NRL Report 7237

Finite-Time Stability of Linear Differential Equations

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Mathematics Research Center
Report 71-4

Mathematics and Information Sciences Division

July 15, 1971

NAVAL RESEARCH LABORATORY
Washington, D.C.

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NRL Report 7237
Change 1

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Subject: Corrections for NRL Report 7237, "Finite-Time Stability of Linear Differential Equations," by Leonard Weiss and Jong-Sen Lee.

Please make the following changes:

p. 2, first line below Eq. (4); change "Hence, $\| x_0 \| < \beta$" etc to "Hence, $\| x(t) \| < \beta$".

p. 3, six lines above bottom of page; change "Finally, we . . . (see [24]) . . ." to "Finally, we . . . (see [23]) . . ."

p. 8, top line; change "and, by Lemma 4," to "and, by Lemma 5."

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Finite-Time Stability of Linear Differential Equations

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Mathematics Research Center
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Abstract: A unified approach to finite-time stability of linear autonomous differential equations is developed in this report. Linear algebraic machinery is used to derive new and computationally feasible results, and also to obtain elegant proofs of known theorems. Connections are made to the Liapunov-like theory of finite-time stability. Corresponding results are also derived for a class of stochastic linear systems.

INTRODUCTION

Finite-time stability differs from classical stability in that one is interested in the behavior of system trajectories which originate within an *a priori* fixed region in the state space over a given fixed time interval (which may be finite or infinite). Over the past 25 years, exploration of various facets of this concept has been made by a number of workers (see Refs [1-21]), many of them motivated by the fact that in a variety of practical situations, the finite-time concept of stability is more pertinent than the classical concept. The theory, in its early stages, was mainly (though not entirely) concerned with linear differential equations ([1-8]), proceeded through some preliminary probing toward a qualitative nonlinear theory ([9-12]), and then underwent systematic development using a particular Lyapunov-like approach initiated in [13-15] and continued in [16-21].

In this paper, we bring the development full circle by presenting some new results in the theory of finite-time stability for linear autonomous differential equations. Not only are these results computationally attractive, but the techniques used to obtain them allow elementary and elegant proofs of known theorems to be given, and enable succinct characterizations of finite-time stability to be made for certain classes of linear differential equations.

Extensions of the aforementioned results to the case of linear systems driven by "white noise" are made, and these also result in easily computable criteria for finite-time stability.

PRELIMINARIES

The symbol $\|\cdot\|$ denotes the euclidian norm on R^n. If A is an $n \times n$ real matrix, the set of eigenvalues of A is denoted by $\{\lambda(A)\}$. If the eigenvalues are real, $\hat{\lambda}(A) \triangleq \max\{\lambda(A)\}$. The spectral norm of A is denoted by $\|A\|^* \triangleq [\hat{\lambda}(A'A)]^{1/2}$ where A' is the transpose of A.

Consider the system of linear differential equations

$$\dot{x}(t) = Ax(t), t \geqslant 0 \tag{1}$$

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NOTE: A presentation of the results in this manuscript was given at the SIAM 1970 Fall Meeting, Oct. 12-14, Boston, Mass.

where A is $n \times n$, real and $x(0) \triangleq x_0$. The unique solution of (1) at time t is given by

$$x(t, x_0) \triangleq x(t) = e^{At} x_0 . \tag{2}$$

Our objective is to investigate (1) for the following type of behavior.

Definition 1. The system (1) is stable with respect to $(\alpha,\beta,T), \alpha \leqslant \beta$, if $\| x_0 \| < \alpha$ implies $\| x(t) \| < \beta$ for all $t \in [0,T)$.

A FUNDAMENTAL RESULT

The theorem given below will be used in conjunction with the linear algebraic techniques developed in the next section to generate further stability results.

THEOREM 1. *A necessary and sufficient condition for the system (1) to be stable (Def. 1) is*

$$\| e^{At} \|^* \leqslant \frac{\beta}{\alpha}, \text{ for all } t \in [0, T) . \tag{3}$$

Proof. for each fixed t,

$$\| x(t) \| = \| e^{At} x_0 \| \leqslant \| e^{At} \|^* \| x_0 \| \tag{4}$$

with equality achieved for some x_0 (independent of $\| x_0 \|$). Hence, $\| x_0 \| < \beta \;\; \forall\, t \in [0, T)$ if and only if $\| e^{At} \|^* \| x_0 \| < \beta \;\; \forall\, t \in [0, T)$ and the latter holds on $[0, T)$ with $\| x_0 \| < \alpha$ if and only if (3) holds.

Since our aim is to obtain results which are computationally tractable, our effort in the succeeding two sections is mainly devoted to translating (3) into computable conditions on the coefficient matrix A.

SOME USEFUL LEMMAS

The statements designated as Lemmas 1, 2, and 4 below are well known. Lemma 3 is less well known (see Dahlquist [22] for the inequality), and we provide a novel proof for it.

LEMMA 1. *If $f(A)$ is a well-defined function of a matrix A, then $\lambda \in \{\lambda(A)\}$ implies $f(\lambda) \in \{\lambda(f(A))\}$.*

Proof. See [23].

LEMMA 2. *If A is a symmetric matrix, then* $\| e^A \|^* = e^{\hat{\lambda}(A)}$.

Proof. Follows from Lemma 1.

LEMMA 3. *For any $n \times n$ real matrix A,*

$$\| e^A \|^* \leqslant e^{\hat{\lambda}((A' + A)/2)} \tag{5}$$

with *equality if and only if A is normal.*

Proof. Any $n \times n$ matrix A has a unique decomposition $A = A_1 + A_2$ where $A_1 = (A + A')/2$ and $A_2 = (A - A')/2$. Now, $e^A = e^{A_1 + A_2} = e^{A_1} e^{A_2}$ if and only if A_1 and A_2 commute. (A_1 and A_2 commute if and only if A is normal.) Hence, under such conditions, $\|e^A\|^* = \|e^{A_1} e^{A_2}\|^*$ and since A_2 is skew-symmetric, $A_2' = -A_2$. Therefore, we have

$$(\| e^A \|^*)^2 = \hat{\lambda}(e^{-A_2} e^{A_1} e^{A_1} e^{A_2}) = \hat{\lambda}(e^{2A_1}) = e^{\hat{\lambda}(A'+A)}.$$

This proves the equality part. To prove the inequality, we start with the Rayleigh quotient [24] representation of $\| \cdot \|^*$, i.e.,

$$\begin{aligned}
\hat{\lambda}(A' + A) &= \max_{x \neq 0} \frac{x'(A' + A)x}{\|x\|^2} \\
&= \max_{x(\cdot)} \int_0^1 \frac{x'(t)(A' + A)x(t)}{\|x(t)\|^2}\, dt \\
&\geq \max_{\left(\substack{x(\cdot) \\ \dot{x} = Ax}\right)} \int_0^1 \frac{\dot{x}'(t)\ x(t) + x'(t)\ \dot{x}(t)}{\|x(t)\|^2}\, dt, \\
&\geq \max_{x_0} \int_{x_0}^{x(1)} \frac{d(\|x(t)\|^2)}{\|x(t)\|^2}\ . \\
&\geq \max_{x_0} ln \left(\frac{\|x(1)\|^2}{\|x_0\|^2} \right) = \max_{x_0} ln \left(\frac{\|e^A x_0\|^2}{\|x_0\|^2} \right) = ln \left(\max_{x_0} \frac{\|e^A x_0\|^2}{\|x_0\|^2} \right) \\
&\geq ln(\|e^A\|^*)^2,
\end{aligned}$$

which implies

$$\| e^A \|^* \leq e^{\hat{\lambda}((A' + A)/2)}.$$

Finally, we shall make use of the well-known relationship (see [24]) between the spectral radius of an $n \times n$ matrix A and its spectral norm, namely,

LEMMA 4. $\max \{ |\lambda(A)| \} \leq \|A\|^*$.

APPLICATIONS OF THEOREM 1 AND LEMMAS 1-4

The following result is new and characterizes finite-time stability for a class of linear systems via an easily computable criterion.

THEOREM 2. *Let A in (1) be normal. Then (1) is stable (Def. 1) if and only if*

$$\hat{\lambda}\left(\frac{1}{2}(A+A')\right) \leqslant \frac{1}{T} \ln(\beta/\alpha) \ . \tag{6}$$

Proof. By Theorem 1 and Lemmas 2 and 3 we have

$$\text{Stability (Def. 1)} \Leftrightarrow \| e^{At} \|^* \leqslant \beta/\alpha \ , \ \text{for all } t \in [0,T)$$

$$\Leftrightarrow e^{\hat{\lambda}((A'+A)t/2)} \leqslant \beta/\alpha, \text{ for all } t \in [0,T)$$

Using $\alpha \leqslant \beta$, then a simple calculation shows that (6) is necessary and sufficient.

This generalizes, via our simpler and more direct argument, an earlier result by Kaplan [21] for the case when A is symmetric. The techniques developed in the last section also allow other known results in finite-time stability to be obtained directly. For example, as an immediate corollary of Lemma 3 and Theorem 1, we obtain

THEOREM 3. (*Dorato* [8]). *A sufficient condition for (1) to be stable (Def. 1) is*

$$\hat{\lambda}\left(\frac{1}{2}(A+A')\right) \leqslant \frac{1}{T} \ln(\beta/\alpha) \ . \tag{7}$$

Even more striking is the simple and direct proof which can be given of the following necessary condition, originally established (see Kaplan [21] by means of a long induction argument.

THEOREM 4. *The system (1) is stable (Def. 1) only if*

$$\max\{Re(\lambda(A))\} \leqslant \frac{1}{T} \ln(\beta/\alpha) \ . \tag{8}$$

Proof. By Theorem 1 and Lemma 4 we obtain, as a necessary condition for stability (Def. 1),

$$\max\{|\lambda(e^{At})|\} \leqslant \beta/\alpha \ \text{ for all } t\epsilon[0,T) \ . \tag{9}$$

By Lemma 1, this implies

$$e^{\max\{Re(\lambda(A))\}t} \leqslant \beta/\alpha \ \text{ for all } t\epsilon[0,T)$$

which implies, in turn, that

$$e^{\max\{Re(\lambda(A))\}(T-\epsilon)} \leqslant \beta/\alpha \ \text{ for all arbitrarily small } \epsilon > 0 \ . \tag{10}$$

Taking the logarithm and the limit as $\epsilon \to 0$, yields (8).

CONNECTIONS TO NONLINEAR THEORY

Let $V: R^n \times [0,T) \to R$ be a continuous function with continuous first partial derivatives. Let

$$\dot{V}(x,t) = \left(\text{grad } V \cdot \frac{dx}{dt}\right) + \frac{\partial V}{\partial t},$$

$$V_m^a(t) \triangleq \min_{\|x\|=a} V(x,t),$$

$$V_M^a(t) \triangleq \max_{\|x\|=a} V(x,t)$$

$$B(\alpha) \triangleq \{x \in R^n : \|x\| < \alpha\}$$

and let $\bar{B}(\alpha)$ be the closure of $B(\alpha)$.

Consider the following theorem*, which holds for certain nonlinear as well as linear systems.

THEOREM 5. *The system (1) is (uniformly†) stable (Def. 1) if and only if there exists a real-valued function $V(x,t)$, as above, and a real-valued integrable function $\varphi(t)$ such that*

(i) $\dot{V}(x,t) \leqslant \varphi(t)$, for all $x \in (B(\beta) - \bar{B}(\alpha))$, all $t \in [0,T]$)

(ii) $\int_{t_1}^{t_2} \varphi(t)dt < V_m^\beta(t_2) - V_M^{\alpha-\epsilon}(t_1)$, all $t_2 > t_1$, and $t_1, t_2 \in [0,T)$ for $\epsilon > 0$ arbitrarily small.

To relate Theorem 5 to previous results, we note first that $V(x,t)$ can be chosen as $V(x)$ since (1) is autonomous. Consider

$$V(x) = ln\,\|x\|. \tag{11}$$

Then

$$\dot{V}(x) = \frac{x'Ax}{\|x\|^2}.$$

Now, with A_1 and A_2 defined as in the proof of Lemma 3, we have $x'A_2x = 0$, and so

$$\dot{V}(x) = \frac{x'A_1x}{\|x\|^2} \leqslant \hat{\lambda}(A_1).$$

Let $\varphi(t) = \hat{\lambda}(A_1)$. Then Theorem 5 yields the result

$$\int_0^T \hat{\lambda}(A_1)dt < ln\left(\frac{\beta}{\alpha-\epsilon}\right),$$

or

$$\hat{\lambda}(A_1) < \frac{1}{T}\,ln\left(\frac{\beta}{\alpha-\epsilon}\right) \text{ for } \epsilon > 0 \text{ arbitrarily small.}$$

---

*This is actually a trivial perturbation of Theorem 1 in [15], the difference being that $V_M^{\alpha-\epsilon}$ appears in this case rather than V_M^α as in [15]. This occurs because the definition of finite-time stability in [15] allowed an initial condition $\|x\| = \alpha$. The proof is exactly the same.

†Uniform finite-time stability is defined in [15] and is equivalent to stability (Def. 1) for systems of the form (1).

Taking the limit as $\epsilon \to 0$, we get

$$\hat{\lambda}(A_1) \leqslant \frac{1}{T} \ln\left(\frac{\beta}{\alpha}\right)$$

which is the result designated as Theorem 3.

Remarks: 1. An extension of Theorem 5 to the case of nonuniform finite-time stability has been made by various people working independently (c.f., for sufficient conditions, [18] and [19], and for converse theorems, [20] and [21]). The extension replaces hypothesis (ii) (assuming initial time 0) by

$$\int_0^T \varphi(t)dt < V_m^\beta(T) - \max_{\|x\| \leqslant \alpha - \epsilon} V(x,0).$$

With this extension, the function $V(x) = \ln \|x\|$ yields a known sufficient condition (see [8]) for stability (Def. 1) of the linear time-varying system $\dot{x}(t) = A(t)x(t), t \geqslant 0$, namely,

$$\int_0^T \hat{\lambda}\left(\frac{1}{2}(A'(s) + A(s))\right) ds \leqslant \ln\left(\frac{\beta}{\alpha}\right).$$

2. If A in (1) is normal, then the Lyapunov-like function (11) will *always* be conclusive in the test for stability (Def. 1).

3. It is interesting (though not surprising once one sees the pattern of results) that the natural Lyapunov-like function associated with finite-time stability of linear systems is *not* a quadratic form, but the logarithm of the latter.

LINEAR SYSTEMS DRIVEN BY WHITE NOISE

In this section, we consider linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), t \geqslant 0 \tag{12}$$

where A is $n \times n$, B is $n \times m$, and $u(\cdot)$ is a vector white noise with zero mean and covariance matrix $Q(t)$ (i.e., $E\{u(t)\} = 0$ and $E\{u(t)\, u'(s)\} = Q(t)\,\delta(t-s)$, where E is the Expectation, and δ is the Dirac delta.)

For any $n \times n$ matrix G, let $tr(G)$ be the trace of G. Then we have

Definition 2. The system (12) is mean-square stable with respect to $(\alpha, \beta, \gamma, T)$, $\alpha \leqslant \beta$, if the conditions $E\{\|x(t)\|^2\} < \alpha^2$ and $tr(Q(t)) \leqslant \gamma^2$, for all $t \in [0,T]$, imply that $E\{\|x(t)\|^2\} < \beta^2$ for all $t \in [0,T]$.

The main result in this section depends on the following Lemma.

LEMMA 5. *Let F be an nxn symmetric matrix and let $\mathbb{P}$ denote the set of nxn nonnegative definite matrices.*

Then

$$\hat{\lambda}(F) = \max_{P \in \wp} \frac{tr(PF)}{tr(P)} \tag{13}$$

Proof: Let S be an $n \times n$ orthogonal matrix such that $S'FS = \Lambda = \text{diag}\,(\lambda_i)$. Also let $S'PS = D$.

Then

$$\begin{aligned} tr(PF) &= tr(PS\Lambda\, S') \\ &= tr(S'PS\Lambda) \\ &= tr(D\Lambda) \\ &= \lambda_1\, d_{11} + \lambda_2\, d_{22} + \ldots\ldots\ldots + \lambda_n\, d_{nn} \\ &\leqslant \hat{\lambda}(F)\,(d_{11} + d_{22} + \ldots + d_{nn}),\ d_{ii} \geqslant 0 \\ &\leqslant \hat{\lambda}(F)\; tr\,(D) \end{aligned}$$

Therefore it is possible to choose $\{d_{ii} \mid i = 1, \ldots, n\}$ such that

$$\hat{\lambda}(F) = \max_{d_{ii}} \frac{tr(D\Lambda)}{tr(D)} = \max_{P \in \wp} \frac{tr(PF)}{tr(P)}$$

Now consider

THEOREM 6. *The system (12) is mean-square stable (Def. 2) if and only if*

$$\alpha^2 \left(\| e^{At} \|^*\right)^2 + \gamma^2 \int_0^t \left(\| e^{A\xi} B \|^*\right)^2 d\xi \leqslant \beta^2 \text{ , for } t \in [0, T) \tag{14}$$

Proof. Let $P(t) = E\,\{x(t)\,x'(t)\}$. Then $P(t) \in \wp$ for each t, and differentiation yields

$$\dot{P}(t) = AP(t) + P(t)A' + BQ(t)B'. \tag{15}$$

The solution to (15) is given by (see [26])

$$P(t) = e^{At}\,P(0)\,e^{A't} + \int_0^t e^{A(t-s)}\,BQ(s)B'\,e^{A'(t-s)}\,ds.$$

Then

$$tr\,(P(t)) = tr\left(P(0)\,e^{A't}\,e^{At}\right) + \int_0^t tr\left(Q(s)B'\,e^{A'(t-s)}\,e^{A(t-s)}\,B\right) ds$$

and, by Lemma 4,

$$tr(P(t)) \leqslant \left(\| e^{At} \|^*\right)^2 \; tr(P(0)) + \int_0^t \left(\| e^{A(t-s)} B \|^*\right)^2 \; tr(Q(s))\, ds. \tag{16}$$

Sufficiency of (14) now follows by substituting, for any fixed $t \epsilon [0,T]$, $tr(P(0)) < \alpha^2$, and $tr(Q(s)) \leqslant \gamma^2$ for all $s \epsilon [0,t]$ into (16). To prove necessity, we first note that there exists a $P(0)$ and a $Q(s)$ such that equality occurs in (16), and such that $tr(P(0)) = \alpha^2$ and $tr(Q(s)) = \gamma^2$ for all $s \epsilon [0,t]$. Suppose (14) does not hold at some $t = t_1 \epsilon [0,t]$. Then with $P(0)$ and $Q(s)$ chosen as indicated, (16) yields $tr(P(t_1)) > \beta^2$, in which case $tr(P(t_1)) \geqslant \beta^2$ as long as $tr(P(0)) < \alpha^2$ and $tr(Q(s)) = \gamma^2$ for all $s \epsilon [0,t_1]$. Hence the negation of (14) implies the negation of stability (Def. 3) and the theorem is proved.

We now develop results for the system (12) analogous to our earlier ones for (1). Let

$$\rho^2 = \frac{\gamma^2 (\| B \|^*)^2}{\hat{\lambda}\,(A' + A)}. \tag{17}$$

COROLLARY 1. *A sufficient condition for the system (12) to be mean-square stable (Def. 2) is*

$$\hat{\lambda}\left(\frac{1}{2}(A' + A)\right) \leqslant \frac{1}{2T}\, ln\left(\frac{\beta^2 + \rho^2}{\alpha^2 + \rho^2}\right). \tag{18}$$

Proof. Let

$$U(t) = \left(\| e^{At} \|^*\right)^2 \; tr(P(0)) + \int_0^t \left(\| e^{A(t-s)} B \|^*\right)^2 \; tr(Q(s))\, ds.$$

Then stability (Def. 2) occurs if $U(t) \leqslant \beta^2 \; \forall \; t \epsilon [0,T]$ for $tr(P(0)) < \alpha^2$ and $tr(Q(s)) \leqslant \gamma^2$ for all $s \in [0,T]$. Now

$$U < \alpha^2 \left(\| e^{At} \|^*\right)^2 + \gamma^2 \int_0^t \left(\| e^{A\xi} B \|^*\right)^2 d\xi.$$

By Lemmas 2 and 3 and the fact that $\| GH \|^* \leqslant \| G \|^* \| H \|^*$,

$$U(t) < \alpha^2 e^{\lambda(A' + A)t} + \gamma^2 \left(\| B \|^*\right)^2 \int_0^t e^{\hat{\lambda}(A' + A)\xi}\, d\xi$$

or

$$U(t) < \alpha^2 e^{\hat{\lambda}(A' + A)t} + \frac{\gamma^2 (\| B \|^*)^2}{\hat{\lambda}\,(A' + A)} \left[e^{\hat{\lambda}(A' + A)t} - 1\right] \triangleq V(t)$$

Then stability (Def. 2) is implied by $V(t) \leqslant \beta^2$. Taking the log of both sides, and using (17), yields (18).

COROLLARY 2. *Consider the system (12) and suppose that* $B = I$ *and* A *is normal. Then (18) is a necessary and sufficient condition for (12) to be mean-square stable (Def. 2).*

Proof. From Theorem 6, we obtain the following necessary and sufficient condition:

$$\alpha^2 (\| e^{At} \|^*)^2 + \gamma^2 \int_0^t (\| e^{A\xi} \|^*)^2 \, d\xi \leqslant \beta^2 \quad \text{for all } t \in [0,T]$$

Applying Lemma 3 allows the integral to be evaluated, and taking the log of the resulting inequality with $t = T$ yields (18).

The close correspondence of these results with those for deterministic undriven systems is completed by given the following necessary condition for stability (Def. 2). Let

$$Re(\bar{\lambda}) = \max \{ Re(\lambda(A)) \}$$

and let

$$\mu^2 = \frac{\gamma^2}{2(Re(\bar{\lambda}))} \quad . \tag{19}$$

Then we have

THEOREM 7. *Let* $B = I$ *in (12). Then the system (12) is mean-square stable (Def. 2) only if*

$$Re(\bar{\lambda}) \leqslant \frac{1}{2T} \ln \frac{\beta^2 + \mu^2}{\alpha^2 + \mu^2} . \tag{20}$$

Proof. The proof is similar in structure to that of Theorem 4 and follows from applying Lemma 3 to the necessary condition (14).

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Security Classification

UNCLASSIFIED

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

FINITE-TIME STABILITY OF LINEAR DIFFERENTIAL EQUATIONS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

This is the final report on one phase of a continuing problem.

5. AUTHOR(S) *(First name, middle initial, last name)*

Leonard Weiss and Jong-Sen Lee

6. REPORT DATE July 15, 1971	7a. TOTAL NO. OF PAGES 13	7b. NO. OF REFS 25

8a. CONTRACT OR GRANT NO. NRL Problems B01-10 and B01-11 b. PROJECT NO. RR 003-02-41-6153 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7237 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* Mathematics Research Center Report 71-4

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Dept. of the Navy (Office of Naval Research), Arlington, Virginia 22217

13. ABSTRACT

A unified approach to finite-time stability of linear autonomous differential equations is developed in this report. Linear algebraic machinery is used to derive new and computationally feasible results, and also to obtain elegant proofs of known theorems. Connections are made to the Liapunov-like theory of finite-time stability. Corresponding results are also derived for a class of stochastic linear systems.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Stability						
Finite-time stability						
Linear differential equations						
White-noise driven linear systems						
Stability *via* Lyapunov-like functions						
White-noise perturbation stability						
Mean-square stable						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Security Classification