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Identifying an Unknown Process by Using Randomly Chosen Inputs

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CONTENTS

ABSTRACT

We investigated the problem of identifying the input-output relationship of an unknown deterministic process. In particular we investigated the ramifications of identifying solely on the basis of a finite number of input-output data pairs. Under this rather severe but realistic limitation we have found identifications can be made by using randomly chosen inputs. Furthermore we have found that if two conditions are met, then the input-output relationship can always be constructed and therefore identified by using randomly chosen inputs. One condition is that the unknown input-output relationship be contained in a class of functions such that any two functions of the class have equal outputs for only an input subset of measure zero. The other condition is that the class of functions be decomposable into a countable sequence of subclasses such that $n + 1$ input-output data pairs are sufficient for determining at most one member of the nth subclass. Classes of functions which meet these conditions include polynomial functions, roots of polynomial functions, roots of the sum of two polynomial functions, and the exponential functions that originate in diffusion or decay problems. Hopefully our results can later be applied to problems cursed with observational errors in the input-output data pairs.

PROBLEM STATUS

A final report on one phase of the problem.

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IDENTIFYING AN UNKNOWN PROCESS BY USING RANDOMLY CHOSEN INPUTS

INTRODUCTION

The problem of identifying an unknown process—a black box—is basically to determine its input-output relationship by analyzing input-output data pairs. From a practical viewpoint or otherwise, this problem is extremely challenging, partly due to undesirable, but nevertheless realistic features. For instance, as Zadeh [1,2] points out, the identification problem is occasionally complicated by such features as the lack of knowledge of the initial state of the process, the presence of noise in observations of the inputs and outputs, and the limited freedom in selecting test inputs. However, a basic complicating feature of all identification problems is the limitation on the number of input-output data pairs that can be *obtained* and *used* for identifying purposes. This number is always *finite.*

In determining the identity of a black box, this limitation alone provides an immense difficulty. Consider a black box whose input-output relationship, denoted by f, is a mapping from the input space of real numbers into the output space of real numbers. Having no further a priori knowledge of the function f, we need no analysis to realize that it is impossible to determine f from only a *finite* number of data pairs. Consequently, to make this problem appear tractable, suppose it is known a priori that f is a polynomial function but of unknown degree. And for simplicity suppose after applying the inputs $x_0, x_1, ..., x_n$ the number 5 is observed each time as the output. Gathering up candidates, we obtain readily the polynomial functions $\Pi_{k=0}^{m}(x - x_k) + 5$ for all $m \geqslant n$, where $x_{n+1}, ..., x_m$ are arbitrary, along with the constant polynomial function whose value is 5. Thus in even this simpler case involving only polynomial functions we find an uncountable number of candidates for the identity, regardless of the cardinality of the finite number of available input-output data pairs. We cannot determine the correct polynomial for the identity unless we can develop an algorithm better than guessing.

Because of this indeterminacy we are forced to consider the two basic questions to which our investigation is addressed.* First, having a candidate function which continues to satisfy additional input-output data pairs, by what method can we test if it is the identity of the process? Second, if we have such a testing method which uses only a finite number of input-output data pairs, how do we construct a candidate that is the identity? Certainly, as we found even in the simple case of a polynomial, one is not apt to obtain the identity by guessing.

*These questions, along with several others, precipitated out of several stimulating discussions on the problem of identification with Professor George Leitmann of the University of California at Berkeley.

In this report we give a means of testing a candidate function for its validity of being the identity. This is the technique of administering randomly chosen inputs. The basic idea of this technique is simple. Given a particular candidate function, we test its validity of being the identity by comparing for a randomly chosen input the output values of the candidate function and of the black box. This test becomes highly reliable whenever the input-output relationship belongs to a class of functions having the property that any two distinct functions of the class have equal values for equal arguments over at most a subset of measure zero. Furthermore, for construction purposes, a class of functions can often be decomposed into a countable sequence of subclasses having the property that $n + 1$ input-output data pairs are sufficient to determine at most one member of the nth subclass in the sequence. An input-output relationship belonging to such a class can be constructed by analyzing input-output data pairs.

With these remarks as a point of departure, we will first define the problem precisely. Afterward we will apply our ideas to the simple problem of identifying a black box whose input-output relationship is an unknown polynomial function. We will then give general results and address some applications.

STATEMENT OF PROBLEM

For the purposes of this report we define a black box B to be a triplet $\{R^m, f, R^n\}$, where R^m and R^n are the sets of all possible inputs and outputs respectively and f is a mapping, usually unknown, from R^m into R^n. (We are letting R denote the space of real numbers, with a superscript denoting the number of dimensions of the space.) Such black boxes can be referred to as static in contradistinction to dynamic black boxes whose outputs evolve with time. The mapping f is the input-output relationship of the black box B and is called the black-box mapping of B. We tacitly assume that a member x of R^m can be administered as an input to the black box B and that the corresponding output $f(x)$ can be observed without error. We say that a black box B is identified if the mapping f is known. We are now in a position to state the problem.

Problem. Let F be a class of functions having domain R^m and range R^n. Given a black box $B = \{R^m, f, R^n\}$, where f is an unknown member of F, identify the black box B by analyzing a finite number of input-output data pairs.

Our method of attack on the problem is to use randomly chosen inputs, namely, inputs obtained from a random sampling of the input space R^m. Various probability distributions can be used in obtaining a random sampling of R^m. Examples are the normal probability distributions on R^m and the uniform probability distribution defined on a cube having unit volume in R^m. We will use an asterisk on a letter to denote a randomly chosen input. We will tacitly assume that members of a finite sequence of randomly chosen inputs $\{x_0^*, x_1^*, ..., x_n^*\}$ are independently generated, with x_i^* for $i = 1, 2, ..., n$ being randomly chosen from the set $R^m - \{x_0^*, ..., x_{i-1}^*\}$.

IDENTIFICATION IN THE CLASS OF POLYNOMIAL FUNCTIONS

Before giving general results, we consider the simple problem of identifying black boxes of the form $\{R^1, f, R^1\}$, where f is an unknown polynomial function. We will show in Lemma 1 that randomly chosen inputs are useful in testing the identity of such black boxes. For any prior probability distribution P on the set of polynomial functions, we have the following lemma and theorem.

Lemma 1. *Suppose* $f\colon R^1 \to R^1$ *is an unknown function belonging to the class of all polynomial functions. Let* p *be a known polynomial function and suppose* p *is a candidate for the mapping* f. *If a randomly chosen input* x^* *results in the output* $f(x^*)$ *equaling* $p(x^*)$, *then the probability is* 1 *that* f *and* p *are identical; that is,* $P(p = f \mid p(x^*) = f(x^*))$ *is equal to* 1.

Proof. Let $A(f, p)$ denote the set of all x contained in R^1 such that $f(x) - p(x) = 0$. The set $A(f, p)$ is unknown, since f is unknown. However, the set $A(f, p)$ does exist and is well defined. It is even possible that $A(f, p)$ is empty.

We are to show that $P(p = f \mid x^* \in A(f, p))$ is equal to 1. This is the probability of p and f being identical conditioned on a randomly chosen input belonging to the set $A(f, p)$. Using Bayes's rule (Feller [3]), we can express this conditional probability as

$$P(p = f \mid x^* \in A(f, p)) = \frac{P(x^* \in A(f, p) \mid f = p)\, P(f = p)}{[P(x^* \in A(f, p) \mid f = p)\, P(f = p)] + [P(x^* \in A(f, p) \mid f \neq p)\, P(f \neq p)]}.$$

The assertion of the lemma follows upon showing that $P(x^* \in A(f, p) \mid f = p) \neq 0$ and $P(x^* \in A(f, p) \mid f \neq p) = 0$. These are the probabilities of a randomly chosen input belonging to $A(f, p)$ conditioned on f and p being identical and not being identical respectively.

If $f = p$, then $A(f, p) = R^1$. Thus, conditioned on $f = p$, the probability is clearly 1 that a randomly chosen input x^* belongs to $A(f, p)$; that is, $P(x^* \in A(f, p) \mid f = p) = 1 \neq 0$.

If $f \neq p$, then it follows from the Fundamental Theorem of Algebra that the polynomial $f - p$ has at most a finite number of zeros; that is, the set $A(f, p)$ has finite cardinality. Whenever $f \neq p$, the conditional probability is zero that a randomly chosen input x^* belongs to $A(f, p)$; that is, $P(x^* \in A(f, p) \mid f \neq p) = 0$. This completes the proof of the lemma.

Lemma 1 gives a method for testing a candidate for the polynomial function f. However, an algorithm is needed for obtaining the identity of B, since the likelihood is zero of guessing the right candidate for the identity. Hence we will now give a simple algorithm for identifying B with high probability through the construction of polynomial functions from input-output data pairs.

Theorem 1. *If* $f\colon R^1 \to R^1$ *is an unknown function belonging to the class of all polynomial functions of one variable, then the probability is* 1 *that* f *can be identified. In other words, one can construct a polynomial function* p *such that the probability is* 1 *that* f *and* p *are identical.*

Proof. Using the assertion of Lemma 1, the probability is 1 that f can be identified by means of the following algorithm.

Choose randomly $x_0^* \in R^1$ and observe the output $f(x_0^*)$. Let p_0 be the constant polynomial function such that $p_0(x) = f(x_0^*)$ for all $x \in R^1$. By the following steps, test this candidate and, if necessary, construct further candidates for testing.

Step 1. Choose randomly $x_1^* \in R^1 - \{x_0^*\}$ and observe the output $f(x_1^*)$. If $f(x_1^*) = p_0(x_1^*)$ then from Lemma 1, it follows that our assertion holds. If $f(x_1^*) \neq p_0(x_1^*)$, then proceed to the next step after constructing p_1 to be the unique polynomial function of the first degree satisfying $p_1(x_i^*) = f(x_i^*)$ for $i \in \{0, 1\}$.

Step n. In general the nth step proceeds as follows. There are n points $\{x_0^*, x_1^*, ..., x_{n-1}^*\}$, and there has been constructed a polynomial function p_{n-1} of the $(n-1)$th degree satisfying $p_{n-1}(x_i^*) = f(x_i^*)$ for $i \in \{0, 1, ..., n-1\}$. The nth step begins by choosing randomly $x_n^* \in R^1 - \{x_0^*, x_1^*, ..., x_{n-1}^*\}$. The output $f(x_n^*)$ is observed, and the condition $f(x_n^*) = p_{n-1}(x_n^*)$ is checked. If $f(x_n^*) = p_{n-1}(x_n^*)$, then this is the last step. If $f(x_n^*) \neq p_{n-1}(x_n^*)$, then we proceed to the $(n+1)$th step after constructing p_n, which is the unique polynomial function of the nth degree satisfying $p_n(x_i^*) = f(x_i^*)$ for $i = \{0, 1, ..., n\}$.

By the preceding construction we assure that $p_N(x_{N+1}^*) = f(x_{N+1}^*)$, where N is the unknown degree of f. By envoking Lemma 1, we see that the probability is 1 that $p_N = f$. Since $N < \infty$, we can construct p_N in a finite number of steps. Thus, for any prior distribution P on the set of polynomial functions, the above algorithm yields the true function f with probability 1. This completes the proof.

TESTING THE IDENTITY IN CLASSES OF FUNCTIONS MORE GENERAL THAN POLYNOMIAL FUNCTIONS

Let F be a class of functions with domain R^m and range R^n. Consider two black boxes $B_1 = \{R^m, f_1, R^n\}$ and $B_2 = \{R^m, f_2, R^n\}$, where f_1 and f_2 are unknown members of the class F. We seek a test from which we can assert with a high level of confidence that f_1 and f_2 are identical. No test is possible for an arbitrary class of functions. Nevertheless, consider the test of randomly choosing an input x^* and observing the difference $f_1(x^*) - f_2(x^*)$. If the difference is nonzero, then f_1 and f_2 are not identical. However, if the difference is zero, then the level of confidence to be placed in f_1 and f_2 being identical depends on the probability of x^* belonging to the intersection of f_1 and f_2, that is, the set of points x having $f_1(x) - f_2(x) = 0$. For some classes of functions a prior statement can be made on the nature of an intersection of two belonging members.

Looking back at the proof of Lemma 1, we note that two distinct polynomial functions intersect in at most a finite number of points. This raises the question of there being other classes of functions which have a similar relationship. From the theory of analytic functions we recall that two distinct analytic functions (with domain R^1) intersect in at most a finite number of points on any compact subset of R^1, and in general their intersection on R^1 is at most countable. The class of polynomial functions and the class of analytic functions are examples of classes F which possess the relationship of two distinct members intersecting in a set which has m-dimensional Lebesque measure zero, where m is the dimension of the domain space R^m. This motivates the usefulness of the following condition.

Condition I. Let F be a class of functions with domain R^m and range R^n. The class F is said to meet Condition I if and only if for distinct members f_1 and f_2 of F the set $A(f_1, f_2) \triangleq \{x \in R^m: f_1(x) - f_2(x) = 0\}$ has m-dimensional Lebesque measure zero.

The probability is zero that a randomly chosen input belongs to the intersection of two distinct members of a class F meeting Condition I. Consequently, for such classes there should be a high level of confidence in f_1 and f_2 being identical whenever the difference $f_1(x^*) - f_2(x^*)$ is zero.

Theorem 2. Let F be a class of functions with domain R^m and range R^n. Suppose that F meets Condition I. Let $B = \{R^m, f, R^n\}$ be a black box such that f is an unknown member of the class F. Let $B_1 = \{R^m, g, R^n\}$ be another black box such that g belongs to the class F. If a randomly chosen input x^* results in the output $f(x^*)$ equaling the output $g(x^*)$, then the probability is 1 that f and g are identical; moreover, if g is known a priori, then the probability is 1 that f is identified.

Proof. It is to be shown that the conditional probability $P(f = g|x^* \in A(f, g))$ is equal to 1. (Here P represents any prior probability distribution on the class of functions F.) Using Bayes's rule, this conditional probability can be expressed as

$$P(f = g|x^* \in A(f, g)) = \frac{P(x^* \in A(f, g)|f = g)\, P(f = g)}{[P(x^* \in A(f, g)|f = g)\, P(f = g)] + [P(x^* \in A(f, g)|f \neq g)\, P(f \neq g)]}.$$

Therefore our assertion follows after showing that $P(x^* \in A(f, g)|f = g) \neq 0$ and $P(x^* \in A(f, g): f \neq g) = 0$.

We choose randomly an input x^* and observe the difference $f(x^*) - g(x^*)$. If f and g are indeed identical, then this difference will be zero, since $A(f, g) = R^m$. Therefore, conditioned on $f = g$, the probability is 1 that x belongs to $A(f, g)$; that is, $P(x^* \in A(f, g)|f = g) = 1 \neq 0$.

If f and g are distinct members of the class F, then the set $A(f, g)$ has m-dimensional measure zero. Conditioned on f and g not being identical, the probability is zero that x^* belongs to $A(f, g)$; that is, $P(x^* \in A(f, g)|f \neq g) = 0$. this concludes the proof of the theorem.

MAIN IDENTIFICATION ALGORITHM USING RANDOMLY CHOSEN INPUTS

Theorem 2 focuses our attention on testing the identity of a black box whose mapping belongs to a class of functions F. However, before we can fruitfully test the identity of a black box, we must be able to construct it, perhaps by means of an algorithm. For this purpose we introduce an additional condition on the class F.

Condition II. Let F be a class of functions with domain R^m and range R^n. The class F is said to meet Condition II if and only if the class F can be decomposed into a sequence of disjoint subclasses F_i, $i = 0, 1, 2, \ldots$, such that any $i + 1$ input-output data pairs are sufficient for determining at most one member of F_i.

Theorem 3. *Let F be a class of functions with domain R^m and range R^n. Suppose that F meets Conditions I and II. Then the probability is* 1 *that any black box $B = \{R^m, f, R^n\}$ having $f \in F$ can be identified by using randomly chosen inputs.*

Proof. Let the decomposition $\{F_i: i = 0, 1, 2, \ldots\}$ of F be as specified in Condition II. Let $B = \{R^m, f, R^n\}$ be a black box such that f is an unknown member of F. Since

$$f \in F = \bigcup_{i=0}^{\infty} F_i,$$

there is a least integer K such that f is contained in the subclass F_K.

For $i = 0, 1, \ldots, N$, where N is arbitrary but finite, we can generate a sequence of randomly chosen inputs $\{x_0^*, x_1^*, \ldots, x_i^*\}$ and observe the sequence of outputs $\{f(x_0^*), f(x_1^*), \ldots, f(x_i^*)\}$. Therefore we obtain the $i + 1$ input-output data pairs

$$\{[x_0^*, f(x_0^*)], [x_1^*, f(x_1^*)], \ldots, [x_i^*, f(x_i^*)]\}$$

from which we determine the unique member f_i of F_i, if one exists, which satisfies these data pairs. Thus we obtain a sequence of functions $\{f_0, f_1, \ldots, f_i\}$, providing they all exist. The process stops whenever the condition $f_i(x_{i+1}^*) - f(x_{i+1}^*) = 0$ is fulfilled. From Theorem 2 we have that the probability is 1 that the process stops if and only if $i = K$; therefore we will have $f_K = f$. This completes the proof of the theorem.

The decomposition $\{F_i: i = 0, 1, 2, \ldots\}$ described for the class F required that any $i + 1$ input-output data pairs be *sufficient* for determining at most one member of F_i. It was not required that any $i + 1$ input-output data pairs be *necessary* in determining a member of F_i. It is possible that only k input-output data pairs are necessary, where $k < i + 1$. As an example, consider a class F which has been decomposed into subclasses such that F_{10} contains, among others, all polynomial functions of the 5th degree. It is necessary to have six input-output data pairs to determine a unique polynomial function of the 5th degree, whereas it is sufficient to have 11 input-output data pairs, to determine at most one polynomial function of the 5th degree. Of course it is possible that no polynomial function of the 5th degree is compatible with a pregiven 11 input-output data pairs.

APPLICATIONS

The class of polynomial functions easily meets Conditions I and II; it provides a simple class for the application of Theorem 3. If we let F be the class of all polynomial functions and F_i be the subclass of all polynomial functions of the ith degree, then the decomposition $\{F_i: i = 0, 1, 2, \ldots\}$ of F is of the type we described in Condition II. Moreover we gave an algorithm for identifying a polynomial in the proof of Theorem 1.

As a second application, we give an algorithm for the class of all roots of polynomial functions having the complex numbers as domain and range. We let $\hat{F}$ denote the class of all functions of the form $\sqrt[k]{p}$, where p is a polynomial function and k is a positive integer. We let C denote the space of complex numbers and let $z \triangleq r(\cos\theta + i \sin\theta)$ denote a complex number having r as the modulus and θ as the phase, where $i = \sqrt{-1}$. We let $B = \{C, \sqrt[k]{p}, C\}$ be a black box such that $\sqrt[k]{p}$ is a member of F. Here the space of complex numbers C may be interpreted as the two-dimensional plane R^2. If z is an input and $p(z) = z_1 \equiv r_1(\cos\theta_1 + i \sin\theta_1)$, then to have uniqueness of output we assume that $\sqrt[k]{p(z)}$ is equal to the root $\sqrt[k]{r_1}\,[\cos(\theta_1/k) + i \sin(\theta_1/k)]$.

One can easily verify that F meets Condition I. We shall show that F meets Condition II. For i and j contained in $\{1, 2, 3, \ldots\}$ we let $\hat{F}^i_j$ denote the subclass containing the ith-root functions of all polynomial functions of the jth degree. That is, if p_j is a polynomial function of the jth degree, then the function $\sqrt[i]{p_j}$ belongs to $\hat{F}^i_j$. One can also easily verify that any $j + 1$ input-output data pairs are necessary and sufficient to determine uniquely a member of $\hat{F}^i_j$ for i contained in $\{1, 2, 3, \ldots\}$. We let $\hat{F}^1_0$ denote the subclass of all constant polynomial functions. Thus

$$\hat{F} = \hat{F}^1_0 \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \hat{F}^i_j.$$

For j contained in $\{1, 2, 3, \ldots\}$ we consider the finite sequence $\{\hat{F}^1_j, \hat{F}^2_j, \ldots, \hat{F}^j_j, \hat{F}^{j+1}_1, \hat{F}^{j+1}_2, \ldots, \hat{F}^{j+1}_j\}$. By combining these finite sequences back-to-back as j increases in $\{1, 2, 3, \ldots\}$, noting that

$$\hat{F} = \hat{F}^1_0 \cup \bigcup_{j=1}^{\infty} \left(\bigcup_{i=1}^{j} \left(\hat{F}^i_j \cup \hat{F}^{j+1}_i \right) \right),$$

we form a decomposition of $\hat{F}$ of the type described in Condition II. We consider the subclass $\hat{F}^1_0$ as the first member of the decomposition. Since F meets both Conditions I and II, Theorem 3 is applicable. Therefore the probability is 1 that any member of $\hat{F}$ can be identified by using randomly chosen inputs. For comparison we will give an algorithm in the proof of Theorem 4.

Theorem 4. If $B = \{C, f, C\}$ is a black box in which f is the root of some polynomial function, then the probability is 1 that f can be identified by using randomly chosen inputs.

Proof. The proof follows from Theorem 2 and the following algorithm. Recall the class $\hat{F}$ and the subclasses $\hat{F}^i_j$.

Choose randomly $z_0^* \in C$ and observe the output $f(z_0^*)$. Define p_0^1 to be the unique constant polynomial function satisfying $p_0^1(z_0^*) = f(z_0^*)$. By the following steps, test this candidate and, if necessary, construct further candidates for testing.

Step 1. Choose randomly $z_1^* \in C - \{z_0^*\}$ and observe the output $f(z_1^*)$. If $f(z_1^*) = p_0^1(z_1^*)$, then stop the process. If $f(z_1^*) \neq p_0^1(z_1^*)$, then for $k \in \{1, 2\}$ define p_1^k to be the unique polynomial function of the first degree satisfying $p_1^k(z_j^*) = [f(z_j^*)]^k$ for $j \in \{0, 1\}$. Note that $\sqrt[k]{p_1^k}$ belongs to $\hat{F}^k_1$. Proceed to Step 2.

Step n (for $n \geqslant 2$). Choose randomly $z_n^* \in C - \{z_0^*, z_1^*, ..., z_{n-1}^*\}$ and observe the output $f(z_n^*)$. If there exists a least integer k_0 in $\{1, 2, ..., n\}$ such that $p_{n-1}^{k_0}(z_n^*) = [f(z_n^*)]^{k_0}$, then stop the process. If for $\ell \in \{1, ..., n-1\}$ there exists a polynomial function p_ℓ^n of the ℓth degree satisfying $p_\ell^n(z_j^*) = [f(z_\ell^*)]^n$ for $j \in \{0, 1, ..., n\}$, then stop the process. If no such k_0 or no such ℓ exists, then for $k \in \{1, 2, ..., n+1\}$ define p_n^k to be the unique polynomial function of the nth degree satisfying $p_n^k(z_j^*) = [f(z_j^*)]^k$ for $j \in \{0, 1, ..., n\}$. Note that $\sqrt[k]{p_n^k}$ belongs to $\hat{F}_n^k$ and that you have checked in the subclasses $\hat{F}_\ell^n$ for $\ell \in \{1, ..., n-1\}$. Thus, during the nth step you have checked in all of the subclasses contained in the finite sequence $\{\hat{F}_{n-1}^1, \hat{F}_{n-1}^2, ..., \hat{F}_{n-1}^{n-1}, \hat{F}_1^n, ..., \hat{F}_{n-1}^n\}$ to identify B. Proceed to Step $n+1$. Since f is contained in $U_{i=1}^{n-1}\left(\hat{F}_{n-1}^i \; U \; \hat{F}_i^n\right)$ for some finite n, this completes the proof of the theorem.

As a third application, we consider functions $f\colon\; C^2 \to C$ having the form $f(z_1, z_2) = \sqrt[k]{p_i(z_1) + p_j(z_2)}$, where p_i and p_j are polynomial functions and k is a positive integer. If $B = \{C^2, f, C\}$ is a black box such that f is an unknown function of the above form, then one can use the algorithm given in the proof of Theorem 4 to identify B. First, one fixes an input z_1 and identifies the function $f(z_1, .) = \sqrt[k]{p_i(z_1) + p_j(.)}$. Second, one fixes an input z_2 and identifies the function $f(., z_2) = \sqrt[k]{p_i(.) + p_j(z_2)}$. Last, one observes the output $f(z_1, z_2)$. The function f is then determined; that is, the probability is 1 that the function obtained in this manner is the black box mapping f. Consequently, if one knew that the relationship between the length of the hypotenuse of a right triangle and the lengths of its legs belonged to the above class of functions, then the probability is 1 that one could identify this relationship by using randomly chosen inputs.

As a final application, we will place our results in proper perspective with the current state of the art of solving identification problems which have a biological or physical significance. We consider the problem of identifying an input-output relationship of the form

$$f(x) = \sum_{i=1}^{n} a_i \exp\,(-a_{n+i}x), \quad x \in (-\infty, \infty),$$

when $n, a_1, ..., a_{2n}$ are unknowns. As in the previous applications, our general results apply equally well to this problem; that is, Conditions I and II are satisfied. Concerning a physical background of various origins of this problem, the work of Gardner et al. [4] points out that this problem arises in the study of the radioactive decay in a mixture of isotopes, dielectric properties of certain compounds, rates at which injected materials disseminate in a living organism, and diffusion problems.

Also connected with this problem, Bellman [5] gives a method for finding the unknown number n without first determining the unknowns $a_1, ..., a_{2n}$. However, as Bellman carefully points out, his method contains a most unsatisfactory condition, a condition which must be checked for large values of x. The unsatisfactory part is twofold. First, it calls for large values of x. And second, it is not known beforehand exactly how large is "large." This situation can be amended easily by using randomly chosen inputs; the condition is checked for one randomly chosen value rather than for large values of x.

REMARKS

In Theorems 1, 3, and 4 we used a finite sequence of randomly chosen inputs in determining the identity of a black box. It is not necessary, however, to administer more than one randomly chosen input to obtain the results given there. *Essentially, the idea is to have the testing independent from the construction of a candidate function.* The testing can be kept independent by bringing in a second party to randomly choose an input, to observe the output of the black box, and to record this information in secrecy. The first party is to generate input-output data pairs by any manner he chooses and to construct candidate functions therefrom. The identification search stops whenever the first party constructs a candidate function which satisfies the secret information of the second party, who, by the way, makes all comparisons.

As a second remark, if classes F_α, α contained in $\{1, 2, ..., m\}$ with m finite, meet Conditions I and II, then the class $F = U_{\alpha=1}^{m} F_\alpha$ also meets Conditions I and II. Thus the general results of Theorems 2 and 3 apply to classes of functions having more than one functional form. For instance, F_1 may contain polynomial functions, F_2 may contain exponential functions, and F_α, $\alpha \in \{3, ..., m\}$, may contain other functional forms. *It is unnecessary to know beforehand the functional form of a black box in order to identify its input-output relationship.*

DISCUSSION AND CONCLUSION

The identification of an unknown process has many avenues of approach. At one extreme is the avenue of providing a sufficient number of acceptable postulates with which a mathematically deductive logic can be used to prove rigorously what the identity is. For example, Euclid's postulates provide a framework with which to prove deductively the Pythagorean theorem. At the other extreme is the avenue of selecting within a restricted class that function which in the sense of some given criterion best approximates the available input-output data pairs [6]-[10]. We took the avenue in between of using input-output data pairs to obtain the identity rather than to approximate it.

As we alluded to in the introduction, identification problems are occasionally complicated because the input-output data pairs are unavoidably cursed with observational errors. We have not concerned our efforts directly with this undesirable feature, although we have been constantly aware of it. Hopefully our results will later prove useful in the study of such problems. We investigated the identification problem in its most ideal situation in order to gain sufficient insight into handling the finiteness-of-available-data limitation, so that we could adequately deal with this limitation in pursuing identification. As a result, we found that randomly chosen inputs provide utility for solving identification problems, in particular those problems in which the classes of functions satisfy Conditions I and II. It is not in general possible to solve just any identification problem, especially when the input-output relationship is not known to belong to a class of functions meeting Condition II. For instance, an earlier investigation [11] gives necessary conditions for a black box to be representable as a dynamical system governed by ordinary differential equations. Those conditions cannot be verified solely on the basis of finite data. And, indeed, it is partly due to the input-output relationship belonging to too large a class of possible representations.

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13. ABSTRACT

We investigated the problem of identifying the input-output relationship of an unknown deterministic process. In particular we investigated the ramifications of identifying solely on the basis of a finite number of input-output data pairs. Under this rather severe but realistic limitation we have found identifications can be made by using randomly chosen inputs. Furthermore we have found that if two conditions are met, then the input-output relationship can always be constructed and therefore identified by using randomly chosen inputs. One condition is that the unknown input-output relationship be contained in a class of functions such that any two functions of the class have equal outputs for only an input subset of measure zero. The other condition is that the class of functions be decomposable into a countable sequence of subclasses such that $n + 1$ input-output data pairs are sufficient for determining at most one member of the nth subclass. Classes of functions which meet these conditions include polynomial functions, roots of polynomial functions, roots of the sum of two polynomial functions, and the exponential functions that originate in diffusion or decay problems. Hopefully our results can later be applied to problems cursed with observational errors in the input-output data pairs.

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