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Smooth Limiting of Two Signals in a Narrowband System

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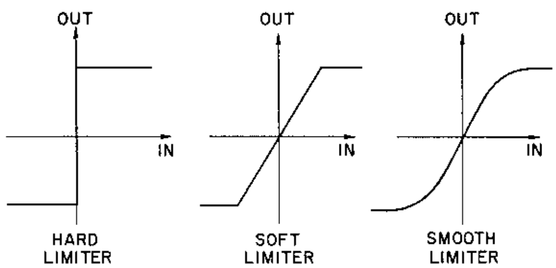
SMOOTH LIMITING OF TWO SIGNALS IN A NARROWBAND SYSTEM

INTRODUCTION

The hard limiter has seen widespread use in acoustic signal processing. It has been used as a two-level quantizer; it has been used to remove random amplitude fluctuations while preserving phase information; it has even been used to enhance signal detectability (signal-to-noise ratio). For example, in an impulsive noise environment the two-input polarity coincidence correlator, which employs hard limiting on both inputs, actually outperforms a correlator without hard limiters (1).



Hard limiting of signals has been studied by numerous investigators (2-6).* Perhaps because much is known about hard limiting of signals, limiting circuits are frequently characterized as ideal hard limiters even when such a characterization is not fully appropriate. For example, limiting action is observed in a saturated amplifier — a circuit with a linear small-signal region and a gradual saturation. The saturated amplifier has also been characterized as a "soft limiter" which limits abruptly at some threshold and passes, without distortion, signals below the threshold. Soft limiters too are not entirely realistic since practical amplifiers rarely display this abrupt limiting property. In order to analyze circuits with a gradual saturation, it would seem desirable to represent the circuit transfer function with a smooth curve.

It is the intention of this report to consider a more realistic model of limiters in general with the input consisting of the sum of two signals. A particularly realistic representation of a practical limiter is a "smooth" transfer function in the form of the error function

$$\operatorname{erf}\,(a/\sqrt{2}\beta) = 2/\sqrt{\pi}\int_0^{a/\sqrt{2}\beta} \exp\,(-t^2)\;dt\;,$$

as shown in Fig. 1, where a is the input signal amplitude and β is the parameter which determines the "hardness" of the limiter.

*The references cited here are by no means exhaustive.

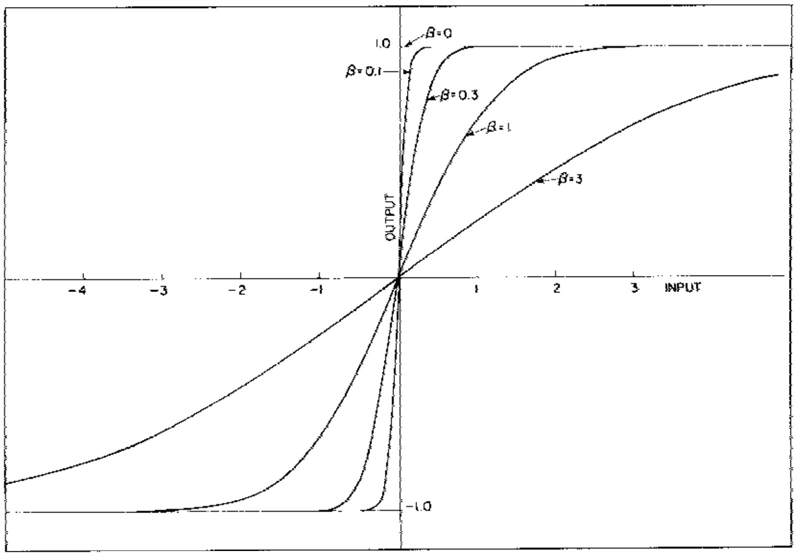


Fig. 1 - Transfer characteristics of error function limiter.

Lieberman (7) and Baum (8) considered an error function limiter and computed the output autocorrelation function in a closed form when the input is Gaussian noise. Galejs (9) examined the output signal-to-noise ratio of a narrow-bandpass error function limiter for an input of a periodic signal plus random noise, and the results were related to those of a corresponding linear system (limiter is replaced by a linear amplifier). Lee (10) obtained the expression for the signal-power-to-crosstalk-power ratio at the output of an error function limiter when the input is a multichannel frequency division multiplex signal.

We consider an error function limiter with two input cases: sine wave plus sine wave and sine wave plus narrowband Gaussian noise. Corresponding to various input ratios, output ratios of inband signals are obtained as a function of $q \equiv a/\beta$ where the parameter q specifies the degree of the limiter "hardness" relative to the input signal amplitude. An advantage of the error function representation of the limiter is that the analysis leads to results which correspond to a wide range of practical limiter shapes, ranging from the linear amplifier to the hard limiter. It is shown that the results corresponding to $q = \infty$ are those known for the ideal hard limiter.

The analysis in this report is based on the method of analyzing "bandpass nonlinearities" suggested by Blachman (11), a method which has proven to be extremely powerful (12,13).

The expressions obtained in this study were evaluated numerically on the Honeywell DDP-24 digital computer and plotted by the Calcomp 563 plotter.

GENERAL FORMULATION

We are considering a bandpass nonlinear device which is memoryless. The system presentation is given in Figure 2. The input and output bandpass filters are essentially identical. The input filter is shown only to emphasize that we are considering inputs whose spectral occupancy lies within the passband of the output filter.

Fig. 2 - Bandpass nonlinear device.

The output $y(t)$ of the nonlinear device is a function of the input $x(t)$ and we may write

$$y(t) = g[x(t)] \tag{1}$$

where the function g relates the input to the output instantaneously. Now let us assume that the input is given by

$$x(t) = A(t) \cos [2\pi f t + \Phi(t)] \tag{2}$$

where $A(t)$ and $\Phi(t)$ may be specified appropriately if $x(t)$ is to represent (a) AM-DSB/SC signal, (b) phase-modulated (PM) signal, (c) frequency-modulated (FM) signal, (d) single-tone carrier, or (e) narrowband Gaussian process. If we let

$$\theta(t) = 2\pi f t + \Phi(t) \tag{3}$$

and substitute Eq. (2) into Eq. (1), we obtain

$$y(t) = g(A \cos \theta) \tag{4}$$

where A and θ are both functions of the time t.

Since for any A, Eq. (4) is an even function of θ, we may expand it in a Fourier series (11,13):

$$y(t) = (1/2)\, B_0(A) + B_1(A) \cos \theta + B_2(A) \cos 2\theta + \ldots \tag{5}$$

where

$$B_k(A) = 1/\pi \int_{-\pi}^{\pi} g(A \cos \theta) \cos k\theta \, d\theta \ . \tag{6}$$

It is apparent from Eq. (5) that $B_k(A)$ is the kth harmonic amplitude. Now let us make the assumption that the second term in Eq. (5) is the only one present in the passband of the output filter. In other words, the bandwidth of $B_1(A)$ is narrow compared to the frequency f_1. Furthermore, the bandwidths of the other $B_k(A)$, $k \neq 1$, terms are narrow enough so that in the region of the spectrum occupied by $B_1(A) \cos \theta$ there is no significant contribution from any other term. Thus the output of the bandpass filter centered at f_1, and assumed to have an ideal rectangular passband of sufficient width to pass only the $B_1(A)$ term, is

$$Z(t) \approx B_1(A) \cos \theta = B_1(A) \cos [2\pi f t + \Phi(t)] \ . \tag{7}$$

From Eq. (2) above we have

$$\cos [2\pi f t + \Phi(t)] = x(t)/A(t) \ , \tag{8}$$

and putting Eq. (8) into Eq. (7) we get

$$z(t) = \left(B_1(A)/A\right) x(t) = \left(B_1(A)/A\right) \times (\text{input}). \tag{9}$$

We can omit the subscript from B_1 since we will henceforth consider only the case of $k = 1$. It is clear that Eq. (9) is the formula to be used for any particular input. In the ensuing analysis, we will consider the two-input-signal case, but the necessity of identifying the nonlinear device with a specific transfer characteristic, i.e., specifying $B(A)$, does not arise at this time.

TWO SINUSOIDAL INPUTS

General

When the input is assumed to be the sum of two sinusoids of constant magnitudes a_1 and a_2 and frequencies f_1 and f_2, respectively, we have

$$x(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t) \tag{10}$$

where f_1 and f_2 both lie within the passband of the system. Using the law of cosines, the amplitude of $x(t)$ is obtained as (see Fig. 3)

$$A(t) = [a_1^2 + a_2^2 + 2a_1 a_2 \cos 2\pi(f_2 - f_1)t]^{1/2}. \tag{11}$$

Thus the output of the bandpass filter is, from Eq. (9),

$$Z(t) = \left(B(A)/A\right) \times (\text{input})$$

$$= \frac{B\left(\sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos 2\pi(f_2 - f_1)t}\right)}{\sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos 2\pi(f_2 - f_1)t}} \times (a_1 \cos 2\pi f_1 t + a_2 \cos 2\pi f_2 t). \tag{12}$$

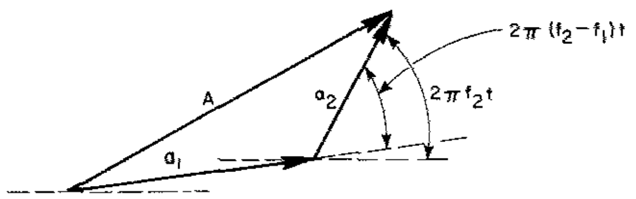


Fig. 3 - Instantaneous phasor representation of two signals.

The next step may be suggested by examining the implications of Eq. (12). Equation (12) represents two amplitude modulations (AM-DSB), each with a "modulation signal" of fundamental frequency $f_2 - f_1$, and the two carriers are f_1 and f_2. One might note that it is a highly unusual amplitude modulation that Eq. (12) represents since the modulation signal appears to be subjected to a complex nonlinear process (which produces all-order harmonics of $f_2 - f_1$) before being multiplied by the carriers. But that is not the point of our concern, for it is not the AM-DSB system that we are interested in as far as Eq. (12) is concerned. The fact is, however, that the AM-DSB signal yields frequency components having frequencies $\pm(nf_1 - (n-1) f_2)$ at the nonlinear device output, where n takes some integer (not necessarily positive) value.

Since Eq. (12) is an even function of t, we can express the output as a sum of cosine terms:

$$Z(t) = \sum_{n=-\infty}^{\infty} c_n \cos 2\pi(nf_1 - (n-1) f_2)\, t\ . \tag{13}$$

Since this is not an orthogonal expansion in general, that is, $\cos 2\pi(nf_1 - (n-1) f_2)\, t$ is not necessarily orthogonal to $\cos 2\pi\,[mf_1 - (m-1) f_2]\, t$, the method by which the c_n is evaluated may not be obvious at this point. However, the appendix contains a derivation based on a complex Fourier series representation of the complex envelope of the analytic signal,

$$\tilde{Z}(t) = Z(t) + j\hat{Z}(t) \tag{14}$$

where $\hat{Z}(t)$ is the Hilbert transform $Z(t)$. From the appendix we thus obtain for the coefficient of the component term of frequency $nf_1 - (n-1) f_2$

$$c_n \approx \frac{1}{\pi} \int_0^{\pi} \frac{B\left(\sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos\gamma}\right)}{\sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos\gamma}} \left(a_1 \cos(n-1)\gamma + a_2 \cos n\gamma\right) d\gamma \tag{15}$$

where

$$\gamma = 2\pi(f_2 - f_1)\, t\ .$$

We have succeeded in obtaining a prediction formula for the amplitude of the signals that exist in the passband of the filter output. Equation (15) may be simplified when we consider the two cases:

(a) $a_2 << a_1$ (one signal is weaker than the other)

(b) $a_1 = a_2 = a$ (equal-amplitude signals).

One Input Signal Relatively Weak

Let us first consider the case where $a_2 << a_1$. This condition implies that

$$1 \pm (a_2/a_1)^2 \approx 1\ ,$$

and hence Eq. (11) simplifies to

$$\begin{aligned} A(t) &= \left[a_1^2 + a_2^2 + 2a_1a_2 \cos 2\pi(f_2 - f_1)\, t\right]^{1/2} \\ &\approx a_1 + a_2 \cos 2\pi(f_2 - f_1)\, t\ , \end{aligned} \tag{16}$$

and

$$\frac{B(A)}{A} \approx \frac{B\,[a_1 + a_2 \cos 2\pi(f_2 - f_1)\, t]}{a_1 + a_2 \cos 2\pi(f_2 - f_1)\, t}\ . \tag{17}$$

The numerator of Eq. (17) can be expanded in a Taylor series about a_1. Using the series expansion up to and including the first-order term, Eq. (17) becomes

$$\frac{B(A)}{A} \approx \frac{B(a_1) + B'(a_1)\, a_2 \cos 2\pi (f_2 - f_1)\, t}{a_1 \left[1 + \frac{a_2}{a_1} \cos 2\pi (f_2 - f_1)\, t\right]}$$

$$\approx \left(B(a_1) + B'(a_1) a_2 \cos 2\pi (f_2 - f_1)\, t\right) \frac{1}{a_1} \left(1 - \frac{a_2}{a_1} \cos 2\pi (f_2 - f_1)\, t\right)$$

$$= \frac{B(a_1)}{a_1} + a_2 \cos 2\pi (f_2 - f_1)\, t \left(\frac{B'(a_1)}{a_1} - \frac{B(a_1)}{a_1^2}\right) - \left(\frac{a_2}{a_1}\right)^2 B'(a_1) \cos^2 2\pi (f_2 - f_1)\, t$$

$$\approx \frac{B(a_1)}{a_1} + a_2 \cos 2\pi (f_2 - f_1)\, t \left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \tag{18}$$

where the prime indicates the derivative with respect to the whole argument. Thus

$$Z(t) = (B(A)/A) \times \text{input}$$

becomes

$$Z(t) \approx \left(\frac{B(a_1)}{a_1} + a_2 \left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \cos 2\pi (f_2 - f_1)\, t\right) \times \left(a_1 \cos 2\pi f_1 t + a_2 \cos 2\pi f_2 t\right)$$

$$= B(a_1) \cos 2\pi f_1 t + a_2 \frac{B(a_1)}{a_1} \cos 2\pi f_2 t$$

$$+ a_2 a_1 \left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \cos 2\pi (f_2 - f_1)\, t \cos 2\pi f_1 t$$

$$+ a_2^2 \left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \cos 2\pi (f_2 - f_1)\, t \cos 2\pi f_2 t$$

$$= B(a_1) \cos 2\pi f_1 t + \frac{a_2 B(a_1)}{a_1} \cos 2\pi f_2 t$$

$$+ \frac{a_2 a_1}{2} \left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \cos 2\pi (2f_1 - f_2)\, t$$

$$+ \frac{a_2 a_1}{2} \left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \cos 2\pi f_2 t$$

$$+ \left(\frac{a_2}{a_1}\right)^2 \left(a_1 B'(a_1) - B(a_1)\right) \cos 2\pi (f_2 - f_1)\, t \cos 2\pi f_2 t \tag{19 (Cont.)}$$

$$\approx B(a_1)\cos 2\pi f_1 t + \left[\frac{a_2 B(a_1)}{a_1} + \frac{a_2 a_1}{2}\left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right)\right]\cos 2\pi f_2 t$$

$$+ \frac{a_2 a_1}{2}\left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right)\cos 2\pi(2f_1 - f_2)t$$

$$= B(a_1)\cos 2\pi f_1 t + \frac{a_2}{2a_1}\left\{\left(\frac{d}{da_1}\right)[a_1 B(a_1)]\right\}\cos 2\pi f_2 t$$

$$+ \frac{a_2 a_1}{2}\left[\left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right)\right]\cos 2\pi(2f_1 - f_2)t \; . \tag{19}$$

Equation (19) reveals the amplitudes of the fundamentals (f_1 and f_2) and third-order intermodulation ($2f_1 - f_2$) quite explicitly, and hence it is not necessary to use Eq.(15) for this case of one signal relatively weak.

Until now we have discussed the bandpass nonlinearity without specifying the form of the nonlinear device. We will apply the general result obtained to a specific nonlinearity, namely, an error function limiter. By an error function limiter, we mean

$$B(A) = \pi \operatorname{erf}\left(\frac{A}{\sqrt{2}\,\beta}\right), \tag{20}$$

where β is a parameter which specifies the degree of hardness in the limiter shape. The error function is defined by

$$\operatorname{erf}(z) \triangleq \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)\,dt \; ;$$

thus

$$B(A) = 2\sqrt{\pi}\int_0^{A/\sqrt{2}\beta} \exp(-t^2)\,dt \, . \tag{21}$$

From Eq. (19), which gives the expression for $z(t)$ when $a_2 << a_1$, we have

$$z(t) = \frac{B(A)}{A}\,x(t)$$

$$= C_1 \cos 2\pi f_1 t + C_0 \cos 2\pi f_2 t + C_2 \cos 2\pi(2f_1 - f_2)t \tag{22}$$

where

$$C_1 = B(a_1) \quad \text{is the output amplitude at frequency } f_1 \, , \tag{23}$$

$$C_0 = \frac{a_2}{2a_1}\left(\frac{d}{da_1}\right)[a_1 B(a_1)] \quad \text{is the output amplitude at frequency } f_2 \, , \tag{24}$$

and

$$C_2 = \frac{a_1 a_2}{2}\left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \quad \text{is the amplitude of third-order intermodulation term .} \tag{25}$$

Now by Eq. (20) we have

$$C_1 = B(a_1) = \pi \operatorname{erf}\left(\frac{a_1}{\sqrt{2}\beta}\right), \tag{26}$$

and

$$\begin{aligned} C_0 &= \frac{a_2}{2a_1}\left(\frac{d}{da_1}\right)[a_1 B(a_1)] \\ &= \frac{a_2}{2}\left(\frac{B(a_1)}{a_1} + B'(a_1)\right) \\ &= \frac{\pi a_2}{2a_1}\left\{\operatorname{erf}\left(\frac{a_1}{\sqrt{2}\beta}\right) + a_1\left(\frac{d}{da_1}\right)\left[\operatorname{erf}\left(\frac{a_1}{\sqrt{2}\beta}\right)\right]\right\}. \end{aligned} \tag{27}$$

But

$$\begin{aligned} \left(\frac{d}{da_1}\right)\left[\operatorname{erf}\left(\frac{a_1}{\sqrt{2}\beta}\right)\right] &= \left(\frac{d}{da_1}\right)\left(\frac{2}{\sqrt{\pi}}\int_0^{a_1/\sqrt{2}\beta} \exp(-t^2)\, dt\right) \\ &= \frac{2/\sqrt{\pi}}{\sqrt{2}\beta}\exp\left(-\frac{a_1^2}{2\beta^2}\right) \\ &= \frac{\sqrt{2/\pi}}{\beta}\exp\left(\frac{-a_1^2}{2\beta^2}\right). \end{aligned} \tag{28}$$

Therefore

$$C_0 = \frac{\pi a_2}{2a_1}\left[\operatorname{erf}\left(\frac{a_1}{\sqrt{2}\beta}\right) + \sqrt{\frac{2}{\pi}}\,\frac{a_1}{\beta}\exp\left(-\frac{a_1^2}{2\beta^2}\right)\right]. \tag{29}$$

Then from Eqs. (26) and (29) we obtain

$$\frac{C_0}{C_1} = \frac{1}{2}\left(\frac{a_2}{a_1}\right)\frac{\operatorname{erf}(q/\sqrt{2}) + \sqrt{\frac{2}{\pi}}\, q \exp(-q^2/2)}{\operatorname{erf}(q/\sqrt{2})}, \tag{30}$$

where

$$q \triangleq a_1/\beta .$$

The parameter q specifies the relative "hardness" of the smooth limiter. Since the limiting action of the device depends on the signal amplitude level, the parameter q may be termed the "limiter level parameter."

Note that C_0/C_1 is the output ratio of the two signals corresponding to the input ratio a_2/a_1. Define the "suppression factor" ρ_1 by

$$\rho_1 \triangleq (C_0/C_1)/(a_2/a_1) \ .$$

Then Eq. (30) gives

$$\rho_1 = \frac{1}{2}\,\frac{\operatorname{erf}(q/\sqrt{2}) + \sqrt{\frac{2}{\pi}}\, q \exp(-q^2/2)}{\operatorname{erf}(q/\sqrt{2})} \ . \tag{31}$$

Equation (31) is plotted in Fig. 4 as a function of q. Note that as $q \to \infty$, which is satisfied when $\beta \to 0$ (hard limiter), $\rho_1 \to 1/2$ (-6 dB) — a well-known result (4). In fact, for any weak signal including a Gaussian process in the presence of strong unmodulated sine wave interference, the signal-to-interference ratio at the output of the hard limiter is always 6 dB worse than that of the input (14). This effect is often referred to as the "limiter capture effect."

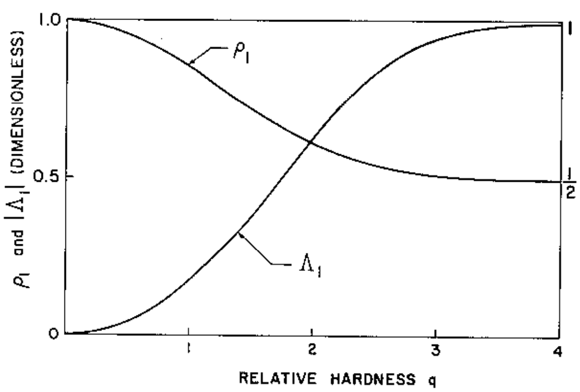


Fig. 4 - Weak signal suppression ρ_1 and normalized intermodulation $|\Lambda_1|$ vs relative limiter hardness q for two sinusoidal inputs, one relatively weak.

The intermodulation-to-weak-signal ratio is also a quantity of interest. Referring to Eq. (25), the intermodulation amplitude C_2 is

$$C_2 = \frac{a_2}{2}\left(B'(a_1) - \frac{B(a_1)}{a_1}\right) \ .$$

If we define Λ_1 as the ratio of intermodulation amplitude to the weak signal amplitude, i.e., $\Lambda_1 \triangleq C_2/C_0$, we have

$$\Lambda_1 = \frac{\frac{a_2}{2}\left(B'(a_1) - \frac{B(a_1)}{a_1}\right)}{\frac{a_2}{2}\left(B'(a_1) + \frac{B(a_1)}{a_1}\right)}$$

$$= \frac{\sqrt{2\pi}\, q \exp(-q^2/2) - \pi \operatorname{erf}(q/\sqrt{2})}{\sqrt{2\pi}\, q \exp(-q^2/2) + \pi \operatorname{erf}(q/\sqrt{2})} \ . \tag{32}$$

Figure 4 shows also a plot of the magnitude of Λ_1 vs $q = a_1/\beta$. It is interesting to note that $|\Lambda_1|$ asymptotically approaches unity for large arguments, indicating that the intermodulation never exceeds the suppressed weak signal.

Equal Amplitude Sinusoids

Let us consider now the case where the two sinusoidal input signals are of equal amplitude. The input $x(t)$ is again given by Eq. (10) but now $a_1 = a_2 = a$. From Eq. (15) the coefficient C_n of the component term of frequency $nf_1 - (n-1)f_2$ is

$$C_n = \frac{1}{\pi}\int_0^{\pi} \frac{B\ \sqrt{2a^2 + 2a^2\cos\gamma}}{\sqrt{2a^2 + 2a^2\cos\gamma}} \left[a\cos(n-1)\gamma + a\cos n\gamma\right] d\gamma$$

$$= \frac{1}{\pi}\int_0^{\pi} B\left(2a\cos\frac{\gamma}{2}\right)\cos(2n-1)\frac{\gamma}{2}\, d\gamma$$

$$= \frac{2}{\pi}\int_0^{\pi/2} B(2a\cos\gamma)\cos(2n-1)\gamma\, d\gamma\ . \tag{33}$$

The specific form of Eq. (33) is obtained via Eq. (6) when a specific device is considered. Therefore, Eq. (33) is a prediction formula for the "inband" signal amplitudes for the case of two, equal-amplitude, sinusoidal inputs.

From Eq. (20) we have

$$B(A) = \pi\,\mathrm{erf}\left(\frac{A}{\sqrt{2}\beta}\right);$$

thus

$$B(2a\cos\gamma) = \pi\,\mathrm{erf}\left(\frac{2a\cos\gamma}{\sqrt{2}\beta}\right). \tag{34}$$

Therefore C_n in Eq. (33) above must be evaluated in the light of Eq. (34):

$$C_n = 2\int_0^{\pi/2} \mathrm{erf}\left(\frac{\sqrt{2}a\cos\gamma}{\beta}\right)\cos(2n-1)\gamma\, d\gamma\ . \tag{35}$$

Now it is clear that we must determine from Eq. (35) the amplitudes of signals corresponding to f_1, f_2, and $2f_1 - f_2$. They are determined by giving an appropriate value for n in the term $nf_1 - (n-1)f_2$.

Amplitude of $f_1(n=1)$ and $f_2(n=0)$ — Note that when $n = 0$ and $n = 1$, Eq. (35) results in the same expression (as expected), and thus

$$C_0 = C_1 = 2\int_0^{\pi/2} \mathrm{erf}\left(\frac{\sqrt{2}a\cos\gamma}{\beta}\right)\cos\gamma\, d\gamma\ . \tag{36}$$

Amplitudes of the third-order intermodulation component — The amplitudes corresponding to $2f_1 - f_2$ and $2f_2 - f_1$ are found from Eq. (35) by setting $n = 2$ and $n = -1$, respectively. Also we note that, as expected, the results for $n = 2$ and $n = -1$ will be identical:

$$C_2 = C_{-1} = 2\int_0^{\pi/2} \operatorname{erf}\left(\frac{\sqrt{2}a\cos\gamma}{\beta}\right)\cos 3\gamma \; d\gamma \; . \tag{37}$$

Equations (36) and (37) give the "responses" of the inband signals (two fundamentals and a third-order intermodulation product) as a function of the input voltage level a and the limiter characteristic β.

If we compare Eq. (36) with $\pi \operatorname{erf}(a/\sqrt{2}\beta)$, we can determine how much each of the signals at f_1 and f_2 is suppressed by the other:

$$\rho_2 \triangleq (C_0/\pi)\operatorname{erf}(a/\sqrt{2}\beta)$$

$$= \frac{2\int_0^{\pi/2} \operatorname{erf}(\sqrt{2}q\cos\gamma)\cos\gamma \; d\gamma}{\pi \operatorname{erf}(q/\sqrt{2})} . \tag{38}$$

Equation (38) was evaluated numerically and is plotted as a function of $q = a/\beta$ in Fig. 5. Examining the asymptotic behavior of Eq. (38) for $q \to \infty$ (corresponding to the hard limiter), since $\operatorname{erf}(\infty) = 1$, we have

$$\lim_{q\to\infty} \rho_2 = \frac{2}{\pi} . \tag{39}$$

This means that each signal undergoes $2/\pi$ (or -3.92 dB) loss due to the presence of the other. This is a well-known result for the hard limiter (12).

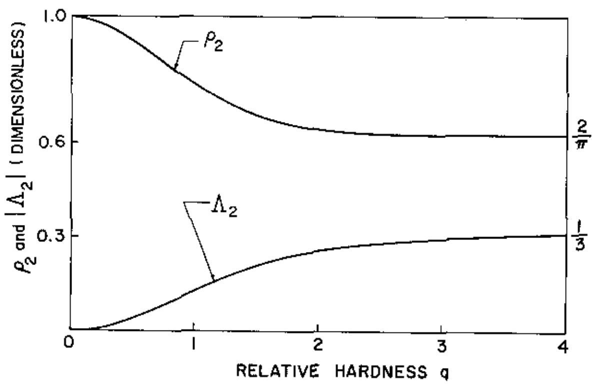


Fig. 5 - Signal suppression ρ_2 and normalized intermodulation $|\Lambda_2|$ vs relative limiter hardness q for two equal-amplitude sinusoidal inputs.

Another quantity of interest is the ratio of intermodulation to suppressed signal, namely,

$$\Lambda_2 \triangleq \frac{C_2}{C_0} = \frac{\int_0^{\pi/2} \operatorname{erf}(\sqrt{2}q \cos \gamma) \cos 3\gamma \, d\gamma}{\int_0^{\pi/2} \operatorname{erf}(\sqrt{2}q \cos \gamma) \cos \gamma \, d\gamma} . \tag{40}$$

A numerical evaluation of Eq. (40) is also plotted in Fig. 5; $|\Lambda_2|$ asymptotically approaches 1/3 (or -9.54 dB) for large q, again the expected result for the hard limiter (10,12).

If we compare Eq. (37) with $\pi \operatorname{erf}(a/\sqrt{2}\beta)$, we determine the intermodulation amplitude with respect to the single input response:

$$\Lambda_3 \triangleq \frac{C_2}{\pi \operatorname{erf}\left(\frac{a}{\sqrt{2}\beta}\right)} = \frac{2 \int_0^{\pi/2} \operatorname{erf}(\sqrt{2}q \cos \gamma) \cos 3\gamma \, d\gamma}{\pi \operatorname{erf}(q/\sqrt{2})} . \tag{41}$$

The asymptotic value of $|\Lambda_3|$ is

$$\lim_{q\to\infty} |\Lambda_3| = \frac{2}{3\pi} . \tag{42}$$

This means that the intermodulation is down by $2/3\pi$ (or -13.46 dB) from the single input response level, or

$$20 \log (2/3\pi) - 20 \log (2/\pi) = -9.54 \text{ dB} \tag{43}$$

down from the suppressed output level as Eq. (40) shows. (See Fig. 6).

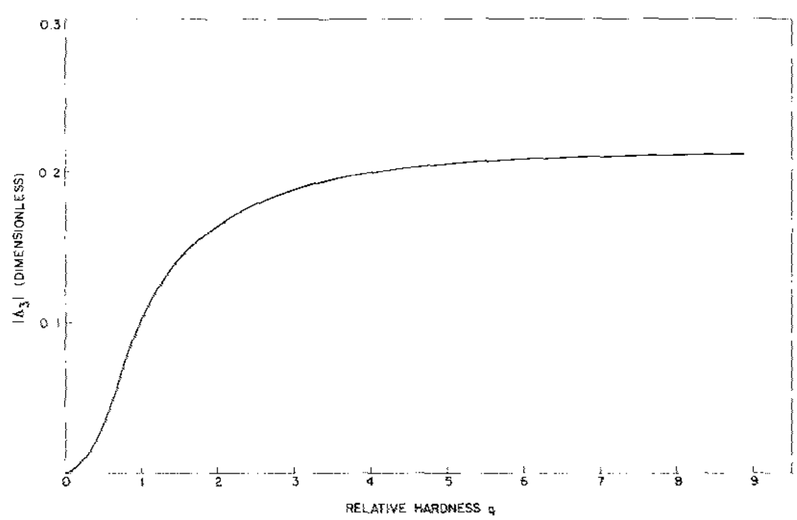


Fig. 6 - Ratio $|\Lambda_3|$ of intermodulation amplitude to single input response vs relative limiter hardness q in a smooth limiter for two equal-amplitude sinusoidal inputs.

SINE WAVE PLUS GAUSSIAN NOISE

Input Signal-to-Noise Ratio Relatively Small

If we replace Eq. (10) with

$$x(t) = a_1(t)\cos(2\pi f_1 t + \alpha) + a_2 \cos 2\pi f_2 t \;, \tag{44}$$

where the first term on the right-hand side represents a narrowband Gaussian random process with α uniformly distributed over $(0, 2\pi)$, then following the preceding development for $a_2 \ll a_1$, we can express the output of the bandpass smooth limiter as

$$Z(t) \approx C_1 \cos(2\pi f_1 t + \alpha) + C_0 \cos 2\pi f_2 t + C_2 \cos\left(2\pi(2f_1 - f_2)t + 2\alpha\right) \tag{45}$$

where

$$C_0 = \frac{a_2}{2a_1}\left(\frac{d}{da}\right)[a_1 B(a_1)] \;, \tag{46}$$

$$C_1 = B(a_1) \;, \tag{47}$$

and

$$C_2 = \frac{a_1 a_2}{2}\left(\frac{d}{da_1}\right)\left(\frac{B(a_1)}{a_1}\right) \;. \tag{48}$$

The "noise" is assumed to be Gaussian with zero mean and a variance σ^2. The noise envelope is then characterized by the Rayleigh density function

$$p_A(a_1) = \frac{a_1}{\sigma^2}\exp\left(-\frac{a_1^2}{2\sigma^2}\right), \quad a_1 \geq 0 \;. \tag{49}$$

To be consistent with Blachman (11) and Cahn (4), the output signal power is defined as $S_{\text{out}} = (1/2)\,\bar{C}_0^2$; thus we must average the signal envelope over all possible values of noise:

$$\begin{aligned}\bar{C}_0 &= \int_0^\infty C_0 p_A(a_1)\, da_1 \\ &= \frac{a_2}{2}\int_0^\infty \frac{1}{a_1}\left(\frac{d}{da_1}\right)[a_1 B(a_1)]\; p_A(a_1)\, da_1 \;.\end{aligned} \tag{50}$$

Integrating by parts and using Eq. (49) we obtain

$$\bar{C}_0 = \frac{a_2}{2\sigma^2}\int_0^\infty a_1 B(a_1)\; p_A(a_1)\, da_1 \;. \tag{51}$$

The output noise power is

$$N_{\text{out}} = \frac{1}{2}\bar{C}_1^2 = \frac{1}{2}\int_0^\infty B^2(a_1)\; p_A(a_1)\, da_1 \tag{52}$$

where we have neglected the C_2 term due to its insignificant contribution to the total noise output. The output signal-to-noise power ratio is

$$\frac{S_{out}}{N_{out}} = \frac{\left(\frac{a_2}{2\sigma^2}\int_0^\infty a_1 B(a_1)\, p_A(a_1)\, da_1\right)^2}{\int_0^\infty B^2(a_1)\, p_A(a_1)\, da_1}$$

$$= \frac{a_2^2}{2\sigma^2}\,\frac{\left(\int_0^\infty a_1 B(a_1)\, p_A(a_1)\, da_1\right)^2}{2\sigma^2\int_0^\infty B^2(a_1)\, p_A(a_1)\, da_1}\,. \tag{53}$$

Considering $a_2^2/2\sigma^2$ to be the input signal-to-noise power ratio, we can define the signal suppression factor ρ_3 as follows:

$$\rho_3 \triangleq \frac{S_{out}}{N_{out}}\bigg/\frac{S_{in}}{N_{in}}$$

$$= \frac{\left(\int_0^\infty a_1 B(a_1)\, p_A(a_1)\, da_1\right)^2}{2\sigma^2\int_0^\infty B^2(a_1)\, p_A(a_1)\, da_1}\,. \tag{54}$$

Letting $p \triangleq \sigma/\beta$ and $\alpha = a/\sigma$, Eq. (54) becomes

$$\rho_3 = \frac{\left(\int_0^\infty \alpha^2 \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}p}\right)\exp\left(-\frac{\alpha^2}{2}\right) d\alpha\right)^2}{2\int_0^\infty \alpha\left(\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}p}\right)\right)^2 \exp\left(-\frac{\alpha^2}{2}\right) d\alpha}\,. \tag{55}$$

Figure 7 shows a numerical evaluation of Eq. (55). This plot differs from the others in that the independent variable is σ/β rather than a_1/β. The change was desirable because this parameter should lend itself to experimental determination. However, $a_1(t)$ is a sample function of a random process and hence not a measurable quantity, whereas σ, the rms value of the input noise, is certainly measurable. The significance of σ/β is the same as the a/β used previously. Both parameters are a measure of limiter hardness relative to the input level.

Note that as the smooth limiter becomes a hard limiter ($p \to \infty$) we obtain the well-known result (2)

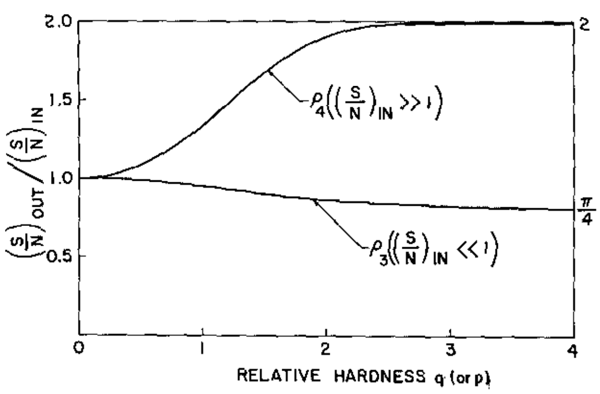


Fig. 7 - Signal-to-noise power transfer ratio for weak (curve ρ_3) and large (curve ρ_4) input signal-to-noise vs relative limiter hardness (p for ρ_3 and q for ρ_4) for sine wave plus Gaussian noise input.

$$\lim_{p\to\infty} \rho_3 = \frac{\pi}{4} \ (-1.05 \text{ dB}) \ . \tag{56}$$

Input Signal-to-Noise Ratio Relatively Large

Let us now consider the case where the input signal-to-noise ratio is very large compared to unity. In order to use the previous development let us alter Eq. (44) so that the first term will represent the signal and the second term the narrowband Gaussian noise. In this manner, we can easily make use of the previous results. Then,

$$x(t) = a_1 \cos 2\pi f_1 t + a_2(t) \cos (2\pi f_2 t + \alpha) \ . \tag{57}$$

Note that the case of "input signal-to-noise ratio relatively large" implies $a_1 >> a_2$.

The output expression is then given by

$$Z(t) = C_1 \cos 2\pi f_1 t + C_0 \cos (2\pi f_2 t + \alpha) + C_2 \cos \left(2\pi (2f_1 - f_2) t - \alpha\right) \tag{58}$$

where C_0, C_1, and C_2 are given by Eqs. (46), (47), and (48), respectively.

Using the same definitions of output power, the output signal power is

$$S_{\text{out}} = \frac{1}{2} \bar{C}_1^2 = \frac{1}{2} \int_0^\infty B^2(a_1) \, p_A(a_2) \, da_2 = \frac{1}{2} B^2(a_1) \tag{59}$$

where $p_A(a_2)$ is again the Rayleigh density function of the noise envelope. The total output noise power in this case becomes*

$$N_{\text{out}} = \frac{1}{2} \bar{C}_0^2 + \frac{1}{2} \bar{C}_2^2 \tag{60}$$

where

*Note that here we do take into consideration the effect of the "signal-and-noise cross term" C_2.

$$\bar{C}_0^2 = \frac{1}{4}\left[\frac{B(a_1)}{a_1} + B'(a_1)\right]^2 \int_0^\infty a_2^2\, p_A(a_2)\, da_2 = \left[\frac{B(a_1)}{a_1} + B'(a_1)\right]^2 \frac{\sigma^2}{2} \,, \tag{61}$$

and likewise

$$\bar{C}_2^2 = \left[B'(a_1) - \frac{B(a_1)}{a_1}\right]^2 \frac{\sigma^2}{2} \,. \tag{62}$$

Equation (60) becomes

$$N_{\mathrm{out}} = \frac{\sigma^2}{2}\left\{\frac{B^2(a_1)}{a_1^2} + [B'(a_1)]^2\right\} . \tag{63}$$

The output signal-to-noise power ratio is

$$\frac{S_{\mathrm{out}}}{N_{\mathrm{out}}} = \frac{B^2(a_1)}{\sigma^2\left\{\frac{B^2(a_1)}{a_1^2} + [B'(a_1)]^2\right\}} \,. \tag{64}$$

Defining the signal-to-noise degradation ρ_4,

$$\rho_4 = \frac{S_{\mathrm{out}}}{N_{\mathrm{out}}}\bigg/\frac{S_{\mathrm{in}}}{N_{\mathrm{in}}}$$

$$= \frac{2B^2(a_1)}{a_1^2\left\{\frac{B^2(a_1)}{a_1^2} + [B'(a_1)]^2\right\}}$$

$$= \frac{2\,[\mathrm{erf}\,(q/\sqrt{2})]^2}{[\mathrm{erf}(q/\sqrt{2})]^2 + \frac{2}{\pi} q^2 \exp(-q^2)} \,. \tag{65}$$

The asymptotic value of ρ_4 for $q \to \infty$ again agrees with the known results for a hard limiter (2):

$$\lim_{q\to\infty} \rho_4 = 2 \,. \tag{66}$$

Figure 7 also shows a plot of Eq. (66).

SUMMARY AND CONCLUSIONS

We have analyzed the effect of a smooth limiter characterized by the error function on signal interactions for the case where the input consists of two sinusoidal signals, one relatively weak (case a) and both of equal amplitude (case b). We have also analyzed the output signal-to-noise ratio when the input consists of a sine wave plus Gaussian noise for (a) the input signal to noise small compared to unity and (b) the input signal to noise large compared to unity. In the graphical presentation of the results it can be observed that when the error function limiter becomes a hard limiter, i.e., when the limiter level parameter approaches infinity, the result is in perfect agreement with the known results for the ideal hard limiter.

The results presented here* can be applied to an existing circuit by the determination of the hardness parameter β. By considering normalized transfer characteristic data extended into the overload or limiting region, a value of β is chosen which gives the best approximation to the transfer curve by the error function curve $\mathrm{erf}\,(a/\sqrt{2}\beta)$ where a is the peak amplitude of the input. This specifies an error function model for the circuit. For a particular situation of interest, e.g., two sinusoidal inputs, one relatively weak, a value of the peak amplitude of the large sinusoid is determined (either by direct measurement or hypothesis); the appropriate graph is then entered at the abscissa corresponding to $q = a/\beta$ and the resultant is read off the ordinate. This procedure must be modified slightly in the case of signal-plus-noise input for small signal-to-noise ratios. Here the limiter level parameter is σ/β where σ is the rms value of the input noise.

*These results were also the subject of a technical presentation (15).

REFERENCES

1. Wolff, S.S., and Gastwirth, J.L., "Robust Two-Input Correlators," J. Acoust. Soc. Am., 41(No. 5:1212-1219 (May 1967).

2. Davenport, W.B., Jr., "Signal-to-Noise Ratios in Band-pass Limiters," J. Appl. Phys., 24:720-727 (June 1953).

3. Blachman, N.M., "The Output Signal-to-Noise Ratio of a Power-Law Device," J. Appl. Phys., 24:783-785 (June 1953).

4. Cahn, C.R., "A Note on Signal-to-Noise Ratio in Band-Pass Limiters," IRE Trans. Inform. Theory, IT-7:39-43 (Jan. 1961).

5. Jones, J.J., "Hard-limiting of Two Signals in Random Noise," IEEE Trans. Inform. Theory, IT-9:34-42 (Jan. 1963).

6. Sollfrey, W., "Hard Limiting of Three and Four Sinusoidal Signals," IEEE Trans. Inform. Theory, IT-15:2-7 (Jan. 1969).

7. Lieberman, G., Report of NRL Progress, pp. 48-51, Jan. 1955.

8. Baum, R.F., "The Correlation Function of Smoothly Limited Gaussian Noise," IRE Trans. Inform. Theory, IT-3:193-197 (Sept. 1957).

9. Galejs, J., "Signal-to-Noise Ratios in Smooth Limiters," IRE Trans. Inform. Theory, IT-5:79-85 (June 1959).

10. Lee, J.S., "Signal-to-Crosstalk Power Ratio in Smoothly Limited Multichannel FDM Signals, IEEE Trans. Commun. Technol., Com-16:63-67 (Feb. 1968).

11. Blachman, N.M., "Band-Pass Nonlinearities," IEEE Trans. Inform. Theory, IT-10: 162-164 (April 1964).

12. Blachman, N.M., "Two-Signal Interaction in a Logarithmic IF Amplifier," IEEE Trans. Commun. Technol., Comm-15:305-307 (April 1967).

13. Blachman, N.M., "Noise and Its Effect on Communication," New York:McGraw-Hill, 1966, Ch. 5.

14. Berglund, C.N., "A Note on Power-law Devices and Their Effect on Signal-to-Noise Ratio," IEEE Trans. Inform. Theory, IT-10:52-57 (Jan. 1964).

15. Lee, J.S., and Hughen, J.H., "Interactions of Two Signals in a Bandpass Smooth Limiter," paper presented at the 78th Meeting of the Acoustical Society of America, San Diego, California, Nov. 1969.

Appendix

DERIVATION OF C_n

In this appendix we show that the filter output $Z(t)$ can be expressed as

$$Z(t) = \sum_{n=-\infty}^{\infty} C_n \cos 2\pi \, [nf_1 - (n-1) f_2] \, t \tag{A1}$$

where C_n represents the amplitude of the output component corresponding to the frequency $[nf_1 - (n-1) f_2]$. The set $\{\cos 2\pi(nf_1 - (n-1) f_2) t\}$, where n takes some integer value, is not an orthogonal set unless the frequencies f_1 and f_2 are commensurable. Thus we cannot employ the usual method of determining generalized Fourier coefficients by multiplying $Z(t)$ by $\cos 2\pi \, [nf_1 - (n-1) f_2] \, t$ and integrating over the interval of orthogonality. In order to determine C_n, we employ an analytic (complex) signal representation of the output. Let $\tilde{Z}(t)$ be the complex representation of the output. Then

$$\tilde{Z}(t) = Z(t) + j\hat{Z}(t) \tag{A2}$$

where $\hat{Z}(t)$ is the Hilbert transform of the real signal output $Z(t)$. The Hilbert transform of the $Z(t)$ can be determined by inspection from Eq. (12) by noting that since $Z(t)$ is expressly required to be a narrowband signal, the bandwidth of $B(A)/A$ must be small compared to either f_1 or f_2. Thus

$$\hat{Z}(t) \cong \frac{B(A)}{A} (a_1 \sin 2\pi f_1 t + a_2 \sin 2\pi f_2 t) \ . \tag{A3}$$

Substituting Eq. (A3) into Eq. (A2) we obtain the analytic signal $\tilde{Z}(t)$:

$$\tilde{Z}(t) \cong \frac{B(A)}{A} \left(a_1 e^{j 2\pi f_1 t} + a_2 e^{j 2\pi f_2 t}\right)$$

$$= \frac{B(A)}{A} \left(a_1 e^{j 2\pi (f_1 - f_2) t} + a_2 \ e^{j 2\pi f_2 t}\right). \tag{A4}$$

If we define the complex envelope function $\tilde{S}(t)$ as

$$\tilde{S}(t) \triangleq \frac{B(A)}{A} \left(a_1 e^{j 2\pi (f_1 - f_2) t} + a_2\right) , \tag{A5}$$

we can then expand $\tilde{S}(t)$ in a complex Fourier series and substitute the series form of Eq. (A5) into Eq. (A4) obtaining

$$\tilde{Z}(t) = \left(\sum_{n=-\infty}^{\infty} \alpha_n e^{j 2\pi n (f_1 - f_2) t}\right) e^{j 2\pi f_2 t} \tag{A6}$$

where

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{S}(t)\, e^{-j2\pi n(f_1 - f_2)t}\, dt \tag{A7}$$

and

$$T = 1/(f_1 - f_2) \ .$$

Now,

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} \frac{B(A)}{A}\left(a_1 e^{j2\pi(f_1-f_2)t} + a_2\right) e^{-j2\pi n(f_1-f_2)t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \frac{B(A)}{A}\left(a_1 e^{-j2\pi(n-1)(f_1-f_2)t} + a_2 e^{-j2\pi n(f_1-f_2)t}\right) dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \frac{B(A)}{A}\left(a_1 \cos 2\pi(n-1)(f_1-f_2)\,t + a_2 \cos 2\pi n(f_1-f_2)\,t\right) dt$$

$$-j\,\frac{1}{T}\int_{-T/2}^{T/2} \frac{B(A)}{A}\left(a_1 \sin 2\pi(n-1)(f_1-f_2)\,t + a_2 \sin 2\pi n(f_1-f_2)\,t\right) dt \ . \tag{A8}$$

Since $B(A)/A$ is an even function of time, the second integral in Eq. (A8) will vanish. It is thus shown that α_n is a real quantity. Substituting $T = 1/(f_1 - f_2)$ and $\gamma = 2\pi(f_2 - f_1)\,t$ into Eq. (A8) gives

$$\alpha_n = \frac{1}{\pi}\int_0^{\pi} \frac{B(A)}{A}\left(a_1 \cos(n-1)\gamma + a_2 \cos n\gamma\right) d\gamma \ . \tag{A9}$$

Now having established that the α_n's are real, it is easy to show that the analytic signal of Eq. (A6) leads to the desired form of $Z(t)$:

$$Z(t) = \frac{1}{2}\,\mathrm{Re}\,[\tilde{Z}(t) + \tilde{Z}^*(t)]$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n \cos 2\pi\,[nf_1 - (n-1)\,f_2]\,t \ . \tag{A10}$$

Comparing this expression term by term with Eq. (A1) we conclude that the α_n and c_n are identical. Thus we have

$$C_n = \frac{1}{\pi}\int_0^{\pi} \frac{B(A)}{A}\left(a_1 \cos(n-1)\gamma + a_2 \cos n\gamma\right) d\gamma \tag{A11}$$

which is the form given in Eq. (15).