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Fluid Models for Tokamak Plasmas

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It does not seem possible to explain the experimental data by using the present two-fluid model applying the usual (classical) transport coefficients. Two major model expansions are obvious: (a) increase the number of fluids in the model, and (b) take into account as many spatial dimensions as possible.

A fluid model that includes neutrals, electrons, and ions with arbitrary charge Z is derived. Cylindrical symmetry is imposed, although the transport coefficients include corrections for toroidal geometry. Assumptions are discussed under which this model can be applied to describe a Tokamak plasma consisting of neutral hydrogen, protons, electrons, and nine ionization stages of oxygen impurities. Some methods are outlined for the numerical (difference) solution of the resultant system of highly nonlinear partial differential equations.

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This is a final report on one phase of a continuing problem.

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FLUID MODELS FOR TOKAMAK PLASMAS

INTRODUCTION

The Tokamak system (Fig. 1) is a toroidal electromagnetic field plasma configuration in which the magnetic field ratio B_z/B_θ is large. The duration of the B_θ pulse is short compared with the characteristic time period for B_z. The plasma density is typically of the order of 10^{13} cm^{-3}, the temperatures can exceed 1 keV (1), and the energy containment time reaches 10 ms (2). Because of these qualities the Tokamak plasma has gained considerable interest for controlled thermonuclear fusion research.

The growing amount of experimental data to be explained, as well as data that will accrue from Tokamak devices being planned, reveals the need for complicated theoretical plasma models (magnetohydrodynamic models) similar to those which have been applied to pinch plasmas over the past several years. These models can be evaluated only by numerical techniques. A two-fluid system comprising heat conduction of electrons and ions, ohmic heating, temperature equipartition, and field diffusion has been solved by Y. N. Dnestrovskii et al. (3,4). Calculations for the same combination of effects, but with different assumptions for the transport coefficients, have been reported by H. Luc, C. Mercier, and Soubbaramayer (5,6) and by R.A. Dory and M.M. Widner (7). The influence of impurities has been included by the present author (8).

It does not seem possible to explain the experimental data by a two-fluid model applying the usual (classical) transport coefficients. One improvement might be expected from deriving new transport coefficients which incorporate possible turbulence effects due to instabilities. This does not affect, however, the basic fluid model for the plasma.

On the level of the fluid description, two major model expansions are obvious. The first one concerns the number of fluids. In the experiment some influence of neutral particles and of impurities like oxygen has been observed (9). The second expansion refers to the geometrical dimensions taken into consideration in order to include deviations from symmetry and macroscopic instabilities.

In the following sections a fluid model that includes neutrals, electrons, and ions with arbitrary charge Z is derived. Cylindrical symmetry is imposed, although the transport coefficients include corrections for toroidal geometry (neoclassical theory). Assumptions are discussed under which this model can be applied to describe a Tokamak plasma consisting of neutral hydrogen, protons, electrons, and nine ionization stages of oxygen impurities.

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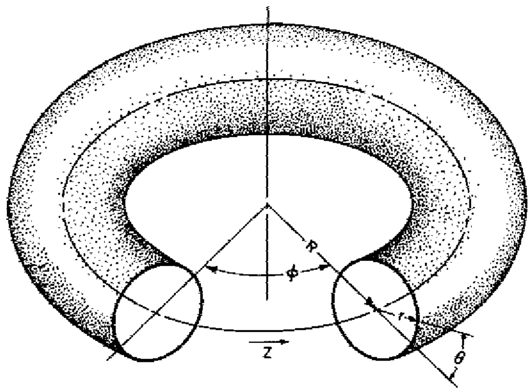

Fig. 1 — Co-ordinates for a Tokamak plasma configuration. The ratio of B_z to B_θ (magnetic field along the toroid axis to magnetic field perpendicular to the axis) is large for such devices.

Finally, some methods are outlined for the numerical solution of the system of partial differential equations which corresponds to the multifluid model.

On the basis of this report, a computer program has been developed. It will be applied to the data of existing and planned Tokamak devices, and the results will be published in future reports and papers (10).

FLUID DESCRIPTION OF PLASMA

Considering a fluid model for the description of a plasma, we make the basic assumption that we understand the plasma sufficiently well as soon as its densities n_k, its flow velocities $\mathbf{v}_k$, its temperatures T_k, and the electromagnetic field quantities $\mathbf{E}$ and $\mathbf{B}$ are given as functions of space and time. The subscript k indicates that we might have to deal with several kinds of fluids, such as electrons and ions with varying charges, or neutrals. Each type of fluid (component) obeys the laws of conservation of mass (Eq. (1)), momentum (Eq. (2)), and internal energy (Eq. (3)). For the electromagnetic field, we have to satisfy Maxwell's equations (Eqs. (4) and (5)). The conservation equations are

$$\frac{\partial n_k}{\partial t} + \mathrm{div}(n_k \mathbf{v}_k) = \Delta n_k \tag{1}$$

$$\frac{\partial}{\partial t}(n_k m_k \mathbf{v}_k) + (\mathbf{v}_k \cdot \mathrm{grad})(n_k m_k \mathbf{v}_k) + n_k m_k \mathbf{v}_k \; (\mathrm{div}\, \mathbf{v}_k) = -\,\mathrm{grad}\, p_k + \mathbf{F}_k + \Delta \mathbf{P}_k \tag{2}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_k k T_k\right) + \mathrm{div}\left(\frac{3}{2} n_k k T_k \mathbf{v}_k\right) = -\,\mathrm{div}\, \mathbf{q}_k - p_k \; \mathrm{div}\, \mathbf{v}_k + \Delta E_k \tag{3}$$

and Maxwell's equations are

$$\frac{\partial \mathbf{B}}{\partial t} = -c \; (\mathrm{curl}\, \mathbf{E}) \tag{4}$$

and for the neutrals the forces are

$$\mathbf{F}_0 = 0. \tag{11}$$

The electrons lose momentum in recombining collisions and exchange it elastically with neutrals and ions:

$$\Delta\mathbf{P}_e = -n_e n_i ZQ m_e \mathbf{v}_e - n_0 n_e \,(m_e m_0/m_e + m_0)\, \left\langle \sigma_{e0} v_{e0} \right\rangle (\mathbf{v}_e - \mathbf{v}_0) + \mathbf{P}_{ei}. \tag{12}$$

According to Spitzer (11b) we can write

$$\mathbf{P}_{ei} = n_e e \eta_{Sp}\, \mathbf{j} = -\mathbf{P}_{ie}. \tag{13}$$

For the ions, the ionization and charge exchange collisions are important, in addition to the kinds of collisions mentioned for the electrons.

Applying Eq. (13), we obtain

$$\begin{aligned}\Delta\mathbf{P}_i = n_0 n_e\, S\, m_i \mathbf{v}_0 - n_e n_i\, Q\, m_i \mathbf{v}_i - n_e e \eta_{Sp} \mathbf{j} \\ - n_0 n_i\, m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle (\mathbf{v}_i - \mathbf{v}_0).\end{aligned} \tag{14}$$

In these equations σ_{e0} (σ_{i0}) denotes the cross section for elastic electron (ion)-neutral collisions; σ_{CE} is the cross section for charge exchange. The rate of change for the momentum of the neutrals can be expressed as

$$\begin{aligned}\Delta\mathbf{P}_0 = -n_0 n_e\, S\, m_0 \mathbf{v}_0 + n_e n_i\, Q\, m_0 \mathbf{v}_i \\ + n_0 n_i m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle (\mathbf{v}_i - \mathbf{v}_0) \\ + n_0 n_e m_e \left\langle \sigma_{e0}\, v_{e0} \right\rangle (\mathbf{v}_e - \mathbf{v}_0).\end{aligned} \tag{15}$$

In a similar way, we construct the expressions for the energy rates of change. The coefficient of equipartition between electron and ion temperature is denoted by c_{eq}, and ionization energy by χ. The remaining symbols are self-explanatory. The electron and ion energy rates are given by

$$\begin{aligned}\Delta E_e = -\frac{3}{2} kT_e\, n_e n_i\, Q\, Z - n_0 n_e\, S\, \chi \\ + \eta j^2 - P_r - n_0 n_e\, \alpha(kT_e - kT_0) \\ - \frac{3}{2} n_e n_i \left(\frac{kT_e - kT_i}{c_{eq}} \right)\end{aligned} \tag{16}$$

and

$$\Delta E_i = \frac{3}{2} kT_0 \; n_0 n_i \; S - \frac{3}{2} kT_i \; n_i n_e \; Q$$

$$+ \frac{3}{2} n_e n_i \left(\frac{kT_e - kT_i}{c_{eq}} \right) - n_0 n_i \, \beta \, (kT_i - kT_0)$$

$$+ r_i \, (\mathbf{v}_i - \mathbf{v}_0)^2. \tag{17}$$

The energy loss due to radiation is denoted by P_r. Frictional heating is described by r_i and r_0 for ions and neutrals, respectively. For neutrals,

$$\Delta E_0 = \frac{3}{2} kT_i \; n_i n_e \; Q - \frac{3}{2} kT_0 \; n_e n_0 \; S$$

$$+ n_0 n_e \; \alpha(kT_e - kT_0) + n_0 n_i \; \beta(kT_i - kT_0)$$

$$+ r_0 \; (\mathbf{v}_i - \mathbf{v}_0)^2. \tag{18}$$

The collision cross sections, averages (e.g., $\langle \sigma_{e0} \, v_{e0} \rangle$), and the functions S, Q, α, β, r_i, and r_0 are collected and investigated in more detail in Ref. 12.

GENERAL SIMPLIFYING ASSUMPTIONS

We have arrived at a rather formidable number of equations for the variables n_e, n_i, n_0, $\mathbf{v}_e$, $\mathbf{v}_i$, $\mathbf{v}_0$, T_e, T_i, T_0, and $\mathbf{B}$. It is quite obvious that the system is much too complicated for analytical methods. Also, a numerical treatment of such a set in three dimensions and in time exceeds the capacity of the largest computers which are available today.

In order to simplify the system of equations we will make several assumptions; we will not, however, try to justify these assumptions.

The quasi-neutrality condition

$$n_e \approx Z n_i \tag{19}$$

eliminates the continuity equation for the ions.

Inserting Eq. (1), we can rewrite the left-hand side of Eq. (2) as

$$n_k m_k \left(\frac{\partial \mathbf{v}_k}{\partial t} + (\mathbf{v}_k \cdot \text{grad}) \mathbf{v}_k \right) + m_k \mathbf{v}_k \; \Delta n_k .$$

For the following we will neglect the inertia terms, i.e.,

$$n_k m_k \left(\frac{\partial \mathbf{v}_k}{\partial t} + (\mathbf{v}_k \cdot \mathrm{grad})\, \mathbf{v}_k \right) \approx 0, \tag{20}$$

as being small compared to the rest of the equations. In certain cases (e.g., charge-exchange-created hot neutrals of very low density in Tokamak discharges) this simplification might have to be revised.

As the remainder of the equation of motion for the neutrals, we obtain

$$\begin{aligned} 0 = &- m_0 \mathbf{v}_o\, (n_e n_i\, Q - n_e n_0\, S) - \mathrm{grad}\, p_0 \\ &- n_0 n_e m_0 \mathbf{v}_0\, S + n_i n_e m_0 \mathbf{v}_i\, Q \\ &+ n_0 n_i m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle (\mathbf{v}_i - \mathbf{v}_0) \\ &+ n_0 n_e m_e \left\langle \sigma_{e0} v_{e0} \right\rangle (\mathbf{v}_e - \mathbf{v}_0). \end{aligned} \tag{21}$$

We resolve this equation for the particle flux $n_0 \mathbf{v}_0$:

$$\begin{aligned} n_0 \mathbf{v}_0 = \Bigg[&- \mathrm{grad}\, p_0 + n_0 n_e m_e \left\langle \sigma_{e0} v_{e0} \right\rangle \mathbf{v}_e \\ &+ \left(n_e m_0\, Q + n_0 m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle \right) n_i \mathbf{v}_i \Bigg] \Bigg/ \Bigg[\frac{n_i n_e}{n_0} m_0\, Q \\ &+ n_i m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle + n_e m_e \left\langle \sigma_{e0} v_{e0} \right\rangle \Bigg]. \end{aligned} \tag{22}$$

This expression for the particle flux has to be used in the continuity equation for the neutrals. The first term can be recognized as the usual particle diffusion term.

In order to arrive at a generalized Ohm's law, we multiply the electron and ion equations of motion by m_i and m_e, respectively, and form the difference of both equations. We use

$$\frac{m_e}{m_i} \ll 1 \text{ and } \frac{Z\, m_e}{m_i} \ll 1 \tag{23}$$

and the definition of the current density

$$\mathbf{j} = n_i\, e\, Z\, \mathbf{v}_i - n_e\, e\, \mathbf{v}_e = n_e e\, (\mathbf{v}_i - \mathbf{v}_e). \tag{24}$$

Then, Ohm's law becomes

$$
\begin{aligned}
\mathbf{E} = {} & \frac{m_e}{m_i n_e e}\,\mathrm{grad}\,p_i - \frac{1}{n_e e}\,\mathrm{grad}\,p_e - \frac{1}{c}\,(\mathbf{v}_i \times \mathbf{B}) \\
& + \frac{1}{n_e\, e\, c}\,(\mathbf{j} \times \mathbf{B}) - \frac{m_e\, Z}{n_e\, e}\, n_0 n_e\, S\, \mathbf{v}_i \\
& + \frac{m_e}{n_e\, e}\, n_0 \left[n_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle + n_e (S - \langle \sigma_{e0} v_{e0} \rangle) \right] (\mathbf{v}_i - \mathbf{v}_0) \\
& + \left[\eta_{Sp} + \frac{m_e}{n_e\, e^2}\, n_0\, (ZS + \langle \sigma_{e0} v_{e0} \rangle) \right] \mathbf{j}.
\end{aligned} \tag{25}
$$

The last expression suggests a small correction for Spitzer's resistivity due to the presence of the neutrals. (Reference 12 presents the conditions for which it can be omitted.) This correction is

$$
\eta_c = \eta_{Sp} + \frac{m_e}{n_e e^2}\, n_0 (ZS + \langle \sigma_{e0} v_{e0} \rangle). \tag{26}
$$

Resolving Eq. (25) for $n_e e \,(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B})$, and inserting it into the equation of motion for the ions, results in

$$
\begin{aligned}
& n_0 n_i \left[ZS\, m_i + m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle + m_e Z \langle \sigma_{e0} v_{e0} \rangle \right] (\mathbf{v}_i - \mathbf{v}_0) \\
& \quad = -\,\mathrm{grad}\,(p_e + p_i) + \frac{1}{c}\,(\mathbf{j} \times \mathbf{B}) \\
& \quad\quad + \frac{m_e}{e}\, n_0\, (ZS + \langle \sigma_{e0} v_{e0} \rangle)\, \mathbf{j}.
\end{aligned} \tag{27}
$$

In principle this equation could be utilized for computing the ion particle flux $n_i \mathbf{v}_i$. This direct approach, however, breaks down for $n_0 \to 0$.

ASSUMPTION OF CYLINDRICAL SYMMETRY

In the following, we restrict ourselves to cylindrically symmetric plasmas (and boundary conditions). At first we will prove that the radial velocities of electrons and ions are equal under these conditions. Applying Eqs. (19) and (7), and constructing the difference between the continuity equations of electrons and ions, we obtain

$$
\frac{1}{r}\,\frac{\partial}{\partial r}\left[r\, n_e \;(v_r^i - v_r^e) \right] = 0. \tag{28}
$$

Since r is bounded,

$$r\, n_e\, (v_r^i - v_r^e) = \text{const}$$

must be valid. At $r = 0$, n_e and the velocities cannot be infinite; therefore,

$$\text{const} = 0,$$

and

$$v_r^i = v_r^e \equiv v_r. \tag{29}$$

This equation will be useful for the transport of thermal energy in the equations for T_e and T_i.

From Maxwell's Eqs. (4) and (5), we find

$$\frac{\partial \mathbf{B}}{\partial t} \equiv \left\{ 0;\, c\, \frac{\partial E_z}{\partial r};\, -c\, \frac{1}{r}\, \frac{\partial}{\partial r}(rE_\theta) \right\} \tag{30}$$

(which suggests that $B_r = 0$), and

$$\frac{4\pi}{c}\, \mathbf{j} \equiv \left\{ 0;\, \frac{\partial B_z}{\partial r};\, \frac{1}{r}\, \frac{\partial}{\partial r}\, (rB_\theta) \right\}. \tag{31}$$

With respect to our special Tokamak field configuration (see the Introduction), we can assume that

$$\frac{\partial B_z}{\partial t} = -\, c\, \frac{1}{r}\, \frac{\partial}{\partial r}(rE_\theta) \approx 0. \tag{32}$$

A procedure analogous to the one which leads to Eq. (28) produces

$$E_\theta = 0. \tag{33}$$

Inserting this result into Eq. (25) yields an expression for the radial ion particle flux which also holds for $n_0 \to 0$:

$$\begin{aligned} 0 &= \frac{1}{c}\, v_r^i\, B_z - \frac{m_e Z}{n_e e}\, n_0 n_e\, S\, v_\theta^i \\ &+ \frac{m_e}{n_e e}\, n_0 \left[n_i\, \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right)\, v_{i0} \right\rangle + n_e (S - \langle \sigma_{e0} v_{e0} \rangle\,) \right]\, (v_\theta^i - v_\theta^0) + \eta_c j_\theta. \end{aligned} \tag{34}$$

In order to present v_r^i in its more familiar form, we multiply Eq. (33) by B_z and evaluate Eq. (27) for $j_\theta B_z$ by multiplying the z-component of Eq. (25) by B_θ and find

$$v_r^i \left(B_z^2 + B_\theta^2 + c^2\eta_c\, n_0 n_i \left(ZS\, m_i + m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE}\right) v_{i0} \right\rangle + m_e\, Z\, \langle \sigma_{e0} v_{e0} \rangle \right)\right)$$

$$= \frac{m_e\, c\, n_0}{e}\, S\, (v_\theta^i B_z - v_z^i B_\theta)$$

$$- \frac{m_e\, c\, n_0}{e} \left(\frac{n_i}{n_e} \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE}\right) v_{i0} \right\rangle + S - \langle \sigma_{e0} v_{e0} \rangle \right) [B_\theta\, (v_z^i - v_z^0)$$

$$- B_z\, (v_\theta^i - v_\theta^0)] + c^2 \eta_c\, n_0 n_i v_r^0 \left(ZS\, m_i + m_i \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE}\right) v_{i0} \right\rangle \right.$$

$$\left. + m_e Z\, \langle \sigma_{e0} v_{e0} \rangle \right) - cE_z B_\theta - \eta_c\, c^2 \frac{\partial}{\partial r}(p_i + p_e). \tag{35}$$

What we did here was actually construct the Poynting flux $\mathbf{E} \times \mathbf{B}$ from Ohm's law (Eq. (25)) and insert $\mathbf{j} \times \mathbf{B}$ from Eq. (27). Without the corrections for neutral gas ($n_0 \to 0$), we arrive at the usual expression

$$v_r^i = -c^2 \eta_c \left(\frac{1}{B_\theta^2 + B_z^2}\right) \frac{\partial}{\partial r}(p_e + p_i) - \left(\frac{cE_z\, B_\theta}{B_z^2 + B_\theta^2}\right) \tag{36}$$

where the two terms constitute the familiar particle diffusion and pinch effect.

It might be worth mentioning that for the temperature Eq. (3) the knowledge of v_r^i is sufficient in cylindrical symmetry since $\mathbf{v}$ occurs only in the "div" operators. The friction terms in Eqs. (17) and (18) are usually negligible or well enough represented by the radial components of the velocities.

INCLUSION OF EFFECTS RESULTING FROM TORUS GEOMETRY

Modifications

Up to this point we have developed a system of equations for the ion and neutral densities, for their radial velocities, for the temperatures T_e, T_i, and T_o, and for the electromagnetic field components. We have imposed the conditions of cylindrical symmetry so that our plasma parameters depend only on radius r and time t. We intend, however, to apply this model to a toroidal Tokamak plasma. There, the torus geometry induces several important effects. The radial ion velocity, for example, is strongly influenced by the Galeev and Sagdeev trapped particle motion (13) and by A.A. Ware's (14) pinch effect. We also will miss the Pfirsch-Schlüter corrections (15). Fortunately, taking into account these effects is possible without increasing the number of parameters and coordinates—that is, these corrections can be expressed by variables of our present cylindrical model.

We will use the so-called neo-classical transport theory (16). It has been worked out for an electron-ion plasma. This theory distinguishes between three different regimes according to the collision frequency ν:

(a) the "banana regime," where

$$\frac{\nu R}{v_T}\left(\frac{r\,B}{R\,B_\theta}\right) < \left(\frac{r}{R}\right)^{3/2} \tag{37}$$

(b) the "plateau regime," where

$$\left(\frac{r}{R}\right)^{3/2} < \frac{\nu R}{v_T}\left(\frac{r\,B}{R\,B_\theta}\right) < 1 \tag{38}$$

(c) and the "classical regime," where

$$\frac{\nu R}{v_T}\left(\frac{r\,B}{R\,B_\theta}\right) > 1. \tag{39}$$

The major radius of the torus is denoted by R; the quantity B represents the magnitude of the field, and the thermal velocity is defined by

$$v_T = \sqrt{\frac{2kT}{m}}. \tag{40}$$

The collision frequency has to be treated separately for ion-ion and for electron-ion collisions. In the latter case we find that (17)

$$\nu_{ei} = \frac{1}{\tau_{ei}} = \frac{4\sqrt{2\pi}\,Z^2 e^4\, n_i \ln \Lambda_{ei}}{3\sqrt{m_e}(kT_e)^{3/2}}\ [\mathrm{sec}^{-1}] \tag{41}$$

where (using cgs-units)

$$\Lambda_{ei} = \begin{cases} \dfrac{3(kT_e)^{3/2}}{2\sqrt{\pi}\,Z\,e^3\sqrt{n_e}}, & \text{for } kT_e < 36.19\,eV \\[2ex] \dfrac{3 \times 7.61\times 10^{-6} \times kT_e}{2\sqrt{\pi}\,Z\,e^3\sqrt{n_e}}, & \text{for } kT_e \geqslant 36.19\,eV. \end{cases} \tag{42}$$

The analogous formula for the ions is

$$\nu_{ii} = \frac{1}{\tau_{ii}} = \frac{4\sqrt{\pi}\,e^4 Z^4\, n_i \ln \Lambda_{ii}}{3\sqrt{m_i}\,(kT_i)^{3/2}}\ [\mathrm{sec}^{-1}] \tag{43}$$

where

$$\Lambda_{ii} = \frac{3\,kT_i}{2\,Z^2\,e^3}\left(\frac{kT_i}{\pi\,n_e}\right)^{1/2} \tag{44}$$

Particle Flux

It has been shown (18) that electron-ion collisions determine the particle flux in the present problem. Recently M.N. Rosenbluth et al. (16) derived for the "banana regime" the relation

$$v_r^b = \rho_{e\theta}^2 \, \nu_{ei} \sqrt{\frac{r}{R}} \left[-1.12 \left(1 + \frac{kT_i}{kT_e}\right) \frac{1}{n} \frac{\partial n}{\partial r} + 0.43 \left(\frac{1}{kT_e} \frac{\partial kT_e}{\partial r}\right) + 0.19 \left(\frac{1}{kT_e} \frac{\partial kT_i}{\partial r}\right)\right] - 2.44 \left(\frac{E_z c}{B_\theta} \sqrt{\frac{r}{R}}\right). \tag{45}$$

In this formula, $\rho_{e\theta}$ is defined by

$$\rho_{e\theta}^2 = \frac{2m_e \, kT_e \, c^2}{e^2 \, B_\theta^2} \, [cm^2]. \tag{46}$$

The expression in brackets corresponds to the diffusion term in Eq. (36), and the last term describes the Ware pinch effect.

In the "plateau regime" we use (19) the relation

$$v_r^p = -\sqrt{\frac{\pi}{2}} \left(\frac{r}{R}\right)^2 \left(\frac{B_\theta v_{Te}}{B \, r \, \nu_{ei}}\right) \left(0.9 \left(\frac{35}{8} \frac{cE_z}{B_\theta}\right) + \rho_{e\theta}^2 \, \nu_{ei} \frac{1}{n} \frac{\partial n}{\partial r}\right). \tag{47}$$

For the classical case the anomalous pinch effect may be neglected (19); however, we must apply the Pfirsch-Schlüter factor as a correction for torus geometry:

$$\psi = 1 + 2 \frac{\eta_{||}}{\eta_\perp} \left(\frac{r B}{R B_\theta}\right)^2. \tag{48}$$

Using Eq. (48), with $\eta_{||}/\eta_\perp = 1/2$, we obtain

$$v_r^c = -\nu_{ei} \left(\frac{2 \, kT_e \, m_e \, c^2}{e^2 \, B^2}\right) \psi \, \frac{1}{n} \frac{\partial n}{\partial r}. \tag{49}$$

Heat Flux for Electrons

Reviewing our basic system of Eq. (1)-(5), we notice that the heat flux constitutes the fourth velocity moment of Boltzmann's equation. Since we want to close our system of variables with the temperatures, we always have to make assumptions for the heat fluxes. In the present problem we again utilize, of course, the results of the neo-classical transport theory.

We assume again that electron-ion collisions are dominant for the electron heat flux. For the "banana regime," Ref. 16 obtains

$$q_e^b = \frac{n_e\, kT_e\, \rho_{e\theta}^2}{\tau_{ei}} \sqrt{\frac{r}{R}} \left[-1.81 \left(\frac{1}{kT_e} \frac{\partial kT_e}{\partial r} \right) - 0.27 \left(\frac{1}{kT_e} \frac{\partial kT_i}{\partial r} \right) + 1.53 \frac{1}{n} \frac{\partial n}{\partial r} \left(1 + \frac{kT_i}{kT_e} \right) \right] + 1.75\, n\, kT_e \left(\frac{E_z c}{B_\theta} \right) \sqrt{\frac{r}{R}} . \tag{50}$$

If the collision frequency belongs to the "plateau regime," we have

$$q_e^p = -\frac{3}{2} \sqrt{\pi} \frac{r}{R^2} \left(\frac{n_e kT_e c}{e\, B_\theta} \right) \rho_e \frac{\partial kT_e}{\partial r} \tag{51}$$

where ρ_e is formed by replacing B_θ^2 with B^2 in Eq. (46).

The classical heat conductivity must also be corrected by the Pfirsch-Schlüter factor, Eq. (48), which has here a slightly different numerical factor of 1.6. The basic formula is taken from Ref. 17:

$$q_e^c = -\frac{n_e kT_e\, \tau_{ei}\, (\gamma_1\, X^2 + \gamma_0)}{m_e\, X^4 + \delta_1\, X^2 + \delta_0} \left[1 + 1.6 \left(\frac{rB}{RB_\theta} \right)^2 \right] \frac{\partial kT_e}{\partial r} \tag{52}$$

where $X = \omega_e \tau_{ei}$. The cyclotron frequency is

$$\omega_e = \frac{eB}{m_e c} . \tag{53}$$

The coefficients γ_1, γ_0, δ, and δ_0 depend on the ion charge Z and are given by a table in Braginskii's article (17).

Heat Flux for Ions

The most important part of the plasma heat flux is carried by ions. In the "banana regime" we use (16)

$$q_i^b = -0.68 \left(\frac{n\, \rho_{i\theta}^2}{\tau_{ii}} \right) \sqrt{\frac{r}{R}} \left(\frac{\partial kT_i}{\partial r} \right) \tag{54}$$

with

$$\rho_{i\theta}^2 = \frac{2\, m_i\, kT_i\, c^2}{Z^2\, e^2\, B_\theta^2}$$

in analogy to Eq. (46).

In the "plateau regime," we find (19)

$$q_i^p = -\frac{3}{2}\sqrt{\pi}\left(\frac{r\,\rho_i\,n_i\,kT_i\,c}{B_\theta\,R^2\,Z\,e}\right)\frac{\partial kT_i}{\partial r} \tag{55}$$

where

$$\rho_i^2 = \frac{2\,m_i\,kT_i\,c^2}{Z^2\,e^2\,B^2} \tag{56}$$

is obtained by replacing B_θ by B in the similar expression for $\rho_{i\theta}$.

The classical case has again been given by Braginskii (17)

$$q_i^c = -\left(\frac{n_i\,kT_i\,\tau_{ii}\,X\left(\frac{5}{2}X^2 + 4.65\right)}{m_i\,(X^4 + 2.70\,X^2 + 0.677)}\right)\psi\,\frac{\partial kT_i}{\partial r} \tag{57}$$

where $X = \omega_i \tau_{ii}$, and

$$\omega_i = \frac{Z\,e\,B}{m_i\,c}. \tag{58}$$

The correction ψ is the same as in Eq. (52).

Modifications Due to Toroidal Averaging

The formulae for the banana regions are obtained by averaging over toroidal surfaces. Unfortunately, the "adiabatic compression" terms do not emerge from this procedure in a form as simple as in the purely cylindrical case. For the ions, we have to write

$$\begin{aligned} -p_i\,\mathrm{div}\,\mathbf{v}_i \to &-\frac{1}{r}\frac{\partial}{\partial r}(r\,n\,kT_i\,v_r) \\ &+ v_r\left(kT_i\frac{\partial n}{\partial r} - 0.17n\,\frac{\partial kT_i}{\partial r}\right), \end{aligned} \tag{59}$$

and in the electron temperature equation we have

$$\begin{aligned} -p_e\,\mathrm{div}\,\mathbf{v}_e \to &-\frac{1}{r}\frac{\partial}{\partial r}(r\,n\,kT_e\,v_r) \\ &- v_r\left(kT_i\frac{\partial n}{\partial r} - 0.17n\,\frac{\partial kT_i}{\partial r}\right). \end{aligned} \tag{60}$$

For similar reasons we have to modify Ohm's law. Equation (25) would yield for the axial component of the electric field

$$E_z = -\frac{1}{c} v_r B_\theta + \eta j_Z$$

if we omit the corrections due to neutral gas. From neoclassical theory, however, we arrive at

$$E_Z = \eta j_Z \left(\frac{1}{1 - 1.9\sqrt{\frac{r}{R}}} \right) - \eta \left(\frac{1}{1 - 1.9\sqrt{\frac{r}{R}}} \right) \sqrt{\frac{r}{R}} \frac{c}{B_\theta}$$

$$\times \left(-2.45\, (kT_e + kT_i) \frac{\partial n}{\partial r} - 0.7\, n \frac{\partial kT_e}{\partial r} + 0.3\, n \frac{\partial kT_i}{\partial r} \right). \qquad (61)$$

The ohmic heating rate H is given by

$$H = E_Z j_Z. \qquad (62)$$

It might be worth keeping in mind that M. Rosenbluth's expressions are derived only for the banana collision frequency regime and only in the limit of small aspect ratio r/R.

ELECTRICAL RESISTIVITY AND EQUIPARTITION TIME

In order to complete the compilation of coefficients, we note here also the electrical resistivity. Basically, we use Spitzer's formula

$$\eta = \frac{\beta\, m_e\, \nu_{ei}}{n_e\, e^2}. \qquad (63)$$

For $Z = 1$, the quantity β assumes the value of 0.51; for $Z > 1$, it decreases slightly and can be represented by the formula

$$\beta = \frac{0.457}{1.077 + Z} + 0.29. \qquad (64)$$

Some experiments indicate an enhancement of the resistivity as soon as the electron drift velocity

$$v_D = \frac{|j|}{n_e\, e}$$

exceeds the sound speed

$$v_{is} = \sqrt{\frac{2\,kT_e}{m_i}}.$$

Approximating the experimental results we would have to multiply η by the function

$$\gamma = 1.222 \left(\frac{v_D}{v_{is}}\right)^2 - 1.888 \left(\frac{v_D}{v_{is}}\right) + 1.766 \tag{65}$$

for $v_D/v_{is} > 1$.

In Eqs. (16) and (17), we defined the equipartition term to be

$$\frac{3}{2}\, n_e n_i \left(\frac{kT_e - kT_i}{c_{eq}}\right).$$

In Spitzer's book (11) we find for the equipartition coefficient needed here

$$c_{eq} = \frac{3m_e\, m_i}{8\sqrt{2\pi}\, Z^2\, e^4 \ln \Lambda_{ei}} \left(\frac{kT_e}{m_e} + \frac{kT_i}{m_i}\right)^{3/2}. \tag{65}$$

Usually we can neglect the ion contribution in the temperature dependent term.

ESTIMATES FOR CORRECTIONS DUE TO NEUTRAL HYDROGEN

In the previous two sections we have compiled the transport coefficients for a fully ionized plasma consisting of electrons and ions of charge Ze. As indicated in the third and fourth sections (e.g., Eq. (26)), these coefficients might be modified by the presence of neutral hydrogen. In this section we will estimate the importance of such corrections in the low-temperature regime ($kT_e \approx 10\, eV$) either at the beginning of the discharge or near the walls where neutrals are likely to exist. We start with Eq. (26) and insert the following quantities:

$$\ln \Lambda \approx 10,$$

$$\sigma_{e0} \approx 5\text{X}10^{-16}\ [\text{cm}^2]$$

$$S \approx 2.3\text{X}10^{-8}\, e^{-13.5/kT_e} \sqrt{\frac{kT_e}{13.5}}\ [\text{cm}^3\ s^{-1}].$$

In these and the following expressions of this section the temperature kT has to be in units of electrovolts. Equation (26) can now be written in the form

$$\eta_c \approx \frac{m_e\, n_i}{e^2\, n_e} \sqrt{kT_e} \left[\frac{1.5\text{X}10^{-5}}{(kT_e)^2} + \frac{n_0}{n_i} (6.3\text{X}10^{-9} e^{-13.5/kT_e} + 3\text{X}10^{-8})\right].$$

Depending on kT_e and the degree of ionization, the correction term exceeds the first one for

$$(n_0/n_i)(kT_e)^2 > 500. \tag{66}$$

Under conditions near ionization equilibrium the correction, therefore, is negligible.

In view of the neutral gas terms in Eq. (38) we will now estimate the ion velocities v_i^θ and v_i^z. To this end we assume that the center-of-mass velocity is zero in the θ and z directions, which provides

$$v_i = \frac{m_e}{e\, n_i\, (m_i + Z\, m_e)}\, j. \tag{67}$$

For the angular component

$$v_i^\theta = -\left(\frac{m_e}{e\, n_i\, m_i}\right) \frac{c}{4\pi} \frac{\partial B_z}{\partial r}.$$

We need the B_z field, which for a Tokamak can be roughly represented by

$$B_z \approx \frac{B_{z0}}{1 + \dfrac{r}{R} \cos\theta}.$$

In the limit where $r/R \ll 1$, we obtain

$$\frac{\partial B_z}{\partial r} = -\frac{B_{z0} \cos\theta}{R\left(1 + \dfrac{r}{R}\cos\theta\right)^2} \approx -\frac{B_{z0}}{R}.$$

Inserting typical data (R = 100 cm, B_{z0} = 30 kG, and $n_i = 10^{13}$ cm^{-3}) we obtain

$$v_i^\theta \approx 5 \times 10^3 \,[\text{cm sec}^{-1}].$$

The order of magnitude of

$$v_i^z = \frac{m_e}{e\, n_i\, m_i}\, j_z$$

might be calculated from both the total current I_z (electrostatic units) and the discharge radius R_a (centimeters) through the relation

$$j_z = I_z/(\pi R_a^2).$$

With I_z = 100 kA and R_a = 15 cm, we estimate

$$v_i^z \approx 5 \times 10^4 \,[\text{cm } s^{-1}].$$

For comparison, the thermal speeds for protons are of the order $v^i_{\mathrm{th}} \approx 10^7$ cm $\sec^{-1}$.

We turn now to Eq. (35) and compare $|B|^2$ with

$$L_1 = c^2 \eta_c n_0 n_i m_i \left[S + \left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle + \frac{m_e}{m_i} \langle \sigma_{e0} v_{e0} \rangle \right].$$

The sum of the ion-neutral collision rates is about 5×10^{-8} cm^3 sec^{-1}. With $\eta_c \approx 6 \times 10^{14} (kT_e)^{-3/2}$, we arrive at

$$L_1 \approx 5 \times 10^{-24} n_0 n (kT_e)^{-3/2} \; [\mathrm{cm}^{-1} \, g \, \sec^{-2}].$$

Since B has a magnitude of several kilogauss, we conclude that

$$L_1 \ll B^2.$$

It seems quite unlikely that v^r_0 becomes much bigger than v^r_i; this also allows us in Eq. (35) to drop the term proportional to v^r_0.

Finally we have to compare the terms

$$L_1 = m_e \, c \, Z \, n_0 \, S \, v^\theta_i \, B_z / e$$

and

$$L_2 = \frac{m_e \, c \, n_0}{e} \left[\left\langle \left(\frac{\sigma_{i0}}{2} + \sigma_{CE} \right) v_{i0} \right\rangle + S - \langle \sigma_{e0} v_{e0} \rangle \right] B_z \, (v^\theta_i - v^\theta_0)$$

with the term

$$L_3 = c \, \eta_c \, j_z \, B_\theta .$$

The last expression arises from $cE_z B_\theta$. Application of the above-mentioned estimates and approximations leads to

$$L_1 \approx 1.7 \times 10^{12} \, n_0 \, B_z,$$

$$L_2 \approx 1.4 \times 10^{-11} \, n_0 \, B_z,$$

and

$$L_3 \approx 1.7 \times 10^6 \frac{I_z \, B_\theta}{R_a^2 \, (kT_e)^{3/2}}$$

where I_Z is in amperes, kT_e in electrovolts, B_θ in gauss, and R_a in centimeters. The ratio of $L_1 + L_2$ and L_3 provides us with the criterion

$$\frac{n_0 B_z R_a^2 (kT_e)^{3/2}}{I_z B_\theta} \geqslant 10^{17}, \tag{68}$$

which specifies the limit above which the neutral gas terms become dominant. At the center ($r = 0$), B_θ always vanishes, but so does v_i^θ in L_1 and L_2. For a typical set of parameter values ($n_0 = 10^{13}$, $kT_e = 10$, $I_z = 10^4$, $B_\theta = 50$, $R_a = 15$ and $B_z = 3 \times 10^4$) in the initial phase of the discharge, the correction from L_1 and L_2 amounts to a few percent. Similar results are obtained for increasing I_z and B_θ in the interior of the plasma, as well as near the wall.

Thus we can conclude that for the particle flux given by Eq. (35), all corrections due to neutral gas are negligible.

Since v_i^z is only about one order of magnitude higher than v_i^θ, we might extend this conclusion to the electric field component E_z (Eq. (25)) and thus simplify Ohm's law considerably.

TWO NEUTRAL GAS COMPONENTS

Under conditions where recombination is small, we are in the present model left with the processes of ionization and charge exchange in order to extinguish or create neutrals. In a discharge, we will start with cold neutrals being ionized at relatively low temperatures. In later phases only, near the (cold) boundary, neutrals will be left at noticeable densities. On the other hand, neutrals will arise from charge exchange collisions displaying the temperature of the ion. According to this picture we will assume *two* components for the neutrals. Since the temperature of the cold neutrals is not expected to vary much, we neglect Eqs. (3) and (18) and replace them by a constant

$$kT_c = \text{const.}^* \tag{69}$$

For the hot component we mentioned already,

$$kT_h = kT_i. \tag{70}$$

We have to consider, however, two equations of continuity for which we take into account the following source terms:

$$\Delta n_h = -n_h\, n_e\, S + n_i\, n_e\, Q + \left\langle \sigma_{CE}\, v_{i0} \right\rangle n_c\, n_i \tag{71}$$

and

$$\Delta n_c = -n_c\, n_e\, S - \left\langle \sigma_{CE}\, v_{i0} \right\rangle n_c\, n_i. \tag{72}$$

*We use the subscript h for "hot component," the subscript c for "cold component."

With $n_0 = n_h + n_c$, Eq. (8) is still valid, of course.

Applying the estimates of the foregoing section and the formula $Q \approx 10^{-13} (13.5/kT_e)^{1/2}$ cm^3 s^{-1}, we can reduce the radial component of Eq. (22) to

$$n_0 v_0^r = -\left(\frac{1}{n_i m_i \left\langle [(\sigma_{i0}/2) + \sigma_{CE}]\, v_{i0} \right\rangle}\right) \frac{\partial p_0}{\partial r} + n_0 v_r \tag{73}$$

as long as

$$\frac{n_e}{n_0\, kT_e} < 2\times10^4 \tag{74}$$

with kT_e given in electrovolts. This condition is practically always fulfilled.

From a review of Eqs. (15) and (21), we learn that we can make use of Eq. (73) for both neutral species by replacing n_0 with n_c or n_h and employing the appropriate temperatures.

IMPURITIES (OXYGEN)

There exists experimental evidence (8) to the effect that the walls of the discharge vessel or the pumping system release high-Z elements in such quantities that their influence on the plasma is not negligible. Mainly, oxygen contributes to losses through radiation and ionization. Ionization also modifies the electron density due to the great number of ionization stages, even though the impurity concentration is considered to be a small percentage of the plasma density, i.e.,

$$\sum_{i=1}^{9} O_i \ll n_e. \tag{75}$$

We further assume that the impurities have the same temperature as the plasma ions and that their diffusion (flow velocities) is small compared to the ionization and recombination process. In order to describe these impurities, we follow the procedure of Ref. 20:

$$\left.\begin{aligned}
\frac{\partial O_1}{\partial t} &= -n_e O_1 S_1 + n_e O_1 (\alpha_1 + \gamma_2 n_e), \\
\frac{\partial O_k}{\partial t} &= n_e O_{k-1} S_{k-1} - n_e O_k (S_k + \alpha_{k-1} + \gamma_{k-1} n_e) \\
&\quad + n_e O_{k+1} (\alpha_k + \gamma_k n_e), \\
\frac{\partial O_9}{\partial t} &= n_e O_8 S_8 - n_e O_9 (\alpha_8 + \gamma_8 n_e).
\end{aligned}\right\}. \tag{76}$$

The subscript k denotes the stage of ionization and runs here from 1 to 8. Formulae for the radiative recombination α_k and the collisional recombination γ_k are taken from Ref. 20; at the relatively low densities of present Tokamak discharges, the three-body recombination γ_k is negligible. The ionization rate coefficients S_k were updated by approximating graphs presented in a recent publication (21).

In the following formulae, we use the notations $x \equiv \log_{10}(kT_e)$ and $y \equiv \log_{10}(S)$. The percentages in parentheses behind each formula give the root-mean-square and the maximum deviation from the values given in Ref. 21 for the range $1\ eV \leqslant kT \leqslant 10^4\ eV$, and $10^{-16}\ \mathrm{cm}^3\ s^{-1} \leqslant S$. The subscripts follow the spectroscopic notation (1 is associated with neutral oxygen.)

$$\left.\begin{array}{l}
y_1 = -2.9063\,x - 15.342\,e^{-x} + 1.5355\,e^{-x^2} \text{ for } x < 1.2;\ (13\%, 21\%)\\
y_1 = -0.5533\,x - 3.1989\,x^{-1} - 4.3587 \text{ for } x \geqslant 1.2;\ (3\%, 7\%)\\[1ex]
y_2 = -0.94488\,x - 32.134\,e^{-x} + 8.8149\,e^{-x^2} \text{ for } x < 1.2;\ (14\%, 28\%)\\
y_2 = -0.05623\,x^2 - 2.9465\,x^{-2} - 6.6862 \text{ for } x \geqslant 1.2;\ (2\%, 5\%)\\[1ex]
y_3 = -10.827\,x^2 + 27.882\,x - 27.504 \text{ for } x < 1.2;\ (9\%, 17\%)\\
y_3 = -0.05952\,x^2 - 3.9097\,x^{-2} - 6.8502 \text{ for } x \geqslant 1.2;\ (3\%, 8\%)\\[1ex]
y_4 = -1.235\,x - 9.5417\,x^{-1} - 1.1978 \text{ for } x < 1.8;\ (10\%, 20\%)\\
y_4 = -0.45583\,x - 5.6989\,x^{-2} - 6.1611 \text{ for } x \geqslant 1.8;\ (1\%, 2\%)\\[1ex]
y_5 = -2.3318\,x - 14.108\,x^{-1} + 2.6555 \text{ for } x < 1.8;\ (3\%, 8\%)\\
y_5 = -0.49385\,x - 7.328\,x^{-2} - 6.2156 \text{ for } x \geqslant 1.8;\ (2\%, 5\%)\\[1ex]
y_6 = -7.5127\,x^{-2} - 0.38242\,x^{-1} - 7.4312 \text{ for } x < 1.8;\ (5\%, 9\%)\\
y_6 = -0.46353\,x - 8.0959\,x^{-2} - 6.637 \text{ for } x \geqslant 1.8;\ (2\%, 4\%)\\[1ex]
y_7 = -54.94\,x^{-3} + 10.256\,x^{-1} - 11.447;\ (5\%, 15\%)\\[1ex]
y_8 = -67.015\,x^{-3} + 13.062\,x^{-1} - 12.391;\ (3\%, 8\%).
\end{array}\right\} . \qquad (77)$$

Even the relatively high deviations for y_2 lie well within the estimated error (21) for the curves approximated.

The power (per unit volume) lost through ionization and recombination can be expressed by

$$P_i = \sum_{j=1}^{8} \chi_j n\,(S_j O_j - \gamma_j n\, O_{j+1}) + \frac{3}{2} kT_e\ \alpha_j n\, O_{j+1} \qquad (78)$$

where χ_j is the ionization potential.

For energy loss due to radiation, the most important resonance lines are taken into account [20]:

$$P_r = 2\times10^8\, \chi_H \left(\frac{\chi_H}{kT_e}\right)^{1/2} n_e \;\; (O_2\, e^{-15/kT_e} + O_3\, e^{-16/kT_e}$$

$$+ O_4\, e^{-16/kT_e} + O_5\, e^{-20/kT_e} + O_6\, (e^{-12/kT} + e^{-83/kT_e})$$

$$+ 2\, O_7\, e^{-575/kT_e} + 2\, O_8\, e^{-655/kT_e}). \tag{79}$$

The radiation power is normalized to the ionization potential χ_H for hydrogen atoms. Bremsstrahlung is important in the keV temperature region and taken into account (11c) by

$$P_{br} = \left(\frac{2\pi\, kT_e}{3\, m_e}\right)^{1/2} \frac{2^5\, \pi\, e^6\, Z^2\, n_i n_e\, 2\sqrt{3}}{3h\, m_e\, c^3\, \pi}$$

$$= 1.334\times10^{-19}\, Z^2\, n_i n_e\, \sqrt{kT_e}\; [\text{ergs cm}^{-3}\, s^{-1}] \tag{80}$$

if kT_e is given in ergs and the densities in cm^{-3}. This formula brings up the question of which Z should be used, since we basically want to treat the plasma as a three-fluid system. We will adopt here the model of an average ion charge

$$Z = \frac{n_H + \sum_{i=2}^{9} Z_i O_i}{n_H + \sum_{i=2}^{9} O_i} = \frac{n_e}{n_e - \sum_{i=2}^{9} (Z_i - 1)\, O_i} \tag{81}$$

because

$$n_H = n_e - \sum_{i=2}^{9} Z_i O_i. \tag{82}$$

This average charge Z varies, of course, with the location, depending on the status of ionization. We already neglected the diffusion of the impurities; the assumption of a small ratio m_e/m_i is not affected by oxygen. For these reasons, we do not introduce an average mass m_i.

SYSTEM OF DIFFERENTIAL EQUATIONS FOR A TOKAMAK PLASMA FLUID MODEL

In this section we collect the equations previously derived and rewrite them in a form which is suitable for numerical treatment. The right-hand side of each equation shows, first, the terms which are common to all three collision frequency regimes, then, following the dots, the terms which are specific to the banana, the plateau, and the classical regimes, respectively. In the actual calculation, the proper regime should be chosen for each time point and each volume element.

The equation of continuity for the electrons assumes the form

$$\frac{\partial n_e}{\partial t} = n_e\ (n_h + n_c)\ S_H - n_e\ n_H\ Q_H + \sum_{i=2}^{9} Z_i\ \frac{\partial O_i}{\partial t} \ldots\ldots$$

where, in addition, we add

(a) "banana regime"

$$\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\alpha_1\ n_e\ \frac{\partial n_e}{\partial r} + \alpha_2\ n_e\ \frac{\partial (rB_\theta)}{\partial r} + \alpha_3\ n_e^2\ \frac{\partial kT_e}{\partial r} - \alpha_4\ n^2\ \frac{\partial kT_i}{\partial r}\right) \tag{83}$$

(b) "plateau regime"

$$\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\beta_1 (kT_e)\ \frac{\partial n_e}{\partial r} + \beta_2\ \frac{\partial (rB_\theta)}{\partial r}\right)$$

and

(c) "classical regime"

$$\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\gamma_1 n_e\ \frac{\partial n_e}{\partial r}\right) .$$

The coefficients are defined in the following way for the banana regime:

$$h_0 = \frac{8\ r\ Z\ \ln \Lambda_{ei}}{3}\left(\frac{ce}{B_\theta}\right)^2 \left(\frac{2\pi\ m_e\ r}{kT_e\ R}\right)^{1/2}$$

$$h_1 = 1.12\ [1 + (kT_i/kT_e)]\ h_0$$

$$h_2 = 0.43\ h_0/kT_e$$

$$h_3 = 0.19\ h_0/kT_e$$

$$g_0 = \frac{2.44 \times 4 \times \beta Z\ (\ln \Lambda_{ei})(ce)^2}{3\ B_\theta\ (\sqrt{R} - 1.9\sqrt{r})}\left(\frac{2\pi\ m_e\ r}{kT_e}\right)^{1/2}$$

$$g_1 = g_0/(4\pi\ kT_e)$$

$$g_2 = 2.44\, g_0 [1 + (kT_i/kT_e)] \sqrt{\frac{r}{R}} \frac{1}{B_\theta}$$

$$g_3 = 0.69\, g_0 \sqrt{\frac{r}{R}} \frac{1}{B_\theta kT_e}$$

$$g_4 = 0.42\, g_0 \sqrt{\frac{r}{R}} \frac{1}{B_\theta\, kT_e}.$$

With these auxiliary definitions, we obtain finally

$$\alpha_1 = h_1 + g_2\, r$$

$$\alpha_2 = g_1$$

$$\alpha_3 = g_3\, r - h_2$$

$$\alpha_4 = h_3 + g_4\, r.$$

In the plateau regime, we find

$$k_1 = \frac{3 \times 0.9 \times 35 \times r\, B_\theta\, (kT_e)^2}{32\sqrt{2}\, R^2\, Z e^4\, n_e\, (\ln\Lambda_{ei}) B_z}$$

$$k_2 = \frac{0.9 \times 35\, \beta}{32\, B_z\, (1-k_1)} \left(\frac{cr}{eR}\right)^2 \left(\frac{m_e\, kT_e}{\pi}\right)^{1/2}$$

$$\beta_1 = \frac{2\, (\pi\, m_e\, kT_e)^{1/2}}{B_z\, B_\theta\, (1 - k_1)} \left(\frac{rc}{Re}\right)^2$$

$$\beta_2 = k_2/r.$$

The classical diffusion is described by

$$\gamma_1 = \frac{1}{r} \frac{8\, Z \ln \Lambda_{ei}}{3} \left(\frac{ec}{r}\right)^2 \left(\frac{2\pi\, m_e}{kT_e}\right)^{1/2} \left[\left(\frac{R}{B_z}\right)^2 + \left(\frac{r}{B_\theta}\right)^2\right].$$

The equation of continuity for cold atoms (kT_0) is quite similarly structured:

$$\frac{\partial n_c}{\partial t} = - n_c n_e S_H - C_H n_c n_H + \frac{1}{r} \frac{\partial}{\partial r} \left(\delta_1 \frac{\partial n_c}{\partial r}\right) \ldots\ldots\ldots$$

where, in addition, we add

(a) "banana regime"

$$\ldots\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\alpha_1\, n_c\,\frac{\partial n_e}{\partial r} + \alpha_2\, n_c\,\frac{\partial}{\partial r}(r\,B_\theta) + \alpha_3\, n_c n_e\,\frac{\partial kT_e}{\partial r} - \alpha_4\, n_c\, n_e\,\frac{\partial kT_i}{\partial r}\right) \tag{84}$$

(b) "plateau regime"

$$\ldots\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\epsilon_1\, n_c\, kT_e\,\frac{\partial n_e}{\partial r} + \epsilon_2\, n_c\,\frac{\partial (rB_\theta)}{\partial r}\right)$$

and

(c) "classical regime"

$$\ldots\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\gamma_1\, n_c\,\frac{\partial n_e}{\partial r}\right).$$

In addition to the previously defined coefficients, we obtain

$$\delta_1 = \frac{r\,kT_c}{n_i m_i\,(½L + C_H)}$$

where L (= $\langle \sigma_{ie} v_{i0} \rangle$) stands for the elastic collisions rate, with kT_c in the argument. Also,

$$\epsilon_1 = k_1/n_e$$

$$\epsilon_2 = \beta_2/n_e.$$

The only major difference in the equation of continuity for the hot neutrals comes from their time- and space-dependent temperature kT_i. Defining

$$\delta_2 = \frac{r}{n_i m_i\,(½L + C_H)},$$

with kT_i in the arguments for L and C_H, the density of hot neutrals n_h is determined from

$$\frac{\partial n_h}{\partial t} = -\,n_h n_e S_H + n_e n_H Q_H + C_H n_c n_H + \frac{1}{r}\frac{\partial}{\partial r}\left(\delta_2 n_h\,\frac{\partial kT_i}{\partial r} + \delta_2\, kT_i\,\frac{\partial n_h}{\partial r}\right) \ldots\ldots\ldots$$

where, in addition, we add

(a) "banana regime"

$$\ldots\ldots\ldots+\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha_1\, n_h\, \frac{\partial n_e}{\partial r}+\alpha_2\, n_h\, \frac{\partial(rB_\theta)}{\partial r}\right.$$

$$\left.+\alpha_3\, n_h n_e\, \frac{\partial(kT_e)}{\partial r}-\alpha_4\, n_h n_e\, \frac{\partial(kT_i)}{\partial r}\right). \qquad (85)$$

(b) "plateau regime"

$$\ldots\ldots\ldots+\frac{1}{r}\frac{\partial}{\partial r}\left(\epsilon_1\, n_h\, kT_e\, \frac{\partial n_e}{\partial r}+\epsilon_2\, n_h\, \frac{\partial(rB_\theta)}{\partial r}\right).$$

and

(c) "classical regime"

$$\ldots\ldots\ldots+\frac{1}{r}\frac{\partial}{\partial r}\left(\gamma_1\, n_h\, \frac{\partial n_e}{\partial r}\right).$$

The equation for the temperature of the electrons contains losses due to recombination Q_H, ionization of hydrogen (χ_H, S_H), bremsstrahlung (Eq. 80)

$$\zeta_2 = P_{br}/n^2,$$

line radiation from oxygen impurities (Eq. 79)

$$\zeta_3 = \frac{P_r}{n},$$

ionization losses from oxygen

$$\zeta_4 = \sum_{j=1}^{8} \chi_j S_j O_j,$$

as well as losses from radiative recombinations

$$\zeta_6 = \sum_{j=1}^{8} \frac{3}{2}\, kT_e\, \alpha_j\, O_{j+1}.$$

The equipartition between electron and ion temperatures is determined by

$$\zeta_1 = \frac{4\, Z\, e^4 \ln \Lambda_{ei}}{m_i} \left(\frac{2\pi\, m_e}{kT_e}\right)^{1/2}.$$

The heat flux is in the banana regime defined by

$$\kappa_0 = \frac{r\, 8\, Z \ln \Lambda_{ei}}{3} \left(\frac{ec}{B_\theta}\right)^2 \left(\frac{2\pi\, m_e\, r}{R\, kT_e}\right)^{1/2}$$

$$\kappa_1 = 1.81\, \kappa_0$$

$$\kappa_2 = 0.27\, \kappa_0$$

$$\kappa_3 = 1.53\, \kappa_0$$

and by

$$\lambda_0 = \frac{1.75\, r\, \beta\, 4\, Z\, (\ln \Lambda_{ei})(ce)^2}{3\,(\sqrt{R} - 1{\cdot}9\,\sqrt{r})\, B_\theta\, kT_e} \left(\frac{2\pi\, m_e\, r}{kT_e}\right)^{1/2}$$

$$\lambda_1 = \lambda_0/(4\pi\, r)$$

$$\lambda_2 = 2.44\, \lambda_0 \sqrt{\frac{r}{R}}\, \frac{1}{B_\theta}$$

$$\lambda_3 = 0.69\, \lambda_0 \sqrt{\frac{r}{R}}\, \frac{1}{B_\theta}$$

$$\lambda_4 = 0.42\, \lambda_0 \sqrt{\frac{r}{R}}\, \frac{1}{B_\theta}.$$

The transport of thermal electron energy is regulated by the coefficients

$$\tau_j = \frac{5}{2}\, \alpha_j \; (j = 1, 4)$$

while the "compression" is taken into account through the terms multiplied to:

$$\nu_1 = (h_1/r) + g_2 \qquad \nu_2 = 0.17\, \nu_1$$

$$\nu_3 = g_1/r \qquad \nu_4 = 0.17\, \nu_3$$

$$\nu_5 = g_3 - (h_2/r) \qquad \nu_6 = 0.17\, \nu_5$$

$$\nu_7 = (h_3/r) + g_4 \qquad \nu_8 = 0.17\, \nu_7.$$

For the ohmic heating terms, the electrical resistivity coefficients have to be defined.

$$\eta_0 = \frac{4\beta\, Z \ln \Lambda_{ei}\, (ce)^2}{3\,(\sqrt{R} - 1.9\sqrt{r})\, kT_e} \left(\frac{R\, 2\pi\, m_e}{kT_e} \right)^{1/2}$$

$$\eta_1 = \frac{1}{4\pi} \eta_0$$

$$\eta_2 = 2.44\, \eta_0 \sqrt{\frac{r}{R}}\, \frac{1}{B_\theta}$$

$$\eta_3 = 0.69\, \eta_0 \sqrt{\frac{r}{R}}\, \frac{1}{B_\theta}$$

$$\eta_4 = 0.42\, \eta_0 \sqrt{\frac{r}{R}}\, \frac{1}{B_\theta}.$$

Using these definitions, we obtain for the ohmic heating

$$\iota_1 = \frac{1}{4\pi\, r^2} \eta_1$$

$$\iota_j = \frac{1}{4\pi\, r} \eta_j \; (j = 2, 4).$$

The equation for the temperature of the electrons, finally, emerges as

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_e\, kT_e\right) = -\frac{3}{2} kT_e\, n_e\, n_H\, Q_H - (n_h + n_c)\, n_e\, \chi_H\, S_H$$
$$- \zeta_1 (1 - kT_i/kT_e)\, n_e^2 - \zeta_2\, n_e^2$$
$$- (\zeta_3 + \zeta_4 + \zeta_6)\, n_e \, \ldots\ldots\ldots \tag{86}$$

where, in addition, we add

(a) "banana regime"

$$\ldots\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left[(\kappa_1 - \lambda_3\, kT_e)\, n_e^2\, \frac{\partial kT_e}{\partial r} \right.$$
$$+ (\kappa_2 + \lambda_4\, kT_e)\, n_e^2\, \frac{\partial kT_i}{\partial r} - \lambda_1\, nkT_e\, \frac{\partial (rB_\theta)}{\partial r}$$
$$\left. - (\kappa_3 + \lambda_2\, kT_e)\, n_e\, (kT_e + kT_i)\, \frac{\partial n_e}{\partial r} \right]$$

$$
\begin{aligned}
&+\frac{1}{r}\frac{\partial}{\partial r}\Bigg[n_e\,kT_e\,\left(\tau_1\,\frac{\partial n_e}{\partial r}+\tau_2\,\frac{\partial(rB_\theta)}{\partial r}\right.\\
&\qquad\left.+\,\tau_3\,n_e\,\frac{\partial kT_e}{\partial r}-\tau_4 n_e\,\frac{\partial kT_i}{\partial r}\right)\Bigg]\\
&+\nu_1\,kT_i\left(\frac{\partial n_e}{\partial r}\right)^2+\nu_3\,kT_i\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial(rB_\theta)}{\partial r}\right)\\
&+\nu_5\,kT_i\,n_e\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial kT_e}{\partial r}\right)-\nu_7\,n_e kT_i\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)\\
&-\nu_2\,n_e\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)-\nu_4\,n_e\left(\frac{\partial(rB_\theta)}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)\\
&-\nu_6\,n^2\left(\frac{\partial kT_e}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)+\nu_8\,n_e^2\left(\frac{\partial kT_i}{\partial r}\right)^2\\
&+\iota_1\left(\frac{\partial(rB_\theta)}{\partial r}\right)^2+\iota_2\,(kT_e+kT_i)\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial(rB_\theta)}{\partial r}\right)\\
&+\iota_3\,n_e\left(\frac{\partial kT_e}{\partial r}\right)\left(\frac{\partial(rB_\theta)}{\partial r}\right)-\iota_4\,n_e\,\frac{\partial kT_i}{\partial r}\,\frac{\partial(rB_\theta)}{\partial r}
\end{aligned}
$$

(b) "plateau regime"

$$
\begin{aligned}
\ldots\ldots\ldots&+\frac{1}{r}\frac{\partial}{\partial r}\left(\kappa_4\,n_e\,kT_e\,\frac{\partial kT_e}{\partial r}\right)\\
&+\frac{1}{r}\frac{\partial}{\partial r}\left(\rho_1\,(kT_e)^2\,\frac{\partial n_e}{\partial r}+\rho_2\,kT_e\,\frac{\partial(rB_\theta)}{\partial r}\right)\\
&+n_e\,kT_e\,\frac{1}{r}\frac{\partial}{\partial r}\left(\pi_0\,(kT_e)\,\frac{\partial n_e}{\partial r}+\pi_1\,\frac{\partial(rB_\theta)}{\partial r}\right)\\
&+\iota_5\left(\frac{\partial(rB_\theta)}{\partial r}\right)^2
\end{aligned}
$$

and

(c) "classical regime"

$$
\ldots\ldots\ldots+\frac{1}{r}\frac{\partial}{\partial r}\left(\kappa_5\,(kT_e)^2\,\frac{\partial kT_e}{\partial r}\right)
$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{3}{2}\gamma_1\, n_e\, kT_e\,\frac{\partial n_e}{\partial r}\right)$$

$$+\,n_e\, kT_e\,\frac{1}{r}\frac{\partial}{\partial r}\left(\gamma_1\,\frac{\partial n_e}{\partial r}\right)$$

$$+\,\iota_5\left(\frac{\partial(rB_\theta)}{\partial r}\right)^2.$$

A few more definitions are necessary for the "plateau" and the "classical" collision regime:

$$\kappa_4 = \frac{3}{2\,B_\theta B_z}\,(2\pi\, m_e\, kT_e)^{1/2}\left(\frac{r\,c}{R\,e}\right)^2$$

$$\rho_1 = \frac{3}{2}\,k_1$$

$$\rho_2 = \frac{3}{2}\frac{1}{r}\,k_2$$

$$\pi_0 = k_1/n_c$$

$$\pi_1 = k_2/(rn_e)$$

$$\iota_5 = \frac{\beta\, Z \ln \Lambda_{ei}}{12\, kT_e\,\pi}\left(\frac{e\,c}{r}\right)^2\left(\frac{2m_e}{\pi\, kT_e}\right)^{1/2}.$$

The coefficient κ_5 for the classical heat conductivity can be computed from Eq. (52). The ion temperature is calculated from the equation

$$\left(\frac{\partial}{\partial t}\;\frac{3}{2}\frac{n_e}{Z}\,kT_i\right) = \frac{3}{2}\,n_H\, S_H(n_c\, kT_c + n_h\, kT_i)$$

$$-\frac{3}{2}\,kT_i\, n_H\, n_e\, Q_H - \frac{3}{2}\,kT_i\, n_c\, n_H\, C_H$$

$$+\,\zeta_1\,(1 - kT_i/kT_e)\, n_e^2 \,\ldots\ldots\ldots \tag{87}$$

where, in addition, we add

(a) "banana regime"

$$\ldots\ldots\ldots + \frac{1}{r}\frac{\partial}{\partial r}\left(\mu_1\, n_e^2\,\frac{\partial kT_i}{\partial r}\right)$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left[n_e\,kT_i\,\left(\theta_1\,\frac{\partial n_e}{\partial r}+\theta_2\,\frac{\partial(rB_\theta)}{\partial r}\right.\right.$$

$$\left.\left.+\theta_3\,n_e\,\frac{\partial kT_e}{\partial r}-\theta_4\,n_e\,\frac{\partial kT_i}{\partial r}\right)\right]$$

$$-\nu_1\,kT_i\,\left(\frac{\partial n_e}{\partial r}\right)^2-\nu_3\,kT_i\,\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial(rB_\theta)}{\partial r}\right)$$

$$-\nu_5\,n_e\,kT_i\,\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial kT_e}{\partial r}\right)+\nu_7\,n_e\,kT_i\,\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)$$

$$+\nu_2\,n_e\,\left(\frac{\partial n_e}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)+\nu_4\,n_e\,\left(\frac{\partial(rB_\theta)}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)$$

$$+\nu_6\,n_e^2\,\left(\frac{\partial kT_e}{\partial r}\right)\left(\frac{\partial kT_i}{\partial r}\right)-\nu_8\,n_e^2\,\left(\frac{\partial kT_i}{\partial r}\right)^2$$

(b) "plateau regime"

$$\ldots\ldots\ldots+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu_2\,n_e\,kT_i\,\frac{\partial kT_i}{\partial r}\right)$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu_3\,(kT_e)(kT_i)\,\frac{\partial n_e}{\partial r}+\mu_4\,kT_i\,\frac{\partial(rB_\theta)}{\partial r}\right)$$

$$+\frac{n_e}{Z}\,kT_i\,\frac{1}{r}\frac{\partial}{\partial r}\left(\pi_0\,kT_e\,\frac{\partial n_e}{\partial r}+\pi_1\,\frac{\partial(rB_\theta)}{\partial r}\right)$$

and

(c) "classical regime"

$$\ldots\ldots\ldots+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu_5\,(kT_i)^4\,\frac{\partial kT_i}{\partial r}\right)$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu_6\,n_e\,kT_i\,\frac{\partial n_e}{\partial r}\right)$$

$$+\frac{n_e}{Z}\,kT_i\,\frac{1}{r}\frac{\partial}{\partial r}\left(\gamma_1\,\frac{\partial n_e}{\partial r}\right).$$

The heat conduction in the banana regime is given by

$$\mu_1 = \frac{r \cdot 0.68 \ln \Lambda ii}{3} \left(\frac{e\,c}{B_\theta}\right)^2 \left(\frac{\pi\, m_i\, r}{kT_i R}\right)^{1/2}.$$

The transport of thermal ion energy is related to that of the electrons through

$$\theta_j = \tau_j/Z \qquad (j = 1, 4).$$

For the plateau regime, we have to define

$$\mu_2 = \frac{3}{2\, B_\theta\, B_z\, Z} \left(\frac{rc}{ZeR}\right)^2 (2\pi\, m_i\, kT_i)^{1/2}$$

$$\mu_3 = \rho_1/Z.$$

$$\mu_4 = \rho_2/Z.$$

The classical heat flux contains

$$\mu_5 = \frac{(2.5\,x^2 + 4.65) \times 9B_z\ [1 + (rB_z/RB_\theta)^2]}{(x^4 + 2.7\,x^2 + 0.677) \times 16\,\pi\, m_i\, c\, e^7\, Z^6\, n_e\, (\ln \Lambda_{ii})^2}$$

where $x = \omega_i \tau_{ii}$. The transport is determined by

$$\mu_6 = 3\gamma_1/(2\,Z).$$

The poloidal magnetic field B_θ in the "banana regime" obeys the field diffusion equation

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial}{\partial r}\left[\eta_1 \frac{1}{r} \frac{\partial (rB_\theta)}{\partial r} + \eta_2\, (kT_e + kT_i)\, \frac{\partial n_e}{\partial r} + \eta_3\, n_e\, \frac{\partial kT_e}{\partial r} - \eta_4\, n_e\, \frac{\partial kT_i}{\partial r}\right].$$

In the "plateau regime" we have

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial}{\partial r}\left(\omega_3\, \frac{\partial (rB_\theta)}{\partial r} - \omega_2\, B_\theta\, \frac{\partial (rB_\theta)}{\partial r} - \omega_1\, B_\theta\, kT_e\, \frac{\partial n_e}{\partial r}\right)$$

and in the "classical regime"

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial}{\partial r}\left(\xi_1\, B_\theta\, \frac{\partial n_e}{\partial r} + \omega_3\, \frac{\partial (rB_\theta)}{\partial r}\right).$$

The η's have been defined already. In the "plateau regime," we have

$$\omega_1 = \pi_0/r$$

$$\omega_2 = \pi_1/r$$

$$\omega_3 = \frac{1}{r}\frac{c^2}{4\pi}\frac{4}{3}\frac{\beta Z (\ln \Lambda_{ei}) e^2}{kT_e}\left(\frac{2\pi m_e}{kT_e}\right)^{1/2}.$$

The last definition needed for the classical regime is

$$\xi_1 = \gamma_1/r.$$

METHODS FOR NUMERICAL SOLUTION OF THE DIFFERENTIAL EQUATIONS

The system of Eqs. (83)-(88) is predominantly of the diffusion type and obviously highly nonlinear. Together with proper boundary conditons, we will solve this system as an initial value problem by finite difference methods.

Structure of Difference Equations

The space-time grid in Fig. 2 helps to clarify the notation for the difference scheme. The function $A(x,t)$ is abbreviated at the grid points, e.g.,

$$A(x_j,t) = A_j$$

$$A(x_j,t+\Delta t) = \hat{A}_j.$$

In the most convenient "explicit" difference scheme, $\hat{A}_j$ is computed from (known) quantities of time level t without involving $\hat{A}_{j+1}$ or $\hat{A}_{j-1}$. For the simple case of a linear diffusion equation with constant coefficient σ, however, the time step Δt is severely constrained by the condition

$$\Delta t \leqslant \frac{(\Delta x)^2}{2\sigma} \tag{89}$$

in order to ensure numerical stability. There exists no indication that, for nonlinear systems of diffusion equations with variable coefficients, the stability conditions would be less stringent.

We investigated, therefore, implicit difference schemes which in principle require the inversion of a matrix with the dimensions of the number of space points $N(j = 1, \ldots N)$ since the equation for $\hat{A}_j$ also contains, e.g., $\hat{A}_{j+1}$ and $\hat{A}_{j-1}$.

It turned out that the difference equation had to be centered carefully. By "centering" we understand the following: the difference approximation for the time derivative on the left-hand side of a differential equation (e.g., $(\hat{A}_j - A_j)/\Delta t$) is centered around the time level $t + (\Delta t/2)$. Therefore, the right-hand side of the equation should be taken at this intermediate level. The analogous procedure holds for spatial derivatives. The diffusion coefficients in our investigation were proportional to powers of $A(x,t)$ and to powers of derivatives $\partial A(x,t)/\partial x$. Detailed results will be reported elsewhere.

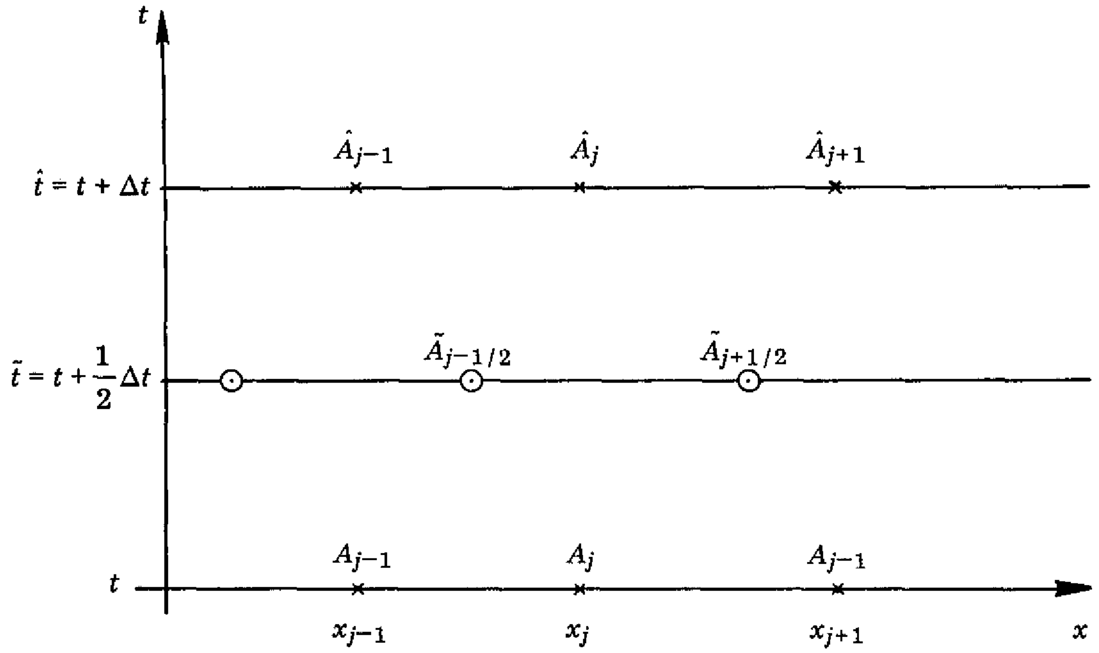

Fig. 2—Space-time grid illustrating the relationships between the functions A_j, $\tilde{A}_j$, and $\hat{A}_j$. These functions are used in the numerical (difference) scheme for solving the set of nonlinear partial differential equations, Eqs. (83)-(88).

Of course, the centered nonlinear difference terms have to be linearized; to do this, the following formulas are useful:

$$\tilde{A} = (\hat{A} + A)/2$$

$$\hat{A}\hat{B} = \hat{A}B + \hat{B}A - AB$$

$$(\widetilde{AB}) = (\hat{A}B + \hat{B}A)/2$$

$$(\widetilde{ABC}) = [C(\hat{A}B + \hat{B}A) + AB(\hat{C} - C)]/2$$

$$(\widetilde{ABCD}) = [CD(\hat{A}B + \hat{B}A) + AB(\hat{C}D + C\hat{D}) - 2ABCD]/2$$

$$(\widetilde{ABCDE}) = [CDE(\hat{A}B + \hat{B}A) + ABC(\hat{D}E + \hat{E}D) + ABDE(\hat{C} - 3C)]/2.$$

We also note the special cases occurring most frequently in the present Tokamak problem:

$$(\widetilde{A^2B}) = A[2\hat{A}B + A(\hat{B} - B)]/2$$

$$(\widetilde{A^2BC}) = A[A(\hat{B}C + \hat{C}B) + 2CB(\hat{A} - A)]/2$$

$$(\widetilde{A^2B^2}) = AB(\hat{A}B + \hat{B}A - AB)$$

$$(\widetilde{A^4B}) = A^3(4\hat{A}B + \hat{B}A - 3AB)/2.$$

Solution of the System of Difference Equations

As we mentioned in the preceding paragraphs implicit schemes require, in principle, matrix inversion. Second spatial derivatives, however, are usually approximated by three adjacent space points using the quantities at the unknown new ($\hat{t}$) and the known old (t) time levels. This produces tridiagonal matrixes for which an efficient method of solution exists (22). It applies to one diffusion equation. In our present problem we have a system of equations, and often the "shear terms" (e.g., $\partial A/\partial t = \partial^2 B/\partial x^2$) are of prime importance. However, it is possible to generalize the mentioned method of solution. We begin with an *ansatz* for m unknown functions $\hat{A}^{(1)}, \ldots \hat{A}^{(m)}$:

$$\left.\begin{aligned}
\hat{A}_j^{(1)} &= F_j^{(11)}\hat{A}_{j+1}^{(1)} + F_j^{(12)}\hat{A}_{j+1}^{(2)} + \ldots + F_j^{(1m)}\hat{A}_{j+1}^{(m)} + E_j^{(1)} \\
\hat{A}_j^{(2)} &= F_j^{(21)}\hat{A}_{j+1}^{(1)} + F_j^{(22)}\hat{A}_{j+1}^{(2)} + \ldots + F_j^{(2m)}\hat{A}_{j+1}^{(m)} + E_j^{(2)} \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
\hat{A}_j^{(m)} &= F_j^{(m1)}\hat{A}_{j+1}^{(1)} + F_j^{(m2)}\hat{A}_{j+1}^{(2)} + \ldots + F_j^{(mm)}\hat{A}_{j+1}^{m} + E_j^{(m)}
\end{aligned}\right\}. \tag{90}$$

With the Eqs. (90) inserted in the difference equations, recursion formulas can be obtained for the auxiliary variables F and E. Utilization of boundary conditions for these recursion formulas, and the computation of the wanted quantities $\hat{A}_j$ ($j = 1,N$), is very similar to the procedure used in Ref. 22, except that matrixes of the dimension $m \times m$ have to be inverted.

The Eqs. (83)-(88) have been written in such a way that the corresponding proper difference equations can be constructed immediately. The outlined methods proved to be numerically stable. Only the truncation errors have to be considered in the choice of time step and number of spatial grid points. Typically we restrict the time step by the requirement that

$$\max_k \left(2 \cdot \frac{|\hat{A}^{(k)} - A^{(k)}|}{\hat{A}^{(k)} + A^{(k)}}\right) \leqslant 0.1 \quad (k = 1, \ldots m). \tag{91}$$

A final remark concerns the treatment of the impurities. The system of equations given by Eqs. (76) constitute a system of ordinary differential equations. In order to limit the number of variables to be stored in the computer, the two-time-level Runge-Kutta method has been applied to obtain a solution. Sometimes the time step from the condition given by Eq. (91) is too large for ionization and recombination processes. Then this time step is subdivided and the Eqs. (76) are solved according to their own characteristic time scale, while the electron temperature and density change only with the bigger step as long as these two parameters are not essentially determined by the influence of the impurities. In the latter case the condition expressed by Eq. (91) enforces a reduction of Δt.

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