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Optimal Distribution of Passive Sensors for Underwater Detection

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Optimal Distribution of Passive Sensors for Underwater Detection

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Abstract: This report considers an operations research problem concerned with the distribution of passive detection devices for submarine detection over an operational area. The number and quality of sensors are parameters to be chosen to minimize a certain cost function subject to the constraint that the total probability of detection over the area is greater than a prescribed value. General optimization techniques are employed to solve this mathematical programming problem. It is found that the optimal solution can be easily obtained graphically without complicated computations.

This report represents a preliminary study of the overall problem involved. However, it is also intended to provide a framework for further research and analysis of this naval decision-and-planning problem.

INTRODUCTION

In the planning and designing stage of an antisubmarine warfare (ASW) project, an important problem which is frequently encountered is to select underwater sensors in the most economical way to achieve detection for a certain operational area. This report attacks this problem by using the techniques of mathematical analysis. A simple statistical detection model is adopted here. Based on this model, a mathematical programming problem is formulated and solved analytically and graphically. It is believed that this simple model can be extended to cover the realistic situation. This report is not complete in its coverage of the problem treated. Rather, it is intended to provide a framework for further analysis.

In the development of the methods to be described herein, the detection system is assumed to consist of passive acoustic detectors scattered over large ocean areas through which a submarine target is passing. All these detectors are assumed to be identical and are operated independently. For simplicity the detectors are assumed to be randomly distributed. However, this assumption is not as restrictive as it appears. Specifically, it has been shown [1] that a random, uniform distribution of detector locations in the field provides essentially the same statistical performance as a regularly spaced distribution.

This report depends on an earlier technical report [1] of Arthur D. Little, Inc. However, to make the present work self-contained, a brief outline of certain background in probability detection laws is contained in the second section. The main results are discussed in the third section, where the optimization problem is formulated and solved by applying the general principle of mathematical programming.

BACKGROUND

In this section the detection probability is briefly reviewed for a noise-emitting target (or submarine) in an exposure to be detected by a single *passive listening device.* Then the detection probability is discussed

NRL Problem No. B01-10; Project No. RR 003-02-41-6152. This is a final report on one phase of the problem; work is continuing on other phases. Manuscript submitted December 31, 1970.

for a group of identical but independently operated listening devices in a randomly distributed field. "Detection" means a "yes" or "no" answer to the question of whether the device has detected a given target. A "no" answer means no detection when the target is actually presented. A "yes" answer in the absence of a target will not be considered (i.e., false alarms will be neglected).

Detection Probability Laws

The detectors are assumed to be omnidirectional. For a certain sensor the probability of detection is taken to be a function of distance only. Two types of detection probability laws, the exponential detection law and normal detection law, are considered and are described below.

Exponential detection law. The probability of detecting a target at a distance R from the detector is given by

$$P(R) = e^{-R^2/C^2}, \tag{1}$$

where C is a parameter which represents the sensitivity of the detector. If R_0 is the range where the detection probability is 0.5, then

$$C^2 = \frac{R_0^2}{\ln 2}. \tag{2}$$

Note that R_0 can be used to indicate the quality (sensitivity) of the detector.

Normal detection law. The detection probability is derived from the well-known sonar equation assumpting a Gaussian distribution. A detailed derivation is given in [1]. The probability of detection at a distance R is given by

$$P(R) = \Phi\left(-\frac{10n}{\sigma}\log\frac{R}{R_0}\right), \tag{3}$$

where

$$\Phi(\gamma) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\gamma} e^{-y^2/2}\, dy \tag{4}$$

and n = the spreading constant; for example n = 1 gives cylindrical spreading and n = 2 gives spherical spreading. σ is a parameter representing the variance of the difference between the sound level of the target and the detection level of the sensor.

Note that the exponential law characterizes a rather sharply decreasing detection probability with increasing range than the normal detection probability law. However, the exponential detection law is a handy approximation because of its mathematical convenience. In some cases of interest it appears to be a reasonably good approximation.

Coverage Area

When a single detector is located in the interior of a large plane area Z of interest and a target is exposed at a random point in this area, the total detection probability, i.e., the average detection probability over the area Z, can be expressed in polar coordinates as

$$D \triangleq \iint_Z P(R,\theta)\frac{R}{Z}\, d\theta dR \ , \tag{5}$$

where $P(R,\theta)$ is the detection probability at (R,θ). If we assume that $P(R,\theta)$ is negligible outside of Z, then

$$D = \frac{\int_0^\infty \int_0^{2\pi} P(R,\theta)R\, d\theta dR}{Z} \triangleq \frac{A}{Z} \ , \tag{6}$$

where

$$A \triangleq \int_0^\infty \int_0^{2\pi} P(R,\theta)R\, d\theta dR \ .$$

A is called the *coverage area.*

Applying the exponential detection law,

$$A = 2\pi \int_0^\infty P(R)R\, dR$$

$$= \frac{\pi R_0^2}{\ln 2} \ . \tag{7}$$

Applying the normal detection law,

$$A = \sqrt{2\pi} \int_0^\infty \int_{-\infty}^{-(10n/\sigma)\log(R/R_0)} e^{-y^2/2}\, dydR$$

After evaluating the integration

$$A = \pi R_0^2\, e^{2a^2} \ ; a = \frac{\sigma \ln 10}{10n} \ . \tag{8}$$

As can be seen in Eqs. (7) and (8), the coverage area for both detection laws is proportional to R_0^2. For the convenience of later development, define

$$A = \beta R_0^2 \ , \tag{9}$$

where

$$\beta = \begin{cases} \dfrac{\pi}{\ln 2}, & \text{if the exponential law is applied ;} \\[2ex] \pi e^{2a^2}, & \text{if the normal detection law is applied.} \end{cases} \tag{10}$$

Random Field

The total detection probability over a large area Z for a single detector is given by A/Z. The detection problem that involves a whole field of detectors scattered over the area Z will now be considered. Since the mathematical analysis of a regular, evenly spaced field is much more complicated than that of randomly distributed detectors, consider only the case where m identical and independently operated detectors are scattered randomly (uniformly distributed) in Z. Fortunately, it has been shown [1] that these two distributions behave very similarly and can be used interchangeably for fields of the same density. Assuming that detectors are operated independently, the probability that exactly k detectors will detect the presence of a target is

$$P_k = \binom{m}{k}\left(\frac{A}{Z}\right)^k \left(1 - \frac{A}{Z}\right)^{m-k}, \tag{11}$$

A field of passive sensors may b
1. Randomly distributed
2. Operated independently
But not statistical independe

and the expected number of detections is

$$\mu = m\frac{A}{Z} \triangleq A\rho,$$

where $\rho \triangleq m/Z$ represents the density (detectors per unit area).

If m is large and A/Z is small, as is usually the case of interest here, the binominal distribution can be approximated by a Poisson distribution

$$P_k = \frac{\mu^k e^{-\mu}}{k!} \tag{12}$$

Therefore, the probability of at least one detection is

$$P = 1 - P_0 = 1 - e^{-\mu},$$

or

$$P = 1 - e^{-A\rho}, \tag{13}$$

where A is given by Eq. (9).

Now the optimization problem can be formulated. Equations (9) and (13) are key equations of this section.

OPTIMIZATION PROBLEM

The problem of interest is to find the amount and quality of identical detectors for submarine detection in the area Z such that the total cost of detectors is minimized yet the probability of detection P (at least one detection) is greater than or equal to a prescribed number P^s, for instance $P^s = 0.9$. The quality of detectors is represented by the parameter R_0, which is the range where the detection probability of a single detector is 0.5. Intuitively, the larger R_0 is, the more sensitive and expensive the detector is. It is reasonable to assume that the cost $v(R_0)$ of a single sensor is a smooth and monotonically increasing function of R_0. The solution for a general $v(R_0)$ is obtained by applying the techniques of nonlinear programming. For ease of understanding, consider first the following three cases (refer to Fig. 1), where for the convenience of later development v is a function of R_0^2.

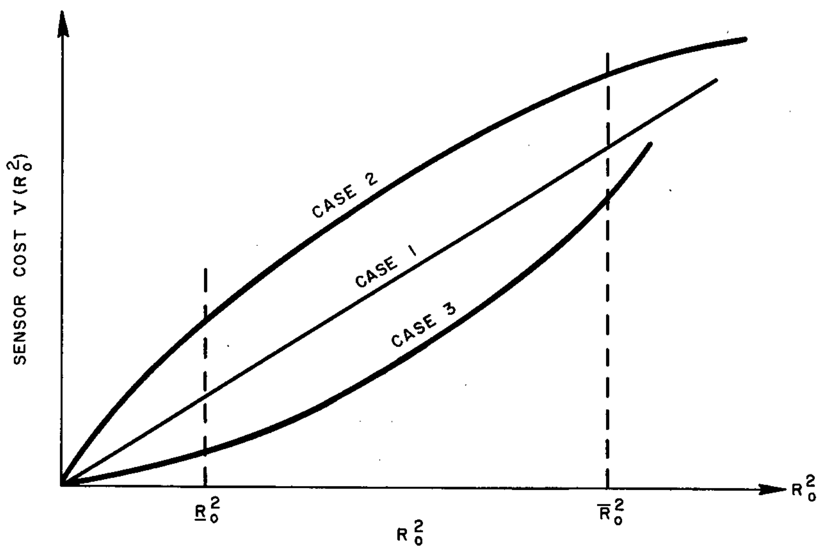

Fig. 1—Sensor cost v as a function of R_0^2 for three cases:
Case 1: v is linear in R_0^2
Case 2: v is a convex function of R_0^2
Case 3: v is a concave function of R_0^2

In Fig. 1 $\underline{R}_0^2$ and $\overline{R}_0^2$ denote the lower and upper bounds of R_0^2. The lower bound $\overline{R}_0^2$ stems from the fact that a detector must perform the operations of detection and must observe the detection probability law. The upper bound $\overline{R}_0^2$ is imposed by the state of the art of electronic technology and environmentally constrained by the target's sound level and background noise level. First these three cases will be considered to gain insight into the problem, and then a general solution will be given. It should be noted that Case 3 is more realistic than the other two.

Formulation of the Problem

The optimization problem is to choose R_0 and m (or ρ) to minimize

$$J = mv(R_0^2) = Z\rho v(R_0^2) \tag{14}$$

subject to the inequality constraints:

$$P = 1 - e^{-A\rho} \geqslant P^s \tag{15}$$

and

$$\underline{R}_0^2 \leqslant R_0^2 \leqslant \overline{R}_0^2. \tag{16}$$

Among the general techniques for solving the optimization problem with inequality constraints are the Lagrange-multiplier method and the penalty-function method [2]. This specific problem, however, can be solved graphically for clarity.

Solutions for Three Special Cases

Case 1: v is linear in R_0^2.

Let $v(R_0^2) = \gamma R_0^2$, where γ is a positive constant. Then

$$J = \gamma Z \rho R_0^2, \tag{17}$$

and using Eq. (9), the constraints are converted into

$$P = 1 - e^{-\beta R_0^2 \rho} \geqslant P^s,$$

$$R_0^2 \rho \geqslant \frac{1}{\beta} \ln\left(\frac{1}{1 - P^s}\right) \tag{18}$$

and

$$\underline{R}_0^2 \leqslant R_0^2 \leqslant \overline{R}_0^2. \tag{19}$$

The optimal solution $(\rho^*, (R_0^2)^*)$ must be located in the admissible region defined by Eqs. (18) and (19) in the (ρ, R_0^2) plane (see Fig. 2).

The contours of constant J (refer to Eq. (17)) are also plotted in Fig. 2. It is seen that J is decreasing as the contour moves toward the boundary of $P = P^s$. Hence, the optimal solution is anywhere on the boundary of $P = P^s$, for R_0^2 between $\underline{R}_0^2$ and $\overline{R}_0^2$. In other words

$$(R_0^2)^* \rho^* = \frac{1}{\beta} \ln \frac{1}{1 - P^s}.$$

The optimal cost is given by

$$J^* = \frac{\gamma Z}{\beta} \ln \frac{1}{1 - P^s}.$$

Case 2: $v(R_0^2)$ is a convex function of R_0^2.

For simplicity consider $v(R_0^2) = \gamma\sqrt{R_0^2}$. Equations (18) and (19) are still applicable to this simple case, but the cost function becomes

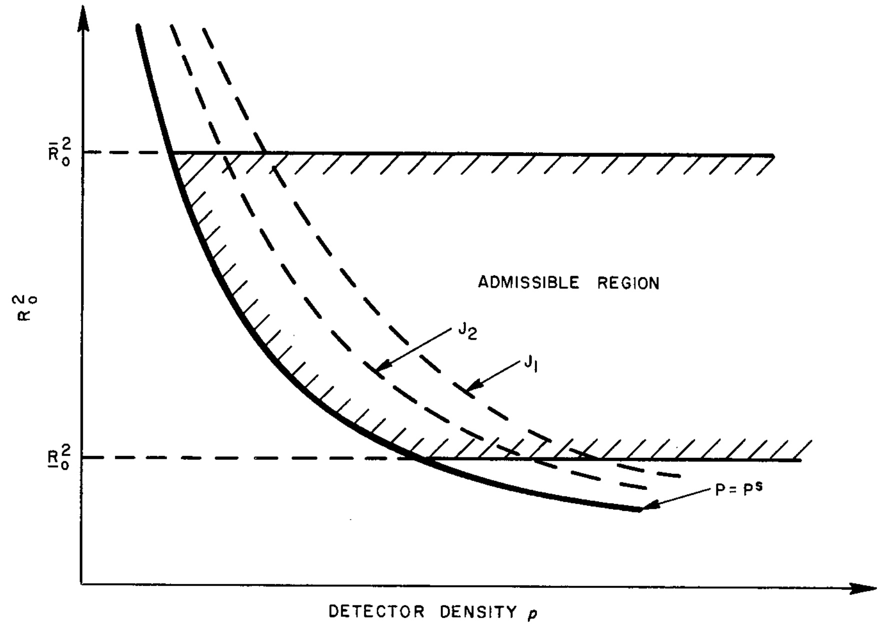

Fig. 2—Admissible region and contours of constant cost J ($J_1 > J_2$) for v linear in R_0^2 (Case 1)

$$J = \gamma Z\rho\sqrt{R_0^2}.$$

The contours of constant J are easily plotted together with the admissible region on the (ρ, R_0^2) plane in Fig. 3. It is directly seen that the optimal solution occurs at the upper corner, i.e., the intersection of $P = P^s$ and $R_0^2 = \bar{R}_0^2$. This implies that the optimal solution is to use high-quality detectors at low density, or

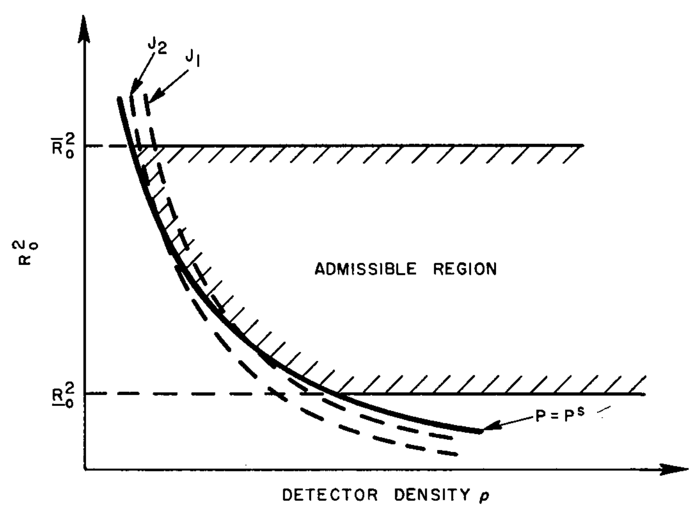

Fig. 3—Admissible region and contours of constant cost J ($J_1 > J_2$) when v is a convex function of R_0^2 (Case 2)

$$R_0^{2*} = \bar{R}_0^2,$$

$$\rho^* = \frac{1}{\beta \bar{R}_0^2} \ln\left(\frac{1}{1 - P^s}\right),$$

and

$$J^* = \frac{\gamma Z}{\beta \bar{R}_0} \ln\left(\frac{1}{1 - P^s}\right).$$

Case 3: $v(R_0^2)$ is a concave function of R_0^2.

As in Case 2, consider a simple concave and monotonically increasing function of R_0^2: $v(R_0^2) = \gamma R_0^4$. The solution is shown in Fig. 4 following the same argument as in Case 2. The optimal solution is at the lower corner, i.e., the intersection of $P = P^s$ and $R_0^2 = \underline{R}_0^2$. This means that the optimal solution is to use low-quality sensors at high density, or

$$R_0^{2*} = \underline{R}_0^2,$$

$$\rho^* = \frac{1}{\beta \underline{R}_0^2} \ln\left(\frac{1}{1 - P^s}\right),$$

and

$$J^* = \frac{\gamma Z \underline{R}_0^2}{\beta} \ln\left(\frac{1}{1 - P^s}\right).$$

Solutions for the General Case

The preceding three cases show that the optimal solution depends on $v(R_0^2)$. It is proved in the appendix for a general case that the optimal solution can be obtained directly from the v vs R_0^2 curve. The following results are derived in the appendix under the reasonable assumption that $v(R_0^2)$ is a positive, smooth, and monotonically increasing function of R_0^2.

(a) The locally optimal solution occurs at the upper corner, that is,

$$R_0^{2*} = \bar{R}_0^2, \rho^* = \frac{1}{\beta \bar{R}_0^2} \ln\left(\frac{1}{1 - P^s}\right). \tag{20}$$

if and only if

$$\frac{v(\bar{R}_0^2)}{\bar{R}_0^2} \geqslant \frac{dv(\bar{R}_0^2)}{d\bar{R}_0^2}. \tag{21}$$

(b) The locally optimal solution is at the lower corner, that is,

$$R_0^{2*} = \underline{R}_0^2, \rho^* = \frac{1}{\beta \underline{R}_0^2} \ln\left(\frac{1}{1 - P^s}\right) \tag{22}$$

if and only if

$$\frac{V(\underline{R}_0^2)}{\underline{R}_0^2} \leqslant \frac{dv(\underline{R}_0^2)}{d\underline{R}_0^2}. \tag{23}$$

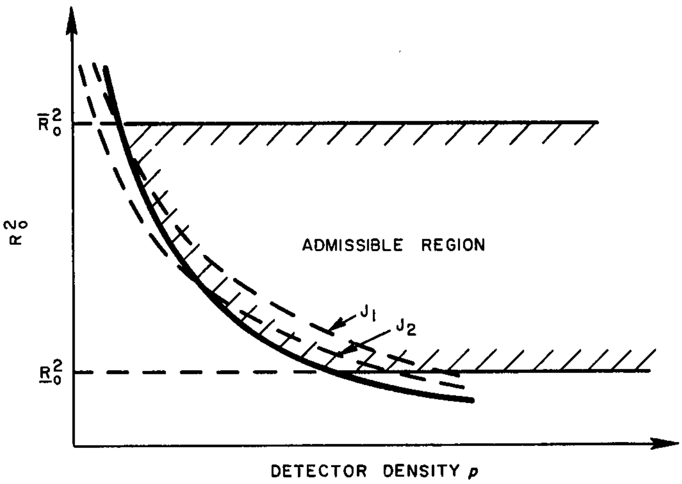

Fig. 4—Admissible region and contours of constant cost J ($J_1 > J_2$) when v is a concave function of R_0^2 (Case 3)

(c) The locally optimal solution is on the $P = P_s^s$ boundary, that is,

$$(R_0^2)^* = (R_0^2)', \rho^* = \frac{1}{\beta(R_0^2)'} \ln\left(\frac{1}{1 - P^s}\right) \tag{24}$$

if and only if

$$\frac{v((R_0^2)')}{(R_0^2)'} = \frac{dv((R_0^2)')}{d(R_0^2)'}, \tag{25}$$

and

$$\frac{d^2\nu((R_0^2)')}{d(R_0^2)^2} > 0, \tag{26}$$

where $\underline{R}_0^2 < (R_0^2)' < \overline{R}_0^2$.

The three criteria can be easily applied graphically as illustrated in the following examples.

Example 1. Given a $v(R_0^2)$ vs R_0^2 curve as in Fig. 5,

it is easily seen that

$$\tan\alpha = \frac{dv(\overline{R}_0^2)}{d\overline{R}_0^2}$$

and

$$\tan\theta = \frac{v(\overline{R}_0^2)}{\overline{R}_0^2}$$

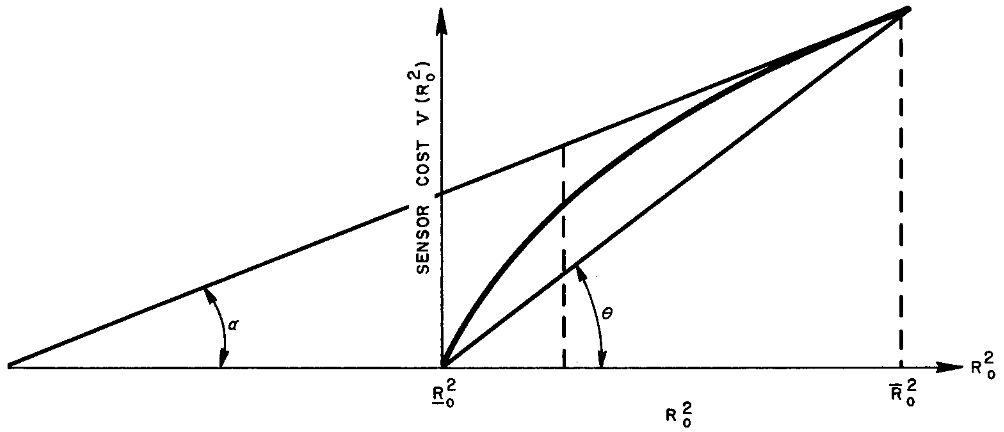

Fig. 5–Example 1

Since $\theta > \alpha$, condition (21) is satisfied. The optimal solution is at the upper corner. But conditions (23) and (25) are not satisfied.

Example 2. For the curve given in Fig. 6,

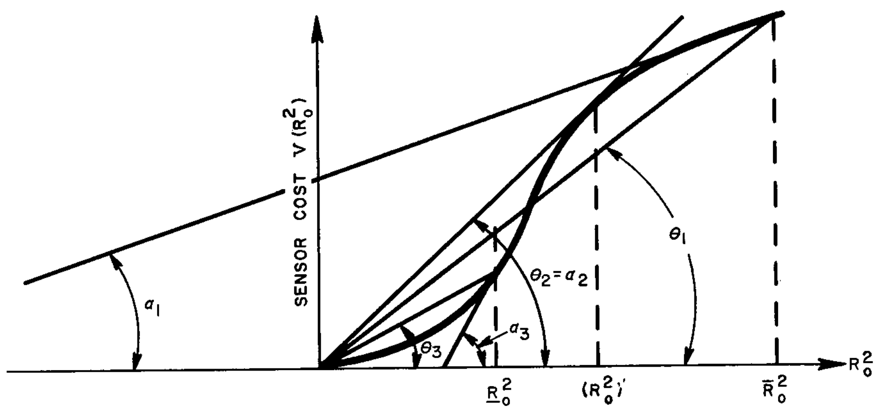

Fig. 6–Example 2

conditions, (21), and (23), are satisfied, since $\theta_1 > \alpha_1$, and $\theta_3 < \alpha_3$. However $(R_0^2)'$ is not a local minimum solution, since Equation (26) is not satisfied at $(R_0^2)'$. This means that there are two locally optimal points. The absolute optimal solution is found by comparing the optimal cost at these two locally optimal points.

Example 3. For a certain ocean operational area the following cost curve is obtained. The cost per sensor includes purchase cost, installation cost and maintenance cost for a period of several years. The cost curve is shown in Figure 7.

Applying the three criteria the minimum solution, $(R_0^2)^* = 13$, is easily found. The number of sensors to be distributed can be obtained from Equation (24).

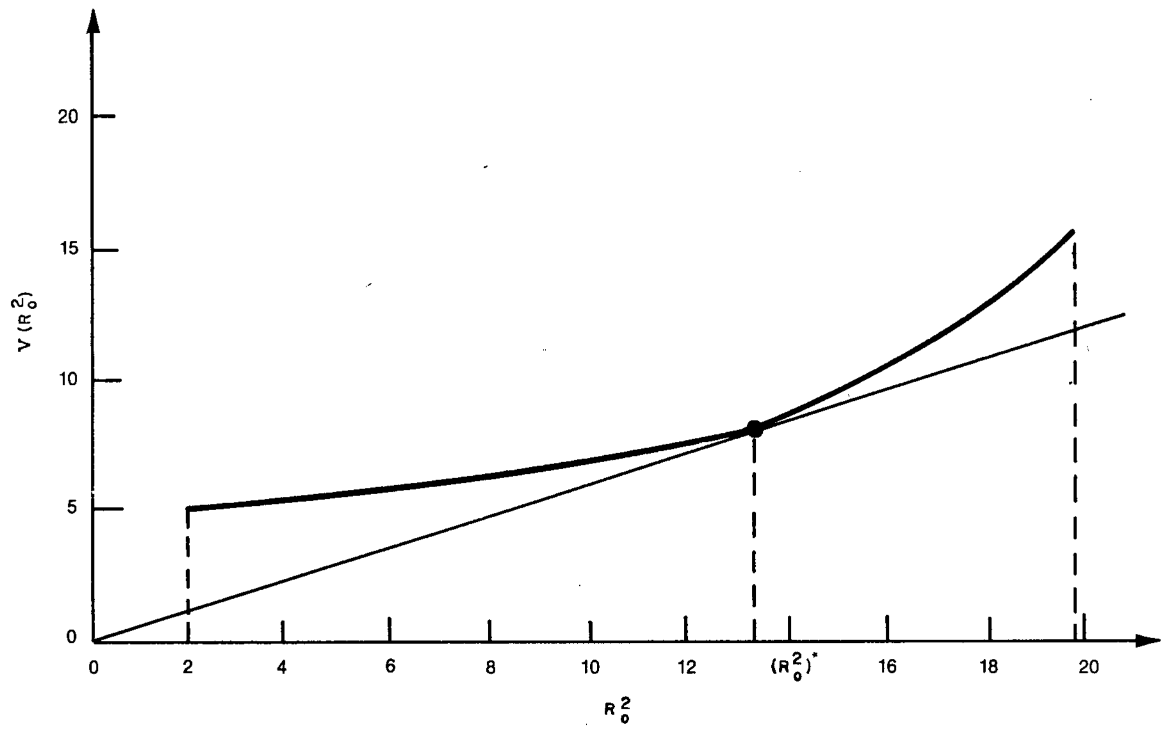

Fig. 7—Example 3

DISCUSSION

Generally the cost function J is a nonlinear function of ρ and R_0, say $f(\rho,R_0)$, instead of being linear in ρ. This occurs when some of the important factors, such as installation cost, reliability, and cost of processing false alarms are taken into consideration. No general technique is available to reach a solution in closed form. A numerical solution can be obtained by using techniques such as the steepest-descent method and the conjugate-gradient method [3].

In the submarine tracking problem, it is preferable to distribute lower quality detectors at a higher density than to distribute higher quality detectors at a lower density. The reason is that the target position is estimated by the first moment of the positions of those detectors which simultaneously detect the presence of the submarine. If low-quality detectors are distributed at high density, only those detectors which are close to the target make detections. Hence, a better estimation of target position will result. Therefore, in realistic situation, the tracking ability should be considered as another constraint in the problem of optimization.

Although we have assumed that detectors are distributed randomly in the area Z, practical installation can be made by arranging the detectors in a square grid (i.e., regular field). It has been mentioned that computer simulation results prove that the statistical performances of these two fields of distribution show only modest differences.

The techniques used in this section can be easily extended to the case of a target moving through the operational area.

APPENDIX
LOCAL OPTIMAL SOLUTION FOR THE GENERAL CASE

This appendix derives the three criteria for locating the optimal solution in the (ρ, R_0^2) plane. For clarity, the problem is restated:

Minimize

$$J = Z\rho v(R_0^2) \tag{A1}$$

with

$$R_0^2\rho \geqslant \frac{1}{\beta}\ \ln\left(\frac{1}{1-P^s}\right) \tag{A2}$$

and

$$\underline{R}_0^2 \leqslant R_0^2 \leqslant \overline{R}_0^2\ , \tag{A3}$$

where $v(R_0^2)$ is a positive, smooth, and monotonically increasing function of R_0^2.

THEOREM. (a) *The upper corner is a local minimum solution or*

$$\left(\rho^*, (R_0^2)^*\right) = \left(\frac{1}{\beta\overline{R}_0^2}\ \ln\left(\frac{1}{1-P^s}\right), \overline{R}_0^2\right)$$

if and only if

$$\frac{v(\overline{R}_0^2)}{\overline{R}_0^2} \geqslant \frac{dv(\overline{R}_0^2)}{d\overline{R}_0^2}\ . \tag{A4}$$

(b) *The lower corner is a local minimum solution or*

$$\left(\rho^*, (R_0^2)^*\right) = \left(\frac{1}{\beta\underline{R}_0^2}\ \ln\left(\frac{1}{1-P^s}\right), \underline{R}_0^2\right)$$

if and only if

$$\frac{v(\underline{R}_0^2)}{\underline{R}_0^2} \leqslant \frac{dv(\underline{R}_0^2)}{d\underline{R}_0^2}\ . \tag{A5}$$

(c) *The local minimum solution is on the $P = P^s$ boundary, or*

$$\left(\rho^*, (R_0^2)^*\right) = \left(\frac{1}{\beta(R_0^2)'}\ \ln\left(\frac{1}{1-P^s}\right), (R_0^2)'\right)$$

if and only if

$$\frac{v((R_0^2)')}{(R_0^2)'} = \frac{dv((R_0^2)')}{d(R_0^2)'} \tag{A6}$$

and

$$\frac{d^2\nu\,((R_0^2)')}{d(R_0^2)^2} > 0 \cdot$$

Proof. A point on the boundary of the admissible region is a local optimal solution if and only if the gradient of J is oriented such that a decrease in J can occur only by violating the constraints. The proofs of parts (a), (b), and (c) follow directly from this argument. Refer to Fig. A1.

For (a) the upper corner is the local optimal solution if and only if ∇J points toward the direction of Φ. However, since $v(R_0^2)$ is positive and monotonically increasing, ∇J must point toward the first quadrant, or $\varphi \leqslant \omega$, where

$$\tan\varphi = \frac{\dfrac{\partial J}{\partial \bar{R}_0^2}}{\dfrac{\partial J}{\partial \rho}}$$

and

$$\tan\omega = \frac{\rho}{\bar{R}_0^2}\ .$$

Thus

$$\varphi \leqslant \omega \Longleftrightarrow \tan\varphi \leqslant \tan\omega$$

$$\Longleftrightarrow \frac{\rho\,\dfrac{dv(\bar{R}_0^2)}{d\bar{R}_0^2}}{v(\bar{R}_0^2)} \leqslant \frac{\rho}{\bar{R}_0^2}$$

$\Longleftrightarrow$ Equation (A4).

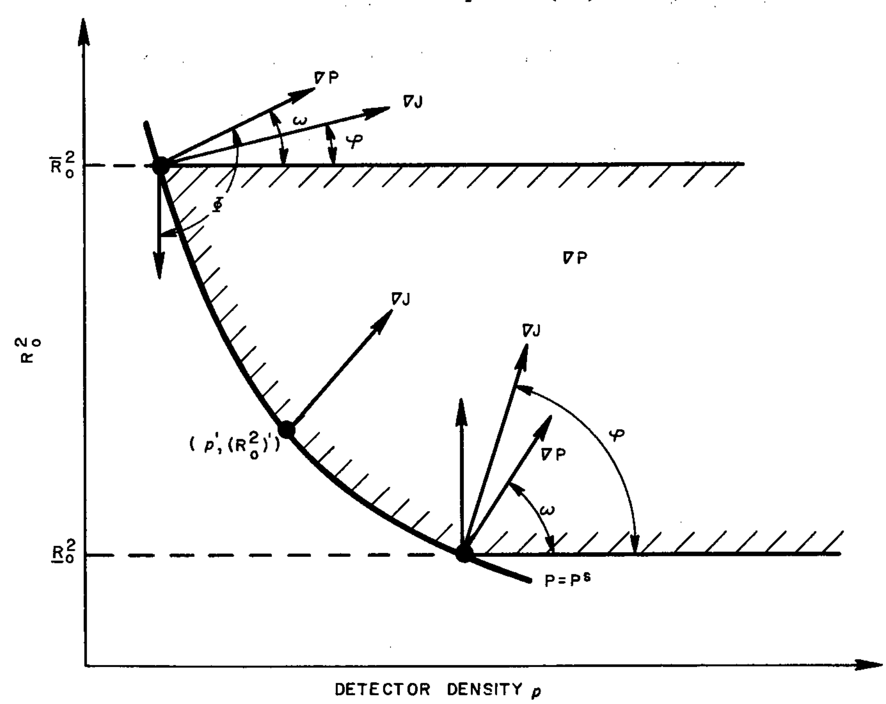

Fig. A1–Graphical proof of general solution

For (b) the proof is analogous to that of (a). From Fig. A1 it immediately follows that

$$\tan \varphi \geqslant \tan \omega$$

or

$$\frac{v(\underline{R}_0^2)}{\underline{R}_0^2} \leqslant \frac{dv(\underline{R}_0^2)}{d\underline{R}_0^2}$$

For (c), $(\rho', (R_0^2)')$ is the local optimal solution if and only if the gradient of J and the gradient of P point in the same direction, and $(\rho', (R_0^2)')$ is not a local maximum solution, that is

$$\frac{v((R_0^2)')}{(R_0^2)'} = \frac{dv((R_0^2)')}{d(R_0^2)'}$$

and

$$\frac{d^2\nu((R_0^2)')}{d(R_0^2)^2} > 0.$$

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This report considers an operations research problem concerned with the distribution of passive detection devices for submarine detection over an operational area. The number and quality of sensors are parameters to be chosen to minimize a certain cost function subject to the constraint that the total probability of detection over the area is greater than a prescribed value. General optimization techniques are employed to solve this mathematical programming problem.

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