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Scattering from a Periodic Corrugated Surface
Part 2 - Semi-Infinite Inhomogeneously Filled Plates with Hard Boundaries

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CONTENTS

ABSTRACT

An incident plane wave is scattered from a periodic corrugated surface consisting of semi-infinite parallel plates, alternately filled with density and wavenumber inhomogeneities, having hard (Neumann) boundaries. Amplitude coefficients of the fields in the various regions are related via sets of linear equations, the latter of which are solved using the modified residue calculus technique. The two examples treated are (a) zero-thickness plates with arbitrary incident angle and no inhomogeneity, and (b) normal incidence with alternate sets of plates filled with a constant wavenumber and density inhomogeneity. The edge condition is derived for these inhomogeneous regions.

PROBLEM STATUS

This is a final report on one phase of the problem; work on the other phases is continuing.

AUTHORIZATION

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SCATTERING FROM A PERIODIC CORRUGATED SURFACE

Part 2—Semi-Infinite Inhomogeneously Filled Plates with Hard Boundaries

1. INTRODUCTION

We wish to solve the problem of the scattering of a plane wave, incident at angel θ_i, from a corrugated surface as illustrated in Fig. 1. The surface consists of infinitesimally thin, semi-infinite, periodically spaced parallel plates. The periodicity interval is separated into two regions by a further parallel plate. A full period (length 2ℓ) consists of an "empty" region (length $2a$) and a region of inhomogeneity ($2\ell - 2a$). The "empty" region has the same wavenumber and density as the medium above the plates, whereas the inhomogeneous region has a different constant wavenumber and density. The plates have hard (Neumann) boundary conditions. References are summarized in previous papers (1, 2) and in a book by Mittra and Lee (3).

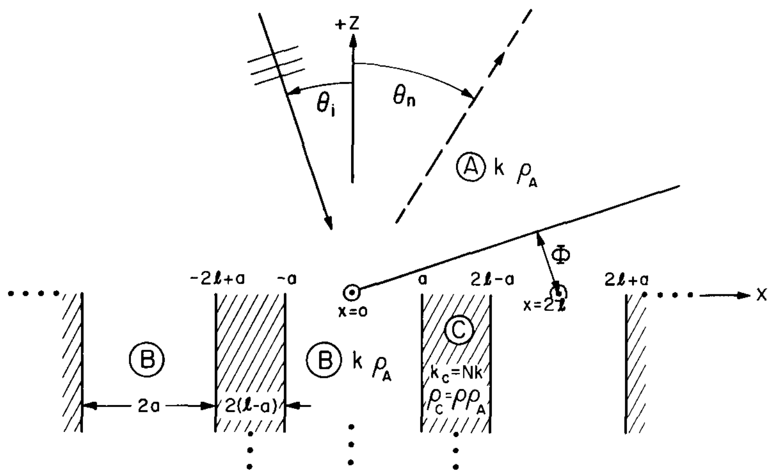

Fig. 1—Plane wave incident at an angle θ_i on an infinite grating of plates which extends to $z = -\infty$ and $y = \pm\infty$ (perpendicular to the plane of the paper). Region C, bounded by two infinitesimally thin plates, is filled with a material having different density and wavenumber values from those of the surrounding media (Regions A and B). The wavenumber k_c is defined by $k_c = Nk$, and the density ρ_c by $\rho_c = \rho\rho_A$. The discrete scattering angles are θ_n, the periodicity is 2ℓ, adjacent empty plates are separated by a distance $2a$, and the parameter t is defined by $t \equiv \ell/a$. Region A is $z \geqslant 0$, and regions B and C are $z \leqslant 0$; Φ is the phase lag.

The basic formalism, which is similar to that contained in Ref. 2, is summarized in Sec. 2. The amplitudes of the assumed velocity potential fields are related by sets of linear equations. The latter are solved using Mittra's modified residue calculus method for two special cases. The flux conservation relation is also derived.

The case of an arbitrary angle of incidence and no inhomogeneity ($t \equiv \ell/a = 1$) is considered in Sec. 3. This is the Carlson-Heins problem (4), for which standard residue calculus methods suffice.

The case of normal incidence ($\alpha_0 = 0$) and arbitrary inhomogeneity ($t \neq 1$) is also investigated in Sec. 3. For this case, the modified residue calculus method is necessary. For both cases the behavior of the field near a plate edge is derived. In particular, the edge effect is illustrated for these density- and wavenumber-inhomogeneous regions.

The summary and conclusions make up Sec. 4. This report is confined to analytic results. The numerical results will be discussed elsewhere.

2. BASIC FORMALISM

Scalar Wave Function

The formalism is similar to that given in NRL Report 7320(2). Details will often be omitted here. A plane wave ψ_i is incident at an angle θ_i on a periodic (period 2ℓ) corrugated surface, as illustrated in Fig. 1. The surface consists of half planes alternately filled with a wavenumber inhomogeneity and different density. The planes extend to $z = -\infty$ and $y = \pm\infty$ (perpendicular to the plane of the figure). The velocity potential or field ψ satisfies the Helmholtz equation*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_i^2\right) \psi(x,z) = 0 \tag{2.1}$$

where

$$\psi(x,z) = \left\{\begin{array}{ll} \psi_A(x,z), & z \geqslant 0 \text{ (region A)} \\ \psi_B(x,z), & z \leqslant 0 \text{ (region B where } -a+2m\ell \leqslant x \leqslant a+2m\ell; \\ & \qquad m = 0,\pm1,\pm2,\ldots) \\ \psi_C(x,z), & z \leqslant 0 \text{ (region C where } a+2m\ell \leqslant x \leqslant 2(m+1)\ell-a; \\ & \qquad m \text{ as above)} \end{array}\right\} \tag{2.2}$$

and

$$k_i = \left\{\begin{array}{l} k, \text{ regions A and B} \\ Nk, \text{ region C} \end{array}\right\}. \tag{2.3}$$

*$e^{-i\omega t}$ is suppressed throughout.

N is a constant representing a wavenumber inhomogeneity; in particular $N = 1$ represents no inhomogeneity and $N = \infty$ represents an impenetrable plate. Regions A, B, and C are filled with densities ρ_A, ρ_B, and ρ_C, respectively. We assume here that $\rho_A = \rho_B$ and $\rho_C \equiv \rho\rho_A$. In addition, ψ satisfies the hard boundary condition

$$\frac{\partial\psi}{\partial n}(x_0,z) = 0 \text{ for } z \leqslant 0 \text{ and } x_0 = \pm a+2m\ell \ (m = 0,\pm1,\pm2,...) \tag{2.4}$$

and the following restrictions:

a. ψ and $\nabla\psi$ are finite in each subregion, except at the plate edges where $|\nabla\psi| = 0(r^{-(1/2)+\epsilon})$ as $r \to 0$ in a polar coordinate system centered on an edge. An explicit form for ϵ is given later where it is shown that $|\epsilon| \leqslant 1/6$.

b. ψ and $\nabla\psi$ are continuous in each subregion, and the pressure and velocity are continuous across the $z = 0$ interface.

c. Apart from the incident wave ψ_i, ψ represents upgoing waves as $z \to +\infty$, and downgoing waves as $z \to -\infty$.

In the corresponding electromagnetic problem, ψ is the y component of the magnetic vector, the latter's only nonzero component.

Fields satisfying the above restrictions have wave functions in the various regions as follows: In region A ($z \geqslant 0$),

$$\psi_A(x,z) = \psi_i(x,z) + \psi_{sc}(x,z) \tag{2.5}$$

and

$$\psi_i(x,z) = \exp[ik(\alpha_0 x-\beta_0 z)] \tag{2.6}$$

where $\alpha_0 = \sin\theta_i$, $\beta_0 = \cos\theta_i$, and

$$\psi_{sc}(x,z) = \sum_{n=-\infty}^{\infty} A_n^h \exp[ik(\alpha_n x+\beta_n z)] \tag{2.7}$$

with

$$\alpha_n = \sin\theta_n$$

and

$$\beta_n = \cos\theta_n = \left\{ \begin{array}{c} \sqrt{1-\alpha_n^2},\ \alpha_n^2 \leqslant 1 \\ +i\sqrt{\alpha_n^2-1},\ \alpha_n^2 > 1 \end{array} \right\}$$

and the grating equation $\alpha_n = \alpha_0+n\Lambda$ where $\Lambda = \lambda/2\ell$. The parameter λ is the incident wavelength. The superscript "h" on the unknown A_n^h amplitudes is used to distinguish them from the A_n amplitudes given for the soft surface boundary situations in Ref. 2. In Region B ($z \leqslant 0$, $-a \leqslant x \leqslant a$), ψ is defined as

$$\psi_B(x,z) = \sum_{j=0}^{\infty} B_j^h \cos\left(\frac{j\pi(x+a)}{2a}\right) e^{-ikq_jz} \tag{2.8}$$

where

$$q_j = \left\{\begin{array}{ll} [1-(j\Lambda t/2)^2]^{1/2}, & (j\Lambda t/2)^2 \leqslant 1 \\ +i[(j\Lambda t/2)^2-1]^{1/2}, & (j\Lambda t/2)^2 > 1 \end{array}\right\}$$

and $t = \ell/a$. The distance between adjacent unfilled plates is $2a$. It is convenient to define $p_j = j\Lambda t/2$. In region C ($z \leqslant 0, a \leqslant x \leqslant 2\ell-a$), ψ is written as

$$\psi_C(x,z) = \sum_{j=0}^{\infty} C_j^h \cos\left(\frac{j\pi(x-a)}{2\ell-2a}\right) e^{-ikr_jz} \tag{2.9}$$

where

$$r_j = \left\{\begin{array}{ll} [N^2-(ju)^2]^{1/2}, & N^2 \geqslant (ju)^2 \\ +i[(ju)^2-N^2]^{1/2}, & N^2 < (ju)^2 \end{array}\right\}$$

and $u = \pi/[k(2\ell-2a)] = \Lambda t/[2(t-1)]$. It is also convenient to define $u_j = r_j|_{N=1}$.

Note that both summations for ψ_B and ψ_C start at $j = 0$ in contrast to the soft case (Ref. 2) which started at $j = 1$. Field representations are completed for $z \leqslant 0$ via the Floquet conditions

$$[\psi(x,z)\exp(-ik\alpha_0 x)]_{x=x_1+2m\ell} = [\psi(x,z)\exp(-ik\alpha_0 x)]_{x=x_1} \tag{2.10}$$

where $m = 0,\pm1,\ldots$, $|x_1| \leqslant a$ for $\psi = \psi_B$, and $a \leqslant x_1 \leqslant 2\ell-a$ for $\psi = \psi_C$.

Linear Equations and Flux Conservation

The amplitude coefficients A_n^h, B_j^h, and C_j^h are related via sets of linear equations. These are derived using the continuity of pressure and velocity across the $z = 0$ interface. First, for the relation between A_n^h and B_j^h ($|x| \leqslant a$) use

$$\psi_A(x,0) = \psi_B(x,0) \tag{2.11}$$

and

$$\frac{\partial \psi_A}{\partial z}(x,0) = \frac{\partial \psi_B}{\partial z}(x,0). \tag{2.12}$$

Substituting the appropriate expressions for ψ_A and ψ_B and projecting out the B_j^h coefficients in the usual manner yields the set of equations

$$\sum_{n=-\infty}^{\infty} \frac{(A_n^h \pm \delta_{n0})\alpha_n I_{nm}}{\beta_n^2 - q_m^2} \begin{Bmatrix} 1 \\ \beta_n \end{Bmatrix} - \frac{i\pi}{\Lambda t} B_m^h \, \tau_m \begin{Bmatrix} 1 \\ q_m \end{Bmatrix} = 0 \tag{2.13±}$$

where the upper (lower) sign is read with the upper (lower) term in brackets, τ_0=2, τ_m=1 (m>0), and

$$I_{nm} = e^{-\pi i \alpha_n/\Lambda t} - (-)^m \, e^{\pi i \alpha_n/\Lambda t}. \tag{2.14}$$

Multiplying Eq. (2.13+) by q_m and successively adding and subtracting the resulting equation from Eq. (2.13−) yields

$$\sum_{n=-\infty}^{\infty} \frac{A_n \alpha_n I_{nm}}{\beta_n \mp q_m} - \frac{\alpha_0 I_{0m}}{\beta_0 \pm q_m} + \frac{2\pi i q_m}{\Lambda t} \tau_m \, B_m^h \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0. \tag{2.15}$$

To derive the equations relating A_n^h and C_j^h, use pressure and velocity continuity (for $a \leqslant x \leqslant 2\ell - a$), i.e.,

$$\rho_A \, \psi_A (x,0) = \rho_C \, \psi_C (x,0) \tag{2.16}$$

and

$$\frac{\partial \psi_A}{\partial z} (x,0) = \frac{\partial \psi_C}{\partial z} (x,0). \tag{2.17}$$

Projecting the C_j^h coefficients from these equations and manipulating the result as above yields

$$\sum_{n=-\infty}^{\infty} \frac{A_n^h \alpha_n J_{nm}}{\beta_n \mp u_m} - \frac{\alpha_0 J_{0m}}{\beta_0 \pm u_m} + \frac{i\pi(t-1)}{\Lambda t} \tau_m C_m^h (r_m \mp \rho u_m) = 0 \tag{2.18±}$$

where

$$J_{nm} = e^{\pi i \alpha_n/\Lambda t} \left(1 - (-)^m e^{\pi i \alpha_n / u}\right) \tag{2.19}$$

and $\rho \equiv \rho_C / \rho_A$. Next define

$$\sigma_m = \frac{r_m - \rho u_m}{r_m + \rho u_m}. \tag{2.20}$$

Multiply Eq. (2.18−) by σ_m and subtract the result from Eq. (2.18+). This yields

$$\sum_{n=-\infty}^{\infty} A_n^h \alpha_n J_{nm} \left(\frac{\sigma_m}{\beta_n + u_m} - \frac{1}{\beta_n - u_m} \right) - \alpha_0 J_{0m} \left(\frac{\sigma_m}{\beta_0 - u_m} - \frac{1}{\beta_0 + u_m} \right) = 0. \tag{2.21}$$

Use Eqs. (2.21) and (2.18−) as the equations relating A_n^h and C_j^h. Special cases of these equations will be solved in later sections.

To determine the conservation of flux condition, use the integral relation

$$\oint_{B+C} d\mu(\rho_0 \psi^*(x,z)\, \overleftrightarrow{\partial}_n \psi(x,z)) = 0 \tag{2.22}$$

where the closed contour $B+C$ is illustrated in Fig. 2, μ is the arc length, n the inward normal, ρ_0 the density in the particular region, and $\phi \overleftrightarrow{\partial}_n \psi \equiv \phi(\partial_n \psi)-(\partial_n \phi)\psi$. Equation (2.22) follows from Green's theorem. Evaluating Eq. (2.22) yields the flux conservation result

$$1 = \sum_n |A_n^h|^2(\beta_n/\beta_0) + \sum_m |B_m^h|^2 \tau_m (q_m/2t\beta_0) + \rho(t-1) \sum_m |C_m^h|^2 \tau_m (r_m/2t\beta_0) \tag{2.23}$$

where summations are over integers such that β_n, q_m, and r_m are real (real propagating orders). The first sum is the reflection coefficient R, and the latter two sums are the transmission coefficient $T(=T_B+T_C)$. Individual spectral reflection and transmission coefficients are obviously defined as

$$R_n = |A_n^h|^2(\beta_n/\beta_0) \tag{2.24}$$

$$T_{Bn} = |B_n^h|^2 \tau_n q_n/(2t\beta_0) \tag{2.25}$$

$$T_{Cn} = \rho |C_n^h|^2 \tau_n r_n (t-1)/(2t\beta_0). \tag{2.26}$$

We next solve two special cases of the general Eqs. (2.15), (2.18−), and (2.21) in the following sections. The general case (arbitrary α_0 and arbitrary t) apparently cannot be solved by the methods we indicate.

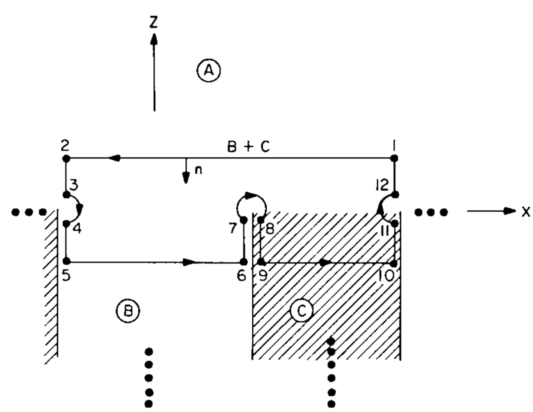

Fig. 2—Contour $B+C$ used with Green's theorem to derive the flux relation. The inward normal is n.

3. DISCUSSION OF THE EQUATIONS

Arbitrary α_0 and $t = 1$

The first case is for infinitesimally thin parallel plates ($t = 1$) with hard boundaries and arbitrary incident angle. This case was first solved by Carlson and Heins (4) using the Wiener-Hopf technique. For $t = 1$, Eqs. (2.15) simplify via the application of

$$I_{nm}\big|_{t=1} = (-)^n I_{0m} \tag{3.1}$$

to the set of equations

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n \alpha_n A_n^h}{\beta_0 \pm q_m} - \frac{\sigma_0}{\beta_0 \mp q_m} + \frac{2\pi i q_m}{\Lambda I_{0m}} \tau_m \begin{Bmatrix} B_m^h \\ 0 \end{Bmatrix} = 0. \tag{3.2}$$

Also for $t = 1$, Eqs. (2.18−) and (2.21) are identically satisfied. Equations (3.2) can be expressed as the residue series of integrals of the form

$$(2\pi i)^{-1} \oint_{C_s} \frac{g(\omega)d\omega}{\omega \pm q_m} = 0 \tag{3.3}$$

where C_s is a contour at infinity and the meromorphic function $g(\omega)$ has the properties:

a. $g(\omega)$ has simple poles at $\omega = \beta_n$ ($n = 0, \pm 1, \pm 2, \ldots$) and $\omega = -\beta_0$.

b. $g(\omega)$ has simple zeroes at $\omega = q_0 (=1)$ and $\omega = q_m$ ($m = 1, 2, \ldots,$).

c. $g(\omega) = 0(\omega^{-1/2})$ as $|\omega| \to \infty$.

d. $\psi = 0(r^{1/2})$ and $|\nabla\psi| = 0(r^{-1/2})$ as a plate edge is approached ($r \to 0$), and $g(\omega)$ is given by

$$g(\omega) = \frac{2\alpha_0\beta_0}{\beta_0^2 - \omega^2} \frac{\Pi(\omega,q)}{\Pi(-\beta_0,q)} \frac{\Pi_{12}(-\beta_0,\beta)}{\Pi_{12}(\omega,\beta)} \frac{1-\omega}{1+\beta_0} e^{iH(\omega+\beta_0)} \tag{3.4}$$

where $H = 2\ln(2)/\Lambda$ and the infinite products are

$$\Pi(\omega,q) = \prod_{m=1}^{\infty} (1-\omega/q_m)(2q_m/im\Lambda)e^{2\omega/im\Lambda} \tag{3.5}$$

$$\Pi_1(\omega,\beta) = \prod_{n=1}^{\infty} (1-\omega/\beta_n)(\beta_n/in\Lambda)e^{\omega/in\Lambda} \tag{3.6}$$

and

$$\Pi_2(\omega,\beta) = \prod_{n=1}^{\infty} (1-\omega/\beta_{-n})(\beta_{-n}/in\Lambda)e^{\omega/in\Lambda} \tag{3.7}$$

with $\Pi_{12} \equiv \Pi_1 \Pi_2$. These products are obviously used to satisfy properties (a) and (b) above. The rest of $g(\omega)$ is constructed as in Ref. 2. Substituting Eq. (3.4) into Eq. (3.3) and performing the integration yields the residue series ($r(\beta)$ is the residue of g at $\omega = \beta$)

$$\sum_{n=-\infty}^{\infty} \frac{r(\beta_n)}{\beta_n \pm q_m} - \frac{r(-\beta_0)}{\beta_0 \mp q_m} + \left\{ \begin{matrix} g(-q_m) \\ 0 \end{matrix} \right\} = 0 \tag{3.8}$$

which is similar to Eq. (3.2) if we identify

$$(-)^n \alpha_n A_n^h = r(\beta_n) \tag{3.9}$$

and

$$B_m^h = \frac{\Lambda I_{0m}}{2\pi i q_m \tau_m} g(-q_m) \tag{3.10}$$

where the condition $r(-\beta_0) = \alpha_0$ has been used to define $g(\omega)$. The explicit calculation of the residue, as in Ref. 2, yields

$$A_n^h = (-)^{n+1} \frac{\beta_0}{\beta_n} \frac{1-\beta_n}{1+\beta_0} \frac{\Pi(\beta_n,q)}{\Pi(-\beta_0,q)} \frac{\Pi_{12}(-\beta_n,\beta)}{\Pi_{12}(\beta_0,\beta)} e^{iH(\beta_0+\beta_n)}. \tag{3.11}$$

For completeness the explicit value of B_m^h is

$$B_m^h = \frac{\Lambda I_{0m}}{\pi i q_m \tau_m} \frac{\alpha_0 \beta_0}{(\beta_0^2 - q_m^2)} \frac{\Pi(-q_m,q)}{\Pi(-\beta_0,q)} \frac{\Pi_{12}(-\beta_0,\beta)}{\Pi_{12}(-q_m,\beta)} \frac{1+q_m}{1+\beta_0} e^{iH(\beta_0 - q_m)}. \tag{3.12}$$

Using A_n^h from Eq. (3.11) in ψ_A [Eq. (2.5)], property (d) can be shown to follow in a way similar to that given in Ref. 2. This completes the solution of the first case.

Normal Incidence ($\alpha_0 = 0$) and $t \neq 1$

The Function $G(w)$—The second case is that of normal incidence on arbitrarily thick* plates ($t \neq 1$), with alternate plates being filled with an inhomogeneous material (depending on the parameter N) and having a different density. For $N = 1$ the geometry is a slight generalization of that of Carlson-Heins (4) since there are superimposed two periodic sets of semi-infinite plates. With $\alpha_0 = 0$ and the result

$$(I_{nm} - I_{-nm})|_{\alpha_0=0} = -2i \sin(\pi n/t)(1+(-)^m),$$

Eqs. (2.15) simplify, for m even, to

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*See footnote on p.2.

$$\sum_{n=1}^{\infty} \frac{nA_n^h \sin(\pi n/t)}{\beta_n \pm q_m} + \frac{\pi}{\Lambda^2 t} \begin{Bmatrix} 1 \\ -A_0^h \end{Bmatrix} \delta_{m0} - \frac{\pi q_m}{2\Lambda^2 t} \tau_m \begin{Bmatrix} B_m^h \\ 0 \end{Bmatrix} = 0 \tag{3.13}$$

where $B_m^h = 0$ for m odd, and we have used the fact that $A_n^h = A_{-n}^h$. The latter follows from the symmetry of both the geometry and the incident field about the $x = 0$ plane. Using a similar development and the result

$$(J_{nm} - J_{-nm})|_{\alpha_0=0} = 2i \sin(\pi n/t)\,(1+(-)^m)$$

in Eqs. (2.21) and (2.18–) reduces them to (m even)

$$\sum_{n=1}^{\infty} nA_n^h \sin(\pi n/t) \left(\frac{\sigma_m}{\beta_n + u_m} - \frac{1}{\beta_n - u_m} \right) - (A_0^h + \sigma_0) \frac{\pi(t-1)}{\Lambda^2 t} \delta_{m0} = 0 \tag{3.14}$$

and

$$\sum_{n=1}^{\infty} \frac{nA_n^h \sin(\pi n/t)}{\beta_n + u_m} - \frac{\pi(t-1)}{\Lambda^2 t} \delta_{m0} + \frac{\pi(t-1)}{4\Lambda^2 t} \tau_m (r_m + \rho u_m) C_m^h = 0 \tag{3.15}$$

where $C_m^h = 0$ for m odd.

We solve Eqs. (3.13)-(3.15) by matching these linear equations to the residue series arising from integrals of a meromorphic function $G(\omega)$ which has the following properties:

a. $G(w)$ has simple poles at $\omega = \beta_n$ ($n = 1,2,3,\ldots$).

b. $G(w)$ has simple zeroes at $\omega = q_m$ ($m = 2,4,6,\ldots$).

c. $G(w)$ has simple zeroes at $\omega = u_m' = u_m + \delta_m$ ($m = 2,4,6,\ldots$) where the δ_m are found from the symmetry relation

$$\sigma_m G(-u_m) = G(u_m) \qquad (m = 2,4,6,\ldots). \tag{3.16}$$

The asymptotic value of δ_m $\left(\delta = \lim_{m\to\infty} \delta_m\right)$ is given by

$$\delta = -(2iu/\pi) \sin^{-1}(\sigma/2) \tag{3.17}$$

where $\sigma = \lim_{m\to\infty} \sigma_m$. The value of δ is derived as in Ref. 2. Note that in Ref. 2, for a soft boundary, δ differed by an overall minus sign from the case here.

d. As $|\omega| \to \infty$, $G(\omega) = 0(\omega^{-(1/2)-\epsilon})$ where $\epsilon \equiv \delta/2iu$. From (c) note that $|\epsilon| \leqslant 1/6$.

e. As an edge is approached ($r \to 0$), $\psi = 0(r^{(1/2)+\epsilon})$ and $|\nabla \psi| = 0(r^{-(1/2)+\epsilon})$.

Now using properties (a)-(d) and Eq. (3.16), residue series resulting from the integrals (for $m = 0,2,4,\ldots$)

$$(2\pi i)^{-1} \oint_{C_s} \frac{G(\omega)d\omega}{\omega \pm q_m} = 0 \tag{3.18}$$

$$(2\pi i)^{-1} \oint_{C_s} G(\omega) \left(\frac{\sigma_m}{\omega + u_m} - \frac{1}{\omega - u_m} \right) d\omega = 0 \tag{3.19}$$

and

$$(2\pi i)^{-1} \oint_{C_s} \frac{G(\omega)d\omega}{\omega + u_m} = 0, \tag{3.20}$$

where C_s is a contour at infinity, match Eqs. (3.13)-(3.15) if we make the following identifications (where $R(\beta)$ is the residue of $G(\omega)$ at $\omega=\beta$):

$$R(\beta_n) = n \sin(\pi n/t) A_n^h \qquad (n \geqslant 1) \tag{3.21}$$

$$G(-q_m) = (\pi/\Lambda^2 t)\, [\delta_{m0} - (q_m \tau_m B_m^h/2)] \tag{3.22}$$

$$G(1) = -\pi A_0^h/\Lambda^2 t \tag{3.23}$$

$$G(1) - \sigma_0 G(-1) = (\sigma_0 + A_0^h)\pi(t-1)/\Lambda^2 t \tag{3.24}$$

$$G(-u_m) = (\pi(t-1)/\Lambda^2 t)((r_m + \rho u_m)(\tau_m C_m^h/4) - \delta_{m0}). \tag{3.25}$$

Equations (3.21)-(3.23) follow from matching the residue series of Eq. (3.18) to Eq. (3.13). Equation (3.24) follows from Eqs. (3.14), (3.16), and (3.19), and Eq. (3.25) follows from Eqs. (3.20) and (3.15). Equations (3.22) and (3.25) for $m=0$ can be equated [both are $G(-1)$], and yield

$$1 - B_0^h = (t-1)\, [(\rho + N)(C_0^h/2) - 1]. \tag{3.26}$$

A further value for $G(-1)$ follows by substituting Eq. (3.23) into Eq. (3.24) and, when combined with Eq. (3.26), yields

$$1 - t - (tA_0^h/\sigma_0) = 1 - B_0^h = (t-1)[(\rho + N)(C_0^h/2) - 1]. \tag{3.27}$$

For $t=1$ note that $B_0^h=1$ and $A_0^h=0$, the latter indicating that, for normal incidence on a $t=1$ surface, there is no backscatter return. Both these results also follow from the first case discussed previously if we set $\alpha_0=0$ in the appropriate equations. It also follows from Eq. (3.27) that if t, ρ, and N are kept as parameters, only one of the terms A_0^h, B_0^h, and C_0^h is independent. Later we choose A_0^h as the independent amplitude and use it to construct $G(\omega)$.

Constructing $G(\omega)$—The meromorphic function $G(\omega)$ is defined using the following infinite products (which are discussed in App. A of Ref. 2) in addition to Eq. (3.6):

$$\Pi_e(\omega,q) = \prod_{m=1}^{\infty} (1-\omega/q_{2m})(q_{2m}/im\Lambda t)e^{\omega/im\Lambda t} \tag{3.28}$$

$$\Pi_e(\omega,u') = \prod_{m=1}^{\infty} (1-\omega/u'_{2m})\left\{u'_{2m}/(2imu+\delta)\right\} e^{\omega/2imu}. \tag{3.29}$$

The product $\Pi_1(\omega,\beta)$ from Eq. (3.6) vanishes at $\omega = \beta_n$ (n = 1,2,...) and is used to satisfy property (a); $\Pi_e(\omega,q)$ is used to satisfy (b) and $\Pi_e(\omega,u')$ is used for (c). Additional terms in the products are to guarantee absolute and uniform convergence and convenient calculation of asymptotic properties. The function $G(\omega)$ which satisfies properties (a)-(c) is given by

$$G(\omega) = E(\omega)\,\Pi_e(\omega,q)\,\Pi_e(\omega,u')/\Pi_1(\omega,\beta) \tag{3.30}$$

where $E(\omega)$ is an entire function to be determined. As $|\omega| \to \infty$ (see App. A of Ref. 2),

$$G(\omega) \approx E(\omega)\,\omega^{-(1/2)-\epsilon}\,e^{i\omega H} \tag{3.31}$$

with

$$H = [t\ln t-(t-1)\ln(t-1)]/\Lambda t \tag{3.32}$$

and ϵ given in the definitions of properties (c) and (d). Choosing $E(\omega)$ as

$$E(\omega) = E\,e^{-i\omega H} \qquad (E = \text{constant}) \tag{3.33}$$

insures (d). The constant E is fixed as follows. Divide Eq. (3.24) by $G(1)$ so that we can write

$$\sigma_0 G = 1 - (\sigma_0+A_0^h)\pi(t-1)/[\Lambda^2 tG(1)] \tag{3.34}$$

where $G \equiv G(-1)/G(1)$. On the right-hand side of Eq. (3.34) substitute for $G(1)$ from Eq. (3.23). We can thus solve for A_0^h in terms of G

$$A_0^h = \frac{\sigma_0(t-1)}{\sigma_0 G-t}. \tag{3.35}$$

G is known from Eqs. (3.30) and (3.33) as

$$G = \frac{\Pi_e(-1,q)}{\Pi_e(1,q)}\,\frac{\Pi_e(-1,u')}{\Pi_e(1,u')}\,\frac{\Pi_1(1,\beta)}{\Pi_1(-1,\beta)}\,e^{2iH}. \tag{3.36}$$

To know G requires that δ_m be known. We will discuss the δ_m calculation later. Assuming the δ_m to be known for the present, G is known, and thus so is A_0^h. Using Eqs. (3.23), (3.30), and (3.33), $G(\omega)$ can be written as

$$G(\omega) = -\frac{\pi A_0^h}{\Lambda^2 t} \frac{\Pi_e(\omega,q)}{\Pi_e(1,q)} \frac{\Pi_e(\omega,u')}{\Pi_e(1,u')} \frac{\Pi_1(1,\beta)}{\Pi_1(\omega,\beta)} e^{i(1-\omega)H}. \tag{3.37}$$

Thus B_m^h and C_m^h are known via Eqs. (3.22) and (3.25). To find A_n^h $(n \geqslant 1)$ from Eq. (3.21) requires a residue calculation similar to that used in Ref. 2. The result can be written as

$$A_n^h = \frac{(-)^n \pi\, A_0^h n}{t\beta_n \sin(\pi n/t)} \frac{\Pi_e(\beta_n,q)}{\Pi_e(1,q)} \frac{\Pi_e(\beta_n,u')}{\Pi_e(1,u')} \frac{\Pi_1(-\beta_n,\beta)}{\Pi_1(-1,\beta)} e^{i(1-\beta_n)H}. \tag{3.38}$$

For completeness we list the other amplitude values. From Eqs. (3.22) and (3.37) (m = 0, 2,4,...)

$$B_m^h = \delta_{m0} + \frac{2A_0^h}{\tau_m q_m} \frac{\Pi_e(-q_m,q)}{\Pi_e(1,q)} \frac{\Pi_e(-q_m,u')}{\Pi_e(1,u')} \frac{\Pi_1(1,\beta)}{\Pi_1(-q_m,\beta)} e^{i(1+q_m)H}, \tag{3.39}$$

and from Eqs. (3.25) and (3.37) (m = 0,2,4,...)

$$C_m^h = \frac{2\delta_{m0}}{\rho+N} - \frac{4A_0^h}{\tau_m(t-1)(r_m+\rho u_m)} \frac{\Pi_e(-u_m,q)}{\Pi_e(1,q)} \frac{\Pi_e(-u_m,u')}{\Pi_e(1,u')} \frac{\Pi_1(1,\beta)e^{i(1+u_m)H}}{\Pi_1(-u_m,\beta)}. \tag{3.40}$$

All these amplitudes are known once the set of δ_m are known. These latter follow from the symmetry condition [Eq. (3.16)] and an iterative procedure similar to that in Ref. 2. The latter is given in the next section.

Zero Shifting—The iterative procedure employed to calculate the δ_m follows from Eq. (3.16). The calculation is similar to that in Ref. 2, and only the final result is listed here:

$$\frac{\delta_m^{(j+1)} e^{2u_m/imu}}{\delta_m^{(j+1)}+2u_m} \prod_{n=1}^{M-1} \frac{\delta_{2n}^{(j+1)}+u_{2n}-u_m}{\delta_{2n}^{(j+1)}+u_{2n}+u_m} e^{u_m/inu}$$

$$= (RHS) \times \prod_{n=M+1}^{\infty} \frac{\delta_{2n}^{(j)}+u_{2n}+u_m}{\delta_{2n}^{(j)}+u_{2n}-u_m} e^{-u_m/inu}, \tag{3.41}$$

where j is the interation index, $2M = m$, and

$$(RHS) = \sigma_m \frac{\Pi_e(-u_m,q)}{\Pi_e(u_m,q)} \frac{\Pi_1(u_m,\beta)}{\Pi_1(-u_m,\beta)} e^{2iu_m H}. \tag{3.42}$$

The iteration is as follows:

1. For m large, $\delta_m \approx \delta = -(2iu/\pi)\sin^{-1}(\sigma/2)$. Assume that $\delta_m^{(0)} = \delta$ and substitute this into the right-hand side of Eq. (3.41). Then calculate first iterations $\delta_2^{(1)},\delta_4^{(1)},\ldots\delta_P^{(1)}$ such that $|\delta_{P+2}^{(1)} - \delta|$, $|\delta_{P+4}^{(1)} - \delta|$, ... are zero to any desired accuracy.

2. Set δ_{P+2}, δ_{P+4}, ... equal zero for all successive iterations. Then δ_2, δ_4, ..., δ_P is the iteration set.

3. Calculate higher iterations until Eq. (3.16) is satisfied to any desired accuracy.

The above iteration procedure was developed by Mittra (3) and used in the present efforts (1,2) with excellent results. Other advantages of the procedure are listed in Ref. 2.

4. SUMMARY

It has been shown possible to solve the problem of scattering from the corrugated structure shown in Fig. 1 using a combination of analytic function theory and a numerical iteration procedure. The steps in the calculation were as follows: first, determine the δ_m zero shifts, then calculate G via Eq. (3.36) and thus A_0^h from Eq. (3.35). Finally, the amplitudes are calculated using Eqs. (3.38)-(3.40). Only the analytic results are presented in this report. Numerical results will be published later.

This report is the second in a series on scattering from corrugated structures. Further surfaces similar to those in Fig. 1, but with finite depth plates, are presently being considered.

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13. ABSTRACT

An incident plane wave is scattered from a periodic corrugated surface consisting of semi-infinite parallel plates, alternately filled with density and wavenumber inhomogeneities, having hard (Neumann) boundaries. Amplitude coefficients of the fields in the various regions are related via sets of linear equations, the latter of which are solved using the modified residue calculus technique. The two examples treated are (a) zero-thickness plates with arbitrary incident angle and no inhomogeneity, and (b) normal incidence with alternate sets of plates filled with a constant wavenumber and density inhomogeneity. The edge condition is derived for these inhomogeneous regions.

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