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LEONARD WEISS AND JONG-SEN LEE

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LEONARD WEISS[1] AND JONG-SEN LEE[2]

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Abstract: A unified approach to finite-time stability of linear discrete-time systems is developed in this paper. Some results from linear algebra are used in deriving new and computationally feasible finite-time stability criteria. Connections are made to the Lyapunov-like nonlinear theory of finite-time stability. Corresponding results are also derived for stochastic linear discrete-time systems.

INTRODUCTION

Difference equations arise and are of utmost importance in the fields of, for example, numerical analysis and sample-data control systems. Stability is of particular interest, and certain classical notions of it have been extensively studied (see [1]-[5] and in particular the comprehensive survey paper by Jury and Tsypkin [6]).

In this paper we consider the concept of finite-time stability of linear discrete-time systems, and we develop a theory which parallels, to some extent, that given in a separate paper on differential equations [7]. Our main objective is to obtain computationally manageable finite-time stability criteria.

We consider finite-time stability of force-free deterministic systems, as well as of linear systems driven by white noise. In addition a complete connection is given between the Lyapunov-like theory of finite-time stability for nonlinear discrete-time systems and the linear theory developed in the sequel. The results should be of interest to workers interested in stability theory per se as well as numerical analysts interested in computation of error bounds.

NOTATION AND DEFINITIONS

The symbol $\|\cdot\|$ denotes the Euclidian norm on R^n; $\|A\|^*$ is the spectral norm of an $n \times n$ matrix A; $\{\lambda(A)\}$ is the set of eigenvalues of A; and if the latter are real, $\hat{\lambda} = \max\{\lambda(A)\}$. The transpose of A is A'.

Consider the system of linear equations

$$x(k+1) = A(k)x(k), \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $A(k)$ is an $n \times n$, real matrix. A solution of (1) is uniquely generated by recursion from a given initial condition $x(0)$, and at the ℓth instant, this solution is denoted by

$$x(\ell;0,x(0)) \triangleq x(\ell) = \Phi(\ell,0)x(0), \tag{2}$$

[1]Department of Electrical Engineering and Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland, and Mathematics Research Center, Naval Research Laboratory, Washington, D.C. Dr. Weiss' research was supported in part by the U.S. Air Force Office of Scientific Research under Grant AFOSR 69-1646.

[2]Operations Research Group, Naval Research Laboratory, Washington, D.C.

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where

$$\Phi(\ell,0)=A(\ell-1)A(\ell-2)\cdots A(0)\triangleq\prod_{k=0}^{\ell-1}A(k),\ \ \ell\geq 1 \tag{3}$$

and

$$\Phi(0,0)=\mathrm{I}\ (\text{the } n\times n \text{ identity matrix}).$$

Definition 1. The system (1) is stable with respect to (α,β,N), $\alpha\leq\beta$, if $\|x(0)\|<\alpha$ implies $\|x(\ell)\|<\beta$ for all $\ell\in\{0,1,\ldots,N\}$.

Definition 2. The system (1) is stable if for every $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that $\|x(\ell)\|<\epsilon$ for all integers $\ell\geq 0$, provided $\|x(0)\|<\delta$.

CRITERIA FOR FINITE-TIME STABILITY

THEOREM 1. *The system (1) is stable (Definition 1) if and only if*

$$\|\Phi(k,0)\|^{*}\leq\frac{\beta}{\alpha},\ k=1,\ldots,N. \tag{4}$$

Proof. From (2)

$$\|x(\ell)\|=\|\Phi(\ell,0)x(0)\|\leq\|\Phi(\ell,0)\|^{*}\,\|x(0)\|$$

with equality for some $x(0)$ (independent of $\|x(0)\|$). Hence, if $\|x(0)\|<\alpha$, then a simple calculation shows that (4) is necessary and sufficient for $\|x(\ell)\|<\beta$ for all $\ell\in\{0,1,\ldots,N\}$. ∎

COROLLARY 1. *Let A in (1) be a constant matrix. Then (1) is stable (Definition 1) if and only if*

$$\|A^{k}\|^{*}\leq\frac{\beta}{\alpha},\ k=1,\ldots,N \tag{5}$$

Consider the following lemma.

LEMMA 1. *For any $n\times n$ matrix A and any positive integer k, $\|A^{k}\|^{*}\leq(\|A\|^{*})^{k}$, with equality if and only if A is normal.*

Proof. The inequality follows directly from a well-known property of the product of bounded, linear operators [8]. To prove the statement regarding equality, write

$$\begin{aligned}\|A^{k}\|^{*}&=\hat{\lambda}^{1/2}\left((A')^{k}(A)^{k}\right)\\&=\hat{\lambda}^{1/2}\left((A'A)^{k}\right)\ \text{if and only if } A \text{ is normal.}\end{aligned}$$

But by the spectral theorem for symmetric matrices [8] it follows that

$$\hat{\lambda}^{1/2}\left((A'A)^{k}\right)=\hat{\lambda}^{k/2}(A'A)\triangleq(\|A\|^{*})^{k}.\ ∎$$

The following results are immediate from Corollary 1 and Lemma 1.

THEOREM 2. *If A in (1) is a constant matrix, then a sufficient condition for stability (Definition 1) of the system (1) is*

$$\|A\|^* \leq \left(\frac{\beta}{\alpha}\right)^{1/N}. \tag{6}$$

THEOREM 3. *If A in (1) is constant and normal, then (6) is a necessary as well as sufficient condition for stability (Definition 1) of the system (1).*

A general necessary condition is given by the next theorem.

THEOREM 4. The system (1) is stable (Definition 1) only if

$$\max\{|\lambda(\Phi(k,0))|\} \leq \frac{\beta}{\alpha},\ k=1,\ldots,N. \tag{7}$$

Proof. The spectral radius and spectral norm of $\Phi(k,0)$ are related by

$$\max\{|\lambda(\Phi(k,0))|\} \leq \|\Phi(k,0)\|^*.$$

Applying Theorem 1 completes the proof. ■

From the fact that $\lambda \in \{\lambda(A)\}$ implies $\lambda^N \in \{\lambda(A^N)\}$, we obtain the following result.

COROLLARY 2. *If (1) is time-invariant, then a necessary condition for stability (Definition 1) is*

$$\max\{|\lambda(A)|\} \leq \left(\frac{\beta}{\alpha}\right)^{1/N}. \tag{8}$$

The above Corollary allows a novel proof of the following classical result.

THEOREM 5. *If the system (1) is stable (Definition 2) and time-invariant, then no eigenvalue of A lies outside the unit circle in the complex plane.*

Proof. If (1) is stable, then for every $\epsilon > 0$, there exists $\delta > 0$ such that (1) is stable with respect to (δ,ϵ,k) for every integer $k \geq 1$. By Corollary 2, $\max\{|\lambda(A)|\} \leq (\epsilon/\delta)^{1/k}$ for integers $k \geq 1$. Then, for some $v \geq 0$, $\max\{|\lambda(A)|\} \leq (1+v)^{1/k}$ for all integers $k \geq 1$, which, on taking the limit, implies $\max\{|\lambda(A)|\} \leq 1$.

FINITE-TIME STABILITY VIA LYAPUNOV-LIKE FUNCTIONS

Our objective is to connect the qualitative approach to finite-time stability of nonlinear discrete-time systems developed in [9] (see also [10]) to the linear theory developed here.

Let $S_N = \{0,1,\ldots,N\}$, and let Δ denote the backward-difference operator, i.e, $\Delta g(k) = g(k+1) - g(k)$. The theorem below, which holds also for nonlinear difference equations, is proved in [9].

THEOREM 6. *The system (1) is stable (Definition 1) if and only if there exist a real-valued function $V(x,k)$, defined for all $k \in S_N$, and a real-valued function $\varphi(k)$, defined on S_{N-1}, such that*

$$\Delta V(x,k)\,|_{(1)} \leq \varphi(k) \text{ for all } k \in S_{N-1}, \text{ all } \|x\| \leq \beta, \tag{9}$$

where $\Delta V|_{(1)}$ denotes the backward difference of V along the trajectories of (1), and

$$\sum_{\substack{j \in S_{N-1} \\ j<k}} \varphi(j) < \left[\min_{\|x\| \geq \beta} V(x,k)\right] - \left[\max_{\|x\| \leq \alpha-\epsilon} V(x,0)\right] \tag{10}$$

for all $k \in S_N$, $\epsilon > 0$ though arbitrarily small.

To apply Theorem 6 to the system (1), we take V as a function of x alone, i.e.,

$$V(x) = \ln \|x\|.$$

Then

$$\begin{aligned}(\Delta V(x)|_{(1)})\,(k) &= \Delta V(x(k)) = V(x(k+1)) - V(x(k)) \\ &= \ln \frac{\|x(k+1)\|}{\|x(k)\|} = \ln \frac{\|A(k)x(k)\|}{\|x(k)\|} \\ &\leq \ln \|A(k)\|^* .\end{aligned}$$

Let $\varphi(k) = \ln \|A(k)\|^*$. Then

$$\begin{aligned}\sum_{\substack{j \in S_{N-1} \\ j<k}} \varphi(j) &= \ln\,(\|A(k-1)\|^*\,\|A(k-2)\|^* \ldots \|A(0)\|^*) \\ &\geq \ln \|\Phi(k,0)\|^* \text{ for all } k \in S_N.\end{aligned} \tag{11}$$

Hence, from (11) and condition (10) of Theorem 6 we get

$$\ln \|\Phi(k,0)\|^* \leq \sum_{\substack{j \in S_{N-1} \\ j<k}} \varphi(j) < \ln \frac{\beta}{\alpha-\epsilon}$$

for all $k \in S_N$, $\epsilon > 0$ though arbitrarily small. Letting $\epsilon \to 0$, we obtain as a sufficient condition for stability (Definition 1) of the system (1),

$$\|\Phi(k,0)\|^* \leq \frac{\beta}{\alpha} \text{ for all } k \in \{0,1,\ldots,N\},$$

which is the sufficiency part of Theorem 1.

STABILITY UNDER WHITE-NOISE PERTURBATION

Consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k),\ k=0,1,2,\ldots, \tag{12}$$

where A is $n \times n$, B is $n \times m$, and $u(\cdot)$ represents a vector white-noise sequence with zero mean and covariance matrix $Q(k)$ (i.e., $E\,\{u(k)\} = 0$, and $E\,\{u(k)u'(j)\} = Q(k)\delta_{kj}$, where δ_{kj} is the kronecker delta and $E =$ Expectation).

For any $n \times n$ matrix G, let $tr(G) =$ trace of G. Then we have the following definition.

Definition 3. The system (12) is mean-square stable with respect to $(\alpha, \beta, \gamma, T)$, $\alpha \leq \beta$, if the conditions $E\{\|x(0)\|^2\} < \alpha^2$ and $tr(Q(k)) \leq \gamma^2$ for all $k \in S_N$ imply $E\{\|x(k)\|^2\} < \beta^2$ for all $k \in S_N$.

The main result in this section depends on the following lemma.

LEMMA 2. *Let F be an $n \times n$ symmetric matrix and let $\mathcal{P}$ denote the set of $n \times n$ nonnegative definite matrices.*

Then

$$\hat{\lambda}(F) = \max_{P \in \mathcal{P}} \frac{tr(PF)}{tr(P)} . \tag{13}$$

Proof. Let S be an $n \times n$ orthogonal matrix such that $S'FS = \Lambda = \text{diag}(\lambda_i)$. Also, let $S'PS = D$.

Then

$$\begin{aligned} tr(PF) &= tr(PS\Lambda S') \\ &= tr(S'PS\Lambda) \\ &= tr(D\Lambda) \\ &= \lambda_1 d_{11} + \lambda_2 d_{22} + \cdots\cdots + \lambda_n d_{nn} \\ &\leq \hat{\lambda}(F)\ (d_{11} + d_{22} + \cdots\cdots + d_{nn}),\ d_{ii} \geq 0 \\ &\leq \hat{\lambda}(F)\ tr\ (D) \end{aligned}$$

Therefore, it is possible to choose $\{d_{ii} \mid i = 1, \ldots, n\}$ such that

$$\hat{\lambda}(F) = \max_{d_{ii}} \frac{tr(D\Lambda)}{tr(D)} = \max_{P \in \mathcal{P}} \frac{tr(PF)}{tr(P)} . \blacksquare$$

THEOREM 7. *The system (12) is mean-square stable (Definition 3) if and only if*

$$\alpha^2 (\|A^n\|^*)^2 + \gamma^2 \sum_{i=0}^{n-1} (\|A^i B\|^*)^2 \leq \beta^2,\ n = 1, \ldots, N \tag{14}$$

Proof. Let $P(k) = E\{x(k)x'(k)\}$. Then $P(k) \in \mathcal{P}$ for each k, and it is easily checked that

$$P(k+1) = A\,P(k)A' + B\,Q\,(k)B',\ k = 0,1,2,\ldots,$$

which has a solution

$$P(n) = A^n P(0)(A^n)' + \sum_{i=0}^{n-1} A^i\,BQ(n-i-1)B'(A^i)',\ n = 1,2,\ldots. \tag{15}$$

Now,

$$tr(P(n)) = tr(P(0)A^n(A^n)') + \sum_{i=0}^{n-1} tr(Q(n-i-1)B'(A^i)'A^iB). \tag{16}$$

But, from Lemma 2

$$\hat{\lambda}(A^n(A^n)') = \max_{P\in\mathcal{P}} \left\{\frac{tr(PA^n(A^n)')}{tr(P)}\right\}. \tag{17}$$

Then (16) and (17) imply

$$tr(P(n)) \leqslant (\|A^n\|^*)^2\, tr(P(0)) + \sum_{i=0}^{n-1} (\|A^iB\|^*)^2\, tr(Q(n-i-1)),\ n = 1,2,\ldots, \tag{18}$$

and the sufficiency of (14) for stability (Definition 3) follows easily from (18) and Definition 3.

To prove necessity we note first that for any fixed value of n (say $n = M \leqslant N$) there exists a $P(0)$ and a sequence $\{Q(k)\,|\,k = 1,\ldots,M\}$ such that equality occurs in (18) for $n = M$, $tr\,P(0) = \alpha^2$, and $tr(Q(k)) = \gamma^2$ for all $k \in S_N$. Suppose (14) does not hold. Then with $P(0)$ and $\{Q(k)\}$ chosen as indicated, (18) yields $tr(P(M)) > \beta^2$, thus negating stability (Definition 3). This proves necessity of (14). ∎

Now let

$$\eta^2 = \frac{\gamma^2\ (\|\beta\|^*)^2}{1-(\|A\|^*)^2}.$$

Then a sufficient condition for stability (Definition 3) is given as follows.

THEOREM 8. *The system (12) is mean-square stable (Definition 3) if*

$$\|A\|^* \leqslant \left(\frac{\beta^2-\eta^2}{\alpha^2-\eta^2}\right)^{1/2N}. \tag{19}$$

Proof. The proof results from a calculation starting with (14) in which we write $\|A^iB\|^* \leqslant \|A^i\|^*\|B\|^*$, then apply Lemma 1, and use the formula

$$\sum_{i=0}^{k-1} \rho^i = \frac{1-\rho^k}{1-\rho}\quad (\rho \text{ real}).\ \blacksquare \tag{20}$$

THEOREM 9. *If $B = I$ in (12) and A is normal, then (19) is necessary and sufficient for the system (12) to be mean-square stable (Definition 3).*

Proof. The proof follows directly from Theorem 7 and Lemma 1. ∎

Finally, we present a necessary condition for stability. Let

$$\bar{\lambda} = \max\{|\lambda(A)|\}$$

and

$$\omega^2 = \frac{\gamma^2}{1-\bar{\lambda}^2}.$$

Then we have the following result.

THEOREM 10. If $B = I$ in (12), then a necessary condition for the system (12) to be mean-square stable (Definition 3) is

$$\bar{\lambda} \leqslant \left(\frac{\beta^2-\omega^2}{\alpha^2-\omega^2}\right)^{1/2N}. \tag{21}$$

Proof. The proof follows from (14), from the fact that $\bar{\lambda} \leqslant \|A\|^*$, and from (18). ∎

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