{Bb1}> + <U_{Aa2} U^*_{Bb2}> + <U_{Aa1} U^*{}_{Bb3}> + <U_{Aa3} U^*_{Bb1}>], \tag{4.7}$$

and (from (4.6) with all zero-order terms set equal to zero)

$$<\Delta\hat{\hat{U}}_{AaBb}(t,\Delta t)> \simeq \left|<U_{Aa1} U^*_{Bb1}>\right|^2. \tag{4.8}$$

Applications of Averages

Let us now consider the applications of (4.5). With the aid of (4.5.1,...7)′, it can be used to obtain the following quantities of possible interest.

(1) The crosscovariance function (or crosscorrelation function; abbreviated as CCF in what follows) between vertically (or horizontally) polarized return signal voltages with possibly different incident polarizations. If the incident polarizations are the same and $\Delta t = 0$, then this quantity is the average power in the vertically (or horizontally) polarized return.

(2) The CCF between right-handed (or left-handed) circularly polarized signal voltages with possibly different incident polarizations. If the incident polarizations are the same and $\Delta t = 0$, then this quantity is the average power in the left-handed (or right-handed) circularly polarized return.

(3) The CCF between vertically and horizontally polarized signal voltages, with possibly different incident polarizations.

(4) The CCF between left and right-handed circularly polarized signal voltages with possibly different incident polarizations.

Eq. (4.6) with the aid of (4.5.2,...,7)′ can be used to determine the following quantities of possible interest:

(5) The CCF between deviations from average power (or quadratically rectified voltage) in two vertically (or horizontally) polarized return signals with possibly different incident polarizations. If the incident polarizations are the same and $\Delta t = 0$, then this quantity is the variance of the vertically (or horizontally) polarized signal power.

(6) The CCF between deviation from average power (or quadratically rectified voltage) in vertically and horizontally polarized returns with possibly different incident polarizations.

(7) The CCF between deviation from average power or quadratically rectified voltage in two left-handed (or right-handed) circularly polarized returns with possibly different incident polarizations. If the incident polarizations are the same and $\Delta t = 0$, this quantity is the variance of the left-handed (or right-handed) circularly polarized signal power.

(8) The CCF between deviation from average power (or quadratically rectified voltage) in left-handed and right-handed circularly polarized returns with possibly different incident polarizations.

Transmit-Receive Polarization Modes

There are 16 transmit-receive polarization modes, any one of which may be of interest in a particular application. These are given in the chart below.

	Receive polarization	Transmit polarization	Designation
(1)	Linear-vertical	Linear-vertical	VV
(2)	Linear-vertical	Linear-horizontal	VH
(3)	Linear-vertical	Circular-left-handed	VL
(4)	Linear-vertical	Circular-right-handed	VR
(5)	Linear-horizontal	Linear-vertical	HV
(6)	Linear-horizontal	Linear-horizontal	HH
(7)	Linear-horizontal	Circular-left-handed	HL
(8)	Linear-horizontal	Circular-right-handed	HR
(9)	Circular-left-handed	Linear-vertical	LV
(10)	Circular-left-handed	Linear-horizontal	LH
(11)	Circular-left-handed	Circular-left-handed	LL
(12)	Circular-left-handed	Circular-right-handed	LR
(13)	Circular-right-handed	Linear-vertical	RV
(14)	Circular-right-handed	Linear-horizontal	RH
(15)	Circular-right-handed	Circular-left-handed	RL
(16)	Circular-right-handed	Circular-right-handed	RR

The factors appearing in the terms of (4.5) and (4.6) are all either average voltages $<U_{A\ell}>$ or $<U_{B\ell}>$, where $\ell = 0,2$, or 4 (the case $\ell = 0$, of course, does not require ensemble averaging), or averages of products of two voltages, of the form $<U_{A\ell}U^*_{Bm}>$ or complex conjugates of such quantities. We note that in a quantity $<U_{A\ell}>$ (or $<U_{B\ell}>$), A (or B) can represent any one of the 16 polarization cases listed in the chart. In a quantity like $<U_{A\ell}U^*_{Bm}>$, A and B can represent any of the 16 cases listed in the charts; hence, there are a total of $16 \times 16 = 256$ possibilities for such a term. The ability to choose the desired case, of course, can easily be programmed into a computer. The quantity $S_{A\ell}$ appearing in (3.7.0,...,4) is calculated for $Aa = VV$, VH, HV or HH. Extension of these results to include circular polarization can be accomplished through an extension of (3.9) and the observation that the linearly polarized response to incident circular polarization is given by

$$\begin{array}{cccccc} U_{VR}(t) & = & U_{VV}(t) & + & jU_{VH}(t). & \\ VL & & VV & - & VH & \\ HR & & HV & + & HH & \\ HL & & HV & - & HH & \end{array} \tag{4.9}$$

The circularly polarized response to linearly polarized incident radiation is

$$\begin{array}{cccccc} U_{LV}(t) & = & U_{VV}(t) & + & jU_{HV}(t). & \\ LH & & VH & + & HH & \\ RV & & VV & - & HV & \\ RH & & VH & - & HH & \end{array} \tag{4.10}$$

From (4.9) or (4.10),

$$\begin{aligned} U_{LL}(t) &= U_{VL}(t) + jU_{HL}(t) = U_{LV}(t) - jU_{LH}(t) \\ &= [U_{VV}(t) + U_{HH}(t)] + j[U_{HV}(t) - U_{VH}(t)], \end{aligned} \tag{4.11.a}$$

$$\begin{aligned} U_{RR}(t) &= U_{VR}(t) - jU_{HR}(t) = U_{RV}(t) + jU_{RH}(t) \\ &= [U_{VV}(t) + U_{HH}(t)] + j[U_{VH}(t) - U_{HV}(t)], \end{aligned} \tag{4.11.b}$$

$$\begin{aligned} U_{LR}(t) &= U_{VR}(t) + jU_{HR}(t) = U_{LV}(t) + jU_{LH}(t) \\ &= [U_{VV}(t) - U_{HH}(t)] + j[U_{VH}(t) + U_{HV}(t)], \text{ and,} \end{aligned} \tag{4.11.c}$$

$$\begin{aligned} U_{RL}(t) &= U_{VL}(t) - jU_{HL}(t) = U_{RV}(t) - jU_{RH}(t) \\ &+ [U_{VV}(t) - U_{HH}(t)] - j[U_{VH}(t) + U_{HV}(t)]. \end{aligned} \tag{4.11.d}$$

Equations (4.11.a,...d) can be used to form the averages needed in (4.5) and (4.6) from the basic calculated quantities U_{VV}, U_{VH}, U_{HV}, U_{HH}.

Inside the integral in each of these quantities (see 3.7.0,...,4) is the quantity $S_{Aa\ell}$, where $Aa = VV$, VH, HV or HH. On the computer, cases such as LL, LR, etc. will be implemented by programming the statement

$$
\begin{array}{lcccccc}
S_{LL\ell} = & (S_{VV\ell} & + S_{HH\ell}) & + j(S_{HV\ell} & - & S_{VH\ell}). & \\
RR\ell & & + & - & & - & \\
LR\ell & & - & + & & + & \\
RL\ell & & - & - & & + &
\end{array}
\tag{4.12}
$$

This is the easiest way to handle cases involving circular polarization on both transmission and reception. The averages required in (4.5) and (4.6) could be (and have been) calculated explicitly for these cases but most of the expressions obtained for them are very long, are not needed in subsequent calculations, and do not lend themselves to easy physical interpretation. Hence, they will not be included in this report.

RESULTS WITH FIRST-ORDER FIELDS ONLY

In this section we will summarize and discuss the results obtained in Appendix IV for the case where only the first-order term in the polarization matrix is considered. Specular reflection terms, target returns and second, third and fourth order terms are not included.

We can obtain a certain amount of information from these results but unfortunately cannot really address the problem of cross-polarized components. However, a certain "quasi-cross-polarization" effect exists and can be studied through these results. This point will be elaborated upon later.

Polarization Matrix Element

The polarization matrix element corresponding to a [1] is calculated in Appendix IV. The result, given in Appendix IV, is

$$
\begin{array}{l}
[S_{VV}(-\hat{\underset{\to}{\alpha}}_o)]_{a(1)} \\ VH \\ HV \\ HH
\end{array}
= \frac{2j\frac{\omega}{c}\alpha_{oz}(\nu^2-1)}{[\Delta(|\hat{\underset{\to}{\alpha}}_o|)]^2(\underset{\to}{\ell}'_V \times \underset{\to}{\ell}'_H)_z}
\left\{ \begin{array}{l}(\underset{\to}{\ell}'_H x \, \underset{\to}{\ell}'_V)_z [(1-\alpha_{oz}^2)\nu^2 + \gamma_{oz}^2] \\ \quad\; V \quad\;\; V \\ \quad\; H \quad\;\; H \\ \quad\; V \quad\;\; H \end{array}\right.
$$
$$
\left. \begin{array}{l} - 2\gamma_{oz}(\alpha_{oz}+\gamma_{oz})(\underset{\to}{\alpha}_o x \, \underset{\to}{\ell}'_V)_z (\hat{\underset{\to}{\alpha}}_o \cdot \underset{\to}{\ell}'_H) \\ \qquad\qquad\qquad\qquad\quad V \qquad\qquad\;\; V \\ \qquad\qquad\qquad\qquad\quad H \qquad\qquad\;\; H \\ \qquad\qquad\qquad\qquad\quad H \qquad\qquad\;\; V \end{array}\right\} .
\tag{5.1}
$$

(Note that the first subscript V or H refers to received signal polarization and the second subscript refers to transmitted signal polarization, where V and H denote vertical and horizontal respectively.)

The quantities in (5.1) are defined below:

$$\nu = \text{complex refractive index of the medium} \tag{5.2.a}$$

$$\omega = \text{(angular) radar frequency} \tag{5.2.b}$$

$$c = \text{free space velocity of light} \tag{5.2.c}$$

$$\Delta(|\hat{\underset{\to}{\alpha}}_o|) = (\gamma_{oz} - \alpha_{oz})(1 - \alpha_{oz}^2 - \alpha_{oz}\gamma_{oz}) \tag{5.2.d}$$

$$\underset{\to}{\alpha}_o = \underset{\to}{i}_{r_3'} = \begin{bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oz} \end{bmatrix} \text{ (from I.22)} \tag{5.2.e}$$

$$\hat{\underset{\to}{\alpha}}_o = \begin{bmatrix} \alpha_{ox} \\ \\ \alpha_{oy} \end{bmatrix} \text{ (from I.27.a)} \tag{5.2.f}$$

$$\begin{aligned} \alpha_{ox} = {} & \sin\theta_3' \cos\phi_3' (\cos\delta \cos\gamma \cos\Phi - \sin\delta \sin\gamma) \\ & + \sin\theta_3' \sin\phi_3' (-\cos\delta \sin\Phi) \qquad (5.2.g.x) \\ & + \cos\theta_3' (\cos\delta \sin\gamma \cos\Phi + \sin\delta \cos\gamma); \text{ (from I.28)} \end{aligned}$$

$$\begin{aligned} \alpha_{oy} = {} & \sin\theta_3' \cos\phi_3' (-\cos\gamma \sin\Phi) + \sin\theta_3' \sin\phi_3' (-\cos\Phi) \\ & + \cos\theta_3' (-\sin\gamma \sin\Phi); \text{ (from I.28)} \end{aligned} \tag{5.2.g.y}$$

$$\begin{matrix} \alpha_{oz} \\ \gamma_{oz} \end{matrix} = \mp\sqrt{\begin{matrix} 1 \\ \nu^2 \end{matrix} - (\alpha_{ox}^2 + \alpha_{oy}^2)} \tag{5.2.g.z}$$

where α_{ox} and α_{oy} are defined in (5.2.g.x,y)

$$\gamma = \text{angle of peak of antenna beam relative to the vertical} \tag{5.2.h}$$

(θ_3', ϕ_3') = spherical angles of the illuminated point on the surface relative to the radar in the S_3' system (whose z_3' axis is along the peak of the antenna beam)

$$\cos\delta = \frac{1}{\sqrt{1 + |\nabla' H|^2}} \quad \text{(from III.15.a)} \tag{5.2.j.1}$$

$$\sin\delta = \frac{|\nabla' H|}{\sqrt{1 + |\nabla' H|^2}} \quad \text{(from III.15.b)} \tag{5.2.j.2}$$

$$\cos\Phi = \frac{H_{x'}}{|\nabla' H|} \quad \text{(from III.15.c)} \tag{5.2.j.3}$$

$$\sin\Phi = \frac{H_{y'}}{|\nabla' H|} \quad \text{(from III.15.d)} \tag{5.2.j.4}$$

where

$H = H(x', y')$ = Mean surface as a function of (x', y'), the horizontal coordinates in the S' system (whose z' axis is perfectly vertical and whose (x', y') plane is along the horizontal sea surface) (5.2.j.5)

$$H_{x'} = \frac{\partial H}{\partial x'}(x', y') \tag{5.2.j.6}$$

$$H_{y'} = \frac{\partial H}{\partial y'}(x', y') \tag{5.2.j.7}$$

$$\nabla' H = \sqrt{(H_x')^2 + (H_y')^2} \tag{5.2.j.8}$$

$$\begin{array}{llll} \ell'_{Vx} = & \ell_{Vx} - \dfrac{1}{\alpha_{oz}} & \ell_{Vz}\,\alpha_{ox} & \text{(from I.28 with } \underset{\to}{\beta} = -\underset{\to}{\hat{\alpha}}_o,\ \beta_z = -\alpha_{oz}) \\ \quad Vy & \quad Vy & \quad V \quad oy & \\ \quad Hx & \quad Hx & \quad H \quad ox & \\ \quad Hy & \quad Hy & \quad H \quad oy & \end{array} \tag{5.2.k}$$

$$\ell_{Vx} = \cos\delta\cos\gamma\cos\Phi - \sin\delta\sin\gamma \tag{5.2.ℓ.1}$$

$$\ell_{Vy} = -\cos\gamma\sin\Phi \tag{5.2.ℓ.2}$$

$$\ell_{Vz} = \sin\delta\cos\gamma\cos\Phi + \cos\delta\sin\gamma \tag{5.2.ℓ.3}$$

$$\ell_{Hx} = - \cos \delta \sin \Phi \tag{5.2.\ell.4}$$

$$\ell_{Hy} = - \cos \Phi \tag{5.2.\ell.5}$$

$$\ell_{Hz} = - \sin \delta \sin \Phi \tag{5.2.\ell.6}$$

(Eqs. (5.2.ℓ.1,...,6) are from (I.27))

The vector and scalar products in, (5.1) are as follows:

$$(\underset{\to}{\ell}'_H x \underset{\to}{\ell}'_V)_z = - (\underset{\to}{\ell}'_V x \underset{\to}{\ell}'_H)_z = \ell'_{Hx} \ell'_{Vy} - \ell'_{Hy} \ell'_{Vx} \tag{5.3.a}$$

$$(\underset{\to}{\ell}'_V x \underset{\to}{\ell}'_V)_z = (\underset{\to}{\ell}'_H x \underset{\to}{\ell}'_H)_z = 0 \tag{5.3.b}$$

$$(\underset{H}{\underset{\to}{\alpha}_o} x \underset{\to}{\ell}'_V)_z = \underset{H}{\alpha_{ox}} \ell'_{Vy} - \underset{H}{\alpha_{oy}} \ell'_{Vx} \tag{5.3.c}$$

$$\underset{H}{\underset{\to}{\hat{\alpha}}_o} \cdot \underset{\to}{\ell}'_V = \underset{H}{\alpha_{ox}} \ell'_{Vx} + \underset{H}{\alpha_{oy}} \ell'_{Vy} \tag{5.3.d}$$

In the degenerate case where the mean surface is perfectly horizontal, some of the quantities defined in (5.2.a,...ℓ) and (5.3.a,...,d) are significantly simplified. In this case

$$H(x',y') = 0. \tag{5.4}$$

Choosing to set $H_{y'}$ identically to zero and allowing $H_{x'}$ to become arbitrarily small (just an arbitrary choice of coordinate orientation which does not affect the end results), we obtain from (5.2.j.1,...,4):

$$\cos \Phi = 1 \tag{5.5.a}$$

$$\sin \Phi = 0 \tag{5.5.b}$$

$$|\nabla' H| = 0 \tag{5.5.c}$$

$$\cos \delta = 1 \tag{5.5.d}$$

$$\sin \delta = 0 \tag{5.5.e}$$

Using (5.5.a,...,e) in (5.2.g.x.y.z), (5.2.k), and (5.2.ℓ.6), we obtain

$$\alpha_{ox} = \sin \theta'_3 \cos \phi'_3 \cos \gamma + \cos \theta'_3 \sin \gamma \tag{5.6.a}$$

$$\alpha_{oy} = - \sin \theta'_3 \sin \phi'_3 \tag{5.6.b}$$

$$\begin{matrix}\alpha_{oz} \\ \gamma_{oz}\end{matrix} = \mp \sqrt{\left\{ \frac{1}{\nu^2} - [\sin^2 \theta_3' (\cos^2 \phi_3' \cos^2 \gamma + \sin^2 \phi_3') + \cos^2 \theta_3' \sin^2 \gamma \dots \right.} \tag{5.6.c}$$

$$\left. + 2 \sin \theta_3' \cos \theta_3' \cos \phi_3' \cos \gamma \sin \gamma] \right\}$$

$$\ell_{Vx} = \cos \gamma \tag{5.7.a}$$

$$\ell_{Vy} = 0 \tag{5.7.b}$$

$$\ell_{Vz} = \sin \gamma \tag{5.7.c}$$

$$\ell_{Hx} = 0 \tag{5.7.d}$$

$$\ell_{Hy} = -1 \tag{5.7.e}$$

$$\ell_{Hz} = 0 \tag{5.7.f}$$

To check our results against those previously obtained by other workers, we further specialize to the case where the surface is perfectly horizontal and illumination is in the $(x_3' - z_3')$ plane and along the peak of the beam, corresponding to the conditions

$$\begin{aligned} \theta_3' &= 0 \\ \phi_3' &= 0 \end{aligned} \tag{5.8}$$

Using (5.8) in (5.6.a,b,c), (5.2.k), (5.7.a,...,f) and (5.3.a,b,c,d), we obtain:

$$\alpha_{ox} = \sin \gamma \tag{5.9.a}$$

$$\alpha_{oy} = 0 \tag{5.9.b}$$

$$\alpha_{oz} = -\sqrt{1 - \sin^2 \gamma} = -\cos \gamma \tag{5.9.c}$$

$$\gamma_{oz} = \sqrt{\nu^2 - \sin^2 \gamma} \tag{5.9.d}$$

$$\ell_{Vx} = \cos \gamma \tag{5.9.e}$$

$$\ell_{Vy} = 0 \tag{5.9.f}$$

$$\ell_{Vz} = \sin\gamma \tag{5.9.g}$$

$$\ell_{Hx} = 0 \tag{5.9.h}$$

$$\ell_{Hy} = -1 \tag{5.9.i}$$

$$\ell_{Hz} = 0 \tag{5.9.j}$$

$$\ell'_{Vx} = \sec\gamma \tag{5.9.k}$$

$$\ell'_{Vy} = 0 \tag{5.9.ℓ}$$

$$\ell'_{Hx} = 0 \tag{5.9.m}$$

$$\ell'_{Hy} = -1 \tag{5.9.n}$$

$$(\underset{\to}{\alpha}_o x \underset{\to}{\ell}'_V)_z = 0 \tag{5.9.0}$$

$$(\underset{\to}{\hat{\alpha}}_o \cdot \underset{\to}{\ell}'_V) = \tan\gamma \tag{5.9.p}$$

$$(\underset{\to}{\alpha}_o x \underset{\to}{\ell}'_H)_z = -\sin\gamma \tag{5.9.q}$$

$$(\underset{\to}{\hat{\alpha}}_o \cdot \underset{\to}{\ell}'_H) = 0 \tag{5.9.r}$$

$$(\underset{\to}{\ell}'_H x \underset{\to}{\ell}'_V)_z = \sec\gamma \tag{5.9.s}$$

$$\begin{aligned}\Delta(|\hat{\alpha}_o|) = (\gamma_{oz} - \alpha_{oz})(1 - \alpha_{oz}{}^2 - \alpha_{oz}\,\gamma_{oz}) = [\sqrt{\nu^2 - \sin^2\gamma} + \cos\gamma]\\ [\sin^2\gamma + \cos\gamma\sqrt{\nu^2 - \sin^2\gamma}] = \nu^2\cos\gamma + \sqrt{\nu^2 - \sin^2\gamma}.\end{aligned} \tag{5.9.t}$$

Substitution of (5.9.a,...,t) and (5.3.b) into (5.1) results in:

$$[S_{VV}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} = \frac{2j\frac{\omega}{c}(\nu^2 - 1)\cos^2\gamma[(\nu^2 - 1)\sin^2\gamma + \nu^2]}{(\nu^2\cos\gamma + \sqrt{\nu^2 - \sin^2\gamma})^2} \tag{5.10.a}$$

$$[S_{VH}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} = 0 \tag{5.10.b}$$

$$[S_{HV}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} = 0 \tag{5.10.c}$$

$$[S_{HH}(-\alpha_o)]_{a(1)} = \frac{-2j\frac{\omega}{c}(\nu^2 - 1)\cos\gamma}{(\cos\gamma + \sqrt{\nu^2 - \sin^2\gamma})^2} . \tag{5.10.d}$$

The results given in (5.10.a,b,c,d) are consistent with those reported in the literature, e.g. Barrick and Peak [46]; Guinard and Daley [10]; Guinard, Ransone and Daley [14]; and Valenzuela, Laing and Daley [13]. Notation and definitions of angles differ from those used here but when the meanings of the symbols in those papers were interpreted in terms of the author's notation and definitions, the results were found to be in exact agreement with those reported previously.

Let us refer back to the definition of vertical and horizontal polarization used in this report. We define vertically polarized transmission or return (labeled V) as that whose $\vec{E}$ vector is in a given direction on the antenna aperture and horizontally polarized transmission or return (labeled H) as that whose $\vec{E}$ vector is perpendicular to the V direction. But the $\vec{E}$ vector corresponding to a plane wave of either V or H type (according to the above definition) launched at the aperture and striking the sea surface at a given point has components both vertically and horizontally polarized (according to the standard definition of vertically polarized radiation as that whose $\vec{E}$ vector is in the plane of incidence and horizontally polarized radiation as that whose $\vec{E}$ vector is normal to the plane of incidence) and these components are acted upon differently by the surface reflection. Hence the return corresponding to a V-type transmitted wave has an H-component and the return corresponding to an H-type transmitted wave has a V-component. If the return is considered as a superposition of plane waves from all angles contained within the antenna beam pattern, then each such plane wave has both V and H components regardless of whether their corresponding transmitted waves were of V or H type. The total $\vec{E}$ field on the aperture resulting from this superposition of returns has a different ratio of V-component to H-component than that resulting from the superposition of $\vec{E}$-fields of all the transmitted waves. This constitutes a kind of "quasi-cross polarization" effect which appears in the return even in first order, although it is very well known that no real cross polarization (defined in the standard way, i.e., referred to polarization at the sea surface, not at the antenna aperture) exists in first order return.

The exception to this is the case where all of the radiation goes out in a single direction, i.e., along the antenna beam axis, in which case the components of the $\vec{E}$-vector of the transmitted or returned plane wave correspond to the components of the $\vec{E}$-vector on the antenna aperture. This is the case covered by Eqs. (5.8), (5.9.a,...,t) and (5.10.a,b,c,d).

As soon as a small departure from this simple situation occurs, then the return has the feature indicated above, i.e. its "polarization" as defined at the antenna aperture is different from that of its transmitted wave.

This is true even for a perfectly horizontal mean sea surface. Of course, the wider the beam, the greater is this "quasi-cross polarization" effect, because a wider beam has much more wave energy propagated at angles far off the beam axis. With a narrow beam, e.g. only a degree or two wide, there should be very little of the effect, because at small angles off of the beam axis, the $\vec{E}$-field components defined as vertically and horizontally polarized at the antenna aperture come much closer to being the vertically and horizontally polarized components at the reflecting surface.

For any given beamwidth, "quasi-cross polarization" effect would be further enhanced by a rough mean-surface, because waves propagating out at different angles would encounter a wide range of angles of incidence at the surface and hence the probability of a vast difference between the polarization defined at the antenna and that defined at the surface would be much greater, even at angles not far off the beam axis.

To summarize this point, our concept of polarization, which defines vertical and horizontal polarization in terms of two mutually perpendicular directions of the $\underset{\rightarrow}{E}$-field on the antenna aperture, does not necessarily correspond to the standard definition of vertical and horizontal polarization, which relates to the $\underset{\rightarrow}{E}$-field in the plane of incidence or normal to the plane of incidence respectively. Suppose we think of the return as a superposition of plane waves reflected from the random surface. If the mean surface is horizontal, the only one of these waves for which these two definitions coincide is the wave along the beam axis. In the case in which the results (5.10.a,b,c,d) are obtained, that is the only wave propagating; cross polarization terms vanish. It can easily be shown that the VH and HV terms also vanish (with a perfectly horizontal mean surface) in the case where the illumination is in the $(x_3' - z_3')$ plane $(\phi_3' = 0)$. In this case, a V type wave (*our* definition) has no components that are horizontally polarized (*standard* definition) at the sea surface and an H-type wave (*our* definition) has no components vertically polarized (*standard* definition) at the sea surface. This is a degenerate case in our analysis because it covers only one plane of illumination and we wish this analysis to cover a superposition of waves from an arbitrary solid angle.

To show the vanishing of VH and HV terms in this case, we need only observe that setting ϕ_3' equal to zero in (5.6.a,b), and (5.7.a,b,d,e) and invoking (5.3.b) results in:

$$\alpha_{ox} = \sin(\theta_3' - \gamma) \tag{5.11.a}$$

$$\alpha_{oy} = 0 \tag{5.11.b}$$

$$\ell_{Vx}' = \cos\gamma - \frac{\sin\gamma \sin(\theta_3' - \gamma)}{\alpha_{oz}} \tag{5.11.c}$$

$$\ell_{Vy}' = 0 \tag{5.11.d}$$

$$\ell_{Hx}' = 0 \tag{5.11.e}$$

$$\ell_{Hy}' = -1 \tag{5.11.f}$$

$$(\underset{\rightarrow o}{\alpha} x \underset{\rightarrow V}{\ell'})_z (\underset{\rightarrow o}{\hat{\alpha}} \cdot \underset{\rightarrow V}{\ell'}) = 0 \tag{5.11.g}$$

$$(\underset{\to}{\alpha}_o x \underset{\to}{\ell}'_H)_z (\hat{\underset{\to}{\alpha}}_o \cdot \underset{\to}{\ell}'_H) = 0. \tag{5.11.h}$$

Eqs. (5.3.b) and (5.11.g,h) imply that

$$[S_{VH}(-\hat{\underset{\to}{\alpha}}_o)]_{a(1)} = [S_{HV}(-\hat{\underset{\to}{\alpha}}_o)]_{a(1)} = 0 \tag{5.12}$$

for the special case where $\phi'_3 = 0$.

Averages Involving Only First-Order Terms

The generic average of the product of two polarization matrix elements each of which contains only first-order terms is the term $<U_{Aa1} U^*_{Bb1}>$ in (4.2.b)′, which is of second order. The generic average of power (or quadratically rectified voltage) products in the case where the polarization matrix contains only first-order terms is the term $<U_{Aa1} U^*_{Aa1} U_{Bb1} U^*_{Bb1}>$ in (4.4.d)′, which is of fourth order.

Summarizing the applicable results in the previous section, we have (from (4.2.b)′)

$$<\hat{U}_{AaBb2}> = \frac{1}{2} Re(<U_{Aa1} U^*_{Bb1}> e^{j\omega_o \Delta t}) \tag{5.13}$$

and from (4.8)

$$<\Delta \hat{\hat{U}}_{AaBb}(t,\Delta t)> = \left| <U_{Aa1} U^*_{Bb1}> \right|^2, \tag{5.14}$$

where (from (4.5.5)′ with a slight notational change)

$$\begin{aligned} <U_{Aa1} U^*_{Bb1}> = (\frac{c}{2\omega})^2 \int_o^{\pi} d\theta'_3 \sin\theta'_3 \int_o^{2\pi} d\phi'_3 \, W(-\frac{2\omega}{c} \hat{\underset{\to}{\alpha}}_o (\theta'_3,\phi'_3,t))\ldots \\ F_{Aa}(\theta'_3,\phi'_3,t) F^*_{Bb}(\theta'_3,\phi'_3,t+\Delta t) \\ [S_{Aa}(-\hat{\underset{\to}{\alpha}}_o(\theta'_3,\phi'_3,t))]_{a(1)} [S_{Bb}(-\hat{\underset{\to}{\alpha}}_o(\theta'_3,\phi'_3,t+\Delta t))]^*_{a(1)}. \end{aligned} \tag{5.15}$$

Here F_{Aa} and F_{Bb} are given by (3.5); the various constituents of F_{Aa} and F_{Bb} are in turn obtained from (3.6) and (3.6.a,b,c,d)′. The quantities $[S_{Aa}]$ and $[S_{Bb}]$ are of course the polarization matrix elements (5.1), or combinations thereof. The pairs of subscripts (Aa) or (Bb) on the $[S]$'s may be (VV), (VH), (HV) or (HH) if attention is confined to linearly polarized transmission and reception. They may also denote (LL), (LR), (RL) or

(RR) if circularly polarized transmission and reception is of interest. In the latter case, we would invoke (4.12), noting that

$$\begin{aligned}
[S_{\substack{LL\\RR\\LR\\RL}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} &= ([S_{VV}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} + [S_{HH}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)}) \cdots \\
&\substack{+\\-\\+\\-}\, j([S_{HV}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} \substack{-\\-\\+\\+} [S_{VH}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)}).
\end{aligned} \tag{5.16}$$

If linearly polarized response to circularly polarized transmissions is of interest, then from (4.9), we can write

$$[S_{\substack{VR\\VL\\HR\\HL}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} = [S_{\substack{VV\\VV\\HV\\HV}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} \substack{+\\-\\+\\-}\, j[S_{\substack{VH\\VH\\HH\\HH}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)}. \tag{5.17}$$

Finally if we wish to study circularly polarized response to incident linear polarization, we can use (4.10) to obtain

$$[S_{\substack{LV\\LH\\RV\\RH}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} = [S_{\substack{VV\\VH\\VV\\VH}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)} \substack{+\\+\\-\\-}\, j[S_{\substack{HV\\HH\\HV\\HH}}(-\underset{\to}{\hat{\alpha}}_o)]_{a(1)}. \tag{5.18}$$

The subscripts (Aa) and (Bb) can each denote any of the possibilities covered by (5.16), (5.17) and (5.18) as well as the standard cases involving only linear polarization in both transmission and reception.

LISTING OF SELECTED RESULTS
(in order of appearance)

1. General matrix expressions for fields of zero-order; Eqs. (2.16.0), (2.16.0)′, p. 19.
2. General matrix expression for fields of first order; Eq. (2.16.1), p. 19; second order; Eq. (2.16.2), p. 20 and n^{th} order; Eq. (2.16.n), pp. 20-22.
3. General forms for radar return signals of first, second, third and fourth orders; Eqs. 3.7.0, 3.7.1, 3.7.2, 3.7.3, 3.7.4, pp. 25-26. (Definitions of quantities used; Eqs. (3.5), (3.6), (3.6-a,b,c,d), pp. 24-25.)
4. Averages of voltage products (in general); Eqs. (4.2), (4.2.a,b,c,d)′, p. 28.
5. Averages of power (or quadratically rectified voltage) products (in general); Eqs. (4.3), (4.4), (4.4.a,b,c,d,e)′, (4.5), (4.5.1,2,3,4,5,6,7)′, (4.6), (4.7), (4.8), pp. 29-33.
6. List of quantities calculable from the results obtained in report, pp. 33-34.
7. Table of transmit-receive polarization modes (in general), p. 34.
8. General forms of radar returns with circular polarization on transmission and linear polarization on reception or vice versa; Eqs. (4.9), (4.10), p. 35.
9. General forms of radar returns with circular polarization on both transmission and reception; Eqs. (4.11.a,b,c,d), p. 35.
10. First order polarization matrix for radar return with linear polarization on transmission and reception, (VV, VH, HV, HH) with arbitrary form for mean sea surface; Eqs. (5.1), (5.2.a,...ℓ.6), (5.3.a,b,c,d), pp. 37-39.
11. Specialization of first-order polarization matrix to case of perfectly flat sea surface; Eqs. (5.4), (5.5,a,...e), (5.6.a,b,c), (5.7.a,...f), pp. 39-40.
12. Specialization of first-order polarization matrix in terms of polarization with respect to the sea surface; sea surface assumed perfectly flat; Eqs. (5.8) (5.9.a,...†), (5.10.a,...d), pp. 40-42.
13. Generic averages of products of two first-order polarization matrix elements; Eqs. 5.13, 5.14, 5.15, p. 44.
14. Results on first-order received signal fields for arbitrary scattering angles (applicable to bistatic radar or communication systems); Eq. IV-17, p. 84 with supporting calculation detail contained within Eqs. (IV-10) through (IV-16)′, pp. 79-84.

Appendix I

COORDINATE SYSTEMS

The basic coordinate system used in solving the boundary value problem is a right-handed rectangular system $S = (x,y,z)$, with basis vectors $(\underset{\to}{i}_x, \underset{\to}{i}_y, \underset{\to}{i}_z)$. The system's (x,y) plane is tangent to the tilted mean surface at its origin of coordinates, denoted by 0. We define another coordinate system $S' = (x',y',z')$ with basis vectors $(\underset{\to}{i}_x', \underset{\to}{i}_y', \underset{\to}{i}_z')$, its (x', y') plane along the horizontal "perfectly calm" sea surface, its origin at an arbitrary point $0'$ on that surface, and its z' axis vertically upward. We are interested in the transformation between these two systems.

Given that the origin 0 of the (x,y,z) system is located at a point (x_0', y_0', z_0') in the (x', y', z') system (see Fig. I-1), we can construct an intermediate system $S'' = (x'', y'', z'')$ with basis vectors $(\underset{\to}{i}_x'', \underset{\to}{i}_y'', \underset{\to}{i}_z'')$, whose coordinates are parallel to those of system S' and whose origin is at 0. A point with coordinates (x', y', z') in system S' will have coordinates $(x' - x_0', y' - y_0', z' - z_0')$ in system S'', i.e.,

$$\begin{aligned} x'' &= x' - x_0', \\ y'' &= y' - y_0', \text{ and} \\ z'' &= z' - z_0'. \end{aligned} \tag{I.1}$$

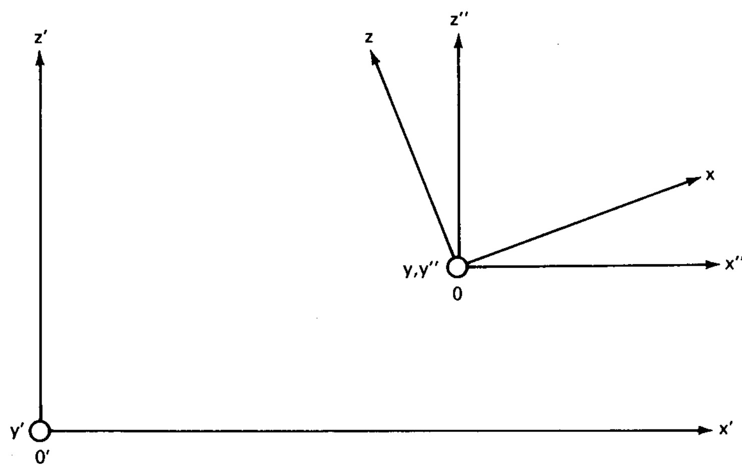

Fig. I-1 — Systems S, S', S''

Referring to Fig. I-2, we construct a second intermediate system $S''' = (x''', y''', z''')$ with origin at 0, whose z''' axis is parallel to the z'' axis of system S'' and that is rotated by an angle Φ with respect to S'' such that the x''' axis is in the direction of the local tilt, i.e., by rotation in the (x''', z''') plane we will be able to transform to the system S. The transformation between S''' and S'' is (from Fig. I-2)

$$\begin{aligned} x''' &= x'' \cos \Phi + y'' \sin \Phi, \\ y''' &= -x'' \sin \Phi + y'' \cos \Phi, \text{ and} \\ z''' &= z''. \end{aligned} \tag{I.2}$$

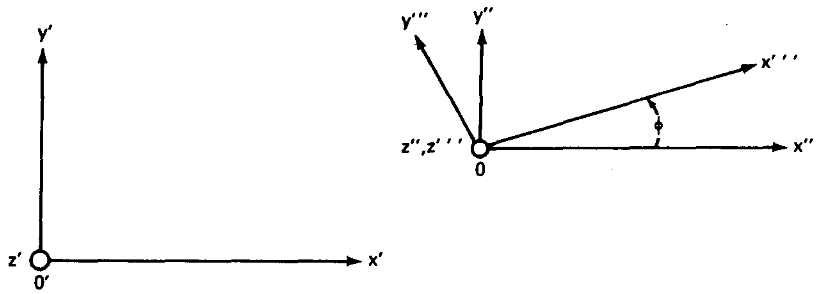

Fig. I-2 — Systems S', S'', S'''

To transform from S''' to S, we now tilt along the (x''', z''') plane (see Fig. I-3) through an angle δ, leading to

$$\begin{aligned} x &= x''' \cos \delta - z''' \sin \delta, \\ y &= y''', \text{ and} \\ z &= x''' \sin \delta + z''' \cos \delta. \end{aligned} \tag{I.3}$$

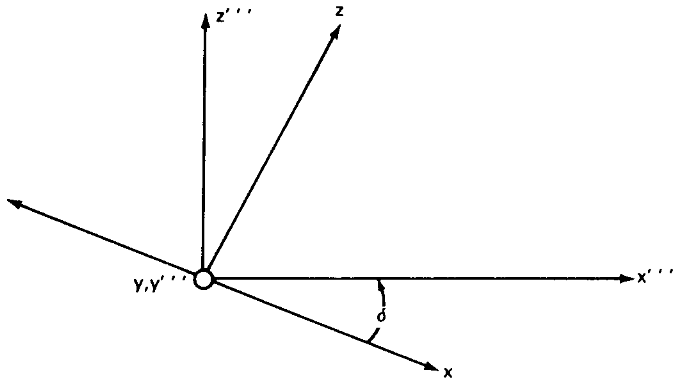

Fig. I-3 — Systems S, S'''

To obtain the transformation between S' and S, we first use (I.2) and (I.3) to express (x,y,z) in terms of (x'', y'', z'') resulting in

$$
\begin{aligned}
x &= x'' \cos\delta \cos\Phi + y'' \cos\delta \sin\Phi - z'' \sin\delta,\\
y &= -x'' \sin\Phi + y'' \cos\Phi, \text{ and}\\
z &= x'' \sin\delta \cos\Phi + y'' \sin\delta \sin\Phi + z'' \cos\delta.
\end{aligned}
\tag{I.4}
$$

We then use (I.1) and (I.4) to obtain (x, y, z) in terms of (x', y', z'), as follows:

$$
\begin{aligned}
x &= (x' - x_0') \cos\delta \cos\Phi + (y' - y_0') \cos\delta \sin\Phi - (z' - z_0') \sin\delta,\\
y &= -(x' - x_0') \sin\Phi + (y' - y_0') \cos\Phi, \text{ and}\\
z &= (x' - x_0') \sin\delta \cos\Phi + (y' - y_0') \sin\delta \sin\Phi + (z' - z_0') \cos\delta.
\end{aligned}
\tag{I.5}
$$

Inversion of (I.5) is straightforward and leads to

$$
\begin{aligned}
x' &= x_0' + x \cos\delta \cos\Phi - y \sin\Phi + z \sin\delta \cos\Phi,\\
y' &= y_0' + x \cos\delta \sin\Phi + y \cos\Phi + z \sin\delta \sin\Phi, \text{ and}\\
z' &= z_0' - x \sin\delta + z \cos\delta.
\end{aligned}
\tag{I.6}
$$

It is important to have the transformation between the sets of basis vectors $(\underset{\rightarrow}{i}_x, \underset{\rightarrow}{i}_y, \underset{\rightarrow}{i}_z)$ and $(\underset{\rightarrow}{i}_x{}', \underset{\rightarrow}{i}_y{}', \underset{\rightarrow}{i}_x{}')$. These are obtained by writing

$$
\underset{\rightarrow}{r} = \underset{\rightarrow}{i}_x x + \underset{\rightarrow}{i}_y y + \underset{\rightarrow}{i}_z z = \underset{\rightarrow}{i}_x{}'(x' - x_0') + \underset{\rightarrow}{i}_y{}'(y' - y_0') + \underset{\rightarrow}{i}_z{}'(z' - z_0')
\tag{I.7}
$$

and then using (I.5) or (I.6) to solve for $(\underset{\to}{i}_x, \underset{\to}{i}_y, \underset{\to}{i}_z)$ in terms of $(\underset{\to}{i}_x', \underset{\to}{i}_y', \underset{\to}{i}_z')$, or vice versa. The results are

$$\begin{aligned}
\underset{\to}{i}_x &= \underset{\to}{i}_x' \cos\delta \cos\Phi + \underset{\to}{i}_y' \cos\delta \sin\Phi - \underset{\to}{i}_z' \sin\delta, \\
\underset{\to}{i}_y &= -\underset{\to}{i}_x' \sin\Phi + \underset{\to}{i}_y' \cos\Phi, \text{ and} \\
\underset{\to}{i}_z &= \underset{\to}{i}_x' \sin\delta \cos\Phi + \underset{\to}{i}_y' \sin\delta \sin\Phi + \underset{\to}{i}_z' \cos\delta,
\end{aligned} \tag{I.8}$$

or the inverse of (I.8),

$$\begin{aligned}
\underset{\to}{i}_x' &= \underset{\to}{i}_x \cos\delta \cos\Phi - \underset{\to}{i}_y \sin\Phi + \underset{\to}{i}_z \sin\delta \cos\Phi, \\
\underset{\to}{i}_y' &= \underset{\to}{i}_x \cos\delta \sin\Phi + \underset{\to}{i}_y \cos\Phi + \underset{\to}{i}_z \sin\delta \sin\Phi, \text{ and} \\
\underset{\to}{i}_z' &= -\underset{\to}{i}_x \sin\delta + \underset{\to}{i}_z \cos\delta
\end{aligned} \tag{I.9}$$

We now construct another coordinate system $S_2' = (x_2', y_2', z_2')$, whose (x_2', y_2') plane is parallel to the (x', y') plane of system S', whose origin is at the radar, and whose z_2' axis is vertically *downward*. (See Fig. I.4.)

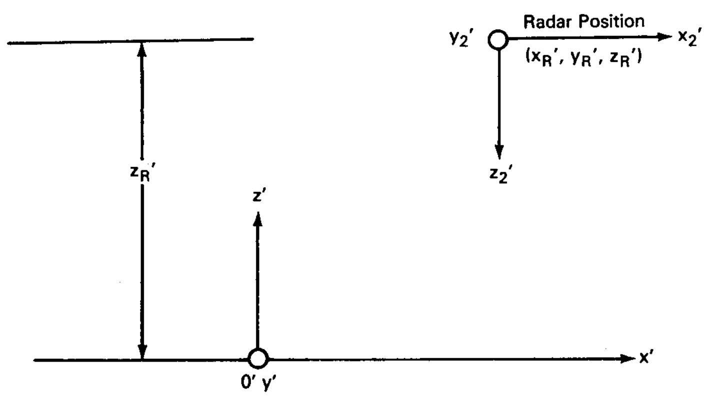

Fig. I-4 — Systems S', S_2'

The transformation between S_2' and S' is

$$\begin{aligned}
x_2' &= -x_R' + x', \\
y_2' &= y_R' - y', \text{ and} \\
z_2' &= z_R' - z',
\end{aligned} \tag{I.10}$$

where (x_R', y_R', z_R') are the radar coordinates in S'.

Inverting (I.10) we have

$$
\begin{aligned}
x' &= x_2' + x_R', \\
y' &= -y_2' + y_R', \text{ and} \\
z' &= -z_2' + z_R'.
\end{aligned}
\tag{I.11}
$$

From (I.11), (I.5) and (I.6);

$$
\begin{aligned}
x &= (x_2' + x_R' - x_o') \cos\delta \cos\Phi - (y_2' - y_R' + y_0') \cos\delta \sin\Phi \\
&\quad + (z_2' - z_R' + z_0') \sin\delta, \\
y &= -(x_2' + x_R' - x_0') \sin\Phi - (y_2' - y_R' + y_0') \cos\Phi, \text{ and} \\
z &= (x_2' + x_R' - x_0') \sin\delta \cos\Phi - (y_2' - y_R' + y_0') \sin\delta \sin\Phi \\
&\quad - (z_2' - z_R' + z_0') \cos\delta,
\end{aligned}
\tag{I.12}
$$

or

$$
\begin{aligned}
x_2' &= (-x_R' + x_0') + x \cos\delta \cos\Phi - y \sin\Phi + z \sin\delta \cos\Phi, \\
y_2' &= (y_R' - y_0') - x \cos\delta \sin\Phi - y \cos\Phi - z \sin\delta \sin\Phi, \text{ and} \\
z_2' &= (z_R' - z_0') + x \sin\delta - z \cos\delta.
\end{aligned}
\tag{I.13}
$$

It is evident that

$$
\begin{aligned}
\underset{\rightarrow}{i}_{x_2}' &= \underset{\rightarrow}{i}_x', \\
\underset{\rightarrow}{i}_{y_2}' &= -\underset{\rightarrow}{i}_y', \text{ and} \\
\underset{\rightarrow}{i}_{z_2}' &= -\underset{\rightarrow}{i}_z'.
\end{aligned}
\tag{I.14}
$$

Let us now construct another coordinate system (x_3', y_3', z_3'). The origin is at the radar and the z_3' axis points in the direction of the antenna beam. (See Fig. I-5.) We can construct this system by a rotation about the y_2' axis. The coordinate transformation is

$$
\begin{aligned}
x_3' &= x_2' \cos\gamma - z_2' \sin\gamma, \\
y_3' &= y_2', \text{ and} \\
z_3' &= x_2' \sin\gamma + z_2' \cos\gamma,
\end{aligned}
\tag{I.15.a}
$$

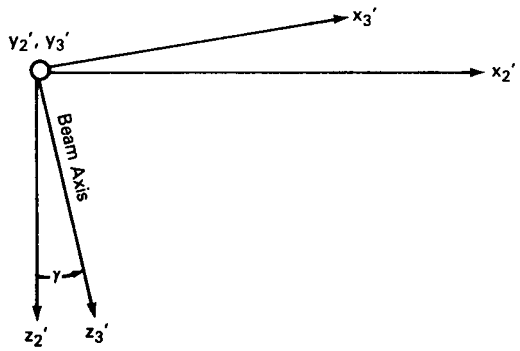

Fig. I-5 — Systems S_2', S_3'

or

$$x_2' = x_3' \cos\gamma + z_3' \sin\gamma,$$
$$y_2' = y_3', \text{ and} \qquad \text{(I.15.b)}$$
$$z_2' = -x_3' \sin\gamma + z_3' \cos\gamma,$$

and for the unit basis vectors

$$\underset{\rightarrow}{i}_{x_3}' = \underset{\rightarrow}{i}_{x_2}' \cos\gamma - i_{z_2}' \sin\gamma,$$
$$\underset{\rightarrow}{i}_{y_3}' = \underset{\rightarrow}{i}_{y_2}', \text{ and} \qquad \text{(I.16.a)}$$
$$\underset{\rightarrow}{i}_{z_3}' = \underset{\rightarrow}{i}_{x_2}' \sin\gamma + i_{z_2}' \cos\gamma,$$

or

$$\underset{\rightarrow}{i}_{x_2}' = \underset{\rightarrow}{i}_{x_3}' \cos\gamma + \underset{\rightarrow}{i}_{z_3}' \sin\gamma,$$
$$\underset{\rightarrow}{i}_{y_2}' = \underset{\rightarrow}{i}_{y_3}', \text{ and} \qquad \text{(I.16.b)}$$
$$\underset{\rightarrow}{i}_{z_2}' = -\underset{\rightarrow}{i}_{x_3}' \sin\gamma + \underset{\rightarrow}{i}_{z_3}' \cos\gamma.$$

From (I.16.a,b), (I.14), and (I.8) or (I.9) we obtain

$$\underset{\rightarrow}{i}_x = \underset{\rightarrow}{i}_{x_3}{}' (\cos\delta \cos\gamma \cos\Phi - \sin\delta \sin\gamma) + \underset{\rightarrow}{i}_{y_3}{}' (-\cos\delta \sin\Phi)$$
$$+ \underset{\rightarrow}{i}_{z_3}{}' (\cos\delta \sin\gamma \cos\Phi + \sin\delta \cos\gamma)$$
$$\underset{\rightarrow}{i}_y = \underset{\rightarrow}{i}_{x_3}{}' (-\cos\gamma \sin\Phi) + \underset{\rightarrow}{i}_{y_3}{}' (-\cos\Phi) + \underset{\rightarrow}{i}_{z_3}{}' (-\sin\gamma \sin\Phi), \text{ and} \tag{I.17.a}$$
$$\underset{\rightarrow}{i}_z = \underset{\rightarrow}{i}_{x_3}{}' (\sin\delta \cos\gamma \cos\Phi + \cos\delta \sin\gamma) + \underset{\rightarrow}{i}_{y_3}{}' (-\sin\delta \sin\Phi)$$
$$+ \underset{\rightarrow}{i}_{z_3}{}' (\sin\delta \sin\gamma \cos\Phi - \cos\delta \cos\gamma),$$

or

$$\underset{\rightarrow}{i}_{x_3} = \underset{\rightarrow}{i}_x (\cos\delta \cos\gamma \cos\Phi - \sin\delta \sin\gamma) + \underset{\rightarrow}{i}_y (-\cos\gamma \sin\Phi)$$
$$+ \underset{\rightarrow}{i}_z (\sin\delta \cos\gamma \cos\Phi + \cos\delta \sin\gamma),$$
$$\underset{\rightarrow}{i}_{y_3}{}' = \underset{\rightarrow}{i}_x (-\cos\delta \sin\Phi) + \underset{\rightarrow}{i}_y (-\cos\Phi) + \underset{\rightarrow}{i}_z (-\sin\delta \sin\Phi), \text{ and} \tag{I.17.b}$$
$$\underset{\rightarrow}{i}_{z_3}{}' = \underset{\rightarrow}{i}_x (\cos\delta \sin\gamma \cos\Phi + \sin\delta \cos\gamma) + \underset{\rightarrow}{i}_y (-\sin\gamma \sin\Phi)$$
$$+ \underset{\rightarrow}{i}_z (\sin\delta \sin\gamma \cos\Phi - \cos\delta \cos\gamma).$$

From (I.17.a,b) we can obtain the relationship between the (x,y,z) components and (x_3', y_3', z_3') components of an arbitrary vector. We begin with the observation that

$$\underset{\rightarrow}{a} = a_x \underset{\rightarrow}{i}_x + a_y \underset{\rightarrow}{i}_y + a_z \underset{\rightarrow}{i}_z = a_{x_3}{}' \underset{\rightarrow}{i}_{x_3}{}' + a_{y_3}{}' \underset{\rightarrow}{i}_{y_3}{}' + a_{z_3}{}' \underset{\rightarrow}{i}_{z_3}{}'. \tag{I.18}$$

Substituting (I.17.a) into the LHS of (I.18), we obtain the matrix equation

$$\begin{bmatrix} a_{x_3}{}' \\ a_{y_3}{}' \\ a_{z_3}{}' \end{bmatrix} = \begin{bmatrix} (\cos\delta \cos\gamma \cos\Phi - \sin\delta \sin\gamma) & (-\cos\gamma \sin\Phi) & (\sin\delta \cos\gamma \cos\Phi + \cos\delta \sin\gamma) \\ (-\cos\delta \sin\Phi) & (-\cos\Phi) & (-\sin\delta \sin\Phi) \\ (\cos\delta \sin\gamma \cos\Phi + \sin\delta \cos\gamma) & (-\sin\gamma \sin\Phi) & (\sin\delta \sin\gamma \cos\Phi - \cos\delta \cos\gamma) \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}. \tag{I.19.a}$$

Substituting (I.17.b) into the RHS of (I.18) or equivalently inverting the 3 X 3 matrix in (I.19.a), we obtain

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} (\cos\delta\cos\gamma\cos\Phi - \sin\delta\sin\gamma) & (-\cos\delta\sin\Phi) & (\cos\delta\sin\gamma\cos\Phi + \sin\delta\cos\gamma) \\ (-\cos\gamma\sin\Phi) & (-\cos\Phi) & (-\sin\gamma\sin\Phi) \\ (\sin\delta\cos\gamma\cos\Phi + \cos\delta\sin\gamma) & (-\sin\delta\sin\Phi) & (\sin\delta\sin\gamma\cos\Phi - \cos\delta\cos\gamma) \end{bmatrix} \begin{bmatrix} a_{x_3}' \\ a_{y_3}' \\ a_{z_3}' \end{bmatrix} . \quad \text{(I.19.b)}$$

It is of interest to have the transformations between the (x,y,z) system and the (x_3', y_3', z_3') system. To develop these transformations, we first invoke (I.12) and (I.15.b) to obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos\gamma\cos\delta\cos\Phi - \sin\gamma\sin\delta) & (-\cos\delta\sin\Phi) & (\sin\gamma\cos\delta\cos\Phi + \cos\gamma\sin\delta) \\ (-\cos\gamma\sin\Phi) & (-\cos\Phi) & (-\sin\gamma\sin\Phi) \\ (\cos\gamma\sin\delta\cos\Phi + \sin\gamma\cos\delta) & (-\sin\delta\sin\Phi) & (\sin\gamma\sin\delta\cos\Phi - \cos\gamma\cos\delta) \end{bmatrix} \begin{bmatrix} x_3' \\ y_3' \\ z_3' \end{bmatrix} \quad \text{(I.20.a)}$$

$$+ \begin{bmatrix} (\cos\delta\cos\Phi) & (\cos\delta\sin\Phi) & (-\sin\delta) \\ (-\sin\Phi) & (\cos\Phi) & 0 \\ (\sin\delta\cos\Phi) & (\sin\delta\sin\Phi) & (\cos\delta) \end{bmatrix} \begin{bmatrix} (x_R' - x_0') \\ (y_R' - y_0') \\ (z_R' - z_0') \end{bmatrix} .$$

Then from (I.15.a) and I.13), we obtain

$$\begin{bmatrix} x_3' \\ y_3' \\ z_3' \end{bmatrix} = \begin{bmatrix} (\cos\gamma\cos\delta\cos\Phi - \sin\gamma\sin\delta) & (-\cos\gamma\sin\Phi & (\cos\gamma\sin\delta\cos\Phi + \sin\gamma\cos\delta) \\ (-\cos\delta\sin\Phi) & (-\cos\Phi) & (-\sin\delta\sin\Phi) \\ (\sin\delta\cos\delta\cos\Phi + \cos\gamma\sin\delta) & (-\sin\gamma\sin\Phi) & (\sin\gamma\sin\delta\cos\Phi - \cos\gamma\cos\delta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{(I.20.b)}$$

$$+ \begin{bmatrix} -\cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} (x_R' - x_0') \\ (y_R' - y_0') \\ (z_R' - z_0') \end{bmatrix}$$

The unit basis vectors along the spherical coordinate axes in the S_3' system are given by

$$\begin{aligned}
\underset{\rightarrow}{i}_{r_3}' &= \underset{\rightarrow}{i}_{x_3}' \sin\theta_3' \cos\phi_3' + \underset{\rightarrow}{i}_{y_3}' \sin\theta_3' \sin\phi_3' + \underset{\rightarrow}{i}_{z_3}' \cos\theta_3', \\
\underset{\rightarrow}{i}_{\theta_3}' &= \underset{\rightarrow}{i}_{x_3}' \cos\theta_3' \cos\phi_3' + \underset{\rightarrow}{i}_{y_3}' \cos\theta_3' \sin\phi_3' - \underset{\rightarrow}{i}_{z_3}' \sin\theta_3', \text{ and} \\
\underset{\rightarrow}{i}_{\phi_3}' &= -\underset{\rightarrow}{i}_{x_3}' \sin\phi_3' + \underset{\rightarrow}{i}_{y_3}' \cos\phi_3'.
\end{aligned} \tag{I.21}$$

Substituting (I.17.b) into (I.21) we have

$$\begin{aligned}
\underset{\rightarrow}{i}_{r_3}' = \underset{\rightarrow}{i}_x &[\sin\theta_3' \cos\phi_3' (\cos\delta \cos\gamma \cos\Phi - \sin\delta \sin\gamma) \\
&+ \sin\theta_3' \sin\phi_3' (-\cos\delta \sin\Phi) + \cos\theta_3' (\cos\delta \sin\gamma \cos\Phi \\
&+ \sin\delta \cos\gamma)] + \underset{\rightarrow}{i}_y [\sin\theta_3' \cos\phi_3' (-\cos\gamma \sin\Phi) \\
&+ \sin\theta_3' \sin\phi_3' (-\cos\Phi) + \cos\theta_3' (-\sin\gamma \sin\Phi)] \\
&+ \underset{\rightarrow}{i}_z [\sin\theta_3' \cos\phi_3' (\sin\delta \cos\gamma \cos\Phi + \cos\delta \sin\gamma) \\
&+ \sin\theta_3' \sin\phi_3' (-\sin\delta \sin\Phi) \\
&+ \cos\theta_3' (\sin\delta \sin\gamma \cos\Phi - \cos\delta \cos\gamma)].
\end{aligned} \tag{I.22}$$

Defining x_3' and y_3' as the electric field directions for vertically and horizontally polarized waves respectively, and denoting x_3' and y_3' components with subscripts V and H respectively, we can write

$$\begin{aligned}
\underset{r}{\widetilde{E}_{iV}}(\underset{\rightarrow}{\beta},\omega) &= \underset{\rightarrow}{\ell}_V^T \underset{r}{\underset{\rightarrow}{\widetilde{E}_i}}(\underset{\rightarrow}{\beta},\omega) = \underset{\rightarrow}{\ell}_V \cdot \underset{r}{\underset{\rightarrow}{\widetilde{E}_i}}, \text{ and} \\
\underset{r}{\widetilde{E}_{iH}}(\underset{\rightarrow}{\beta},\omega) &= \underset{\rightarrow}{\ell}_H^T \underset{r}{\underset{\rightarrow}{\widetilde{E}_i}}(\underset{\rightarrow}{\beta},\omega) = \underset{\rightarrow}{\ell}_H \cdot \underset{r}{\widetilde{E}_i}
\end{aligned} \tag{I.23}$$

(where the scalar products on the left are in vector-matrix notation and those on the right are in standard vector notation) where (from (I.19.a))

$$\underset{\rightarrow}{\ell}_V = \begin{bmatrix} \ell_{Vx} \\ \ell_{Vy} \\ \ell_{Vz} \end{bmatrix} = \begin{bmatrix} (\cos\delta \cos\gamma \cos\Phi - \sin\delta \sin\gamma) \\ (-\cos\gamma \sin\Phi) \\ (\sin\delta \cos\gamma \cos\Phi + \cos\delta \sin\gamma) \end{bmatrix}, \text{ and} \tag{I.24.a}$$

$$\underset{\rightarrow}{\ell}_H = \begin{bmatrix} \ell_{Hx} \\ \ell_{Hy} \\ \ell_{Hz} \end{bmatrix} = \begin{bmatrix} (-\cos\delta\sin\Phi) \\ (-\cos\Phi) \\ (-\sin\delta\sin\Phi) \end{bmatrix} . \tag{I.24.b}$$

Using (2.8) in (I.23), we obtain the matrix equation

$$\begin{bmatrix} \tilde{E}_{\substack{iV\\ r}} \\ \tilde{E}_{\substack{iH\\ r}} \end{bmatrix} = \begin{bmatrix} (\ell_{Vx} \pm \frac{\beta_x}{\beta_z}\ell_{Vz}) & (\ell_{Vy} \pm \frac{\beta_y}{\beta_z}\ell_{Vz}) \\ (\ell_{Hx} \pm \frac{\beta_x}{\beta_z}\ell_{Hz}) & (\ell_{Hy} \pm \frac{\beta_y}{\beta_z}\ell_{Hz}) \end{bmatrix} \begin{bmatrix} \tilde{E}_{\substack{ix\\ r}} \\ \tilde{E}_{\substack{iy\\ r}} \end{bmatrix} , \tag{I.25}$$

where $\ell_{V\substack{x\\y\\z}}$ and $\ell_{H\substack{x\\y\\z}}$ are given in (I.24.a,b).

Inversion of (I.25) results in

$$\begin{bmatrix} \tilde{E}_{\substack{i_x\\ r}} \\ \tilde{E}_{\substack{iy\\ r}} \end{bmatrix} = \frac{1}{[(\ell_{Vx} \pm \frac{\beta_x}{\beta_z}\ell_{Vz})(\ell_{Hy} \pm \frac{\beta_y}{\beta_z}\ell_{Hz}) - (\ell_{Hx} \pm \frac{\beta_x}{\beta_z}\ell_{Hz})(\ell_{Vy} \pm \frac{\beta_y}{\beta_z}\ell_{Vz})]}$$

$$\begin{bmatrix} (\ell_{Hy} \pm \frac{\beta_y}{\beta_z}\ell_{Hz}) & -(\ell_{Vy} \pm \frac{\beta_y}{\beta_z}\ell_{Vz}) \\ -(\ell_{Hx} \pm \frac{\beta_x}{\beta_z}\ell_{Hz}) & (\ell_{Vx} \pm \frac{\beta_x}{\beta_z}\ell_{Vz}) \end{bmatrix} \begin{bmatrix} \tilde{E}_{\substack{iV\\ r}} \\ \tilde{E}_{\substack{iH\\ r}} \end{bmatrix} . \tag{I.26}$$

The matrix $\underline{\ell}$ introduced in (2.18.a,b) is (from (I.24.a,b))

$$\underline{\ell} = \begin{bmatrix} \ell_{Vx} & \ell_{Hx} \\ \ell_{Vy} & \ell_{Hy} \\ \ell_{Vz} & \ell_{Hz} \end{bmatrix} = \begin{bmatrix} (\cos\delta\cos\gamma\cos\Phi - \sin\delta\sin\gamma) & (-\cos\delta\sin\Phi) \\ (-\cos\gamma\sin\Phi) & (-\cos\Phi) \\ (\sin\delta\cos\gamma\cos\Phi + \cos\delta\sin\gamma) & (-\sin\delta\sin\Phi) \end{bmatrix}. \tag{I.27}$$

The matrix $L(\underset{\rightarrow}{\alpha}_o)$ appearing in (2.31. 0,1,...,4) is obtained from $L(\underset{\rightarrow}{\beta})$ as given in (2.19) which is equivalent to the 2×2 matrix in (I.26), where $\underset{\rightarrow}{\beta} = \underset{\rightarrow}{\hat{\alpha}}_o$ and $\underset{\rightarrow}{\hat{\alpha}}_o$ is obtained from $\underset{\rightarrow}{i}_{r_3}{}'$ (given by (I.22)), i.e.,

$$\underset{\rightarrow}{\hat{\alpha}}_o = \begin{bmatrix} \alpha_{ox} \\ \alpha_{oy} \end{bmatrix}, \tag{I.27.a}$$

where

$$\begin{aligned} \alpha_{ox} &= \sin\theta_{3S}{}'\cos\phi_{3S}{}'(\cos\delta\cos\gamma\cos\Phi - \sin\delta\sin\gamma) \\ &+ \sin\theta_{3S}{}'\sin\phi_{3S}{}'(-\cos\delta\sin\Phi) \\ &+ \cos\theta_{3S}{}'(\cos\delta\sin\gamma\cos\Phi + \sin\delta\cos\gamma), \text{ and} \\ \alpha_{oy} &= \sin\theta_{3S}{}'\cos\phi_{3S}{}'(-\cos\gamma\sin\Phi) \\ &+ \sin\theta_{3S}{}'\sin\phi_{3S}{}'(-\cos\Phi) + \cos\theta_{3S}{}'(-\sin\gamma\sin\Phi), \end{aligned} \tag{I.28}$$

and where subscript S' refers to the illuminated point on the mean surface.

For some purposes (e.g., characterizing target position), it is desirable to have direct transformations between the S' and $S_3{}'$ systems, which do not involve the angles δ or Φ. To develop these transformations, we first invoke (I.10) and (I.15.a), from which we obtain

$$\begin{aligned} x_3{}' &= (x' - x_R{}')\cos\gamma + (z' - z_R{}')\sin\gamma, \\ y_3{}' &= -(y' - y_R{}'), \text{ and} \\ z_3{}' &= (x' - x_R{}')\sin\gamma - (z' - z_R{}')\cos\gamma. \end{aligned} \tag{I.29.a}$$

From (I.11) and (I.15.b),

$$x' = x_R' + x_3' \cos\gamma + z_3' \sin\gamma,$$
$$y' = y_R' - y_3', \text{ and} \tag{I.29.b}$$
$$z' = z_R' + x_3' \sin\gamma - z_3' \cos\gamma.$$

The transformations of unit basis vectors are obtained from (I.14) and (I.16.a,b) and are as follows:

$$\underset{\to}{i}_x' = \underset{\to}{i}_{x_3}' \cos\gamma + \underset{\to}{i}_{z_3}' \sin\gamma,$$
$$\underset{\to}{i}_y' = -\underset{\to}{i}_{y_3}', \tag{I.30.a}$$
$$\underset{\to}{i}_z' = \underset{\to}{i}_{x_3}' \sin\gamma - \underset{\to}{i}_{z_3}' \cos\gamma,$$

and

$$\underset{\to}{i}_{x_3}' = \underset{\to}{i}_x' \cos\gamma + i_z' \sin\gamma,$$
$$\underset{\to}{i}_{y_3}' = -\underset{\to}{i}_y', \text{ and} \tag{I.30.b}$$
$$\underset{\to}{i}_{z_3}' = \underset{\to}{i}_x' \sin\gamma - \underset{\to}{i}_z' \cos\gamma.$$

Transformations between the spherical coordinates of the S_3' system and the rectangular coordinates of the S' system are obtained from (I.29.a,b) through the standard rectangular-to-spherical relationships. From (I.29.b)

$$x' = x_R' + r_3' (\sin\theta_3' \cos\phi_3' \cos\gamma + \cos\theta_3' \sin\gamma),$$
$$y' = y_R' - r_3' \sin\theta_3' \sin\phi_3', \tag{I.31}$$
$$z' = z_R' + r_3' (\sin\theta_3' \cos\phi_3' \sin\gamma - \cos\theta_3' \cos\gamma),$$

and from (I.29.a)

$$r_3' \sin\theta_3' \cos\phi_3' = (x' - x_r') \cos\gamma + (z' - z_R') \sin\gamma, \tag{I.32.a}$$

$$r_3' \sin\theta_3' \sin\phi_3' = -(y' - y_R'), \text{ and} \tag{I.32.b}$$

$$r_3' \cos\theta_3' = (x' - x_R') \sin\gamma - (z' - z_R') \cos\gamma. \tag{I.32.c}$$

Substituting r_3' obtained from (I.32.c) into (I.32.a,b) we obtain

$$\tan\theta_3' \cos\phi_3'[(x' - x_R')\sin\gamma - (z' - z_R')\cos\gamma] = (x' - x_R')\cos\gamma + (z' - z_R')\sin\gamma, \text{ and} \tag{I.33.a}$$

$$\tan\theta_3' \sin\phi_3'[(x' - x_R')\sin\gamma - (z' - z_R')\cos\gamma] = -(y' - y_R'). \tag{I.33.b}$$

Summing and squaring (I.33.a) and (I.33.b), then taking the square root and the arctangent of the result, we have

$$\theta_3' = \tan^{-1}\sqrt{\frac{\{[(x' - x_R')\cos\gamma + (z' - z_R')\sin\gamma]^2 + [y' - y_R']^2\}}{\{[(x' - x_R')\sin\gamma - (z' - z_R')\cos\gamma]\}^2}}. \tag{I.34.a}$$

Taking the ratio of (I.33.b) to (I.33.a), then taking the arctangent, we obtain

$$\phi_3' = \tan^{-1}\left[\frac{-(y' - y_R')}{(x' - x_R')\cos\gamma + (z' - z_R')\sin\gamma}\right]. \tag{I.34.b}$$

Solution of (I.32.c) for r_3' gives us

$$r_3' = \frac{(x' - x_r')\sin\gamma - (z' - z_R')\cos\gamma}{\cos\theta_3'}. \tag{I.34.c}$$

It is useful to have the x' and y' coordinates of a point S on the surface (denoted by x_S' and y_S') in terms of the spherical angles θ_3' and ϕ_3' of that point (denoted by θ_{3S}', ϕ_{3S}') and the radar altitude $(z_R' - z_S') \cong h_R$.

We obtain this relationship from (I.31) and (I.34.c) as follows:

$$x_S' = x_R' + \frac{h_R\cos\gamma\,[\tan\theta_{3S}'\cos\phi_{3S}'\cos\gamma + \sin\gamma]}{[1 - \sin\gamma\,(\tan\theta_{3S}'\cos\phi_{3S}'\cos\gamma + \sin\gamma)]}, \text{ and} \tag{I.35.a}$$

$$y_S' = y_R' - \frac{h_R\cos\gamma\tan\theta_{3S}'\sin\phi_{3S}'}{[1 - \sin\gamma\,(\tan\theta_{3S}'\cos\phi_{3S}'\cos\gamma + \sin\gamma)]}. \tag{I.35.b}$$

Appendix II

SMALL-SCALE SURFACE FLUCTUATIONS

The averages over small-scale surface fluctuations are calculated in this appendix. We note in the second section that the ripple surface is given by $z(\underset{\rightarrow}{\rho},t)$, where $\underset{\rightarrow}{\rho} = (x,y) =$ position coordinates on the mean surface, considered locally as planar

The triple Fourier transform of $z(\underset{\rightarrow}{\rho},t)$ is

$$Z(\underset{\rightarrow}{\Omega}) = \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho} \int_{-\infty}^{\infty} dt\, z(\underset{\rightarrow}{\rho},t)e^{-j\omega t - j\underset{\rightarrow}{k} \cdot \underset{\rightarrow}{\rho}}, \tag{II.1}$$

where $\underset{\rightarrow}{\Omega} = (\underset{\rightarrow}{k},\omega) = (k_x k_y,\omega)$.

We assume that $z(\underset{\rightarrow}{\rho},t)$ is a sample function of a zero-mean, statistically homogeneous*, wide-sense stationary Gaussian random process. From the "zero-mean" assumption it follows that the ensemble average (denoted by $< >$) of $z(\underset{\rightarrow}{\rho},t)$ is

$$<z(\underset{\rightarrow}{\rho},t)> = 0. \tag{II.2}$$

From the assumptions of statistical homogeneity and stationarity, it follows that

$$<z(\underset{\rightarrow}{\rho},t)z(\underset{\rightarrow}{\rho} + \Delta\underset{\rightarrow}{\rho}, t + \Delta t)> = R(\Delta\underset{\rightarrow}{\rho},\Delta t), \text{ both independent of } \underset{\rightarrow}{\rho} \text{ and } t. \tag{II.3}$$

Using (II.2) and (II.3) in (II.1), we have

$$<Z(\Omega)> = \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho} \int_{-\infty}^{\infty} dt <z(\underset{\rightarrow}{\rho},t)>e^{-j\omega t - j\underset{\rightarrow}{k} \cdot \underset{\rightarrow}{\rho}} = 0, \text{ and} \tag{II.4}$$

*An alternative terminology might be "spatially stationary."

$$<Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_2)> = \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho}_1 \int_{-\infty}^{\infty} dt_1 \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho}_2 \int_{-\infty}^{\infty} dt_2 <z(\underset{\rightarrow}{\rho}_1,t_1)z(\underset{\rightarrow}{\rho}_2,t_2)>$$

$$e^{-j(\omega_1 t_1 + \omega_2 t_2 + \underset{\rightarrow}{k}_1 \cdot \underset{\rightarrow}{\rho}_1 + \underset{\rightarrow}{k}_2 \cdot \underset{\rightarrow}{\rho}_2)} = \iint d\underset{\rightarrow}{\rho}_1 dt_1 d(\Delta\underset{\rightarrow}{\rho}) d(\Delta t) R(\Delta\underset{\rightarrow}{\rho},\Delta t) \tag{II.5}$$

$$e^{-j[(\omega_1 + \omega_2)t_1 + (\underset{\rightarrow}{k}_1 + \underset{\rightarrow}{k}_2)\cdot \underset{\rightarrow}{\rho}_1]} e^{-j[\omega_2 \Delta t + \underset{\rightarrow}{k}_2 \cdot \Delta \rho]} .$$

We note that (See (2.7.a,b))

$$\iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho}_1 \, e^{-j(\underset{\rightarrow}{k}_1 + \underset{\rightarrow}{k}_2)\cdot \underset{\rightarrow}{\rho}_1} = (2\pi)^2 \delta(k_{1x} + k_{2x})\, \delta(k_{1y} + k_{2y}), \text{ and} \tag{II.6.a}$$

$$\iint_{-\infty}^{\infty} dt_1 \, e^{-j(\omega_1 + \omega_2)t_1} = 2\pi \, \delta(\omega_1 + \omega_2). \tag{II.6.b}$$

Using (II.6.a,b) in (II.5), we have

$$<Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_2)> =$$

$$\left[(2\pi)^3 \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho}' \int_{-\infty}^{\infty} dt' R(\underset{\rightarrow}{\rho}',t') e^{j(\omega_1 t' + \underset{\rightarrow}{k}_1 \cdot \underset{\rightarrow}{\rho}')} \right] \delta(\underset{\rightarrow}{\Omega}_1 + \underset{\rightarrow}{\Omega}_2), \tag{II.7}$$

where

$$\delta(\underset{\rightarrow}{\Omega}_1 + \underset{\rightarrow}{\Omega}_2) = \delta(k_{1x} + k_{2x})\, \delta(k_{1y} + k_{2y})\, \delta(\omega_1 + \omega_2).$$

The integral in (II.7) is the power-spectrum (in both spatial wave number $\underset{\rightarrow}{k}$ and frequency ω) of $z(\underset{\rightarrow}{\rho},t)$, which may be denoted by $W(\underset{\rightarrow}{\Omega})$ or $W(\underset{\rightarrow}{k},\omega)$; thus,

$$Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_2) = W(\underset{\rightarrow}{\Omega}_1)\, \delta(\underset{\rightarrow}{\Omega}_1 + \underset{\rightarrow}{\Omega}_2), \tag{II.8}$$

where $\underset{\rightarrow}{\Omega}_1 = (\underset{\rightarrow}{k}_1,\omega_1)$ and $W(\underset{\rightarrow}{\Omega})$ has the property $W(\underset{\rightarrow}{\Omega}) = W(-\underset{\rightarrow}{\Omega})$.

Because $z(\rho,t)$ must be real, it is evident from (II.1) that

$$Z^*(\underset{\rightarrow}{\Omega}) = \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho} \int_{-\infty}^{\infty} dt\, z(\underset{\rightarrow}{\rho},t) e^{j\omega t + j\underset{\rightarrow}{k} \cdot \underset{\rightarrow}{\rho}} = Z(-\underset{\rightarrow}{\Omega}). \tag{II.9}$$

As a consequence of (II.8) and (II.9),

$$<Z(\underset{\rightarrow}{\Omega}_1)Z^*(\underset{\rightarrow}{\Omega}_2)> = <Z(\underset{\rightarrow}{\Omega}_1)Z(-\underset{\rightarrow}{\Omega}_2)> = W(\underset{\rightarrow}{\Omega}_1)\,\delta(\underset{\rightarrow}{\Omega}_1 - \underset{\rightarrow}{\Omega}_2). \tag{II.10}$$

A general property of zero-mean Gaussian random functions $X_{1,2,3,4}$ is

$$\begin{aligned} <X_1X_2X_3X_4> &= <X_1X_2><X_3X_4> \\ &+ <X_1X_3><X_2X_4> + <X_1X_4><X_2X_3>. \end{aligned} \tag{II.11}$$

From (II.11) and the fact that $Z(\underset{\rightarrow}{\Omega})$ is a zero-mean Gaussian random function since $z(\rho,t)$, its Fourier transform, is such a function and it is a well known property of Gaussian random functions that their Fourier transforms are also Gaussian, we obtain

$$\begin{aligned} <Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_2)Z(\underset{\rightarrow}{\Omega}_3)Z(\underset{\rightarrow}{\Omega}_4)> &= <Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_2)><Z(\underset{\rightarrow}{\Omega}_3)Z(\underset{\rightarrow}{\Omega}_4)> \\ + <Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_3)><Z(\underset{\rightarrow}{\Omega}_2)Z(\underset{\rightarrow}{\Omega}_4)> &+ <Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_4)><Z(\underset{\rightarrow}{\Omega}_2)Z(\underset{\rightarrow}{\Omega}_3)) \end{aligned} \tag{II.12}$$

Another property of a zero-mean Gaussian random function is that its odd-order averages vanish; consequently,

$$<Z(\underset{\rightarrow}{\Omega}_1)Z(\underset{\rightarrow}{\Omega}_2)Z(\underset{\rightarrow}{\Omega}_3)> = 0. \tag{II.13}$$

We make the assumption that the small scale fluctuations on any part of the mean surface are only very weakly correlated with small scale fluctuations on another part of the mean surface. We use the extreme limit of this condition as an approximation on integrating over the mean surface, i.e., we assume that

$$\begin{aligned} <f(Z(\underset{\rightarrow}{\Omega}_a),\underset{\rightarrow}{r}_a)g(Z(\underset{\rightarrow}{\Omega}_b),\underset{\rightarrow}{r}_b)> &\neq 0 \text{ if } |\underset{\rightarrow}{r}_a - \underset{\rightarrow}{r}_b| \leqslant \epsilon \\ &= 0 \text{ if } |\underset{\rightarrow}{r}_a - \underset{\rightarrow}{r}_b| > \epsilon, \end{aligned} \tag{II.14}$$

where ϵ is a small positive real number that limits the correlation region to a very small patch such that the tilt of the mean surface remains uniform throughout the patch. If ϵ is large enough to allow correlation between two points on parts of the mean surface with different tilt angles, then a condition that will later follow from the ensemble averaging process (namely a correlation only between two points with the same $\underset{\rightarrow}{\alpha}_0$) could possibly apply to two points on widely separated parts of the mean surface but with the same $\hat{\underset{\rightarrow}{\alpha}}_0$. This would not make sense physically. Since the scale of mean surface tilt is large compared with the scale of the small scale fluctuations, this assumption is easily justified.

Another assumption we make is that the fluctuation height $z(\underset{\rightarrow}{\rho},t)$ varies so slowly that we can neglect its time variations during illumination of a patch by the radar, i.e.,

$$z(\underset{\rightarrow}{\rho},t) = z(\underset{\rightarrow}{\rho}). \tag{II.15}$$

It follows from (II.1), (II.15) and (2.7.b) that

$$Z(\underset{\rightarrow}{\Omega}) = Z(\underset{\rightarrow}{k},\omega) = 2\pi\, Z(\underset{\rightarrow}{k})\delta(\omega) \tag{II.16}$$

where

$$Z(k) = \iint_{-\infty}^{\infty} d\underset{\rightarrow}{\rho}\, z(\underset{\rightarrow}{\rho})e^{-jk\cdot\underset{\rightarrow}{\rho}}.$$

From (II.4), (II.8), (II.10), (II.12), (II.13) and (II.16), we can write

$$<Z(\underset{\rightarrow}{k})> = 0, \tag{II.17}$$

$$<Z(\underset{\rightarrow}{k}_1)Z(\underset{\rightarrow}{k}_2)> = W(\underset{\rightarrow}{k}_1)\, \delta(\underset{\rightarrow}{k}_1 + \underset{\rightarrow}{k}_2), \tag{II.18}$$

$$<Z(\underset{\rightarrow}{k}_1)Z^*(\underset{\rightarrow}{k}_2)> = W(\underset{\rightarrow}{k}_1)\, \delta(\underset{\rightarrow}{k}_1 - \underset{\rightarrow}{k}_2), \tag{II.19}$$

$$<Z(\underset{\rightarrow}{k}_1)Z(\underset{\rightarrow}{k}_2)Z(\underset{\rightarrow}{k}_3)> = 0, \text{ and} \tag{II.20}$$

$$\begin{aligned}<Z(\underset{\rightarrow}{k}_1)Z(\underset{\rightarrow}{k}_2)Z(\underset{\rightarrow}{k}_3)Z(\underset{\rightarrow}{k}_4)> &= <Z(\underset{\rightarrow}{k}_1)Z(\underset{\rightarrow}{k}_2)><Z(\underset{\rightarrow}{k}_3)Z(\underset{\rightarrow}{k}_4)> \\ + <Z(\underset{\rightarrow}{k}_1)Z(\underset{\rightarrow}{k}_3)><Z(\underset{\rightarrow}{k}_2)Z(\underset{\rightarrow}{k}_4)> &+ <Z(\underset{\rightarrow}{k}_1)Z(\underset{\rightarrow}{k}_4)><Z(\underset{\rightarrow}{k}_2)Z(\underset{\rightarrow}{k}_3)>\end{aligned} \tag{II.21}$$

where $W(k)$ is the spatial spectrum of the small fluctuations on the surface and each of the two-fold product averages in (II.21) may be obtained from (II.18) or from (II.19) and (II.9) when appropriate. More generally, if complex conjugates $Z^*(\underset{\rightarrow}{k})$ are involved in any of these averages, then (II.9) can be invoked to convert $Z^*(\underset{\rightarrow}{k})$ into $Z(-\underset{\rightarrow}{k})$ and then any of the rules given by (II.17) through (II.21) can be applied.

The ensemble averages needed in the text can be obtained by straight-forward manipulation from (3.7.0,...4) and (II.9, 14, 17, 18, 19, 20, 21). Note that those of (4.2) or (4.5) are all either averages of a voltage, such as $<U_{Aa2}>$ and $<U_{Aa4}>$ (or complex conjugates of these) or two-fold product averages involving a voltage and its complex conjugate, namely $<U_{Aa1}U^*_{Bb2}>$ and $<U_{Aa1}U^*_{Bb3}>$. The first two of these, by virtue of (II.8) and (II.21), contain a factor $\delta(\underset{\rightarrow}{\hat{\alpha}}_0)$ within their integrands that limits contributions to points of normal incidence, i.e., those points for which $\hat{\alpha}_0 = 0$; thus, they become summations of contributions from such points as given in (4.5.3)′ and (4.5.4)′. The two-fold product averages $<U_{Aa\ell}U^*_{Bbm}>$ by virtue of (3.7.1, 2, 3), each contain a factor $F_{Aa}(\theta_3{}',\phi_3{}',t)F^*_{Bb}(\theta_3{}',\phi_3,t)$, which in turn by virtue of (3.5) and (II.16) contains a phase factor

$$e^{\pm 2j\frac{\omega}{c}[r_0(\theta_3{}',\phi_3{}',t) - r_0(\theta_3{}',\phi_3{}',t + \Delta t)]} .$$

Another type of two-fold product average, not appearing in (4.2) or (4.5) but appearing only in (4.4.a,...,e)′ and conspicuously absent from (4.6), is $\langle U_{Aa\ell}U_{Bbm}\rangle$ or its complex conjugate; specific examples being $\langle U_{Aa1}U_{Bb1}\rangle$ as in (4.4.a)′, $\langle U_{Aa1}U_{Bb1}\rangle$ as in (4.4.b)′, $\langle U_{Aa3}U_{Bb1}\rangle$ as in (4.4.c)′ and $\langle U_{Aa2}U_{Bb2}\rangle$ as in (4.4.d)′. This type of average, again by virtue of (3.5) and (II.16), contains a factor $F_{Aa}(\theta_3{}',\phi_3{}',t)F_{Bb}(\theta_3{}',\phi_3{}', t + \Delta t)$ implying a phase factor

$$e^{\pm 2j\frac{\omega}{c}[r_0(\theta_3{}',\phi_3{}',t) + r_0(\theta_3{}',\phi_3{}',t + \Delta t)]} .$$

Finally, there are the three-fold product averages of the type

$$\langle U_{Aa1}U^*_{Bb\underset{2}{1}}U_{Cc\underset{1}{2}}\rangle$$

and their complex conjugates; specific examples being $\langle U^*_{Aa1}U_{Bb1}U_{Bb2}\rangle$ and $\langle U^*_{Bb1}U^*_{Aa1}U_{Bb2}\rangle$ as given in (4.4.b)′, $\langle U^*_{Bb1}U_{Aa1}U_{Aa2}\rangle$ and $\langle U^*_{Aa1}U^*_{Bb1}U_{Aa2}\rangle$ as in (4.4.c)′, and $\langle U_{Bb1}U^*_{Bb1}U^*_{Aa2}\rangle$ and $\langle U_{Aa1}U^*_{Aa1}U^*_{Bb2}\rangle$ as in (4.4.d)′. These averages, again because of (3.5) and (II.16) contain a factor $F_{Aa}(\theta_3{}',\phi_3{}',t_1)F_{Bb}(\theta_3{}',\phi_3{}',t_2)$ $F_{Cc}(\theta_3{}',\phi_3{}',t_3)$, where two of the t_k's inside the arguments of these functions are equal while the other is in general different; e.g., $t_1 = t$, $t_2 = t$, $t_3 = t + \Delta t$. The phase factor, then, has the form

$$e^{\pm 2j\frac{\omega}{c}[r_0(\theta_3{}',\phi_3{}',t) - r_0(\theta_3{}',\phi_3{}',t_2) + r_0(\theta_3{}',\phi_3{}',t_3)]} .$$

The eventual neglect (in (4.6)) of the averages $\langle U_{Aa\ell}U_{Bbm}\rangle$ and all terms of the three-fold product averages except the term of the form $\langle U_{Aa1}U^*_{Bb1}\rangle\langle U_{Cc2}\rangle$ is based on the nature of the phase factors mentioned above. The time separation Δt is sufficiently small to allow the approximation

$$e^{\pm 2j\frac{\omega}{c}[r_0(\theta_3{}',\phi_3{}',t) - r_0(\theta_3{}',\phi_3{}',t + \Delta t)]} \approx 1 \qquad \text{(II.22)}$$

Note that (except when $\Delta t = 0$) (II.22) is not meant to imply that the phase factor is actually or even approximately replaceable by unity; it implies only that it is not so oscillatory as to "wipe out" the $(\theta_3{}',\phi_3{}')$ integral. On the other hand, the phase factors that appear in the averages $\langle U_{Aa1}U_{Bb1}\rangle$ and the three-fold product averages are much more oscillatory because the values of r_0 in the exponents (which of course must be positive) are added rather than subtracted; hence, the integrands in these averages are products of relatively smooth functions and highly oscillatory phase factors and should be extremely small. Based on these qualitative arguments, the following approximations will be made:

$$<U_{Aa\ell}U_{Bbm}> \approx 0 \text{ for } \ell,m = 1,2,3 \tag{II.23}$$

$$<U_{Aa1}U^*_{Bb1}U_{Cc2}> \simeq <U_{Aa1}U^*_{Bb1}><U_{Cc2}>, \tag{II.24}$$

(with the aid of (II.21))

$$<U_{Aa1}U_{Bb1}U^*_{Cc2}> \approx 0, \tag{II.25}$$

(with the aid of (II.21) and (II.23)).

A final average that appears in (4.4.d)′ is (from II.12) with the aid of (II.23)),

$$\begin{aligned} <U_{Aa1}U^*_{Aa1}U^*_{Bb1}U_{Bb1}> &= <U_{Aa1}U^*_{Aa1}><U^*_{Bb1}U_{Bb1}> \\ &+ <U_{Aa1}U_{Bb1}><U^*_{Aa1}U^*_{Bb1}> + <U_{Aa1}U^*_{Bb1}><U^*_{Aa1}U_{Bb1}> \\ &\simeq <|U_{Aa1}|^2><|U_{Bb1}|^2> + |<U_{Aa1}U^*_{Bb1}>|^2. \end{aligned} \tag{II.26}$$

Appendix III

THE MEAN SURFACE

The mean surface or "swell" surface is characterized by specifying angles δ and Φ as functions of $\theta_3{}'$ and $\phi_3{}'$. We will first specify the mean surface as a function of x' and y', denoted by $H'(x',y')$.

Referring to Appendix I, we note that a point on the mean surface must by definition fulfill the condition $z = 0$, because the mean surface is explicitly defined as the $z = 0$ surface. We set $z = 0$ in (I.6) and note that z' in (I.6) is functionally related to (x',y') through the equation

$$z' = H'(x',y') \text{ for } z = 0. \tag{III.1}$$

Referring to Figure (I-2) and Eqs. (I.1) and (I.2), we note that

$$\begin{aligned} x''' &= (x' - x_0{}') \cos \Phi + (y' - y_0{}') \sin \Phi \\ y''' &= - (x' - x_0{}') \sin \Phi + (y' - y_0{}') \cos \Phi \end{aligned} \tag{III.2}$$

Representing x''' and y''' as functions of x' and y' through (III.2), writing $H'(x',y')$ in the form $H'(x'''(x',y'),y'''(x',y'))$, and applying the chain rule, we obtain

$$\frac{\partial H'}{\partial x'} = \frac{\partial H'}{\partial x'''}\frac{\partial x'''}{\partial x'} + \frac{\partial H'}{\partial y'''}\frac{\partial y'''}{\partial x'} \text{, and} \tag{III.3.a}$$

$$\frac{\partial H'}{\partial y'} = \frac{\partial H'}{\partial x'''}\frac{\partial x'''}{\partial y'} + \frac{\partial H'}{\partial y'''}\frac{\partial y'''}{\partial y'} . \tag{III.3.b}$$

But from (III.2)

$$\frac{\partial x'''}{\partial x'} = \cos \Phi \tag{III.4.a}$$

$$\frac{\partial x'''}{\partial x'} = \sin \Phi \tag{III.4.b}$$

$$\frac{\partial y'''}{\partial x'} = - \sin \Phi\text{, and} \tag{III.4.c}$$

$$\frac{\partial y'''}{\partial y'} = \cos \Phi. \tag{III.4.d}$$

Since x''' is the direction of the local tilt of the surface relative to the horizontal and y''' is the direction normal to that tilt, it follows that

$$\frac{\partial H'}{\partial x'''} = \tan \delta, \text{ and} \tag{III.5.a}$$

$$\frac{\partial H'}{\partial y'''} = 0. \tag{III.5.b}$$

It follows from (III.3.a,b), (III.4.a,...,d) and (III.5.a,b) that

$$H_{x'} = \frac{\partial H}{\partial x'} = \tan \delta \cos \Phi, \text{ and} \tag{III.6.a}$$

$$H_{y'} = \frac{\partial H}{\partial y'} = \tan \delta \sin \Phi. \tag{III.6.b}$$

Squaring (III.6.a,b), adding and taking the square root, we have

$$\tan \delta = \left|\nabla' H\right| = \sqrt{(H_{x'})^2 + (H_{y'})^2}, \tag{III.7}$$

or equivalently,

$$\cos \delta = \frac{1}{\sqrt{1 + |\nabla' H|^2}}, \text{ and} \tag{III.8.a}$$

$$\sin \delta = \frac{|\nabla' H|}{\sqrt{1 + |\nabla' H|^2}}. \tag{III.8.b}$$

Dividing (III.6.b) by (III.6.a),

$$\tan \Phi = \frac{H_{y'}}{H_{x'}}, \tag{III.9}$$

or equivalently,

$$\cos \Phi = \frac{H_{x'}}{|\nabla' H|}, \text{ and} \tag{III.10.a}$$

$$\sin \Phi = \frac{H_{y'}}{|\nabla' H|}. \tag{III.10.b}$$

We choose as a model for $H(x',y')$ a sinusoidal function of the form

$$H(x',y') = C \cos (K_1 x' + K_2 y' + \Psi). \tag{III.11}$$

Differentiating (III.11) with respect to x' and y', we have

$$H_{x'} = - K_1 C \sin (K_1 x' + K_2 y' + \psi), \text{ and} \tag{III.12.a}$$

$$H_{y'} = - K_2 C \sin (K_1 x' + K_2 y' + \psi). \tag{III.12.b}$$

Using (III.12.a,b) in (III.8.a,b) and (III.10.a,b),

$$\cos \delta = \frac{1}{\sqrt{1 + C^2(K_1^2 + K_2^2) \sin^2(K_1 x' + K_2 y' + \psi)}}, \tag{III.13.a}$$

$$\sin \delta = \frac{C \sqrt{K_1^2 + K_2^2} \sin (K_1 x' + K_2 y' + \psi)}{\sqrt{1 + C^2(K_1^2 + K_2^2) \sin^2(K_1 x' + K_2 y' + \psi)}}, \tag{III.13.b}$$

$$\cos \Phi = \frac{- K_1}{\sqrt{K_1^2 + K_2^2}}, \text{ and} \tag{III.14.a}$$

$$\sin \Phi = \frac{- K_2}{\sqrt{K_1^2 + K_2^2}}. \tag{III.14.b}$$

To obtain $\cos \delta$, $\sin \delta$, $\cos \Phi$ and $\sin \Phi$ as functions of θ_3' and ϕ_3' for fixed values of γ and h_R, we use (I.35.a,b) in (III.8.a,b) and (III.10.a,b). If we employ the specialized model (III.11), then $\cos \Phi$ and $\sin \Phi$ as given by (III.14.a,b) are independent of x' and y' and $\cos \delta$ and $\sin \delta$ can be expressed as functions of θ_3' and ϕ_3' by substituting (I.35.a,b) into (III.13.a,b). Summarizing these results, first for the general case, we write: (from (III.8.a,b) and (III.10.a,b))

$$\cos \delta = \frac{1}{\sqrt{1 + |\nabla' H|^2}}, \tag{III.15.a}$$

$$\sin \delta = \frac{|\nabla' H|}{\sqrt{1 + |\nabla' H|^2}}, \tag{III.15.b}$$

$$\cos \Phi = \frac{H_{x'}}{|\nabla' H|}, \text{ and} \tag{III.15.c}$$

$$\sin \Phi = \frac{H_{y'}}{|\nabla' H|} \tag{III.15.d}$$

where (from (I.35.a,b), with removal of the subscript S)

$$H_{x'} = \frac{\partial H}{\partial x'}, \ H_{y'} = \frac{\partial H}{\partial y'}, \tag{III.15.e}$$

$$x' = x_R' + \frac{h_R \cos\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]},$$

$$y' = y_R' - \frac{h_R \cos\gamma \tan\theta_3' \sin\phi_3'}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]}, \tag{III.15.f}$$

and $H(x',y')$ is an assigned real function of x' and y'. For the case where (III.11) is used (from (III.13.a,b) and (III.14.a,b)),

$$\cos\delta = \frac{1}{\sqrt{1 + C^2(K_1^2 + K_2^2)\sin^2(K_1x' + K_2y' + \psi)}}, \tag{III.16.a}$$

$$\sin\delta = \frac{C\sqrt{K_1^2 + K_2^2}\,\sin(K_1x' + K_2y' + \psi)}{\sqrt{1 + C^2(K_1^2 + K_2^2)\sin^2(K_1x' + K_2y' + \psi)}}, \tag{III.16.b}$$

$$\cos\Phi = \frac{-K_1}{\sqrt{K_1^2 + K_2^2}}, \text{ and} \tag{III.16.c}$$

$$\sin\Phi = \frac{-K_2}{\sqrt{K_1^2 + K_2^2}}, \tag{III.16.d}$$

where x' and y' are given in (III.15.f).

Some of the terms in our final results, (U_{Aa0}, $<U_{Aa2}>$ and $<U_{Aa4}>$) contain the condition $\hat{\underset{\to}{\alpha}}_o = (\alpha_{ox}, \alpha_{oy}) = 0$. From (I.28) (dropping subscripts) this condition is equivalent to

$$\tan\theta_3'\cos\phi_3'(\cos\delta\cos\gamma\cos\Phi - \sin\delta\sin\gamma) + \tan\theta_3'\sin\phi_3'(-\cos\delta\sin\Phi) \\ = -(\cos\delta\sin\gamma\cos\Phi + \sin\delta\cos\gamma), \tag{III.17.a}$$

$$\tan\theta_3'\cos\phi_3'(\cos\gamma\sin\Phi) + \tan\theta_3'\sin\phi_3'(\cos\Phi) = -\sin\gamma\sin\Phi, \tag{III.17.b.}$$

$$\tan\theta_3'\cos\phi_3' = -\frac{(\sin\delta\cos\gamma\cos\Phi + \cos\delta\sin\gamma)}{(\cos\delta\cos\gamma - \sin\delta\sin\gamma\cos\Phi)} \\ = -\frac{(\tan\delta\cos\Phi + \tan\gamma)}{(1 - \tan\delta\tan\gamma\cos\Phi)}, \text{ and} \tag{III.18.a}$$

$$\tan\theta_3'\sin\phi_3' = \frac{(\sin\delta\sin\Phi)}{(\cos\delta\cos\gamma - \sin\delta\sin\gamma\cos\Phi)} \\ = \frac{(\tan\delta\sin\Phi\sec\gamma)}{(1 - \tan\delta\tan\gamma\cos\Phi)}. \tag{III.18.b}$$

Squaring and summing (III.18.a,b) and then taking the ratio of (III.18.b.) to (III.18.a) we obtain

$$(\theta_3')_{\hat{\underset{\to}{\alpha}}_o = 0} = \tan^{-1}\left\{\frac{\sqrt{(\tan\delta\cos\gamma\cos\Phi + \sin\gamma)^2 + (\tan\delta\sin\Phi)^2}}{(\cos\gamma - \tan\delta\sin\gamma\cos\Phi)}\right\}, \text{ and} \tag{III.19.a}$$

$$(\phi_3')_{\hat{\underset{\to}{\alpha}}_o = 0} = \tan^{-1}\left\{\frac{-\tan\delta\sin\Phi}{\tan\delta\cos\gamma\cos\Phi + \sin\gamma}\right\} \tag{III.19.b}$$

If the surface is perfectly horizontal, i.e., $\delta = 0$, then Eqs. (III.19.a,b) degenerate into

$$(\theta_3')_{\substack{\hat{\underset{\to}{\alpha}}_o = 0 \\ \delta = 0}} = \tan^{-1}\left\{\frac{\sqrt{(\sin\gamma)^2}}{\cos\gamma}\right\} = \gamma, \text{ and} \tag{III.20.a}$$

$$(\phi_3')_{\substack{\hat{\underset{\to}{\alpha}}_o = 0 \\ \delta = 0}} = \tan^{-1}\left\{-\frac{0}{\sin\gamma}\right\} = \pi. \tag{III.20.b}$$

The implication of (III.20.a,b) is that the angle θ_3' corresponds to the vertical direction (see (I.15.a,b) and Fig. I-5). This makes physical sense because the condition $\hat{\underset{\to}{\alpha}}_o = 0$ corresponds to normal incidence on the surface, which would obviously occur only with vertical incidence if the surface were perfectly horizontal.

The condition on the angles (θ_3', ϕ_3') corresponding to $\underset{\to}{\hat{\alpha}}_o = 0$ can be determined from (III.17.a,b) *or* (III.18.a,b) *or* (III.19.a,b) combined with (III.15.a,...,f). The results in the general case are: from (III.18.a,b) and (III.15.a,...,d)

$$(\tan\theta_3' \cos\phi_3')_{\underset{\to}{\hat{\alpha}}_o = 0} = \frac{-(H_{x'} + \tan\gamma)_{\underset{\to}{\hat{\alpha}}_o = 0}}{(1 - H_{x'} \tan\gamma)_{\underset{\to}{\hat{\alpha}}_o = 0}}, \text{ and} \tag{III.21.a}$$

$$(\tan\theta_3' \sin\phi_3')_{\underset{\to}{\hat{\alpha}}_o = 0} = \frac{(H_{y'})_{\underset{\to}{\hat{\alpha}}_o = 0} \sec\gamma}{(1 - H_{x'} \tan\gamma)_{\underset{\to}{\hat{\alpha}}_o = 0}}, \tag{III.21.b}$$

where (from (III.15.f)) $H_{x'}$ and $H_{y'}$ are functions of the two variables

$$(x')_{\underset{\to}{\hat{\alpha}}_o = 0} = x_R' + \frac{h_R \cos\gamma\, (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)_{\underset{\to}{\hat{\alpha}}_o = 0}}{[1 - \sin\gamma\, (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]_{\underset{\to}{\hat{\alpha}}_o = 0}}, \text{ and} \tag{III.21.c}$$

$$(y')_{\underset{\to}{\hat{\alpha}}_o = 0} = y_R' - \frac{h_R \cos\gamma\, (\tan\theta_3' \sin\phi_3')_{\underset{\to}{\hat{\alpha}}_o = 0}}{[1 - \sin\gamma\, (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]_{\underset{\to}{\hat{\alpha}}_o = 0}}. \tag{III.21.d}$$

In the special case where the model (III.11) is used, with the aid of (III.16.a,...,d), Eqs. (III.21.a,b) have the form

$$(\tan\theta_3' \cos\theta_3')_{\underset{\to}{\hat{\alpha}}_o = 0} = \frac{(CK_1 \sin(K_1 x' + K_2 y' + \psi) - \tan\gamma)_{\underset{\to}{\hat{\alpha}}_o = 0}}{[1 + CK_1 \sin(K_1 x' + K_2 y' + \psi) \tan\gamma]_{\underset{\to}{\hat{\alpha}}_o = 0}}, \text{ and} \tag{III.22.a}$$

$$(\tan\theta_3' \sin\phi_3')_{\underset{\to}{\hat{\alpha}}_o = 0} = \frac{-CK_2 [\sin(K_1 x' + K_2 y' + \psi)]_{\underset{\to}{\hat{\alpha}}_o = 0} \sec\gamma}{[1 + CK_1 \sin(K_1 x' + K_2 y' + \psi) \tan\gamma]_{\underset{\to}{\hat{\alpha}}_o = 0}}, \tag{III.22.b}$$

where x' and y' are given by (III.21.c,d).

In describing the position of a point on the surface illuminated by the radar, one can consider this surface as perfectly horizontal *except* where phase differences are involved. The quantity $(z_R' - z')$ has usually been considered equivalent to h_R, the radar altitude, because the difference is usually negligible. If the highest swell were 30 m (100 ft) and the radar altitude were 1.6 km (1 mile), the error in considering $(z_R' - z')$ equivalent to h_R would be about 2 percent. In most cases the error would be even smaller.

The phase factor e^{jkr_o} which appears in the field is "washed out" in all of our averages exept U_{Aao}, $\langle U_{Aa2} \rangle$, and $\langle U_{Aa4} \rangle$. Thus, we need only consider the effect of this factor to calculate those terms. Referring to the diagram of Fig. III-1, the vector $\vec{r}_o$ is that from the illuminated point $[x', y', H(x',y')]$ on the mean surface to the radar. The vector $\hat{\vec{r}}_o$ is that from the point $(x',y',0)$ (directly below the illuminated point) to the radar. The vector $\vec{H}'$ is equal to $\vec{i}_z{}'H'(x',y')$, where $\vec{i}_z{}'$ is the unit basis vector in the z' direction. It is evident that

$$|H| << |\vec{r}_o|, |\hat{\vec{r}}_o|, \tag{III.23}$$

from which it follows that

$$\begin{aligned} r_o &= |\vec{r}_o| = |\hat{\vec{r}}_o - \vec{H}'| = \sqrt{(\hat{\vec{r}}_o - \vec{i}_{z'}H') \cdot (\hat{\vec{r}}_o - \vec{i}_{z'}H')} \\ &= \sqrt{\hat{r}_o^2 + (H')^2 - 2(\hat{\vec{r}}_o \cdot \vec{i}_{z'})H'} = \hat{r}_o \left[1 - \frac{H'}{\hat{r}_o}\left(\frac{\hat{\vec{r}}_o}{\hat{r}_o}\right) \cdot \vec{i}_{z'}\right] + 0\left[\left(\frac{H'}{\hat{r}_o}\right)^2\right] \\ &\simeq \hat{r}_o - H'(x',y') \cos \theta_R{}' = \hat{r}_o - \frac{h_R}{\hat{r}_o} H'(x',y'), \end{aligned} \tag{III.24}$$

where $\theta_R{}'$ is the polar angle of the radar with respect to the point $(x',y',0)$, described in the (x',y',z') coordinate system.

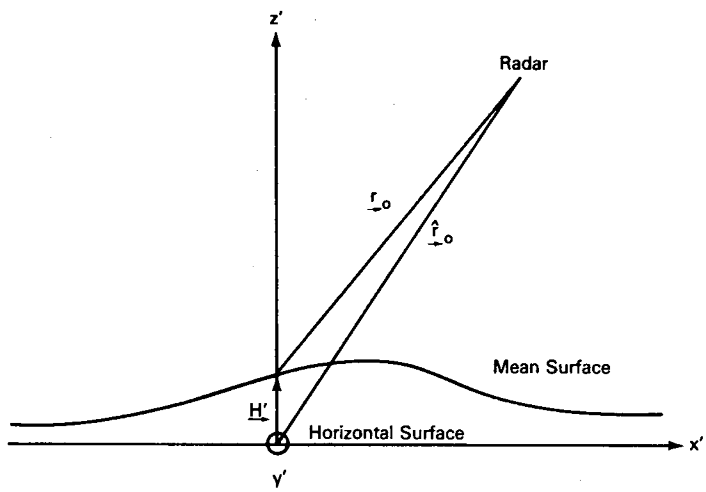

Fig. III-1 — Mean surface geometry

To express the phase factor entirely in terms of the coordinates (θ_3', ϕ_3'), we note that (with the aid of (I.35.a,b) removing the subscript S)

$$\hat{r}_o = \sqrt{(x_R' - x')^2 + (y_R' - y')^2 + h_R^2}$$

$$= \frac{h_R \cos\gamma \sec\theta_3'}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]} \,. \tag{III.25}$$

It follows from (III.24) and (III.25) that

$$r_o \simeq \frac{h_R \cos\gamma \sec\theta_3'}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]} \tag{III.26}$$

$$- \sec\gamma \cos\theta_3' \,[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]\, H(x',y'),$$

where x' and y' in $H(x',y')$ are given by (III.15.f).

The factor e^{jkr_o}/r_o which appears in our averages can be approximated through (III.26) by

$$\frac{e^{jkr_o}}{r_o} \simeq \frac{e^{\left(\frac{jkh_R \cos\gamma \sec\theta_3'}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]}\right)}}{h_R \cos\gamma \sec\theta_3'}$$

$$[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)] \tag{III.27}$$

$$e^{-jk \sec\gamma \cos\theta_3' \,[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]\, H(x',y')},$$

where

$$x' = x_R' + \frac{h_R \cos\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]}\,, \text{ and}$$

$$y' = y_R' - \frac{h_R \cos\gamma \tan\theta_3' \sin\phi_3'}{[1 - \sin\gamma\,(\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]}$$

(again with the aid of (I.35.a,b)).

In the special case where (III.11) is used, (III.27) has the form

$$
\frac{e^{jkr_o}}{r_o} \simeq \frac{e^{\left(\frac{jkh_R \cos\gamma \sec\theta_3'}{[1 - \sin\gamma (\tan\gamma_3' \cos\theta_3' \cos\gamma + \sin\gamma)]}\right)}}{h_R \cos\gamma \sec\theta_3'} \cdot
$$

$$
[1 - \sin\gamma (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)] \cdot
$$

$$
\Bigg\{ e^{-jk \sec\gamma \cos\theta_3' [1 - \sin\gamma (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]} \cdot \tag{III.28}
$$

$$
C \cos \Bigg([K_1 x_R' + K_2 y_R' + \psi]
$$

$$
+ \frac{h_R \cos\gamma \, [K_1 (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma) - K_2 \tan\theta_3' \sin\phi_3']}{[1 - \sin\gamma (\tan\theta_3' \cos\phi_3' \cos\gamma + \sin\gamma)]} \Bigg) \Bigg\}
$$

In the terms where the factor e^{jkr_o}/r_o is not washed out in the averaging process, i.e., U_{Aa}, $\langle U_{Aa2} \rangle$ and $\langle U_{Aa4} \rangle$, the condition $\underset{\to}{\hat{\alpha}}_o = 0$ prevails. Consequently, when (III.27) is used in these terms, the angles θ_3' and ϕ_3' obey the conditions (III.21.a,b,c). When (III.28) is used in these same terms, (i.e., when the model (III.11) is applied) then θ_3' and ϕ_3' obey the conditions (III.22.a,b) and (III.21.c,d).

Appendix IV

CALCULATION OF FIRST-ORDER RESULTS

To obtain the result (5.1), we first specialize to the case where $z(\underset{\rightarrow}{\rho},t)$ does not vary significantly with time during the period of illumination; hence, we can denote $z(\underset{\rightarrow}{\rho},t)$ by $z(\underset{\rightarrow}{\rho})$ and its Fourier transform by $Z(\underset{\rightarrow}{k})$, having eliminated by this specialization the necessity of Fourier transforming with respect to time as well as the spatial variables.

From (2.16.1) (with the assumption indicated above)

$$\hat{\tilde{\underset{\rightarrow}{E}}}^{(1)}(\underset{\rightarrow}{\beta},\omega) = \int\int d\underset{\rightarrow}{\beta}\, d\omega\, d\underset{\rightarrow}{\rho}_1\, dt\, e^{\frac{j}{c}\left[(\omega_1 \underset{\rightarrow}{\beta}_1 - \omega \underset{\rightarrow}{\beta})\cdot \underset{\rightarrow}{\rho}_1 - c(\omega_1 - \omega)t_1\right]} \hat{R}^{(1)}(\underset{\rightarrow}{\beta},\omega/\underset{\rightarrow}{\beta}_1,\omega_1,\underset{\rightarrow}{\rho}_1,t_1)\,\tilde{\tilde{\underset{\rightarrow}{E}}}_i(\underset{\rightarrow}{\beta}_1,\omega_1) \tag{IV.1}$$

where

$$\hat{R}^{(1)}(\underset{\rightarrow}{\beta},\omega/\underset{\rightarrow}{\beta}_1,\omega_1,\underset{\rightarrow}{\rho}_1,t_1) = \frac{1}{(2\pi)^3\,\Delta(|\underset{\rightarrow}{\beta}|)}\, b(\underset{\rightarrow}{\beta})\, a\,(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1),$$

and where $b(\beta)$ is the 4×4 matrix given in (2.16.0)$'$; $\Delta(|\underset{\rightarrow}{\beta}|) = (\beta_z + Y_z)(1 - \beta_z^2 + \beta_z Y_z)$, and $a(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1)$ is a 4×2 matrix defined by

$$a(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1) = \widetilde{B}_1(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1) - \widetilde{A}_1(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1)\,\frac{b(\underset{\rightarrow}{\beta}_1)}{\Delta(|\underset{\rightarrow}{\beta}_1|)}\, B_o(\underset{\rightarrow}{\beta}_1), \tag{IV.2}$$

where $B_o(\underset{\rightarrow}{\beta}_1)$ is obtained from (2.12.a)$''$ and (2.12.b)$''$ with $\underset{\rightarrow}{\beta}$ set to $\underset{\rightarrow}{\beta}_1$, and the elements of the 4×4 matrix $\widetilde{A}_1(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1)$ and the 4×2 matrix $\widetilde{B}_1(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1)$ are obtained from (2.12.a)$'''$ and (2.12.b)$'''$ respectively. The latter elements are the first order terms of the series expansion of (2.12.a)$'''$ and (2.12.b)$'''$. The first order terms of (2.12.a)$'''$ are as follows:

$$(A_{11}{}^{(1)})_1 = \frac{\beta_x}{\beta_z}\, z_y \tag{IV.3.a}$$

$$(A_{12}{}^{(1)})_1 = \frac{\beta_y}{\beta_z}\, z_y - j\frac{\omega}{c}\beta_z z \tag{IV.3.b}$$

$$(A_{13}{}^{(1)})_1 = \frac{\beta_x}{\gamma_z}\, z_y \tag{IV.3.c}$$

$$(A_{14}{}^{(1)})_1 = \frac{\beta_y}{\gamma_z} z_y - j\frac{\omega}{c}\gamma_z z \tag{IV.3.d}$$

$$(A_{31}{}^{(1)})_1 = \beta_y z_y - j\frac{\omega}{c}z(1 - \beta_y{}^2) \tag{IV.3.e}$$

$$(A_{32}{}^{(1)})_1 = - \beta_x [z_y + j\frac{\omega}{c}\beta_y z] \tag{IV.3.f}$$

$$(A_{33}{}^{(1)})_1 = - \beta_y z_y + j\frac{\omega}{c}z(\nu^2 - \beta_y{}^2) \tag{IV.3.g}$$

$$(A_{34}{}^{(1)})_1 = \beta_x [z_y + j\frac{\omega}{c}z\beta_y] \tag{IV.3.h}$$

The first order terms of (2.12.b)''' are:

$$(B_{11}{}^{(1)})_1 = \frac{\beta_x}{\beta_z} z_y \tag{IV.4.a}$$

$$(B_{12}{}^{(1)})_1 = \frac{\beta_y}{\beta_z} z_y - j\frac{\omega}{c}\beta_z z \tag{IV.4.b}$$

$$(B_{31}{}^{(1)})_1 = - \beta_y z_y + j\frac{\omega}{c}z(1 - \beta_y{}^2) \tag{IV.4.c}$$

$$(B_{32}{}^{(1)})_1 = \beta_x [z_y + j\frac{\omega}{c}\beta_y z] \tag{IV.4.d}$$

To obtain the first order results, we follow the analysis using (2.15.0), (2.15.1), (2.16.1), (2.17), (2.18.a,b) and (2.19) with the aid of (2.16.0)$'$. Expressing $z(\underset{\to}{\rho})$, $z_x(\underset{\to}{\rho})$ and $z_y(\underset{\to}{\rho})$ in terms of its Fourier transform, we have

$$z(\underset{\to}{\rho}) = \iint d\underset{\to}{k}\, e^{j\underset{\to}{k} \cdot \underset{\to}{\rho}}\, Z(\underset{\to}{k}), \tag{IV.5.a}$$

$$z_x(\underset{\to}{\rho}) = \iint d\underset{\to}{k} jk_x\, e^{j\underset{\to}{k} \cdot \underset{\to}{\rho}}\, Z(\underset{\to}{k}), \text{ and} \tag{IV.5.b}$$

$$z_y(\rho) = \iint d\underset{\to}{k} jk_y\, e^{j\underset{\to}{k} \cdot \underset{\to}{\rho}}\, Z(\underset{\to}{k}). \tag{IV.5.c}$$

From (IV.5.a,b,c) and (IV.3.a) the element $A^{(1)}$ could be expressed as

$$(A_{11}{}^{(1)})_1 = \frac{\beta_x}{\beta_z} \iint d\underset{\rightarrow}{k}\, e^{j\underset{\rightarrow}{k} \cdot \underset{\rightarrow}{\rho}}\, jk_x\, Z(\underset{\rightarrow}{k})$$

$$= \iint d\underset{\rightarrow}{k}\, e^{j\underset{\rightarrow}{k} \cdot \underset{\rightarrow}{\rho}}\, Z(\underset{\rightarrow}{k})\, (\widetilde{A}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}))_1, \tag{IV.6}$$

where

$$(\widetilde{A}_{11}\,(\underset{\rightarrow}{\beta}_1,\underset{\rightarrow}{k}))_1 = jk_y \frac{\beta_x}{\beta_z} \; .$$

The elements of $(\widetilde{A}_{jk})_1$ and $(\widetilde{B}_{jk})_1$ obtained from (IV.3.a,...,h) and (IV.4.a,...,d) as indicated in (IV.6), are as follows:

$$(\widetilde{A}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_x k_{1y} \beta_z{}^{-1} \tag{IV.3.a$'$}$$

$$(\widetilde{A}_{12}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_y k_{1y} \beta_z{}^{-1}\, j\frac{\omega}{c}\beta_z \tag{IV.3.b$'$}$$

$$(\widetilde{A}_{13}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_x k_{1y} \gamma_z{}^{-1} \tag{IV.3.c$'$}$$

$$(\widetilde{A}_{14}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_y k_{1y} \gamma_z{}^{-1} - j\frac{\omega}{c}\, \gamma_z \tag{IV.3.d$'$}$$

$$(\widetilde{A}_{31}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_y k_{1y} - j\frac{\omega}{c}(1 - \beta_y{}^2) \tag{IV.3.e$'$}$$

$$(\widetilde{A}_{32}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = -\, j\beta_x k_{1y} - j\frac{\omega}{c}\beta_x\beta_y \tag{IV.3.f$'$}$$

$$(\widetilde{A}_{33}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = -\, j\beta_y k_{1y} + j\frac{\omega}{c}\,(\nu^2 - \beta_y{}^2) \tag{IV.3.g$'$}$$

$$(\widetilde{A}_{34}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_x k_{1y} + j\frac{\omega}{c}\beta_x\beta_y \tag{IV.3.h$'$}$$

$$(\widetilde{B}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_x k_{1y} \beta_z{}^{-1} \tag{IV.4.a$'$}$$

$$(\widetilde{B}_{12}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_y k_{1y} \beta_z{}^{-1} - j\frac{\omega}{c}\beta_z \tag{IV.4.b$'$}$$

$$(\widetilde{B}_{31}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = -\, j\beta_y k_{1y} + j\frac{\omega}{c}\,(1 - \beta_y{}^2) \tag{IV.4.c$'$}$$

$$(\widetilde{B}_{32}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}_1))_1 = j\beta_x k_{1y} + j\frac{\omega}{c}\beta_x\beta_y \tag{IV.4.d$'$}$$

The other elements of $\widetilde{A}$ and $\widetilde{B}$ are obtained through the relationships implied by (2.11.a,b)$'$, i.e.,

$$\widetilde{A}_{21} = -\widetilde{A}_{12}', \tag{IV.7.a}$$

$$\widetilde{A}_{22} = -\widetilde{A}_{11}', \tag{IV.7.b}$$

$$\widetilde{A}_{23} = -\widetilde{A}_{14}', \tag{IV.7.c}$$

$$\widetilde{A}_{24} = -\widetilde{A}_{13}', \tag{IV.7.d}$$

$$\widetilde{A}_{41} = \widetilde{A}_{32}', \tag{IV.7.e.}$$

$$\widetilde{A}_{42} = \widetilde{A}_{31}', \tag{IV.7.f}$$

$$\widetilde{A}_{43} = \widetilde{A}_{34}', \tag{IV.7.g}$$

$$\widetilde{A}_{44} = \widetilde{A}_{33}', \tag{IV.7.h}$$

$$\widetilde{B}_{21} = -\widetilde{B}_{12}', \tag{IV.8.a}$$

$$\widetilde{B}_{22} = -\widetilde{B}_{11}', \tag{IV.8.b}$$

$$\widetilde{B}_{41} = \widetilde{B}_{32}', \text{ and} \tag{IV.8.c}$$

$$\widetilde{B}_{42} = \widetilde{B}_{31}', \tag{IV.8.d}$$

where the prime indicates that x and y components are interchanged in the corresponding original element, e.g., if $\widetilde{A}_{jk} = \beta_x k_{1y}$, then $\widetilde{A}_{jk}' = \beta_y k_{1x}$.

We now invoke (2.17), (2.18.a,b) and (2.19) with the aid of which we can use (IV.1) to express the horizontally and vertically polarized components of the electric field of the reflected wave to the first order of perturbation. The result of this operation is:

$$[\tilde{\tilde{E}}_{\substack{rV\\H}}(\underset{\rightarrow}{\beta})]_1 = \underset{\rightarrow}{\varrho}{}^T_{\substack{V\\H}}\, Q(\underset{\rightarrow}{\beta})[\hat{\tilde{\underset{\rightarrow}{E}}}(\underset{\rightarrow}{\beta})]_1, \tag{IV.9}$$

where

$$\underset{H}{\underset{\to}{\ell}{}_V^T} = \text{transpose of } \underset{H}{\underset{\to}{\ell}{}_V},$$

$Q(\underset{\to}{\beta})$ is given in (2.17), and $[\hat{\widetilde{E}}(\underset{\to}{\beta})]_1$ is the first order term of $\hat{\widetilde{E}}(\underset{\to}{\beta},\omega)$ as given by (IV.1). From (IV.1), (IV.9) and (2.19),

$$[\widetilde{E}_{r\underset{H}{V}}(\underset{\to}{\beta})]_1 = \iint d\underset{\to}{\beta}_1 [\widetilde{S}_{\underset{H}{V}}(\underset{\to}{\beta};\underset{\to}{\beta}_1)]_1 \,\widetilde{\widetilde{E}}_i(\underset{\to}{\beta}_1) Z[-\frac{\omega}{c}(\underset{\to}{\beta}_1 - \underset{\to}{\beta})], \qquad \text{(IV.10)}$$

where

$$[\widetilde{S}_{\underset{H}{V}}(\underset{\to}{\beta};\underset{\to}{\beta})]_1 = \frac{1}{2\pi} \underset{H}{\underset{\to}{\ell}{}_V^T} Q(\underset{\to}{\beta}) b_o(\underset{\to}{\beta})(a) L(\underset{\to}{\beta}_1), \qquad \text{(IV.10)}'$$

and where (a) is the quantity $a(\underset{\to}{\beta},\underset{\to}{k}_1)$ defined in (IV.2).

Examining the definition of $a(\underset{\to}{\beta},\underset{\to}{k})$ in (IV.2), we note the matrix product $(\widetilde{A}_1(\underset{\to}{\beta};\underset{\to}{k}_1)b_o(\underset{\to}{\beta}_1))$, which is a 4 X 4 matrix. The first step is the calculation of that product. Calling this matrix product $\widetilde{d}$, using c to denote the matrix $\widetilde{A}_1$, and invoking (IV.3.a,...,h), (IV.4.a,...,d), (IV.7.a,...,h), and (IV.8.a,...,d), we obtain

$$d = \frac{1}{\Delta}\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ -c'_{12} & -c'_{11} & -c'_{14} & -c'_{13} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c'_{32} & c'_{31} & c'_{34} & c'_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ -b'_{12} & -b_{11} & b_{14} & b'_{13} \\ b_{11} & b_{32} & b_{13} & b_{14} \\ -b'_{32} & -b_{11} & b_{14} & b'_{13} \end{bmatrix} = \frac{j[\widetilde{d}_{jk}]}{\Delta} = \frac{\widetilde{d}}{\Delta} \qquad \text{(IV.11)}$$

where $j[\widetilde{d}_{jk}] = j\widetilde{d} = d\Delta$, Δ is $\Delta(|\underset{\to}{\beta}|)$ as defined above (IV.2) and, because of the special properties of the matrices c and b, we have (Note that $b'_{11} = b_{11}$, $b'_{14} = b_{14}$)

$$j\widetilde{d}_{11} = b_{11}(c_{11} + c_{13}) - b'_{12}c_{12} - b'_{32}c_{14},$$

$$j\widetilde{d}_{12} = -b_{11}(c_{12} + c_{14}) + b_{12}c_{11} + b_{32}c_{13},$$

$$j\widetilde{d}_{13} = b_{13}(c_{11} + c_{13}) + b_{14}(c_{12} + c_{14}),$$

$$j\widetilde{d}_{14} = b_{14}(c_{11} + c_{13}) + b'_{13}(c_{12} + c_{14}),$$

$$
\begin{aligned}
d_{21} &= d'_{12}, \\
d_{22} &= d'_{11}, \\
d & \\
d_{23} &= -\,d'_{14}, \\
d_{24} &= -\,d'_{13}, \\
j\tilde{d}_{31} &= b_{11}(c_{31}+c_{33}) - b'_{12}c_{32} - b'_{32}c_{34}, \\
j\tilde{d}_{32} &= -\,b_{11}(c_{32}+c_{34}) + b_{12}c_{31} + b_{32}c_{33}, \\
j\tilde{d}_{33} &= b_{13}(c_{31}+c_{33}) + b_{14}(c_{32}+c_{34}), \\
j\tilde{d}_{34} &= b_{14}(c_{31}+c_{33}) + b'_{13}(c_{32}+c_{34}), \\
d_{41} &= -\,d'_{32}, \\
d_{42} &= -\,d'_{31}, \\
d_{43} &= d'_{34}, \text{ and} \\
d_{44} &= d'_{33}.
\end{aligned}
\qquad \text{(IV.12)}
$$

Calculating the elements of the matrix $\tilde{d}$ from (IV.3.a,...,h), (IV.1.4.a,...,d), (IV.7.a,...,h), (IV.8.a,...,d) and (IV.12), we obtain (explicitly indicating the arguments of the functions):

$$
d_{jk} = \frac{j}{\Delta(|\underset{\to}{\beta}|)}\, \tilde{d}_{jk}\,(\underset{\to}{\beta},\underset{\to}{k})
$$

$$
\begin{aligned}
\tilde{d}_{11}(\underset{\to}{\beta},\underset{\to}{k}) &= \tilde{d}_{11a}(|\underset{\to}{\beta}|)k_y\beta_y + \frac{\omega}{c}\,[\tilde{d}_{11b}(|\underset{\to}{\beta}|)\beta_y^2 + \tilde{d}_{11c}(|\underset{\to}{\beta}|)], \\
\tilde{d}_{12}(\underset{\to}{\beta},\underset{\to}{k}) &= \tilde{d}_{12a}(|\underset{\to}{\beta}|)k_y\beta_x + \frac{\omega}{c}\,\tilde{d}_{12b}(|\underset{\to}{\beta}|)\beta_x\beta_y, \\
\tilde{d}_{13}(\underset{\to}{\beta},\underset{\to}{k}) &= \tilde{d}_{13a}(|\underset{\to}{\beta}|)k_y\beta_x + \frac{\omega}{c}\,\tilde{d}_{13b}(|\underset{\to}{\beta}|)\beta_x\beta_y, \\
\tilde{d}_{14}(\underset{\to}{\beta},\underset{\to}{k}) &= \tilde{d}_{14a}(|\underset{\to}{\beta}|)k_y\beta_y + \frac{\omega}{c}\,[\tilde{d}_{14b}(|\underset{\to}{\beta}|)\beta_y^2 + \tilde{d}_{14c}(|\underset{\to}{\beta}|)], \\
\tilde{d}_{31}(\underset{\to}{\beta},\underset{\to}{k}) &= \tilde{d}_{31a}(|\underset{\to}{\beta}|)k_y\beta_x + \frac{\omega}{c}\,\tilde{d}_{31b}(|\underset{\to}{\beta}|)\beta_x\beta_y,
\end{aligned}
$$

$$\widetilde{d}_{32}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) = \widetilde{d}_{32a}(|\underset{\rightarrow}{\beta}|)k_y\beta_y + \frac{\omega}{c}[\widetilde{d}_{32b}(|\underset{\rightarrow}{\beta}|)\beta_y^2 + \widetilde{d}_{32c}(|\underset{\rightarrow}{\beta}|)],$$

$$\widetilde{d}_{33}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) = \widetilde{d}_{33a}(|\underset{\rightarrow}{\beta}|)k_y\beta_y + \frac{\omega}{c}[\widetilde{d}_{33b}(|\underset{\rightarrow}{\beta}|)\beta_y^2 + \widetilde{d}_{33c}(|\underset{\rightarrow}{\beta}|)],$$

$$\widetilde{d}_{34}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) = \widetilde{d}_{34a}(|\underset{\rightarrow}{\beta}|)k_y\beta_x + \frac{\omega}{c}\widetilde{d}_{34b}(|\underset{\rightarrow}{\beta}|)\beta_x\beta_y,$$

$$d_{21} = \mathrm{d}'_{12},$$

$$d_{22} = d'_{11},$$

$$d_{23} = -d'_{14},$$

$$d_{24} = -d'_{13},$$

$$d_{41} = -d'_{32},$$

$$d_{42} = -d'_{31},$$

$$d_{43} = d'_{34}, \text{ and}$$

$$d_{44} = d'_{33}, \tag{IV.13}$$

where

$$\widetilde{d}_{11\genfrac{}{}{0pt}{}{a}{b}}(|\underset{\rightarrow}{\beta}|) = 1 - \nu^2,$$

$$\widetilde{d}_{11c}(|\underset{\rightarrow}{\beta}|) = 0,$$

$$\widetilde{d}_{12\genfrac{}{}{0pt}{}{a}{b}}(|\underset{\rightarrow}{\beta}|) = \nu^2 - 1,$$

$$\widetilde{d}_{13\genfrac{}{}{0pt}{}{a}{b}}(|\underset{\rightarrow}{\beta}|) = \widetilde{d}_{14\genfrac{}{}{0pt}{}{a}{b}}(|\underset{\rightarrow}{\beta}|) = -(\beta_z + \gamma_z),$$

$$\widetilde{d}_{14c}(|\underset{\rightarrow}{\beta}|) = \Delta(|\underset{\rightarrow}{\beta}|),$$

$$\widetilde{d}_{31\,{a \atop b}}(|\underset{\rightarrow}{\beta}|) = \widetilde{d}_{32\,{a \atop b}}(|\underset{\rightarrow}{\beta}| = \begin{bmatrix} \Delta(|\underset{\rightarrow}{\beta}|) \\ \Delta(|\underset{\rightarrow}{\beta}|) + (1-\nu^2)(\beta_z - \gamma_z) \end{bmatrix},$$

$$\widetilde{d}_{32c}(|\underset{\rightarrow}{\beta}|) = -\nu^2(\beta_z + \gamma_z),$$

$$\widetilde{d}_{33\,{a \atop b}} = \begin{bmatrix} 0 \\ 1-\nu^2 \end{bmatrix},$$

$$\widetilde{d}_{33c} = \beta_z\gamma_z(1-\nu^2), \text{ and}$$

$$\widetilde{d}_{34\,{a \atop b}} = \begin{bmatrix} 0 \\ \nu^2 - 1 \end{bmatrix}. \qquad \text{(IV.13)}'$$

Denoting $\widetilde{B}_1$ by $\widetilde{e}$ and $\widetilde{B}_o$ by f and invoking (IV.12), IV.13) and (IV.13)$'$, we obtain

$$a = j(\widetilde{e} - \frac{d}{\Delta} f) =$$

$$\frac{1}{\Delta}\left\{ \begin{bmatrix} \widetilde{e}_{11} & \widetilde{e}_{12} \\ -\widetilde{e}'_{12} & -\widetilde{e}'_{11} \\ \widetilde{e}_{31} & \widetilde{e}_{32} \\ \widetilde{e}'_{32} & \widetilde{e}'_{31} \end{bmatrix} \Delta - \begin{bmatrix} \widetilde{d}_{11} & \widetilde{d}_{12} & \widetilde{d}_{13} & \widetilde{d}_{14} \\ \widetilde{d}'_{12} & \widetilde{d}'_{11} & -\widetilde{d}'_{14} & -\widetilde{d}'_{13} \\ \widetilde{d}_{31} & \widetilde{d}_{32} & \widetilde{d}_{33} & \widetilde{d}_{34} \\ -\widetilde{d}_{32} & -\widetilde{d}'_{31} & \widetilde{d}'_{34} & \widetilde{d}'_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ f_{31} & f_{32} \\ f_{32} & f'_{31} \end{bmatrix} \right\}, \qquad \text{(IV.14)}$$

where

$$e_{11} = j\widetilde{e}_{11} = j\beta_z^{-1} k_y \beta_x,$$

$$e_{12} = j\widetilde{e}_{12} = j\beta_z^{-1}(k_y\beta_y - \frac{\omega\beta_z^2}{c}),$$

$$e_{31} = j\widetilde{e}_{31} = -j\beta_z^{-1}[k_y\beta_y\beta_z - \frac{\omega\beta_z}{c}(1-\beta_y^{\,2})],$$

$$e_{32} = je_{32} = j\beta_z^{-1}\beta_x[k_y\beta_z + \frac{\omega\beta_z}{c}\beta_y], \qquad \text{(IV.14)}'$$

$$f_{31} = -\left(\frac{1-\beta_y^{\,2}}{\beta_z}\right), \text{ and}$$

$$f_{32} = -\left(\frac{\beta_x\beta_y}{\beta_z}\right).$$

Expanding (IV.14), we have

$$
\begin{aligned}
\Delta a_{11} &= j[\tilde{e}_{11}\Delta + \tilde{d}_{12} - \tilde{d}_{13}f_{31} - \tilde{d}_{14}f_{32}], \\
\Delta a_{21} &= j[-\tilde{e}'_{12}\Delta + \tilde{d}'_{11} + \tilde{d}'_{14}f_{31} + \tilde{d}'_{13}f_{32}], \\
\Delta a_{31} &= j[\tilde{e}_{31}\Delta + \tilde{d}_{32} - \tilde{d}_{33}f_{31} - \tilde{d}_{34}f_{32}], \\
\Delta a_{41} &= j[\tilde{e}'_{32}\Delta - \tilde{d}'_{31} - \tilde{d}'_{34}f_{31} - \tilde{d}'_{33}f_{32}], \\
\Delta a_{12} &= e_{12}\Delta - d_{11} - d_{13}f_{32} - d_{14}f'_{31} = -\Delta a'_{21}, \\
\Delta a_{22} &= -e'_{11}\Delta - d'_{12} + d'_{14}f_{32} + d'_{13}f'_{31} = -\Delta a'_{11}, \\
\Delta a_{32} &= e_{32}\Delta - d_{31} - d_{33}f_{32} - d_{34}f'_{31} = \Delta a'_{41}, \text{ and} \\
\Delta a_{42} &= e'_{31}\Delta + d'_{32} - d'_{34}f_{32} - d'_{33}f'_{31} = \Delta a'_{31}.
\end{aligned}
\tag{IV.15}
$$

From (IV.13), (IV.13)′ and (IV.15) we obtain, after considerable manipulation (again explicitly indicating function arguments),

$$
a_{jk}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) = \frac{1}{\Delta(|\underset{\rightarrow}{\beta}|)}\,\tilde{a}_{jk}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}),
\tag{IV.16}
$$

where

$$
\begin{aligned}
\tilde{a}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= \tilde{a}_{11a}(|\underset{\rightarrow}{\beta}|)k_y\beta_x + \frac{\omega}{c}\tilde{a}_{11b}(|\underset{\rightarrow}{\beta}|)\beta_x\beta_y, \\
\tilde{a}_{21}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= \tilde{a}_{21a}(|\underset{\rightarrow}{\beta}|)k_x\beta_x + \frac{\omega}{c}[\tilde{a}_{21b}(|\underset{\rightarrow}{\beta}|)\beta_x{}^2 + \tilde{a}_{21c}(|\underset{\rightarrow}{\beta}|)], \\
\tilde{a}_{31}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= \tilde{a}_{31a}(|\underset{\rightarrow}{\beta}|)k_y\beta_y + \frac{\omega}{c}[\tilde{a}_{31b}(|\beta|)\beta_y{}^2 + \tilde{a}_{31c}(|\underset{\rightarrow}{\beta}|)], \\
\tilde{a}_{41}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= \tilde{a}_{41a}(|\underset{\rightarrow}{\beta}|)k_x\beta_y + \frac{\omega}{c}\tilde{a}_{41b}(|\underset{\rightarrow}{\beta}|)\beta_x\beta_y, \\
a_{12} &= -a'_{21}, \\
a_{22} &= -a'_{11}, \\
a_{32} &= a'_{41}, \text{ and} \\
a_{42} &= a'_{31},
\end{aligned}
$$

and where

$$
\begin{aligned}
\tilde{a}_{11\substack{a\\b}}(|\underset{\rightarrow}{\beta}|) &= 2(\nu^2 - 1),\\
\tilde{a}_{21\substack{a\\b}}(|\underset{\rightarrow}{\beta}|) &= -2(\nu^2 - 1),\\
\tilde{a}_{31\substack{a\\b}}(|\underset{\rightarrow}{\beta}|) &= \begin{bmatrix} 0 \\ 2(\nu^2 - 1)(\gamma_z - \beta_z) \end{bmatrix},\\
\tilde{a}_{31c}(|\underset{\rightarrow}{\beta}|) &= -2(\nu^2 - 1)\gamma_z,\\
\tilde{a}_{41\substack{a\\b}}(|\underset{\rightarrow}{\beta}|) &= \begin{bmatrix} 0 \\ -2(\nu^2 - 1)(\gamma_z - \beta_z) \end{bmatrix},\\
\tilde{a}_{12}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= -(\tilde{a}_{21}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}))',\\
\tilde{a}_{22}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= (\tilde{a}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}))',\\
\tilde{a}_{32}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= (\tilde{a}_{41}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}))', \text{ and}\\
\tilde{a}_{42}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= (\tilde{a}_{31}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}))'.
\end{aligned}
\tag{IV.16)'}
$$

Using the above results we can write (IV.16) more compactly as follows:

$$
a_{jk}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) = \frac{j}{\Delta(|\underset{\rightarrow}{\beta}|)}\,\tilde{a}_{jk}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}), \tag{IV.17}
$$

where

$$
\begin{aligned}
\tilde{a}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= \beta_x\left\{\tilde{a}_{11a}(|\underset{\rightarrow}{\beta}|)k_y + \frac{\omega}{c}\tilde{a}_{11b}(|\underset{\rightarrow}{\beta}|)\beta_y\right\},\\
\tilde{a}_{21}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= -\beta_x[\tilde{a}_{11}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k})]' + \frac{\omega}{c}\tilde{a}_{21c}(|\underset{\rightarrow}{\beta}|),\\
\tilde{a}_{31}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= -\beta_y[\tilde{a}_{41}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k})]' + \frac{\omega}{c}\tilde{a}_{31c}(|\underset{\rightarrow}{\beta}|), \text{ and}\\
\tilde{a}_{41}(\underset{\rightarrow}{\beta},\underset{\rightarrow}{k}) &= -\beta_y\left\{\tilde{a}_{31a}(|\underset{\rightarrow}{\beta}|)k_x + \frac{\omega}{c}\tilde{a}_{31b}(|\underset{\rightarrow}{\beta}|)\beta_x\right\}.
\end{aligned}
$$

To specialize the results to the backscattering case, we first note that the unit vector $\underset{\to}{\alpha}_o$ is defined as that directed from the radar to the illuminated patch. In vector-matrix notation

$$\underset{\to}{\alpha}_o = \begin{bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oz} \end{bmatrix}, \tag{IV.18.a}$$

where $\alpha_{oz} = -\sqrt{1 - \alpha_{ox}{}^2 - \alpha_{oy}{}^2}$.

We now define a two-dimensional unit vector $\hat{\underset{\to}{\alpha}}_o$, the projection of $\underset{\to}{\alpha}_o$ on the x-y plane, as

$$\hat{\underset{\to}{\alpha}}_o = \begin{bmatrix} \alpha_{ox} \\ \alpha_{oy} \end{bmatrix}. \tag{IV.18.b}$$

For backscattering of an incident plane wave or plane wave pulse, the unit vector corresponding to the direction of the incident wave is $+\underset{\to}{\alpha}_o$, while the unit vector corresponding to the direction of the backscattered wave is $-\underset{\to}{\alpha}_o$ (see 2.1c). Then,

$$\underset{\to}{\beta}_1 = \begin{bmatrix} \beta_{1x} \\ \beta_{1y} \end{bmatrix} = \begin{bmatrix} \alpha_{ox} \\ \alpha_{oy} \end{bmatrix} = \hat{\underset{\to}{\alpha}}_o, \text{ and} \tag{IV.19.a}$$

$$\underset{\to}{\beta} = \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} \equiv \begin{bmatrix} -\alpha_{ox} \\ -\alpha_{oy} \end{bmatrix} = -\hat{\underset{\to}{\alpha}}_o. \tag{IV.19.b}$$

Continuing the specialization to the backscattering case, we invoke (IV.19.a,b) in the expression (IV.10)′ resulting in an expression of the form

$$[\underset{H}{\tilde{E}_{rV}}(-\hat{\underset{\to}{\alpha}}_o)]_1 = [\underset{H}{\tilde{S}_V}(-\hat{\underset{\to}{\alpha}}_o;\hat{\underset{\to}{\alpha}}_o)]_1\, \tilde{\tilde{E}}_i(\hat{\underset{\to}{\alpha}}_o) Z(-\frac{2\omega}{c}\hat{\underset{\to}{\alpha}}_o). \tag{IV.20}$$

To continue the specialization to backscattering, we invoke (IV.19.a,b) in the expression (IV.9) and carry out the matrix operations

$$[\underset{H}{\tilde{E}_{rV}}(-\hat{\underset{\to}{\alpha}}_o)]_k = \underset{H}{\underset{\to}{\ell}{}^T_V}\, Q(-\hat{\underset{\to}{\alpha}}_o) b_o(-\hat{\underset{\to}{\alpha}}_o) B(\hat{\underset{\to}{\alpha}}_o) L(\hat{\underset{\to}{\alpha}}_o), \tag{IV.21}$$

where $\hat{B}(\hat{\underset{\to}{\alpha}}_o)$ is a 4×2 matrix which will be specified below, where $\underset{\to}{\ell}_V$ and $\underset{\to}{\ell}_H$ are given by (I.24.a,b) or (I.27), $Q(-\hat{\underset{\to}{\alpha}}_o)$ is given in (2.17) with

$$\beta_{\substack{x\\y}} = -\alpha_{o\substack{x\\y}},$$

$\beta_z = -\alpha_{oz}$, $b_o(-\hat{\underset{\to}{\alpha}}_o)$ is $[A_o(\underset{\to}{\beta}_1,\omega)]^{-1}$ as given in (2.16-0)$'$ with

$$\beta_{\substack{x\\y}} = \alpha_{o\substack{x\\y}},$$

$\beta_z = -\alpha_{oz}$, and $L(\hat{\underset{\to}{\alpha}}_o)$ is given in (2.19) with

$$\beta_{\substack{x\\y}} = \alpha_{o\substack{x\\y}}, \beta_z = -\alpha_{oz}.$$

The indicated operations lead to an expression of the form (3.7.1). The $\widetilde{E}$'s appearing in (IV.9), (IV.10), and (IV.20) are designated are U's in (3.7.1) and the quantities

$\widetilde{S}_{\substack{V\\H}}$, etc., appearing in (IV.20) are the quantities S_{Aa1}

as given in (3.7.1), where $Aa = VV$, VH, HV or HH, depending on incident and received polarizations.

These quantities S_{Aa1} are obtained by carrying out the calculations indicated in (IV.21) with the aid of (2.16-0)$'$, (2.17), (2.18.a,b) and with the appropriate specialization of

$\beta_{\substack{x\\y\\z}}$, as follows:

$$[S_{\substack{VV\\VH\\HV\\HH}}(-\hat{\underset{\to}{\alpha}}_o)] = \frac{\pm\ 1}{\pm\ \Delta'(\underset{\to}{\ell}'_V x \underset{\to}{\ell}'_H)_z} \begin{bmatrix} (\widetilde{S}_{VV}) - (\widetilde{S}_{VV})' \\ VH \quad\quad VH \\ HV \quad\quad HV \\ HH \quad\quad HH \end{bmatrix}, \tag{IV.22}$$

where (primes on the elements $(\widetilde{B}_k)'_{\ell m}$ indicate x-y interchange)

$$
\begin{aligned}
(\widetilde{S}_{\substack{VV\\VH\\HV\\HH}}) = \ell'_{\substack{H\\V\\H\\V}X} \Bigg[& (\underset{\to}{\alpha}_o x \underset{\to}{\ell}'_{\substack{V\\V\\H\\H}})_z \Big(-(\alpha_{oz} + \gamma_{oz}) \left\{ \alpha_{oy} (\hat{B})'_{11} + \alpha_{ox} (\hat{B})'_{21} \right\} \\
& + \left\{ \alpha_{ox} (\hat{B})'_{31} - \alpha_{oy} (\hat{B})'_{41} \right\} \Big) - \alpha_{oz} \nu^2 \left\{ \ell'_{\substack{V\\V\\H\\H}x} (\hat{B})'_{11} - \ell'_{\substack{V\\V\\H\\H}y} (\hat{B})'_{21} \right\} \\
& - \alpha_{oz}\gamma_{oz} \left\{ \ell'_{\substack{V\\V\\H\\H}y} (\hat{B})'_{31} + \ell'_{\substack{V\\V\\H\\H}x} (B)'_{41} \right\} \Bigg], \text{ and}
\end{aligned}
$$

$$
(\widetilde{S}_{\substack{VV\\VH\\HV\\HH}})' = \text{same as } (\widetilde{S}_{\substack{VV\\VH\\HV\\HH}})
$$

except that x and y are everywhere interchanged.

$$
(\underset{\to}{\alpha}_o x \underset{\to}{\ell}')_z = z \text{ component of cross-product of } \underset{\to}{\alpha}_o \text{and } \underset{\to}{\ell}', \text{ equal to } (\underset{\to}{\alpha}_{ox} \ell'_y - \alpha_{oy} \underset{\to}{\ell}'_x).
$$

$$
\Delta' = (\gamma_{oz} - \alpha_{oz})(1 - \alpha^2_{oz} - \alpha_{oz}\gamma_{oz}) = \Delta(|\underset{\to}{\hat{\alpha}}_o|)
$$

$$
(\underset{\to}{\ell}'_V x \underset{\to}{\ell}'_H)_z = \ell'_{Vx}\ell'_{Hy} - \ell'_{Vy}\ell'_{Hx}.
$$

The quantity $[S_{VV}(-\underset{\to}{\hat{\alpha}})]$ given in (IV.22) is the element of the first order "polarization matrix" corresponding to vertically polarized reception and vertically polarized transmission, where the illuminated surface patch is at an angle $\underset{\to}{\alpha}_o$ with respect to the radar. The subscript VH indicates vertically polarized reception, horizontally polarized transmission, HV indicates horizontally polarized reception, vertically polarized transmission and HH indicates that both reception and transmission are horizontally polarized.

We now turn to the evaluation of the matrix element $\hat{B}$ appearing in (IV.22). In the first order case, these are the elements of the matrix $\widetilde{a}$ given in (IV.17) with $\underset{\to}{k}$ set equal to

$-\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o$ and $\underset{\rightarrow}{\beta}$ set equal to $\underset{\rightarrow}{\hat{\alpha}}_o$. From (IV.16)′, (IV.17), and (IV.22) with the above specifications of the variables, we have

$$
[S_{\begin{smallmatrix}VV\\ VH\\ HV\\ HH\end{smallmatrix}}(-\underset{\rightarrow}{\hat{\alpha}}_o)] = \frac{j(\frac{\omega}{c})}{[\Delta(|\underset{\rightarrow}{\hat{\alpha}}_o|)]^2(\underset{\rightarrow}{\ell}'_V x \underset{\rightarrow}{\ell}'_H)_z}
$$

$$
\Bigg\{ (\underset{\rightarrow}{\alpha}_o x \underset{\rightarrow}{\ell}'_{\begin{smallmatrix}H\\ V\\ H\\ V\end{smallmatrix}})_z(\underset{\rightarrow}{\hat{\alpha}}_o \cdot \ell'_{\begin{smallmatrix}V\\ V\\ H\\ H\end{smallmatrix}})\alpha_{oz}\nu^2[-2\widetilde{a}_{11a}+\widetilde{a}_{11b}] + (\underset{\rightarrow}{\alpha}_o x \underset{\rightarrow}{\ell}'_{\begin{smallmatrix}V\\ V\\ H\\ H\end{smallmatrix}})_z(\underset{\rightarrow}{\hat{\alpha}}_o \cdot \ell'_{\begin{smallmatrix}H\\ V\\ H\\ V\end{smallmatrix}} \tag{IV.23}
$$

$$
\Big([-2\widetilde{a}_{31a}+\widetilde{a}_{31b}][1-\alpha_{oz}^2-\alpha_{oz}\gamma_{oz}] + [\widetilde{a}_{31c}-(\alpha_{oz}+\gamma_{oz})\widetilde{\alpha}_{21c}]\Big)
$$

$$
+ (\underset{\rightarrow}{\ell}'_H x \underset{\rightarrow}{\ell}'_V)_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \alpha_{oz}[\nu^2\widetilde{a}_{21c} - \gamma_{oz}\widetilde{a}_{31c}] \Bigg\},
$$

where

$$
\widetilde{a}_{\begin{smallmatrix}11a\\ 11b\\ 21c\\ 31a\\ 31b\\ 31c\end{smallmatrix}} = 2(\nu^2-1)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ (\gamma_{oz}+\alpha_{oz}) \\ -\gamma_{oz} \end{bmatrix}, \tag{IV.24.a}
$$

$$
-2\widetilde{a}_{11a}+\widetilde{a}_{11b} = -2(\nu^2-1), \text{ and} \tag{IV.24.b}
$$

$$
-2a_{31a}+\widetilde{a}_{31b} = 2(\nu^2-1)(\gamma_{oz}+\alpha_{oz}). \tag{IV.24.c}
$$

We can arrive at the result (5.1) by substituting (IV.24.a,b,c) into (IV.23) or alternatively by using (IV.16)′ in (IV.17) to obtain the $\widetilde{a}$ elements directly with

$$
\underset{\rightarrow}{k} = -\frac{2\omega}{c}\underset{\rightarrow}{\hat{\alpha}}_o, \underset{\rightarrow}{\beta} = \underset{\rightarrow}{\hat{\alpha}}_o
$$

and substituting those results directly into (IV.22) in place of the elements B_{jk}. If we choose the latter course, then the resulting elements are:

$$\hat{B}_{jk} = a_{jk}(\underset{\to}{\hat{\alpha}}_o; -\frac{2\omega}{c}\underset{\to}{\hat{\alpha}}_o) = \frac{j}{\Delta(|\underset{\to}{\hat{\alpha}}|_o)} \underset{\to}{\hat{a}}_{jk}(\underset{\to}{\hat{\alpha}}_o; -\frac{2\omega}{c}\underset{\to}{\hat{a}}_o)$$

where

$$\tilde{a}_{11}(\underset{\to}{\hat{\alpha}}_o; -\frac{2\omega}{c}\underset{\to}{\hat{\alpha}}_o) = -\frac{2\omega}{c}(\nu^2 - 1)\alpha_{ox}\alpha_{oy},$$

$$\tilde{a}_{21}(\underset{\to}{\hat{\alpha}}_o; -\frac{2\omega}{c}\underset{\to}{\hat{\alpha}}_o) = \frac{2\omega}{c}(\nu^2 - 1)\alpha_{ox}^2,$$

$$\tilde{a}_{31}(\underset{\to}{\hat{\alpha}}_o; -\frac{2\omega}{c}\underset{\to}{\hat{\alpha}}_o) = \frac{2\omega}{c}(\nu^2 - 1)[(\alpha_{oz} + \gamma_{oz})\alpha_{oy}^2 - \gamma_{oz}], \text{ and}$$

$$\tilde{a}_{41}(\underset{\to}{\hat{\alpha}}_o; -\frac{2\omega}{c}\underset{\to}{\hat{\alpha}}_o) = -\frac{2\omega}{c}(\nu^2 - 1)[(\alpha_{oz} + \gamma_{oz})\alpha_{ox}\alpha_{oy}].$$

Substitution of (IV.25) into (IV.22) or the alternative approach through (IV.23) and (IV.24.a,b,c) yields (5.1).

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