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# Two-Dimensional Systematic Point Count for Volume Fraction Analysis from a Poisson Theoretic Approach

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## ABSTRACT

In many fields of scientific investigation, the structure of cellular aggregates or random arrays of discrete particles imbedded in some solid is observed on a two-dimensional section and inferences drawn therefrom as to the real structure in three dimensions. A fast, reliable method for the quantitative determination of the percentages of these micro- or macroconstituents would be of great benefit for structural studies in the solid state.

One of the techniques most often used for the estimation of volume fractions from measurements made on a random two-dimensional section is that of the two-dimensional systematic point count, i.e., that the fractional number of regularly dispersed points falling within the boundaries of a two-dimensional feature on a plane provides an unbiased estimate of the areal fraction, and consequently of the volume fraction, of that feature.

The two-dimensional systematic point count is demonstrated here from a Poisson theoretic approach. In addition, two methods of application are investigated: one using a normal approximation, the other, the Poisson distribution. The relationship between the latter and the point-count procedure is also indicated.

## PROBLEM STATUS

This is a final report on one phase of the problem; work is continuing on other phases.

## AUTHORIZATION

NRL General and Administrative Function 78-1601

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## LIST OF SYMBOLS

| | |
|---|---|
| $A$ | Total area examined |
| $A_\alpha$ | Area of $\alpha$ phase |
| $A_f$ | Areal fraction of $\alpha$ phase ($A_\alpha/A$) |
| $a$ | Area of an individual $\alpha$ feature |
| $C, D$ | Constants |
| $d$ | Spacing of a square, two-dimensional lattice |
| $E[\ \ ]$ | Expected value of [ ] |
| $k$ | A constant |
| $m$ | Number of occurrences, an index |
| $N$ | Total number of lattice points applied |
| $N(A)$ | Number of $\alpha$ features in A |
| $N_f$ | Fraction of lattice points occupied by $\alpha$ features, ($Np(A)/N$) |
| $Na$ | Number of $\alpha$ features per unit area |
| $Na_i$ | Number of $\alpha$ features of area $a_i$ per unit area |
| $Na_i(A)$ | Number of $\alpha$ features of area $a_i$ in A |
| $Np$ | Number of lattice points occupied by $\alpha$ features per unit area |
| $Np(A)$ | Number of lattice points occupied by $\alpha$ features in A |
| $Np_i$ | Number of lattice points occupied by $\alpha$ features of area $a_i$ per unit area |
| $Np_i(A)$ | Number of lattice points occupied by $\alpha$ features of area $a_i$ in A |
| $n$ | Sample size |
| $P$ | Probability |
| $r$ | Number of different sizes of $\alpha$ features |
| $V$ | Volume of structure |
| $V_f$ | Volume fraction of $\alpha$ phase |
| $V_\alpha$ | Volume of $\alpha$ phase |
| $\bar{x}$ | Sample mean |
| $Z$ | Chi-square variate |
| $z$ | Standard normal variate |
| $\alpha$ | Phase whose volume fraction is being estimated |
| $\alpha$ | Significance level |
| $\gamma$ | Precision or tolerance |
| $\lambda$ | Parameter of the Poisson distribution |
| $\nu$ | Parameter of the Poisson distribution |
| $\sigma$ | Standard deviation |
| $\sigma^2$ | Variance |
| $\chi^2$ | The Chi-square distribution |

# TWO-DIMENSIONAL SYSTEMATIC POINT COUNT FOR VOLUME FRACTION ANALYSIS FROM A POISSON THEORETIC APPROACH

## INTRODUCTION

In many fields of scientific investigation the structure of cellular aggregates or random arrays of discrete particles imbedded in some solid is observed on a two-dimensional section and inferences are drawn therefrom as to the real structure in three dimensions. The petrologist's thin section, the biologist's microtome slice, and the metallurgist's or chemist's plane-polished or etched sections are well-known examples, although the problem is a general one. A fast, reliable method for the quantitative determination of the percentages of these micro- or macroconstituents would be of great benefit for structural studies in the solid state.

The experimental investigation which prompted this study was an effort to determine the void content in filament-wound composites at low void levels. Standard chemical analyses of the void content of these composites at such low void levels yielded negative results and thus proved to be totally unsatisfactory.

The techniques most often used for estimation of volume fractions from measurements made on a random two-dimensional section are based on one or more of the following principles:

1. For an areal or Delesse* analysis: that the areal fraction of a three-dimensional feature intercepted by a random plane provides an unbiased estimate of the volume fraction of that feature.

2. For a lineal or Rosiwal† analysis: that the fractional intercept on a line passing at random through a two- or three-dimensional feature provides an unbiased estimate of the areal or volume fraction, respectively, of that feature.

3. For a point-count analysis: that the fractional number of randomly or regularly dispersed points falling within the boundaries of a two-dimensional feature on a plane, or within a three-dimensional feature in a volume, provides an unbiased estimate of the areal or volume fraction, respectively, of that feature.

The property of being without bias referred to in these principles implies only that the expected value is equal to the true value, not that an analysis will be free of error.

In this report, only the point-count principle will be considered, and more specifically, the two-dimensional systematic point count as opposed to the one-dimensional or random point count.

The following restrictions or assumptions will underlie the results:

1. That the feature under consideration occur as discrete particles randomly (or uniformly) distributed in three dimensions. This assumption implies that the volume fraction is small.

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*A. Delesse, "Procede Mechanique Pour Determiner la Composition des Roches," Annales des Mines, (1848).
†A. Rosiwal, Verh. Wein geol. Reichs 32:143-175 (1898).

2. The intersections of a feature with the plane of polish will be assumed to occur as discrete areas and to be randomly (or uniformly) distributed on the plane. This follows from item 1.

3. Only statistical errors will be considered; however, in an actual analysis there will be errors in measurement as well. Such errors may lead to a biased estimate of the volume fraction.

## THE TWO-DIMENSIONAL SYSTEMATIC POINT COUNT

Let the phase or constituent whose volume fraction is to be estimated be denoted by $\alpha$ and the remaining phases as a single phase $\beta$.

For this procedure, it is considered preferable to use a systematic array of points, such as that provided by the corners of a two-dimensional lattice. This has the advantage that only the points falling on an $\alpha$ phase need be counted, since the total number of points applied to a structure is predetermined. This method appears to be experimentally expedient and to have statistical advantages as well.

A square, coarse-mesh lattice will be considered here, although the method may be implemented in general. A coarse-mesh lattice may be defined mathematically as one in which the spacing is restricted such that

$$p_0 + p_1 = 1 \text{ and } p_n = 0, \text{ for } n \geqslant 2,$$

where $p_n$ is the probability that an $\alpha$ feature will occupy n lattice points.

### Areal Analysis

It is commonly accepted in the literature and has been shown by Delesse and others that the expected relative area of a given feature is equal to the relative volume of that feature. A mathematical justification of this is given in the Appendix.

The probability $p_1^i$ that a given feature of area $a_i$ will occupy a lattice point is $a_i/d^2$ where $1/d^2$ is the number of points per unit area (for a square lattice, d will be the lattice spacing). See the Appendix.

### Probability Assumptions

1. The probability of the occurrence of an $\alpha$ feature in a given subarea $\Delta A$ on a random plane becomes proportional to that area as $\Delta A \to 0$, i.e., P($\alpha$ feature occurring in $\Delta A$) $\approx \lambda \Delta A$, as $\Delta A \to 0$, where $\lambda$ is the same for all $\Delta A$ in a given plane and for all planes of the same orientation.

2. The probability of more than one $\alpha$ feature occurring in an area $\Delta A$ is zero by comparison.

3. The occurrence of an $\alpha$ feature in an area $\Delta A_i$ is independent of an occurrence in any other area $\Delta Aj$, $i \neq j$. This holds regardless of the respective size of the $\Delta A$ or of the area of the $\alpha$ feature.

A process governed by the preceding probability laws is said to be a Poisson process. Therefore, with these assumptions and that of randomly distributed $\alpha$ features on the plane of polish, the number of $\alpha$ features of a given size within a given area will follow a Poisson distribution. Thus, P(k, $\alpha$ features of area $a_i$ occurring in A) = $[(\lambda_i A)^k e^{-\lambda_i A}]/k!$ for finite area A, when $i = 1, \ldots, r$, the number of different size $\alpha$ features; thus

$$E[Na_i(A)] = \lambda_i A,$$

$$\sigma^2 [Na_i(A)] = \lambda_i A,$$

$$E[Na_i(A)]/A = \lambda_i \text{ regardless of } A;$$

that is,

$$E[Na_i] = \lambda_i.$$

The sum of independently distributed Poisson variates is again Poisson distributed. Thus, the number of $a$ features of all sizes within a given area will be Poisson distributed;

$$P(k, a \text{ features in } A) = \frac{(\lambda A)^k e^{-\lambda A}}{k!},$$

where $\lambda = (\lambda_1 + \ldots + \lambda_r)$, and

$$E[N(A)] = \lambda A,$$

$$\sigma^2 [N(A)] = \lambda A,$$

$$E[N(A)]/A = \lambda \text{ regardless of } A;$$

that is,

$$E[Na] = \lambda .$$

The probability that a given number, say j, of $a$ features of area $a_i$ will occupy lattice points in A, given that there are a specified number k of area $a_i$ in A, is given by

$$\binom{k}{j}\left(\frac{a_i}{d^2}\right)^j \left(1-\frac{a_i}{d^2}\right)^{k-j} .$$

However, in these analyses, the number of $a$ features of area $a_i$ in A is unknown.

Therefore,

P(j, $a$ features of area $a_i$ occupying lattice points) =

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda_i A}(\lambda_i A)^k}{k!} \binom{k}{j}\left(\frac{a_i}{d^2}\right)^j \left(1-\frac{a_i}{d^2}\right)^{k-j} .$$

For all $k < j$, the probability will be zero; thus the sum might just as well run from $k = j \ldots \infty$, i.e.,

$$\sum_{k=j}^{\infty} \frac{e^{-\lambda_i A}(\lambda_i A)^k}{k!} \binom{k}{j}\left(\frac{a_i}{d^2}\right)^j \left(1-\frac{a_i}{d^2}\right)^{k-j}$$

$$= e^{-\lambda_i A} (\lambda_i A)^j \left(\frac{a_i}{d^2}\right)^j \sum_{k=j}^{\infty} \frac{(\lambda_i A)^{k-j}}{k!} \binom{k}{j} \left(1-\frac{a_i}{d^2}\right)^{k-j}$$

$$= e^{-\lambda_i A} \left(\frac{\lambda_i A\, a_i}{d^2}\right)^j \sum_{k=j}^{\infty} \frac{(\lambda_i A)^{k-j}}{k!} \frac{k!}{j!(k-j)!} \left(1-\frac{a_i}{d^2}\right)^{k-j}$$

$$= e^{-\lambda_i A} \left(\frac{\lambda_i A a_i}{d^2}\right)^j \sum_{k=j}^{\infty} \frac{[(\lambda_i A)(1-a_i/d^2)]^{k-j}}{j!(k-j)!}$$

Let $k - j = \nu$;

$$= e^{-\lambda_i A} \left(\frac{\lambda_i A a_i}{d^2}\right)^j \sum_{\nu=0}^{\infty} \frac{[(\lambda_i A)(1-a_i/d^2)]^{\nu}}{j!\nu!}$$

$$= e^{-\lambda_i A + \lambda_i A (1-a_i/d^2)} \left(\frac{\lambda_i A a_i}{d^2}\right)^j \Big/ j!$$

$$= e^{-\lambda_i A a_i/d^2} \left(\frac{\lambda_i A a_i}{d^2}\right)^j \Big/ j! \quad .$$

Thus, the number of lattice points occupied by $\alpha$ features of area $a_i$ will also follow a Poisson distribution:

$$E[Np_i(A)] = \lambda_i a_i A/d^2$$

$$\sigma^2 [Np_i(A)] = \lambda_i a_i A/d^2$$

$$E[Np_i(A)]/A = \lambda_i a_i/d^2 ;$$

that is,

$$E[Np_i] = \lambda_i a_i/d^2 \quad .$$

Again, using the additive property for independently distributed Poisson variates, the total number of lattice points occupied by $\alpha$ features follows a Poisson distribution:

P(j lattice points occupied by $\alpha$ features) =

$$e^{-\lambda a A/d^2} \left(\frac{\lambda a A}{d^2}\right)^j \Big/ j! \quad ,$$

where

$$\lambda a = (\lambda_1 a_1 + \ldots + \lambda_r a_r), \text{ and}$$

$$E[Np(A)] = \lambda aA/d^2$$

$$\sigma^2 [Np(A)] = \lambda aA/d^2$$

$$E[Np(A)]/A = \lambda a/d^2;$$

that is,

$$E[Np] = \lambda a/d^2$$

and

$$E[Np(A)]/N = \frac{\lambda aA}{Nd^2} = \frac{(\lambda_1 a_1 + \ldots + \lambda_r a_r)A}{Nd^2} = E[A_a]/Nd^2 .$$

Since $1/d^2 \approx$ number of lattice points per unit area and equal to the number in the limit (see the Appendix),

$$Nd^2 = A.$$

Thus

$$\frac{E[Np(A)]}{N} = \frac{E[A_a]}{A}, \text{ or } E\left[\frac{Np(A)}{N}\right] = E\left[\frac{A_a}{A}\right].$$

Therefore, $E[N_f] = E[A_f] = V_f$, as indicated in the introduction (item 3), i.e. that the fractional number of regularly dispersed points falling within the boundaries of a two-dimensional feature on a random plane has been shown to provide an unbiased estimate of the volume fraction of that feature.

In addition,

$$E[Np(A)] = \frac{\lambda aA}{d^2} = \lambda aN, \text{ or } E\left[\frac{Np(A)}{N}\right] = \lambda a$$

$$\sigma^2 [Np(A)] = \frac{\lambda aA}{d^2} = \lambda aN, \text{ or } \sigma^2 \left[\frac{Np(A)}{N}\right] = \frac{1}{N^2}\sigma^2 [Np(A)] = \frac{\lambda a}{N}$$

and

$$E[N_f] = \lambda a$$

$$\sigma^2 [N_f] = \frac{\lambda a}{N}.$$

Thus,

$$\frac{\sigma(N_f)}{E[N_f]} = \frac{\sqrt{\lambda a/N}}{\lambda a} = \frac{1}{\sqrt{N\lambda a}} = \frac{1}{\sqrt{E[Np(A)]}} .$$

Therefore, for randomly dispersed features, the variance of the analysis is independent of the size of the features. Of the several methods used for volume fraction analysis, this procedure appears to have minimum variance.

From the above, the relative standard deviation (i.e., $\sigma/V_f$, where $\sigma$ is the standard deviation of $V_f$) in the volume fraction resulting from statistical errors will be approximately the reciprocal of the square root of the number of lattice points occupied by $a$ features. This number does not include experimental errors.

## APPLICATION I

As previously shown, the number of lattice points occupied by $a$ features in area A, Np(A), follows a Poisson distribution with parameter

$$\frac{\lambda aA}{d^2} = \lambda aN \ .$$

When the value of the parameter is sufficiently large, the Poisson distribution may be approximated by the normal distribution with mean and variance equal to the parameter value of the Poisson. A comparison between the two distributions shows that the normal approximation may be used when the parameter value is greater than 9.

Therefore, central confidence intervals for the parameter $\nu$ of a Poisson distribution, with $\nu$ greater than 9, may be given by

$$(\bar{x} - \nu)\sqrt{n/\nu} = \pm z_a \ ,$$

(where $\bar{x}$ is the sample mean, n the number of samples, and $z_a$ is the value of the normal deviate corresponding to a confidence coefficient of $1 - a$). On solving for $\nu$,

$$\nu^2 - \left(2\bar{x} + \frac{z_a^2}{n}\right)\nu + \bar{x}^2 = 0$$

or

$$\nu = \bar{x} + \frac{z_a^2}{2n} \pm \sqrt{\frac{z_a^2 \bar{x}}{n} + \frac{z_a^4}{4n^2}}$$

with the ambiguity in the square root giving upper and lower limits, respectively.

To the order of $n^{-1/2}$ the confidence interval for $\nu$ is equivalent to

$$\nu = \bar{x} \pm z_a\sqrt{\bar{x}/n} \ ,$$

from which upper and lower limits are seen to be equidistant from the mean $\bar{x}$.

Letting $n = 1$, $\bar{x} = \mathrm{Np(A)}$, and $\nu = \lambda aN$ results in

$$Na\lambda = \mathrm{Np(A)} + \frac{z_a^2}{2} \pm \sqrt{z_a^2\,\mathrm{Np(A)} + \frac{z_a^4}{4}}$$

or

$$\lambda a = \frac{Np(A)}{N} + \frac{z_a^2}{2N} \pm \frac{1}{N}\sqrt{z_a^2\, Np(A) + \frac{z_a^4}{4}}\ .$$

For example, if $a = 0.05$, then $z_a = 1.96$, and

$$Na\lambda = Np(A) + 1.92 \pm \sqrt{3.84\, Np(A) + 3.69}\ .$$

If a priori information concerning the approximate volume fraction is known, than an estimate of N required for the normal approximation to be valid may be ascertained. Let this estimate of the volume fraction be $\lambda a$; then $Na\lambda > 9$ implies that $N > 9/\lambda a$.

Assuming the normal approximation to be valid, then an estimate of N for a given precision or tolerance level may be determined; i.e.,

$$Na\lambda \pm k_a \sqrt{Na\lambda}$$

or

$$Na\lambda \left[1 \pm \frac{k_a}{\sqrt{Na\lambda}}\right]$$

where $k_a$ is the value of the normal deviate corresponding to the $1 - a$ confidence coefficient.

For example, if $a = 0.05$, then $Na\lambda[1 \pm 1.96/\sqrt{Na\lambda}]$ corresponds to a 95% confidence interval.

For a given precision or tolerance $\gamma$, let

$$\frac{k_a}{\sqrt{Na\lambda}} = \gamma\ ,$$

and on solving for N, this gives $N = k_a^2/\gamma^2\lambda a$.

Again, if $a = 0.05$, then $k_a = 1.96$, corresponding to a 95% confidence interval, and $N = 3.84/\gamma^2\lambda a$. As before, using the estimate $\lambda a$ for the volume fraction, N may be estimated.

## APPLICATION II

### Description of Method

A somewhat more precise method for estimating the parameter $\nu$ of a Poisson process is the following.

If a Poisson process is observed until a specified number m of events has been counted, then the amount $N_m$ of observations required to obtain the m events can be used to form confidence intervals for $\nu$, the parameter of the Poisson process, using the fact that $2\nu N_m$ is $\chi^2$ distributed with 2m degrees of freedom. Let C and D be values such that if Z has a $\chi^2$ distribution with 2m degrees of freedom, then

$$P(Z < C) = a/2 \text{ and } P(Z > D) = a/2$$

(where $a$ is the chosen significance level). Then

$$1 - a = P(C \leqslant 2\nu N_m \leqslant D) = P\left(\frac{C}{2N_m} \leqslant \nu \leqslant \frac{D}{2N_m}\right).$$

Consequently, $(C/2N_m, D/2N_m)$ is a confidence interval for $\nu$, with confidence coefficient $1 - a$. Thus, letting $\nu = \lambda a$ gives

$$P\left(\frac{C}{2N_m} \leqslant \lambda a \leqslant \frac{D}{2N_m}\right) = 1 - a,$$

where $\lambda a = E[N_f] = V_f$.

By arbitrarily assuming the estimate of $\lambda a$ to be the midpoint of the interval, i.e., $(D + C)/4N_m$, the relative deviation from the midpoint, $(D - C)/(D + C)$, may be considered as a measure of precision or tolerance. Therefore, the estimate of $\lambda a$ may be given by

$$\left(\frac{D+C}{4N_m}\right)\left(1 \pm \frac{D-C}{D+C}\right).$$

Let $\gamma$ denote the precision or tolerance; then

$$\frac{D-C}{D+C} = \gamma .$$

At a given significance level $a$, using standard tables of the $\chi^2$ distribution with 2m degrees of freedom, for any given value of m, the values of D and C can be found. Thus, the tolerance $\gamma$ corresponding to these values of D and C can be determined.

Conversely, for a given tolerance $\gamma$ and a given significance level $a$, the m value required so that the D and C values will yield a tolerance equal to or less than the given value is obtained. Some representative values are given in the following table.

Table 1
Precision Levels

| Number of Events m | Significance Level, $a$ | | |
|---|---|---|---|
| | 0.01 | 0.05 | 0.10 |
| 9 | 0.71 | 0.59 | 0.51 |
| 12 | 0.66 | 0.52 | 0.46 |
| 15 | 0.59 | 0.47 | 0.41 |
| 20 | 0.53 | 0.42 | 0.36 |
| 25 | 0.48 | 0.38 | 0.32 |
| 30 | 0.44 | 0.35 | 0.29 |
| 40 | 0.39 | 0.30 | 0.26 |
| 50 | 0.35 | 0.27 | 0.23 |

For other values of $\alpha$ and m, the corresponding values of $\gamma$ can be determined by using tables of the $\chi^2$ distribution with 2m degrees of freedom.

As in the previous procedure, if a priori information concerning the approximate value of the volume fraction is known, then $N_m$ can be estimated by letting $(D + C)/4N_m$ be an estimate of the volume fraction, i.e., for a given $\alpha$ and $\gamma$, the value of m required so that $(D - C)/(D + C) = \gamma$ can be approximately determined, and consequently the D and C values. Then, the estimate of $N_m$ is given by

$$N_m = \frac{D+C}{4\lambda a}$$

where $\lambda a$ is the estimate of the volume fraction.

## Relation of Application II to the Point-Count Method

The latter appears to be a more precise method than that of the normal approximation, but its relationship to the point-count method is somewhat more obscure. The following procedure will attempt to show the relationship between the second method given and that of the point count.

The $\sqrt{2\chi^2}$ may be approximated by a normal distribution with mean $\sqrt{2n-1}$ and variance 1, where n is the degrees of freedom associated with the $\chi^2$ distribution. Thus, the variable $\sqrt{2\lambda a N_m}$ is normally distributed with mean $\sqrt{4m-1}$ and variance 1, where 2m is the degrees of freedom associated with the $\chi^2$ distribution for the variable $2\lambda a N_m$.

As above, let $(D + C)/4N_m$ be the estimate of the volume fraction. Then by letting

$$\sqrt{2C} = \sqrt{4m-1} - k_\alpha$$

and

$$\sqrt{2D} = \sqrt{4m-1} + k_\alpha,$$

(where $k_\alpha$ is the value of the normal deviate corresponding to a $1-\alpha$ confidence level), and solving for D and C, it can be shown that

$$\frac{D+C}{4N_m} = \frac{m}{N_m} - \frac{1}{4N_m} + \frac{k_\alpha^2}{4N_m} .$$

For $N_m$ sufficiently large, the last two terms on the right may be disregarded, giving

$$\frac{D+C}{4N_m} \approx \frac{m}{N_m}$$

For example, if $\alpha = 0.05$, $k_\alpha = 1.96$,

$$\frac{D+C}{4N_m} = \frac{m}{N_m} - \frac{1}{4N_m} + \frac{3.84}{4N_m} = \frac{m+0.72}{N_m}$$

Similarly, for other values of $\alpha$ and for $N_m$ sufficiently large, $(D + C)/4N_m$ can be shown to be approximately the relative point count.

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## Appendix

## SUPPLEMENTARY EQUATIONS

### VOLUME FRACTION

Delesse (1848) proved mathematically that in a uniform rock the volume proportions of the various minerals are equal to their areal proportions viewed on a random section.

Elementary calculations show that the relative area of $a$ features in an arbitrary cross section is an unbiased estimate of the relative volume of that feature, i.e.,

$$E[A_f] = V_f \ .$$

Consider an arbitrary cross-sectional area of a single $a$ feature $t_a$; then the $E[t_a]$ = (volume of $t_a$)/(height of $t_a$). Likewise, consider an arbitrary cross-sectional area of the structure w; then $E[w]$ = (volume of structure)/(height of structure).

Let the height of the $a$ feature be equal to the height of the structure by zero extension. Then the $E[t_a]/E[w]$ = (volume of $t_a$)/(volume of structure).

If $t_a/w$ and w are independent, then $E[t_a/w] = E[t_a]/E[w]$ = (volume of $t_a$)/(volume of structure), implying $E[A_f] = V_f$. In this paper, the independence of $t_a/w$ and w is implied by assumption 1, page 2.

### PROBABILITY OF COINCIDENCE OF A LATTICE POINT AND A FEATURE

As previously stated, the probability, $p_1^i$ that a given feature of area $a_i$ will occupy a lattice point is given by $a_i/d^2$. To show this, consider a coarse-mesh square lattice applied at random to the plane of polish. Let

M = number of lattice points applied to the plane including boundary points,

$a_i$ represent equal areas of a given feature, and

A = the total area occupied by the lattice.

Then the probability that a given feature with area $a_i$ will occupy a lattice point is given by $p_1^i = Ma_i/A$.

If k is the number of lattice points on each side of the array, then $M = k^2$, $k = \sqrt{M}$, and $(k-1)^2 d^2 = A$, where for a square lattice d is the lattice spacing. Thus, $p_1^i = (Ma_i)/[(k-1)^2 d^2] = (Ma_i)/[(\sqrt{M}-1)^2 d^2]$ and as $M \to \infty$, $p_1^i \to a_i/d^2$.

Since $k^2/[(k-1)^2 d^2]$ is the number of points per unit area, as $k \to \infty$, $1/d^2$ is approximately the number of points per unit area.

Security Classification

**DOCUMENT CONTROL DATA - R & D**

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3. REPORT TITLE

Two-Dimensional Systematic Point Count for Volume Fraction Analysis from a Poisson Theoretic Approach

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A final report on one phase of the problem; work is continuing on other phases.

5. AUTHOR(S) (First name, middle initial, last name)

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13. ABSTRACT

In many fields of scientific investigation, the structure of cellular aggregates or random arrays of discrete particles imbedded in some solid is observed on a two-dimensional section and inferences drawn therefrom as to the real structure in three dimensions. A fast, reliable method for the quantitative determination of the percentages of these micro- or macroconstituents would be of great benefit for structural studies in the solid state.

One of the techniques most often used for the estimation of volume fractions from measurements made on a random two-dimensional section is that of the two-dimensional systematic point count, i.e., that the fractional number of regularly dispersed points falling within the boundaries of a two-dimensional feature on a plane provides an unbiased estimate of the areal fraction, and consequently of the volume fraction, of that feature.

The two-dimensional systematic point count is demonstrated here from a Poisson theoretic approach. In addition, two methods of application are investigated: one using a normal approximation, the other, the Poisson distribution. The relationship between the latter and the point-count procedure is also indicated.

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| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| volume fraction | | | | | | |
| areal fraction | | | | | | |
| point count | | | | | | |
| $\alpha$ phase | | | | | | |
| $\beta$ phase | | | | | | |
| Poisson | | | | | | |
| micro- or macroconstituents | | | | | | |
| random arrays | | | | | | |
| coarse-mesh lattice | | | | | | |
| void content | | | | | | |
| lattice points | | | | | | |
| fractional number | | | | | | |
| Chi Square | | | | | | |
| confidence interval | | | | | | |
| precision | | | | | | |
| degrees of freedom | | | | | | |
| Delesse analysis | | | | | | |
| Rosiwal analysis | | | | | | |