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# The Equations of Motion of a Satellite in a Local Vertical Coordinate System

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## ABSTRACT

The equations of motion of a satellite have been derived in a local vertical coordinate system. The method of derivation and the form of the equations provide a simplified means of visualizing and planning the maneuvering within multiple satellite constellations. The equations are also useful in station keeping, rendezvous, and collision avoidance considerations.

## PROBLEM STATUS

This is an interim report; work is continuing.

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NRL Problem A01-17

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# THE EQUATIONS OF MOTION OF A SATELLITE IN A LOCAL VERTICAL COORDINATE SYSTEM

## INTRODUCTION

The equations of motion derived in this report have been previously published (1,2). However, the method of derivation and the form of the equations presented herein should permit a more intuitive comprehension of the relative motions of orbiting masses. This information should be especially useful to persons concerned with the problems of satellite rendezvous, positioning, and collision avoidance.

The equations are derived and presented in a "local vertical" coordinate system, which is defined as a coordinate system rotating at an angular velocity and phase very nearly equal to that of the orbiting objects under consideration. It is felt that this coordinate system provides an intuitive reference system for considering the relative motion of orbiting objects. The validity of the equations will, of course, require that the objects considered remain in very similar orbits. Thus, the usefulness of the equations is restricted to satellites injected into nearly identical orbits and to the terminal phase of rendezvous maneuvers.

## THE LOCAL VERTICAL COORDINATE SYSTEM

The local vertical coordinate system is defined such that the $z$ axis is always in the direction of the radius vector $\mathbf{R}$ from the center of rotation to a point in the cluster of orbiting objects. This radius vector is constrained to rotate with a constant angular velocity $\boldsymbol{\omega}$ so as to describe a Keplerian circular orbit; thus, the square of the angular velocity must be inversely proportional to the cube of the radius. The direction of the $y$ axis is parallel to the angular velocity $\boldsymbol{\omega}$, and the direction of the $x$ axis forms a right-hand coordinate system. (It is parallel to the velocity of the tip of the radius vector $\mathbf{R}$.) The origin is chosen to be the tip of the radius vector as shown in Fig. 1.

## THE EQUATIONS OF MOTION

The effective force on a point of mass $m$ in the rotating local vertical coordinate system described above is the sum of the external forces, i.e., the gravity force, the centrifugal force, and the Coriolis force. Thus,

$$F = m\ddot{\mathbf{r}} = F(\text{ext}) - \frac{m\nu(\mathbf{R}+\mathbf{r})}{|\mathbf{R}+\mathbf{r}|^3} - m\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{R}+\mathbf{r})] - 2m[\boldsymbol{\omega} \times (\mathbf{R}+\dot{\mathbf{r}})], \quad (1)$$

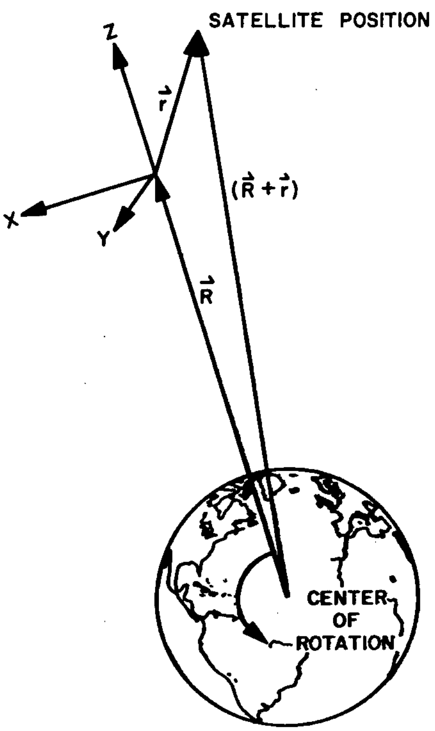


Fig. 1—The rotating local vertical coordinate system

where $\nu$ is the gravity constant. By expanding the vector triple product and $|\mathbf{R} + \mathbf{r}|^{-3}$, it is easily shown that

$$\ddot{\mathbf{r}} = \mathrm{a(ext)} + (\mathbf{R} + \mathbf{r})|\boldsymbol{\omega}|^2\left[1 - \left(1 + \frac{\mathbf{r}}{\mathbf{R}}^{2} + \frac{2z}{|\mathbf{R}|}\right)^{-3/2}\right]$$

$$- (\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}) \; , \qquad (2)$$

where a(ext) = the acceleration due to external forces. Note that $\dot{\mathbf{R}} = 0$ and $\nu/|R|^3 = |\omega|^2$.

The absolute value of **R** is at least 3500 miles; thus, for $|\mathbf{r}|$ less than 350 miles it is certainly reasonable to neglect second-order terms in $|\mathbf{r}|/|\mathbf{R}|$. With this approximation, the equations of motion linearized in terms of the $x$, $y$, and $z$ components of **r** become

$$\ddot{x} = a_1 - 2\omega\dot{z} \qquad (3)$$

$$\ddot{y} = a_2 - \omega^2 y \qquad (4)$$

$$\ddot{z} = a_3 + 3\omega^2 z + 2\omega\dot{x} \; , \qquad (5)$$

where $a_1$, $a_2$, and $a_3$ are respectively the $x$, $y$, and $z$ components of acceleration from external forces and $\omega$ is the magnitude of the orbital angular velocity of the reference coordinate system.

These equations lend themselves nicely to solution by Laplace transforms. The solutions are

$$x = x_0 - \frac{3a_1 t^2}{2} - \left(\frac{2a_3}{\omega} + 3\dot{x}_0 + 6\omega z_0\right)t - \left(\frac{2\dot{z}_0}{\omega} - \frac{4a_1}{\omega^2}\right)(1 - \cos\omega t) \qquad (6)$$

$$- \left(\frac{2a_3}{\omega^2} - 6z_0 - \frac{4\dot{x}_0}{\omega}\right)\sin\omega t$$

$$y = y_0 \cos\omega t + \frac{a_2}{\omega^2}(1 - \cos\omega t) + \frac{\dot{y}_0}{\omega}\sin\omega t \qquad (7)$$

$$z = \frac{2a_1}{\omega}t + z_0\cos\omega t + \left(\frac{a_3}{\omega^2} + 4z_0 + \frac{2\dot{x}_0}{\omega}\right)(1 - \cos\omega t) \qquad (8)$$

$$+ \left(\frac{2a_1}{\omega^2} + \frac{\dot{z}_0}{\omega}\right)\sin\omega t$$

$$\dot{x} = -3a_1 t - \left(\frac{2a_3}{\omega} + 3\dot{x}_0 + 6\omega z_0\right) - \left(2\dot{z}_0 - \frac{4a_1}{\omega}\right)\sin\omega t \qquad (9)$$

$$- \left(\frac{2a_3}{\omega} - 6\omega z_0 - 4\dot{x}_0\right)\cos\omega t$$

$$\dot{y} = \left(\frac{a_2}{\omega} - \omega y_0\right) \sin \omega t + \dot{y}_0 \cos \omega t \tag{10}$$

$$\dot{z} = \frac{2a_1}{\omega} + \left(\frac{a_3}{\omega} + 3\omega z_0 + 2\dot{x}_0\right) \sin \omega t + \left(\frac{2a_1}{\omega} + \dot{z}_0\right) \cos \omega t, \tag{11}$$

where $x_0$, $y_0$, and $z_0$ are the initial coordinates and $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ are the initial velocities. Figures 2 and 3 illustrate the trajectories for various initial conditions. Motion in the $y$ direction is not illustrated because it is always simple harmonic motion which is easily visualized as superimposed on the $x$-$z$ plane motion.

## MULTIPLY LAUNCHED SATELLITES

One of the problems associated with multiple payload launches is the possibility of collision among the payloads. It is generally recognized that the probability of collision of separately launched satellites is extremely small; however, in multiply launched satellites,

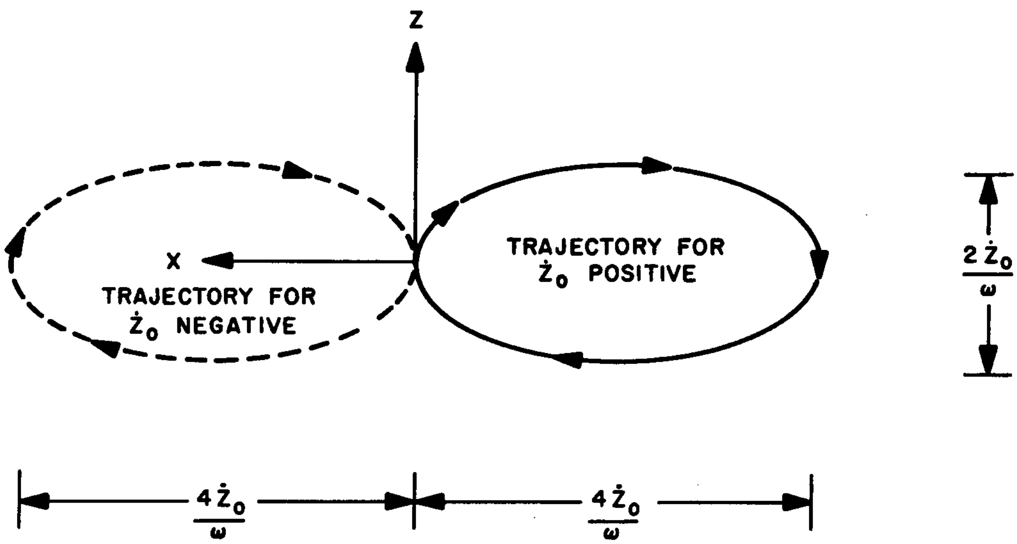


Fig. 2—Elliptical motion of a satellite having only $z$ initial velocity

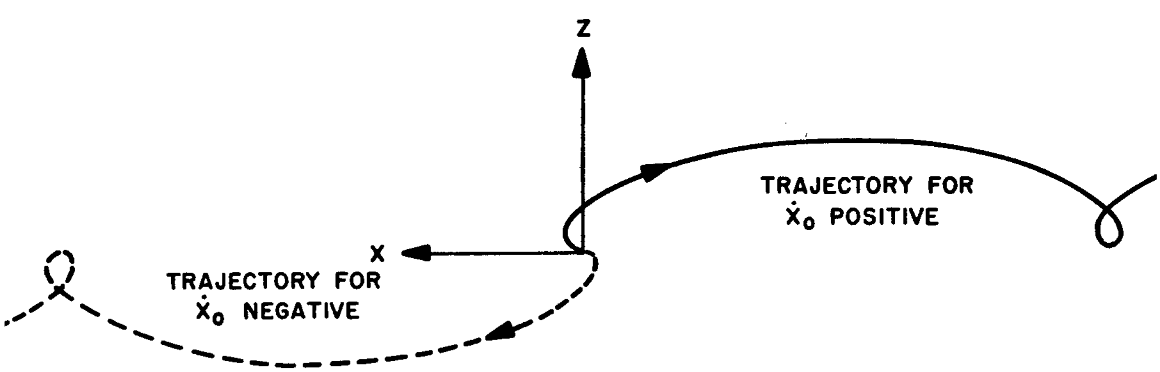


Fig. 3—Cycloidal motion of a satellite having only $x$ initial velocity

which are placed into essentially the same orbit by a single launch vehicle, the probability of collision becomes significant. By judicial selection of the separation velocities of the payloads relative to the launch vehicle, measures can be taken which will minimize the possibility of collision. One obvious measure is to separate the payloads so that drag forces will cause them to continually separate. For instance, consider two satellites, one having a higher drag force. The satellite with the high drag will lose more energy and spiral into higher velocity orbits. Thus, to avoid the possibility of the higher-drag satellite overtaking and passing the low-drag satellite, it should be initially separated into a higher velocity orbit.

Another consideration is that it is possible to separate satellites from the launch vehicle into orbits which have identical periods and differ only in inclination and/or eccentricity. This could result in periodic collisions at the orbital frequency and obviously should be avoided. This problem can be analyzed using the previously derived equations.

Let the local vertical coordinate system have its origin at the center of mass of the system of satellites and launch vehicle. In this coordinate system the trajectories of satellites impulsively ejected with initial velocities $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ are easily obtained from these equations.

Consider the case of an initial velocity purely in the $x$ direction, or flight line, where $\dot{y}_0 = \dot{z}_0 = 0$. Then

$$x = -3\dot{x}_0 t + 4\dot{x}_0 \sin \omega t \tag{12}$$

$$z = \frac{2x_0}{\omega}(1 - \cos \omega t) \tag{13}$$

and

$$y = 0 \ . \tag{14}$$

The resulting trajectory in the $x = z$ plane is a cycloid as shown in Fig. 3.

Eliminating the parameter $\omega t$ between the $x$ and $z$ equations results in

$$\frac{(x + 3\dot{x}_0 t)^2}{\dfrac{16\dot{x}_0^2}{\omega^2}} + \frac{\left(z - \dfrac{2\dot{x}_0}{\omega}\right)^2}{\dfrac{4\dot{x}_0^2}{\omega^2}} = 1 \ , \tag{15}$$

which is an ellipse centered at $x = -3\dot{x}_0 t$ and $z = 2\dot{x}_0/\omega$ with major and minor axes $a = 4\dot{x}_0/\omega$ and $b = 2\dot{x}_0/\omega$.

Thus, the cyloidal motion can also be thought of as an ellipse falling behind at the rate of $3\dot{x}_0 t$. A negative initial $x$ velocity would, of course, cause the ellipse to fall ahead.

Consider the case of an initial velocity only in the $z$ direction:

$$x = \frac{2z_0}{\omega}(1 - \cos \omega t) \tag{16}$$

$$z = \frac{z_0}{\omega} \sin \omega t \tag{17}$$

$$y = 0 \ . \tag{18}$$

Since $x$ and $z$ are truly parametric in $\omega t$, time can be eliminated, which results in a closed orbit described by

$$\frac{(x + 2\dot{z}_0)^2}{\frac{4\dot{z}_0^2}{\omega^2}} + \frac{z^2}{\frac{\dot{z}_0^2}{\omega^2}} = 1 \ . \tag{19}$$

This is an ellipse centered at $z = 0$ and $x = 2\dot{z}_0/\omega$ with $a = 2\dot{z}_0/\omega$ and $b = \dot{z}_0/\omega$ as illustrated in Fig. 2.

Thus, if perturbations such as drag are neglected, a satellite separated with only the $z$ component of velocity will return to its starting point once per orbit and collide with the launch vehicle and any other satellite separated simultaneously with only $z$ velocity. It is easily seen that satellites with only $y$ initial velocity oscillate with a simple harmonic motion along a line in the $y$ direction with the orbital period.

It is also easily deduced that a combined initial velocity of $y$ and $z$ (no $x$) serves only to rotate the ellipse previously described out of the $x$-$z$ plane. Thus, any separation of satellites with no $x$ component of initial velocity will result in a collision with the launch vehicle and, for simultaneous separations, with each other. Simultaneous separations having the same initial $x$ components of velocity will result in collisions at the orbital period, regardless of their initial $y$ and $z$ components.

Thus, to minimize the possibility of collisions, the drag effects must be considered and the separation velocities chosen so that an $x$ component is always greater than zero, unequal to the $x$ component of any other satellite. If there is a zero $x$ component, it must be assured that the launch vehicle is pushed out of the way with an additional impulse.

Figure 4 shows the trajectories relative to the center of mass of a typical multiple payload separation.

## ORBITAL POSITIONING, PARKING, AND STATION KEEPING

If Laplace transforms are used again to solve the equations of motion, it can be shown that a constant acceleration commencing at $t = 0$ and ending at $t = T$ will cause the *change* in motion described by

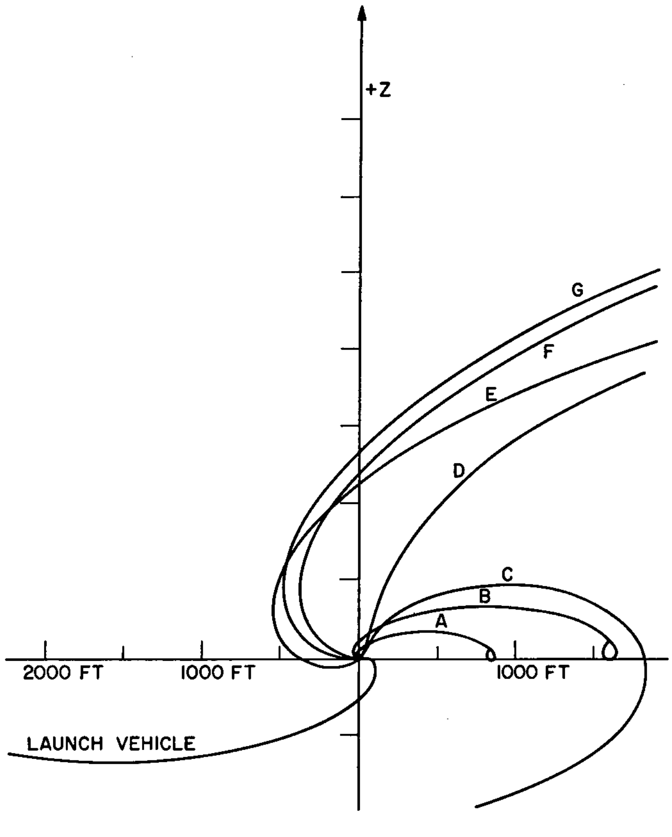


Fig. 4—Trajectories of typical multiply launched satellites relative to the center of mass

$$\Delta x = -\frac{3a_1T^2}{2} - 3a_1T\Delta t - \frac{2a_3T}{\omega} - \frac{4a_1}{\omega^2}[\cos\omega(T+\Delta t) - \cos\omega\Delta t] \tag{20}$$

$$+ \frac{2a_3}{\omega^2}[\sin\omega(T+\Delta t) - \sin\omega\Delta t]$$

$$\Delta y = \frac{-a_2}{\omega^2}[\cos\omega(T+\Delta t) - \cos\omega\Delta t] \tag{21}$$

$$\Delta z = \frac{2a_1T}{\omega^2} - \frac{a_3}{\omega^2}[\cos\omega(T+\Delta t) - \cos\omega\Delta t] \tag{22}$$

$$- \frac{2a_1}{\omega^2}[\sin\omega(T+\Delta t) - \sin\omega\Delta t]$$

$$\Delta\dot{x} = -3a_1T + \frac{4a_1}{\omega}[\sin\omega(T+\Delta t) - \sin\omega\Delta t] \tag{23}$$

$$+ \frac{2a_3}{\omega}[\cos\omega(T+\Delta t) - \cos\omega\Delta t]$$

$$\Delta\dot{y} = \frac{a_2}{\omega}[\sin\omega(T+\Delta t) - \sin\omega\Delta t] \tag{24}$$

$$\Delta\dot{z} = \frac{a_3}{\omega}[\sin\omega(T+\Delta t) - \sin\omega\Delta t] + \frac{2a_1}{\omega}[\cos\omega(T+\Delta t) - \cos\omega\Delta t] \ , \tag{25}$$

where $\Delta t$ is measured after the termination of the thrust.

It should be noted that if the thrust period $T$ is an integral multiple of the orbital period $2\pi/\omega$, the oscillating terms go to zero. This means that the thrusting did not change the eccentricity or inclination of the orbit.

Usually station keeping is accomplished with thrusters directed along the flight line (the $x$ direction). The purpose is to control $\Delta x$ and $\Delta\dot{x}$ since these coordinates most directly affect the angular position of the satellite in orbit. If the other parameters which are usually of little consequence for station-keeping purposes are neglected, the station-keeping equations become

$$\Delta x = -\frac{3a_1T^2}{2} - 3a_1T\Delta t$$

$$\Delta\dot{x} = -3a_1T \ .$$

If the station-keeping equations are divided by the radius of the orbit, $\Delta x/R$ and $\Delta\dot{x}/R$ become an angle and an angular rate respectively, thus providing a good approximation to the equations necessary for the spacing and parking of satellites with a flight-line thruster. Satellites can also be positioned with $z$-axis thrusting; however, the rate of change of $x$ exists only during the period of $z$ thrusting.

The $\Delta x$ for each orbital period of $z$ thrust is $4\pi a_3/\omega^2$. With a three-slug satellite and a 10-micropound $z$ thrust, $\Delta x$ per orbital period of thrust is 40 ft. A 10-micropound $x$ thrust will change the position by 195 ft during one revolution of thrusting and then the $\Delta x$ will continue to change by 390 ft each revolution thereafter.

Although $z$ axis thrusting is not very effective and cannot be used for parking, it may be useful in applications requiring very fine positioning. It does have the advantage of not requiring a three-axis stability.

Considering Eqs. (20) through (25) it is apparent the terminal phase of a rendezvous maneuver consists of measuring the relative positions $\Delta x$, $\Delta y$, and $\Delta z$ and relative velocities $\Delta\dot{x}$, $\Delta\dot{y}$, and $\Delta\dot{z}$ of a target vehicle, then applying thrust so as to make the relative position and velocity zero.

The magnitudes and durations of the thrusts in each of the axes provide six parameters that can be controlled to optimize a rendezvous maneuver. Equations (20) through (25) provide the basis for the necessary computations.

Most spacecraft maneuvers, other than station keeping, are accomplished with short thrusts from relatively large motions. They can be treated as impulsive velocity changes. The rendezvous or positioning equations can thus be simplified to

$$\Delta x = -3\dot{x}_0 t + \frac{4\dot{x}_0}{\omega}\sin\omega t - \frac{2\dot{z}_0}{\omega}(1 - \cos\omega t) \tag{26}$$

$$\Delta y = \frac{y_0}{\omega}\sin\omega t \tag{27}$$

$$\Delta z = \frac{\dot{z}_0}{\omega}\sin\omega t + \frac{2\dot{x}_0}{\omega}(1 - \cos\omega t) \tag{28}$$

$$\Delta\dot{x} = -3\dot{x}_0 + 4\dot{x}_0\cos\omega t - 2\dot{z}_0\sin\omega t \tag{29}$$

$$\Delta\dot{y} = \dot{y}_0\cos\omega t \tag{30}$$

$$\Delta\dot{z} = \dot{z}_0\cos\omega t + 2\dot{x}_0\sin\omega t\ , \tag{31}$$

where $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ are now considered impulsive velocity changes.

These equations have proven to be extremely useful in visualizing and planning maneuvers necessary to position satellites in orbit and provide a means of predicting the complex constellations that can be formed with multiple satellites.

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13. ABSTRACT


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