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Entire Functions of Extreme Rates of Growth

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Entire Functions of Extreme Rates of Growth

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Abstract: It is well known that the rate of growth of an entire transcendental function exceeds that of a polynomial but that, subject to this restriction, there exist such functions of arbitrarily fast and arbitrarily slow rates of growth. In this report, it is shown by construction that there exist entire functions whose upper rates of growth are arbitrarily fast and, simultaneously, whose lower rates of growth are arbitrarily slow, again subject to the above restriction. It is also shown that there exist such functions which have two arbitrarily prescribed rates of growth on different, although unspecified, sequences.

INTRODUCTION

It is well known [1-4] that the rate of growth of an entire transcendental function as measured by its maximum modulus exceeds that of a polynomial but that, subject to this restriction, there exist such functions of arbitrarily fast and of arbitrarily slow rates of growth. F. Gross, in an oral communication, raised the question as to whether or not there exist entire functions whose upper rates of growth are arbitrarily fast and, simultaneously, whose lower rates of growth are arbitrarily slow, again subject to the above necessary restriction. It is the main purpose of this report to give an affirmative answer to this question. It is also shown that there exist such functions which have two arbitrarily prescribed rates of growth on different, although unspecified, sequences.

MAIN THEOREM

Our principal result can be stated as follows, where we denote the maximum modulus as usual by

$$M(r) = \max_{|z| = r} |f(z)|.$$

THEOREM 1. *Let $h(r)$ and $k(r)$ be positive functions of r for $r > 0$ such that* (*) $\log k(r) \neq O(\log r)$ *as $r \to \infty$. Then there exists an entire transcendental function $f(z)$ with nonnegative coefficients and two sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers tending monotonically to infinity such that, for every positive integer n, $M(c_n) > h(c_n)$ and $M(r_n) < k(r_n)$.*

Proof. We first define by induction the sequences $\{c_n\}$ and $\{r_n\}$ and an increasing sequence $\{\lambda_n\}$ of positive integers as follows:

Let $c_1 = 1$. Let λ_1 be any positive integer such that

$$2^{\lambda_1} > h(c_1).$$

Let r_1 be any positive number greater than c_1 such that

$$\left(\frac{2r_1}{c_1}\right)^{\lambda_1} < k(r_1) - 1.$$

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This is possible by inequality (*).

For $n > 1$, let $c_n = 4r_{n-1}$. Let λ_n be any positive integer greater than λ_{n-1} and such that

$$2^{\lambda_n} > h(c_n).$$

Let r_n be any positive number greater than c_n such that

$$\sum_{m=1}^{n} \left(\frac{2r_n}{c_m}\right)^{\lambda_m} < k(r_n) - 1.$$

As above, this is possible by (*).

We now construct the desired function as follows: Let

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{2z}{c_n}\right)^{\lambda_n}.$$

This is clearly an entire function. Since its Taylor coefficients are nonnegative, we have $M(r) = f(r)$. But

$$f(c_n) > \left(\frac{2c_n}{c_n}\right)^{\lambda_n} = 2^{\lambda_n} > h(c_n)$$

and

$$\begin{aligned} f(r_n) &= \sum_{m=1}^{n} \left(\frac{2r_n}{c_m}\right)^{\lambda_m} + \sum_{m=n+1}^{\infty} \left(\frac{2r_n}{c_m}\right)^{\lambda_m} \\ &< k(r_n) - 1 + 2^{-\lambda_{n+1}} + 2^{-\lambda_{n+2}} + \dots \\ &< k(r_n) - 1 + 2^{-1} + 2^{-2} + \dots \\ &= k(r_n), \end{aligned}$$

which completes the proof.

RELATED RESULTS

By means of similar techniques, it is possible to give very simple constructions of entire functions of arbitrarily fast and of arbitrarily slow rates of growth. This may be done as follows:

THEOREM 2. (Poincaré [3, pp. 49-50]; see also Ref. 5, pp. 10-11, and Ref. 6, p. 324.) *Let $h(r)$ be a positive function of r for $r \geqslant 0$ bounded in every finite interval. Then there exists an entire transcendental function $f(z)$ with nonnegative coefficients such that $M(r) > h(r)$ for all r.*

Proof. We may assume that $h(r)$ is nondecreasing. Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that, for every positive integer n, we have

$$\left(\frac{n+1}{n}\right)^{\lambda_n} > h(n+2).$$

Then we define

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$$f(z) = 1 + h(2) + \sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^{\lambda n}.$$

This is an entire function. For $r > 2$, putting $m = [r] - 1$, we have

$$M(r) = f(r) > \left(\frac{r}{m}\right)^{\lambda m} \geqslant \left(\frac{m+1}{m}\right)^{\lambda m}$$

$$> h(m+2) \geqslant h(r),$$

while, for $r \leqslant 2$, $M(r) > h(2) \geqslant h(r)$.

THEOREM 3. *Let $k(r)$ be a positive function having a positive lower bound for $r > 0$ and such that*

$$(**) \quad \frac{\log k(r)}{\log r} \to \infty \text{ as } r \to \infty.$$

Then there exists an entire transcendental function $f(z)$ with nonnegative coefficients such that $M(r) < k(r)$ for all r. (See Ref. 2, pp. 306-7, where $f(z)$ is constructed as a canonical product of unbounded index with negative zeros.)

Proof. Let $\{m_n\}$ be an increasing sequence of positive numbers tending to infinity such that, for every positive integer n, and all $r > 0$, we have

$$m_n^n > \frac{(2r)^n}{k(r)}.$$

This is possible by (**), since $k(r)$ has a positive lower bound. We then define

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{z}{m_n}\right)^n.$$

This is an entire transcendental function with nonnegative coefficients, and therefore

$$M(r) = f(r) = \sum_{n=1}^{\infty} \left(\frac{r}{m_n}\right)^n < \sum_{n=1}^{\infty} \frac{k(r)}{2^n} = k(r),$$

from which the result follows.

PRESCRIBED RATES OF GROWTH

It is possible, under quite general conditions, to replace inequality by equality in Theorem 1. We thus obtain an entire transcendental function whose maximum modulus has two arbitrarily prescribed rates of growth on different, although unspecified, sequences. The exact formulation is as follows:

THEOREM 4. *Let $h(r)$ and $k(r)$ be positive continuous functions of r for $r > 0$ such that* $\log h(r) \neq O(\log r)$ *and* $\log k(r) \neq O(\log r)$ *as $r \to \infty$. Then there exists an entire transcendental function $f(z)$ with nonnegative coefficients such that $M(r) = h(r)$ on some sequence of values of $r \to \infty$, and $M(r) = k(r)$ on another such sequence.*

Proof. Let $p(r) = \min\,[h(r), k(r)]$ and $q(r) = \max\,[h(r), k(r)]$. We first assume that $\log p(r) \neq O(\log r)$ as $r \to \infty$. Then, by Theorem 1, there exists an entire transcendental function $f(z)$ with nonnegative coefficients and two increasing sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers tending to infinity such that, for every positive integer n, $M(c_n) > q(c_n)$ and $M(r_n) < p(r_n)$. Hence $M(c_n) > h(c_n)$, $M(r_n) < h(r_n)$ and $M(c_n) > k(c_n)$, $M(r_n) < k(r_n)$, from which the conclusion follows by continuity.

Suppose now that $\log p(r) = O(\log r)$ as $r \to \infty$. It is easy to find a pair of positive continuous functions of r, $H(r)$ and $K(r)$, bounded away from zero for $r > 0$ such that $\log H(r)/\log r$ and $\log K(r)/\log r$ tend monotonically to infinity as $r \to \infty$ and $H(r) = h(r)$, $K(r) = k(r)$ on sequences of values of $r \to \infty$, say $\{d_n\}$ and $\{s_n\}$ respectively. Let $L(r) = \min\,[H(r), K(r)]$. Then, by Theorem 3, there exists an entire transcendental function $f(z)$ with nonnegative coefficients such that $M(r) < L(r)$ for all $r > 0$. But

$$\lim_{n\to\infty} \frac{\log h(d_n)}{\log d_n} = \lim_{n\to\infty} \frac{\log k(s_n)}{\log s_n} = \infty$$

and

$$\lim_{n\to\infty} \frac{\log p(d_n)}{\log d_n} = \lim_{n\to\infty} \frac{\log p(s_n)}{\log s_n} < \infty;$$

hence,

$$\lim_{n\to\infty} \frac{\log k(d_n)}{\log d_n} = \lim_{n\to\infty} \frac{\log h(s_n)}{\log s_n} < \infty.$$

However,

$$\lim_{n\to\infty} \frac{\log M(d_n)}{\log d_n} = \lim_{n\to\infty} \frac{\log M(s_n)}{\log s_n} = \infty,$$

since $f(z)$ is transcendental; hence, for sufficiently large n, $M(d_n) > k(d_n)$ and $M(s_n) > h(s_n)$. But $M(d_n) < L(d_n) \leq H(d_n) = h(d_n)$ and $M(s_n) < L(s_n) \leq K(s_n) = k(s_n)$, from which the conclusion again follows by continuity, which completes the proof.

An important special case is obtained by equating $h(r)$ and $k(r)$. We then have the following:

COROLLARY: *Let $h(r)$ be a positive continuous function of r for $r > 0$ such that $\log h(r) \neq O(\log r)$ as $r \to \infty$. Then there exists an entire transcendental function $f(z)$ with nonnegative coefficients such that $M(r) = h(r)$ on some sequence of values of $r \to \infty$.*

REMARKS

Theorems 1 through 3 may also be derived with the aid of some fundamental results on asymptotic growth of entire functions obtained recently by Clunie and Kövari [7,8]. It seems preferable, though, to make use of the above direct and elementary constructions. We also note the following interesting result due to Borel and Hardy [See Ref. 5, pp. 28-29], which is closely related to Theorem 4.

THEOREM. *Let*

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n$$

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and

$$\psi(z) = \sum_{n=0}^{\infty} b_n z^n$$

be any two entire functions with positive coefficients. Then there exists an entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where every c_n is equal either to a_n or to b_n, and two increasing sequences $\{x_n\}$ and $\{x_n'\}$ of positive numbers tending to infinity, such that

$$\lim_{n\to\infty} \frac{f(x_n)}{\phi(x_n)} = \lim_{n\to\infty} \frac{f(x_n')}{\psi(x_n')} = 1.$$

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