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On Meromorphic Functions with Three Almost Linear Values

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On Meromorphic Functions with Three Almost Linear Values

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Abstract: In this report we prove that if a meromorphic function f has three distinct values a, b, and c (∞ is permitted) which are distributed almost on a line through the origin and if $N(r, f) = O(1)r^{\alpha}$, $\alpha < 2$, then $T(r, f) = O(1)r^{\alpha}$ (if $2 > \alpha > 1$) or $T(r, f) = O(1)r \log r$ (if $\alpha \leqslant 1$).

INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let $f(z)$ ($z = re^{i\theta}$) be meromorphic for $|z| < \infty$. Consider a system of rays defined by

$$re^{i\theta_1}, re^{i\theta_2}, \ldots re^{i\theta_q}, r \geqslant 0, \tag{1}$$

where

$$0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_q < 2\pi, q \geqslant 1.$$

THEOREM 1 (Edrei [1]). *Let $f(z)$ be meromorphic for $|z| < \infty$ such that all roots except the many finite ones of the three equations*

$$f(z) = 0, \ f(z) = \infty, \ f^{(\ell)}(z) = 1 \quad (\ell \geqslant 0, \ f^{(0)} = f) \tag{2}$$

be distributed on the system of rays (1). Denote the deficiency of the value a of the function $f^{(\ell)}$ by $\delta(a, f^{(\ell)})$, and assume that

$$\delta(0,f) + \delta(1,f^{(\ell)}) + \delta(\infty,f) > 0. \tag{3}$$

Then the order ρ of $f(z)$ is necessarily finite, and

$$\rho \leqslant \beta = \sup\left(\frac{\pi}{\theta_2 - \theta_1}, \frac{\pi}{\theta_3 - \theta_2}, \ldots, \frac{\pi}{\theta_{q+1} - \theta_q}\right), \text{ where } \theta_{q+1} = 2\pi + \theta_1.$$

The above result has been strengthened by Ostrovskii [2]. Throughout the following, f will always denote a meromorphic function for $|z| < \infty$, $\{r_k e^{i\phi_k}\}$ denotes the poles of $f(z)$, and

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$$C(R,\alpha,\beta,f) = 2\sum_{\substack{1<r_k\leqslant R\\ \alpha<\phi_k<\beta}} \left(\frac{1}{r_k^{\pi/\gamma}} - \frac{r_k^{\pi/\gamma}}{R^{2\pi/\gamma}}\right) \sin\frac{\pi}{\gamma}(\phi_k - \alpha),\ 0<\beta-\alpha=\gamma\leqslant 2\pi.$$

$K_1(t)$ and $K_2(t)$ will denote two positive nondecreasing functions of t, and $k_i = \overline{\lim_{t\to\infty}} K_i(t)(\log t)^{-1}$, $i = 1, 2$.

Definition 1 ([2] p. 971). The a-points of f(z) are "near" the system of rays (1) if the following condition holds:

$$\sum_{i=1}^{q} C(R,\theta_i,\theta_{i+1},(f-a)^{-1}) \leqslant K_1(R)K_2(T(R,f)) \tag{4}$$

with k_1 finite and $k_2 \lesssim 1$.

Definition 2. A value c (∞ permitted) is called a $*$-defect value of f if

$$\Delta^*(a) = \sup_{B^c \epsilon K} \ \underline{\lim_{\substack{r\to\infty\\ r\epsilon B}}} \ \frac{m(r,a)}{T(r,f)} > 0,$$

where K denotes the class of subsets of the positive real axis with upper density less than one. (The upper density of a set $E \subset [1,\infty)$ is $\overline{\lim_{r\to\infty}}\ r^{-1}\ \text{mes}(\{E\cap[1,r)\})$. Using Nevanlinna's theory for functions meromorphic in the whole plane as well as in a closed sector, Ostrovskii proved the following:

THEOREM 2 ([2] p. 971). *Let f(z) be a function meromorphic for $|z| < \infty$. Suppose that at least one of the following conditions is satisfied:*
A. (1) The zeros and poles of f(z) are "near" the system of rays (1); (2) at least one of the function $f^{(\ell)}(z)$ ($\ell \geqslant 0$) has one $*$-defect value which is different from 0 and ∞.
B. (1) The poles of f(z) and the a-points of $f^{(\ell)}(z)$ ($a \neq 0, \infty$) are "near" the system of rays (1); (2) zero is a $*$-defect value of f(z).
C. (1) The zeros and poles of f(z) and the a-points of $f^{(\ell)}(z)$ for some $a \neq 0$ or ∞ are "near" the system of rays (1); (2) ∞ is a $*$-defect value of f(z). Then the order of f is finite and does not exceed the value of

$$\chi = \chi(r,k_1,k_2) = (\pi+\gamma k_1)\gamma^{-1}(1-k_2)^{-1},$$

where $\gamma = \min_{1\leqslant i\leqslant q} (\theta_{i+1} - \theta_i)$.

Moreover, if both values $\sigma_i = \overline{\lim_{t\to\infty}} K_i(t)\,t^{-k_i}(i = 1, 2)$ are finite, then the growth of T(R,f) does not exceed the normal type of order χ, and if in addition one of these σ_i values is equal to zero, then T(R,f) is of, at most, minimal type of order χ.

If $k_1 = k_2 = 0$, then the order of f is at most $\pi\gamma^{-1}$. Also T(r,f) is at most of order $\pi\gamma^{-1}$ mean type. In particular the above conclusion holds when $K_i(t) = O(1)$ ($i = 1, 2$). Thus the assertion here is stronger than in Theorem 1.

Since $\Delta^*(\infty) = 1$ for any entire function, the following results from Theorem 2.

THEOREM 3 ([3] p. 487). *Let $f(z)$ be an entire function. Let $\{b_\ell\}_{\ell=1}$ $(b_\ell \neq 0)$ and $\{c_m\}_{m=1}$ $(c_m \neq 0)$ be the 0-points and 1-points of $f(z)$ respectively. Assume that*

$$\sum |\mathrm{Im}(b_\ell)^{-1}| + \sum |\mathrm{Im}(c_m)^{-1}| < \infty. \tag{5}$$

then $f(z)$ is an exponential-type function satisfying

$$\int_{-\infty}^{\infty} \frac{|\log|f(t)||}{1+t^2} dt < \infty.$$

It would be interesting to prove or disprove the hypothesis that (fz) has a $*$-defect value which can be removed from Theorem 2. In this report when the system of rays (1) consists of a line through the origin, some results slightly weaker than those obtained in Theorem 2 will be established without assuming that f possesses some $*$-defect value.

The methods used in this report are based on Nevanlinna's theory, on its analogous theorems for functions meromorphic in an open halfplane mostly developed by Tsuji [6], and on the fundamental result of Levin and Ostravskii [4]. It is assumed that the reader has some acquaintance with the usual notations such as m(r,f), N(r,f), and T(r,f) and with the basic results of Navanlinna's theory or meromorphic functions. For a good account see Nevanlinna [7] or Hayman [5].

Before we give a precise statement, we first introduce some necessary definitions.

Definition 3. A sequence of nonzero complex numbers $\{a_n\}$ forms a generalized A_ϕ set if the following condition is satisfied:

$$\sum_{|\mathrm{Im}(a_n e^{-i\phi})^{-1}| > r^{-1}} |\mathrm{Im}(a_n e^{-i\phi})^{-1}| = O(\log r) \text{ as } r \to \infty.$$

Remarks: (i). If $\phi = 0$ and the summation is bounded as $r \to \infty$, then $\{a_n\}$ is called an A-set [4].

(ii). In particular if all the numbers a_n lie on a straight line through the origin, then $\{a_n\}$ forms a generalized A_Φ -set.

Definition 4. We shall say that a complex number c (∞ permitted) is an almost linear value of f it the roots $\{c_n\}$ of the equation

$$f(z) - c = 0$$

form a generalized A_ϕ-set.

With the above preparation we can now state our main results as follows.

THEOREM 4. *Let f be a meromorphic function in the whole finite plane with its number of poles satisfying the condition:*

$$N(r, F) = O(1)\, r^{\alpha} \quad \text{as } r \to \infty,$$

where α is a constant less than 2. Assume that there are three distinct, almost linear values a_1, a_2, and a_3 (∞ permitted) and that the zeros of $f - a_i = 0$, $i = 1, 2, 3$, form a generalized A_ϕ–set. Then

$$T(r,f) = \begin{cases} O(1) r^{\alpha} & \text{if } 2 > \alpha > 1; \\ O(1) r \log r & \text{if } \alpha \leqslant 1. \end{cases}$$

The following is an immediate consequence of this theorem.

COROLLARY. *Let f be an entire function. Assume that f has two finite, distinct, almost linear values a and b and that the zeros of (f-a) (f-b) = 0 form a generalized A_ϕ –set. Then the order of f is at most one, and in fact*

$$T(r,f) = O(r \log r) \quad \text{as } r \to \infty$$

By using a result which is analogous to Hayman's ([5] p. 60), we can strengthen our results somewhat by taking into consideration not only the almost linear values of the function but also the almost linear values of its derivatives.

THEOREM 5. *Let f be a meromorphic function such that the condition of Theorem 4 is satisfied. Assume that the roots of the equation*

$$f(f^{(\ell)} - c) = 0 \quad (\ell \textit{ is an integer; } c \neq 0, \infty)$$

form a generalized A_α –set. Then

$$T(r,f) = \begin{cases} O(1) r^{\alpha} & \text{if } 2 > \alpha > 1; \\ O(1) r \log r & \text{if } \alpha \leqslant 1. \end{cases}$$

As an immediate consequence of this theorem, we obtain the following known result.

COROLLARY ([1] p. 277). *Let f(z) be an entire function, the zeros of which are real. Furthermore, assume that for some integer ℓ ($\ell \geqslant 0$), the roots of $f^{(\ell)}(z) = 1$ are all real. Then the order of f(z) is finite and does not exceed one.*

NOTATIONS AND PRELIMINARY THEOREMS

In what follows we shall use f to denote a function which is meromorphic in the whole finite plane. Following Tsuji [6] we let

$$m_0(r,\infty) = m_0(r,f) = \frac{1}{2\pi} \int_{\sin^{-1}(r^{-1})}^{\pi-\sin^{-1}(r^{-1})} \log^+ |f(r\sin\theta\, e^{i\theta})| \frac{d\theta}{r\sin^2\theta},$$

$$m_0(r,a) = m_0(r,\frac{1}{f-a}) \quad (a \neq \infty),$$

and

$$N_0(r,\infty) = N_0(r,f) = \int_1^r \frac{n_0(t,\infty)}{t^2}\, dt = \sum_{1 \leqslant r_k \leqslant r\sin\phi_k} (\frac{\sin\phi_k}{r_k} - \frac{1}{r}),$$

where $n_0(t,\infty)$ denotes the number of poles of the function f(z) in the region $\{|z-(it/2)| \leqslant (t/2), |z| \geqslant 1\}$, $\{r_k e^{i\phi_k}\}$ are the poles of f (counting multiplicities),

$$N_0(r,a) = N_0(r,\frac{1}{f-a}) \quad (a \neq \infty),$$

and

$$T_0(r,f) = m_0(r,f) + N_0(r,f).$$

The functional $T_0(r,f)$ is the (upper) half-plane (i.e., Im $z > 0$) Tsuji characteristic of f. Tsuji proved the following two fundamental theorems, which are analogous to Nevanlinna's first and second fundamental theorems.

THEOREM 6. *For any complex number a $(a \neq \infty)$ and any nonconstant f meromorphic in Im $z > 0$,*

$$T_0(r,\frac{1}{f-a}) = T_0(r,f) + O(1).$$

THEOREM 7. *If f is meromorphic in Im $z > 0$ and nonconstant and if $a_1, a_2, \ldots, a_q$ are $q \geqslant 3$ distinct numbers (∞ permitted), then*

$$(q-2)\, T_0(r,f) \leqslant \sum_{k=1}^{q} N_0(r,a_k) + R(r,f),$$

where $R(r,f) = O\{\log r + \log T_0(r,f)\}$ for all r outside a set of r values of finite measure.

The following theorem is an analog of a result of Hayman which will be used to prove Theorem 5.

THEOREM 8. ([4] p. 332). *If f is meromorphic in Im $z > 0$, then for $\ell \geqslant 1$ we have*

$$T_0(r,f) \leqslant (2+\frac{1}{\ell})N_0(r,\frac{1}{f}) + (2+\frac{2}{\ell})N_0(r,\frac{1}{f^{(\ell)}-1}) + R(r,f).$$

PROOF OF THE MAIN RESULTS

In proving the main results we first quote the following lemma, which builds a relation between the quantities $m_{0,\pi}(r,f)$ and $m(r,f)$. The lemma will play an important role in the proof of our theorems.

LEMMA (Levin and Ostrovskii [4]). *If the function $f(z)$ is meromorphic in the halfplane $Im\, z > 0$, then the following inequality is valid:*

$$\int_R^\infty \frac{m_{0,\pi}(r,f)}{r^3}\,dr \leqslant \int_R^\infty \frac{m_0(r,f)}{r^2}\,dr \quad (R \geqslant 1),$$

where

$$m_{\alpha,\beta}(r,f) = \frac{1}{2\pi}\int_\alpha^\beta \log^+ |f(re^{i\theta})|\,d\theta \quad (0 \leqslant \alpha < \beta \leqslant 2\pi).$$

Proof of Theorem 4. First of all we may assume without loss of generality that $\phi = 0$. If follows that the zeros of the equations

$$f(z) - a_i = 0, \ i = 1, 2, 3,$$

form a generalized A_0−set. This and the definition of the N_0 functional imply that the following conditions are satisfied:

$$\sum_{i=1}^{3} N_0(r, \frac{1}{f-a_i}) = O(\log r). \tag{5}$$

Therefore it follows from Theorem 7 and (5) that the following relation holds outside of a set of r values of finite measure in both upper and lower halfplanes:

$$T_0(r,f) = O(\log r) \ \text{ as } r \to \infty. \tag{6}$$

From the monotonicity of $T_0(r,f)$([6] p. 108), it is easily shown that this estimate holds for all sufficiently large r.

Accordingly we have that

$$m_0(r,f) = O(\log r) \ (\text{as } r \to \infty). \tag{7}$$

holds in both upper and lower halfplanes. Here we denote, without ambiguity, the Tsuji functionals $m_0(r,f)$, $N_0(r,f)$, and $T_0(r,f)$ for both upper and lower halfplanes.

Applying the lemma of Levin and Ostrovskii on both upper and lower halfplanes, we obtain

$$\int_R^\infty \frac{m_{0,\pi}(r,f)}{r^3} dr \leqslant \int_R^\infty \frac{O(\log r)}{r^2} dr \quad (R > 1) \tag{8}$$

and

$$\int_R^\infty \frac{m_{\pi,2\pi}(r,f)}{r^3} dr \leqslant \int_R^\infty \frac{O(\log r)}{r^2} dr \quad (R > 1) \tag{9}$$

Combining (8) with (9) and noting that $m_{0,\pi}(r,f) + m_{\pi,2\pi}(r,f) = m(r,f)$, we get

$$\int_R^\infty \frac{m(r,f)}{r^3} dr = O(1) \frac{\log R}{R}. \tag{10}$$

Now from the hypothesis that $N(r,f) = O(1)r^a$ we have

$$\int_R^\infty \frac{N(r,f)}{r^3} dr = \begin{cases} \dfrac{O(1)}{R^{2-a}} & \text{if } 2 > a > 1; \\ \dfrac{O(1)\log R}{R} & \text{if } a \leqslant 1. \end{cases} \tag{11}$$

Adding (10) and (11) we obtain

$$\int_R^\infty \frac{T(r,f)}{r^3} dr = \begin{cases} \dfrac{O(1)}{R^{2-a}} & 2 > a > 1; \\ \dfrac{O(1)\log R}{R} & a \leqslant 1. \end{cases} \tag{12}$$

Since T is a monotonic, increasing function of r, it follows from (12) that

$$T(R,f) \int_R^\infty \frac{dr}{r^3} = \begin{cases} \dfrac{O(1)}{R^{2-a}} & \text{if } 2 > a > 1; \\ \dfrac{O(1)\log R}{R} & \text{if } a \leqslant 1. \end{cases}$$

Consequently,

$$T(R,f) = \begin{cases} O(1)R^a & \text{if } 2 > a > 1; \\ O(1)R\log R & \text{if } a \leqslant 1. \end{cases}$$

Theorem 4 is thus proved.

The proof of Theorem 5 is almost identical to the above argument when one notes that under the hypotheses of Theorem 5 one can obtain the condition (7) from Theorem 8.

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13. ABSTRACT

In this report we prove that if a meromorphic function f has three distinct values, a, b, and c (∞ is permitted) which are distributed almost on a line through the origin and if $N(r,f) = O(1)r^{\alpha}, \alpha < 2$, then $T(r,f) = O(1)r^{\alpha}$ (if $2 > \alpha > 1$) or $T(r,f) = O(1)r \log r$ (if $\alpha \leqslant 1$).

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