NRL Report 7031

Detection Results for Scanning Radars Employing Feedback Integration

G.V. Trunk

Radar Analysis Staff
Radar Division

March 3, 1970

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

DETECTION RESULTS FOR SCANNING RADARS EMPLOYING FEEDBACK INTEGRATION

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

An interim report on one phase of the problem; work on the problem is continuing.

5. AUTHOR(S) *(First name, middle initial, last name)*

G.V. Trunk

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 3, 1970	16	9

8a. CONTRACT OR GRANT NO. NRL Problem R02-46 b. PROJECT NO. A37538/652/70F48111704 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7031 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Department of the Navy (Naval Air Systems Command), Washington, D.C. 20360

13. ABSTRACT

A method has been devised for calculating the detection probabilities for a scanning radar employing feedback integration. The optimal value of K, the feedback value, is determined by equating $1/(1-K)$, the equivalent number of pulses integrated, to 63% of the number of pulses between the 3-dB points of the scanning antenna. This expression holds for small- and large-sample sizes and for fluctuating and nonfluctuating targets. Then, by subtracting an antenna beam-shape factor of 0.3 dB from the midbeam signal-to-noise ratio, one can use the standard detection curves to find the probability of detecting a target as it passes through the beam. The beam-shape factor differs from 0.3 dB by less than 0.2 dB for small-sample sizes.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Scanning radars Probability of detection Feedback integrator Delay-line integrator						

☆U.S. GOVERNMENT PRINTING OFFICE: 1970-397-174/23

CONTENTS

ABSTRACT

A method has been devised for calculating the detection probabilities for a scanning radar employing feedback integration. The optimal value of K, the feedback value, is determined by equating $1/(1-K)$, the equivalent number of pulses integrated, to 63% of the number of pulses between the 3-dB points of the scanning antenna. This expression holds for small- and large-sample sizes and for fluctuating and nonfluctuating targets. Then, by subtracting an antenna beam-shape factor of 0.3 dB from the midbeam signal-to-noise ratio, one can use the standard detection curves to find the probability of detecting a target as it passes through the beam. The beam-shape factor differs from 0.3 dB by less than 0.2 dB for small-sample sizes.

PROBLEM STATUS

This is an interim report on one phase of the problem; work on the problem is continuing.

AUTHORIZATION

NRL Problem R02-46
Project A37538/652/70F48111704

Manuscript submitted November 25, 1969.

DETECTION RESULTS FOR SCANNING RADARS EMPLOYING FEEDBACK INTEGRATION

INTRODUCTION

In the conventional method of computing the performance of a scanning radar, the analysis is based on the existence of an optimum integration angle, integration over this angle yielding the greatest possible improvement in the signal-to-noise (S/N) ratio. This optimum angle is found to be 0.84β for a Gaussian-shaped beam, where β is the half-power beamwidth. If integration is performed over this optimum angle, the actual radar performance can be calculated by subtracting a beam-shape factor of 1.6 dB from the noncoherent integration gain associated with the total number of pulses within the beamwidth β. The factor 1.6 dB is found in the following manner: Integration over 0.84β of a Gaussian-shaped beam yields the same S/N improvement as integration over 0.47β of a rectangular beam, resulting in a "scanning loss" of $10 \log \sqrt{0.47}$, which equals 1.6 dB. This procedure is due to Blake (1-3) and yields results which correlate very well with actual radar data.

Later Hall (4) considered the problem in a slightly different manner. He integrates all the pulses within the 3-dB beamwidth and weights them with the two-way antenna power pattern. For the large-sample case, Hall calculates an antenna beam-shape factor by a method similar to Blake's, except for the nonuniform weighting he introduces. The small-sample behavior is found by direct calculation. That is, the individual probability density functions of the sum of both noise and signal-plus-noise samples, properly weighted, are found by convolving the probability density functions of the individual samples. From these densities for the sums, the probabilities of detection and false alarm can be found. The exact procedure for calculating the probability of detection can be found in his paper.

The previous methods assume that the N pulses are integrated by an ideal postdetection integrator (one with perfect memory). However, in many scanning systems, especially high-resolution systems, the storage requirements dictate the use of a feedback integrator,* such as shown in Fig. 1. This report investigates several questions concerning the feedback integrator, such as, what is the optimum value of the feedback factor K and how does one calculate the probability of detection? To answer these questions a method similar to the methods of Blake and Hall will be adopted.

LARGE-SAMPLE CASE

Since the results of this report depend on computer calculations, the more realistic $\sin(x)/x$ antenna pattern will be used instead of the more manipulatable Gaussian-shaped antenna pattern. That is, the one-way voltage antenna pattern is

$$G(\theta) = \frac{\sin(a\theta)}{a\theta} , \tag{1}$$

*Conventionally, this type of integrator is referred to as a delay-line integrator. However, since it is much easier to stabilize the integrator by using digital components, the integrator will be called a feedback integrator throughout this report.

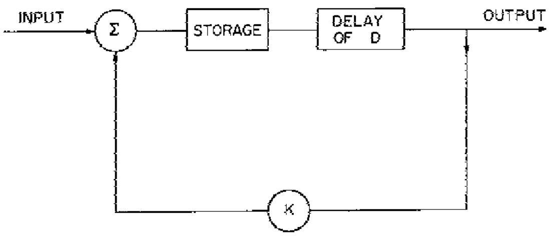

Fig. 1 - Feedback integrator, where K is the feedback factor and D is the interpulse period

where θ is the angle measured from the beam center and $a = 1.3916/\theta_1$, $2\theta_1$ being the 3-dB beamwidth of the antenna.

When a large number of pulses are integrated (this is equivalent to K being very nearly equal to 1), the distributions of noise and signal-plus-noise will be approximately Gaussian because of the Central Limit Theorem. As shown in the Appendix, for two Gaussian densities $G(\mu_1, \sigma_1^2)$ and $G(\mu_2, \sigma_2^2)$ the probability of detection is a monotonic increasing function of a quantity that can be approximated by the S/N ratio $(\mu_2 - \mu_1)/\sigma_1$. For the feedback integrator, this expression equals

$$R(K, \alpha) = \frac{\mu(\alpha|K)}{\sigma_N(K)} = \frac{\sum_{i=0}^{\infty} K^i \left\{\frac{\sin\,[a(\alpha + iD)]}{a(\alpha + iD)}\right\}^4}{2\sigma^2/\sqrt{(1-K^2)}}, \tag{2}$$

where α is the position of the latest received pulse as indicated in Fig. 2, σ^2 is the variance of the Gaussian noise, K is the feedback value, and $2/D$ is the number of pulses lying within the 3-dB beamwidth. Thus, if the scanning rate is ω revolutions per second and if the pulse repetition rate is F pps, $D = \omega\pi/\theta_1 F$. Consequently, the optimal value of K can be found by maximizing $R(K, \alpha)$ with respect to K and α. The expression was maximized with the aid of a computer by first finding the value of α that maximizes the difference in means, $\mu(\alpha|K)$, for a given value of K. This calculation was simplified by making use of the fact that

$$\mu(\alpha - D|K) = \left\{\frac{\sin\,[a(\alpha - D)]}{a(\alpha - D)}\right\}^4 + K\mu(\alpha|K) \tag{3}$$

and assuming that the antenna gain equals zero outside the first null; i.e., $G(\theta) = 0$, if $|\theta| \geq \pi/a$. The results for $D = 0.01$ and K ranging from 0.99 to 0.997 in steps of 0.001

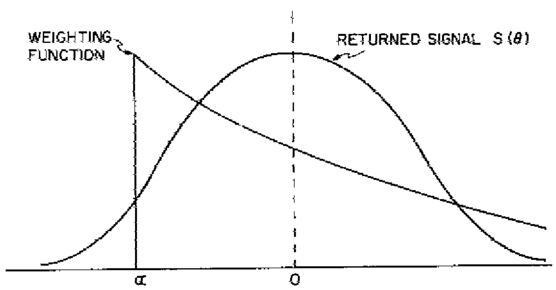

Fig. 2 - Weighting function of feedback integrator

appear in the first three columns of Table 1. As one would expect, the values of α_{opt} and $\mu(\alpha_{opt}|K)$ increase monotonically with K. Referring to the last two columns of Table 1, if $\mu(\alpha_{opt}|K)$ is divided by $2\sigma^2\sqrt{1/(1-K^2)}$, the largest S/N ratio is obtained for $K = 0.992$ with a corresponding $\alpha = 0.625$ and $N = 124$. The symbol N will be referred to as the number of pulses integrated and equals $1/(1-K)$, which is approximately $2/(1-K^2)$. Note that the maximum is rather broad since the values of K in the interval 0.99 to 0.994 will yield S/N ratios less than 0.1 dB from the maximum.

Table 1
Optimization of the Feedback Integrator for $D = 0.01$

K	α_{opt}	$(\alpha_{opt}\|K)$	$1/(1-K)$	$R(K,\alpha)$
0.990	0.565	65.4	100	$4.62/\sigma^2$
0.991	0.595	69.3	111	$4.65/\sigma^2$
0.992	0.625	73.6	124	$4.67/\sigma^2$
0.993	0.665	78.5	142	$4.66/\sigma^2$
0.994	0.715	84.1	166	$4.61/\sigma^2$
0.995	0.765	90.4	200	$4.51/\sigma^2$
0.996	0.825	97.8	250	$4.37/\sigma^2$
0.997	0.965	106.6	333	$4.13/\sigma^2$

To calculate an antenna beam-shape factor (ABF) for this system (the ABF plays the same role as Blake's "scanning loss"), one can reason as follows: If the beam was rectangular, the signal strength would build up as the number of pulses integrated, while the noise would build up as the square root of this number. That is, the S/N voltage improvement for the constant signal would be $\sqrt{1/(1-K)}/2\sigma^2$. The ratio of this number to the largest S/N improvement obtained for the feedback system will be defined as the antenna beam-shape factor:

$$\mathrm{ABF} = 20\ \log\left\{\frac{\sqrt{1/(1-K)}/2\sigma^2}{R(K,\alpha_{opt})}\right\}. \tag{4}$$

The previous procedure was repeated for smaller values of D, and the optimized results are shown in Table 2. From this table, conclusions can be drawn which make it unnecessary to maximize Eq. (2) to find the optimal K. The procedure simply makes use of the approximation

$$N = \frac{1}{1-K} \approx \frac{2(0.63)}{D} + 1\ . \tag{5}$$

Hence, the value of K is given by

$$K = \frac{1.26}{1.26 + D}\ ; \tag{6}$$

the ABF is always taken to be 1.6 dB.

Table 2
Optimized Feedback Parameters for Small Values of D

D	α_{opt}	K	$1/(1-K)$	ABF (dB)
0.01	0.625	0.9920	124	1.52
0.006	0.627	0.9952	208	1.58
0.004	0.630	0.9968	312	1.58
0.003	0.632	0.9976	416	1.58

SMALL-SAMPLE CASE

For the small-sample case, Eq. (2) cannot be used because the distributions of noise and signal-plus-noise are not Gaussian. Consequently, a Monte Carlo method will be used to find the value of K that maximizes the probability of detection. To do this, it is first necessary to find the noise thresholds. The noise value for the nth pulse out of the integrator is

$$S_n = KS_{n-1} + Z_n \ , \tag{7}$$

or equivalently

$$S_n = \sum_{i=0}^{\infty} K^i Z_{n-i} \ , \tag{8}$$

where $\{Z_n\}$ are independent and Rayleigh distributed.

If $C_y(u)$ denotes the characteristic function of y, the characteristic function of S_n is

$$C_{S_n}(u) = \prod_{i=0}^{\infty} C_\lambda(u) \ , \quad \text{where} \quad \lambda = K^i Z_{n-i} \ . \tag{9}$$

Since the infinite product cannot be calculated, the product was truncated to 40 terms; and the Fast Fourier Transform was used to calculate $C_{S_n}(u)$ and its inverse transform, which is the density of S_n. The thresholds for various values of K and for a false alarm probability of 10^{-6} are shown in Table 3.

Table 3
Thresholds for Various Values of K with $P_{fa} = 10^{-6}$

K	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88
T	12.81	13.30	13.84	14.45	15.14	15.92	16.82	17.86

The Monte Carlo was run for $D = 0.24$, with S/N ratios of 3, 5, and 7 dB and for values of K varying from 0.81 to 0.88. One-thousand cases were run for each S/N ratio. For each case the initial signal pulse was uniformly distributed in the first 0 to D interval, so that a pulse did not arrive at the beam center. The results for the Monte Carlo

Table 4
Monte Carlo Results for the Small-Sample Case

S/N Ratio (dB)	Number of Detections in a 1000 Trials for Various Values of K							
	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88
3	60	60	65	63	65	65	66	65
5	373	374	378	381	381	377	367	363
7	851	852	855	861	862	859	856	844

are shown in Table 4. Again there exists a rather broad spread for the optimum value of K; however, the peak value of 0.85 is very close to the value of 0.84 which is predicted by Eq. (6). A value of $K = 0.84$ corresponds to 6.25 integrated pulses. To find the ABF, the characteristic function approach was used to calculate the detection curves for the sum of six and seven Rician distributed pulses. The detection curves are the solid lines in Fig. 3. The detection probability for $K = 0.84$ and S/N = 3 dB is found in Table 4 to be 0.063. If a horizontal line is drawn at $P_d = 0.063$ on Fig. 3, it will intersect the $N = 6$ and $N = 7$ curves. Then, by using linear interpolation between the curves, one sees that a S/N ratio of 2.35 dB is needed to obtain a P_d of 0.063 when $N = 6.25$. The difference between 2.35 dB and the 3 dB appearing in Table 4 is the ABF, i.e., ABF = 0.65 dB. In comparing all the results of Table 4 for $K = 0.84$ with the results of Fig. 3, one sees that the ABF takes on values of 0.65, 0.53, and 0.60 dB, values all less than the large sample value of 1.6 dB. The difference is due to the fact that the probabilities appearing in Table 4 correspond to the cumulative probability of detecting a target as the antenna beam sweeps the target, whereas the 1.6 dB represents a comparison when the feedback integrator is in its optimum position ($\alpha = 0.63$). To make a valid comparison, the Monte Carlo procedure was repeated with the optimal position being uniformly distributed between $0.63 + D/2$ and $0.63 - D/2$. The results of the Monte Carlo appear in Table 5. If one compares the $K = 0.84$ results with Fig. 3 and the $N = 32$ with those of Robertson (5), one obtains the results in Table 6. Thus, the large sample value of 1.6 dB for the ABF is approached very quickly.

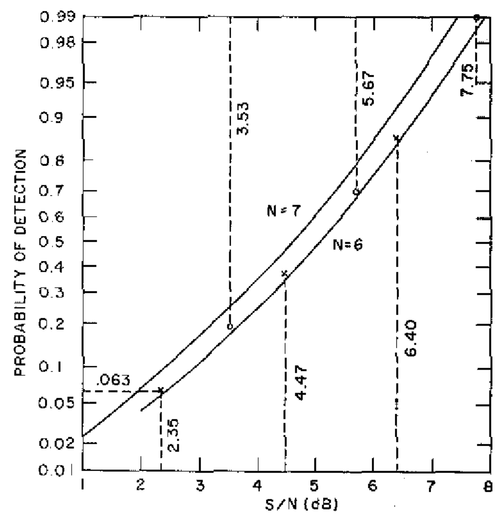

Fig. 3 - Probability of detecting a nonfluctuating target in Rayleigh noise, with $N = 6$ and 7 and $P_{fa} = 10^{-6}$. The dotted lines are used to calculate the ABF, the symbol x refers to cumulative detection probabilities, and the symbol o refers to single-sample detection probabilities.

Table 5
Monte Carlo Results for Small Values of K with $\alpha = 0.63$

$K = 0.84$, $D = 0.24$, and $N = 6.25$		$K = 0.968$, $D = 0.04$, and $N = 32$	
S/N Ratio (dB)	No. of Detections	S/N Ratio (dB)	No. of Detections
3	34	-2	3
5	191	0	73
7	700	2	546
9	990	4	985

Table 6
ABF for Several S/N Ratios and Values of K

$N = 6.25$		$N = 32$	
S/N Ratio (dB)	ABF (dB)	S/N Ratio (dB)	ABF (dB)
5	1.47	0	1.80
7	1.32	2	1.55
9	1.57	4	1.40
Average ABF = 1.45		Average ABF = 1.58	

The previous procedure where the ABF was calculated for the cumulative probability of detection was repeated for $N = 32$. For this case the ABF took on the values 0.38, 0.30, and 0.30 dB.

Summarizing, the detection probabilities for a feedback integrator can be found by the following procedure: First, K is chosen by Eq. (2). Then, the ABF is subtracted from the midbeam S/N ratio; and any of the standard references (5-7) can be used to find the detection probabilities. The ABF equals 0.3 dB, if one is concerned with the cumulative probability of detection, or ABF equals 1.6 dB, if one is concerned with the maximum probability of detection for a single look.

FLUCTUATING TARGETS

The question arises as to whether the previous procedure also applies when the target is fluctuating?* The answer is obviously yes when the sample size is large, because it is well known that the fluctuating results approach the nonfluctuating results as the sample size grows large (8). To investigate the small-sample behavior a Monte

*In this report, only pulse-to-pulse fluctuations are considered.

Carlo, using Swerling II target fluctuations, was run; and the results appear in Table 7 and Fig. 4. Comparing these two items, one obtains ABF values equal to 1.1, 1.38, 1.28, and 1.37 dB for the maximum probability of detection on a single look and ABF values equal to 0.55, 0.45, and 0.35 dB for the cumulative probability of detection. While there is more variation with fluctuating targets, the previous method can still be used with the only difference being that the detection probabilities are found from detection curves for fluctuating targets (9).

Table 7
Monte Carlo Results for K = 0.84 and
Swerling II Fluctuating Targets

Cumulative Detections		Detections for α = 0.63	
S/N Ratio (dB)	No. of Detections	S/N Ratio (dB)	No. of Detections
1	24	2	23
3	86	4	121
5	318	6	332
7	662	8	680
9	911	10	894

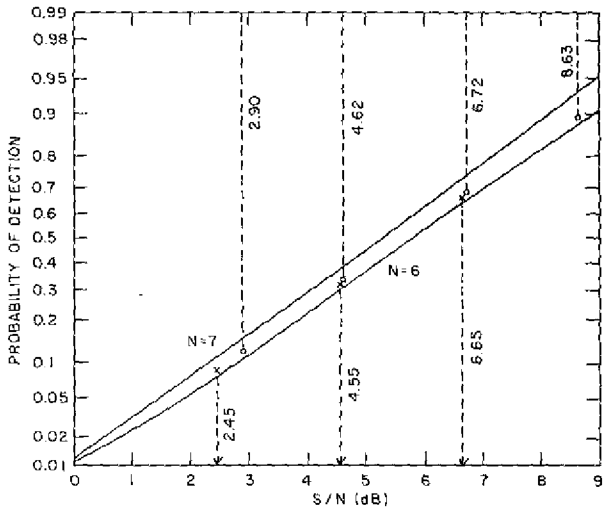

Fig. 4 - Probability of detecting a Swerling II fluctuating target in Rayleigh noise, with N = 6 and 7 and $P_{fa} = 10^{-6}$. The dotted lines are used to calculate the ABF, the symbol × refers to cumulative detection probabilities, and the symbol o refers to single-sample detection probabilities.

CONCLUSIONS

A simple method has been developed to calculate detection probabilities for a feedback integrator. A formula is given for the optimal value of K regardless of whether the sample size is large or small or whether the target is nonfluctuating or fluctuating. By using an ABF of 0.3 or 1.6 dB, the cumulative or single-look probabilities of detection, respectively, can be found from previously calculated detection curves. The values of ABF change by about 0.2 dB for the small sample size. The value of ABF for N less than six pulses has not been considered, since a feedback system would not be usually used in these cases.

ACKNOWLEDGMENT

The author thanks J. D. Wilson for helping with various programming problems which arose during this study.

REFERENCES

1. Blake, L.V., "The Effective Number of Pulses per Beamwidth for a Scanning Radar," Proc. IRE 41:770-774 (1953)

2. Blake, L.V., Addendum to "Pulses per Beamwidth for Radar," Proc. IRE 41:1785 (1953)

3. Blake, L.V., "A Guide to Basic Pulse-Radar Maximum-Range Calculation, Part I - Equations, Definitions, and Aids to Calculation," NRL Report 5868, Dec. 28, 1962

4. Hall, W.M., "Antenna Beam-Shape Factor in Scanning Radars," IEEE Trans. on Aerospace and Electronic Systems AES-4 (No. 3):402-409 (1968)

5. Robertson, G.H., "Operating Characteristics for a Linear Detector of CW Signals in Narrow-Band Gaussian Noise," Bell Sys. Tech. J. 46:755-774 (1967)

6. Marcum, J.I., and Swerling, P., "Studies of Target Detection by Pulsed Radar," IRE Trans. Information Theory IT-6:59-308 (1960)

7. Fehlner, L.F., "Marcum and Swerling's Data on 'Target Detection by a Pulsed Radar'," Applied Physics Lab., Johns Hopkins University, Report TG-451, July 2, 1962

8. Skolnik, M.I., "Introduction to Radar Systems," New York:McGraw-Hill, p. 52, 1962

9. Swerling, P., "Probability of Detection for Fluctuating Targets," IRE Trans. IT-6: 269-308 (1960)

Appendix

CALCULATION OF MONOTONIC QUANTITY

In the detection of a radar signal, one is concerned with the following binary hypothesis

$$
\begin{aligned}
H_0: &\quad Z_i^2 = x_i^2 + y_i^2 \\
H_1: &\quad Z_i^2 = [x_i + G^2(\theta_i)]^2 + y_i^2
\end{aligned}
\tag{A1}
$$

where the noise samples $\{x_i\}$ and $\{y_i\}$ are independent, identically distributed Gaussian random variables with mean zero and variance σ^2 and $G^2(\theta_i)$ is the two-way voltage antenna pattern. Since we are concerned with small S/N ratios, the linear detector can be approximated by a square-law detector.* Consequently, the test statistic S_n will be of the form

$$S_n = KS_{n-1} + Z_n^2 \tag{A2}$$

or

$$S_n = \sum_{i=0}^{\infty} K^i Z_{n-i}^2 \, . \tag{A3}$$

If K is very close to 1, S_n will be Gaussian distributed under both H_0 and H_1. (Strictly speaking, S_n will not be Gaussian unless $K = 1$.) In terms of the two Gaussian densities representing the null hypothesis and alternative, $G(\mu_1, \sigma_1^2)$ and $G(\mu_2, \sigma_2^2)$, respectively, the probability of false alarm is

$$P_{fa} = \int_T^{\infty} G(\mu_1, \sigma_1^2)\, dx \; ; \tag{A4}$$

and the probability of detection is

$$P_d = \int_T^{\infty} G(\mu_2, \sigma_2^2)\, dx \, , \tag{A5}$$

where T is the threshold. Equations (A4) and (A5) can be rewritten as

$$1 - P_{fa} = \int_{-\infty}^{(T-\mu_1)/\sigma_1} G(0,1)\, dx \equiv \Phi[(T-\mu_1)/\sigma_1] \tag{A6}$$

*W. R. Bennett, "Response of a Linear Rectifier to Signal and Noise," Bell Sys. Tech. J. 23:97-113 (1944).

and

$$P_d = \int_{(T-\mu_2)/\sigma_2}^{\infty} G(0,1)\ dx\ . \tag{A7}$$

From Eq. (A7), it can be implied that P_d is a monotonic increasing function of the quantity $Q = (\mu_2 - T)/\sigma_2$. Solving Eq. (A6) for T, it is seen that

$$Q = [\mu_2 - \mu_1 - \sigma_1 \Phi^{-1}(1 - P_{fa})]/\sigma_2\ . \tag{A8}$$

If $\sigma_1 = \sigma_2$, the constant $\Phi^{-1}(1 - P_{fa})$ can be ignored; and then $Q = (\mu_2 - \mu_1)/\sigma_1$, the S/N ratio.

In calculating the mean and variance of S_n, the following well-known information about the moments of a zero mean Gaussian variable will be used:

$$E(x_i) = E(Y_i) = 0,\ E(x_i^2) = E(y_i^2) = \sigma^2,\ E(x_i^3) = E(y_i^3) = 0\ ,$$

and

$$E(x_i^4) = E(y_i^4) = 3\sigma^4\ .$$

Then, the means of Z_i^2 are

$$E(Z_i^2|H_0) = E(x_i^2 + y_i^2) = 2\sigma^2 \tag{A9}$$

$$E(Z_i^2|H_1) = E\{[x_i + G^2(\theta_i)]^2 + y_i^2\} = G^4(\theta_i) + 2\sigma^2\ , \tag{A10}$$

and the second moments are

$$E(Z_i^4|H_0) = E(x_i^2 + y_i^2)^2 = 8\sigma^4 \tag{A11}$$

$$E(Z_i^4|H_1) = E\{[x_i + G^2(\theta_i)]^2 + y_i^2\}^2 = G^8(\theta_i) + 8G^4(\theta_i)\sigma^2 + 8\sigma^4\ . \tag{A12}$$

The variances of Z_i^2 are

$$\mathrm{Var}\,(Z_i^2|H_0) = E(Z_i^4|H_0) - [E(Z_i^2|H_0)]^2 = 4\sigma^4 \tag{A13}$$

$$\mathrm{Var}\,(Z_i^2|H_1) = E(Z_i^4|H_1) - [E(Z_i^2|H_1)]^2 = 4\sigma^2 G^4(\theta_i) + 4\sigma^4\ . \tag{A14}$$

Now, since the mean of the sum of random variables is equal to the sum of the means of the random variables and since the variance of the sum of independent random variables is equal to the sum of the variances of these random variables,

$$\mu_1 = \sum_{i=0}^{\infty} K^i(2\sigma^2) = 2\sigma^2/(1-K) \tag{A15}$$

$$\mu_2 = \sum_{i=0}^{\infty} K^i\left[G^4(\theta_i) + 2\sigma^2\right] = 2\sigma^2/(1-K) + \sum_{i=0}^{\infty} K^i G^4(\theta_i) \tag{A16}$$

$$\sigma_1^2 = \sum_{i=0}^{\infty} K^{2i}(4\sigma^4) = 4\sigma^4/(1-K^2) \tag{A17}$$

$$\sigma_2^2 = \sum_{i=0}^{\infty} K^{2i}\left[4\sigma^2 G^4(\theta_i) + 4\sigma^4\right] = 4\sigma^4/(1-K^2) + 4\sigma^2 \sum_{i=0}^{\infty} K^{2i} G^4(\theta_i) \ . \tag{A18}$$

Substituting these values into Eq. (A8) yields

$$Q = \frac{\sum_{i=0}^{\infty} K^i G^4(\theta_i) - 2\sigma^2 \Phi^{-1}(1-P_{fa})\Big/\sqrt{1-K^2}}{2\sigma\left[\sum_{i=0}^{\infty} K^{2i} G^4(\theta_i) + \sigma^2/(1-K^2)\right]^{1/2}} \ . \tag{A19}$$

However, for the small-signal case $\sigma^2 >> G^4(\theta_i)$, Q reduces to

$$Q = \frac{\sum_{i=0}^{\infty} K^i G^4(\theta_i)}{2\sigma^2/\sqrt{1-K^2}} \ , \tag{A20}$$

which is proportional to the S/N ratio.