NRL Report 7397

UNCLASSIFIED

Jacobian Variational Principles and the Equivalence of Second Order Systems

WILLIAM B. GORDON

Mathematics Research Center
Report 72-1
Mathematics and Information Sciences Division

April 5, 1972

NAVAL RESEARCH LABORATORY
Washington, D.C.

Approved for public release; distribution unlimited.

UNCLASSIFIED

Jacobian Variational Principles and the Equivalence of Second Order Systems

WILLIAM B. GORDON

Mathematics Research Center
Mathematics and Information Sciences Division

Abstract: Two systems of differential equations are said to be equivalent if the trajectories of one coincide with the trajectories of the other modulo a transformation of independent (time) variable. The equivalence of second order systems is discussed, and the results obtained are used to derive a variational principle for the plane restricted three-body problem.

1. INTRODUCTION AND NOTATION

Variational techniques have recently been used to obtain information concerning the existence of periodic solutions to conservative Hamiltonian systems whose potentials have convex or infinitely deep wells. (See Berger [1-3] and Gordon [6-7].) That is, we consider dynamical equations of the form

$$\ddot{x} + \nabla V(x) = 0 \tag{1}$$

and seek periodic solutions in a neighborhood of a point p for which $\nabla V(p) = 0$ or wind around singularities q for which $V(x) \to -\infty$ as $x \to q$.

In this report we shall derive a variational principle which may be of use in obtaining analogous results for the circular, plane restricted, three-body problem, i.e., equations of the type

$$\ddot{x} + 2B\dot{x} + \nabla V(x) = 0 \tag{2}$$

where B is a square matrix which satisfies

$$B^2 = -I \; ; \; B^T = -B. \tag{3}$$

This variational principle is a direct analog to Jacobi's Variational Principle (given later) which holds for systems of type (1) and will be expressed in an isoperimetric form; i.e., it will be shown that every solution to Eg. (2) is an extremal for a certain functional F restricted to a manifold of the type $J =$ constant, where J is a certain other functional. As the title of this report suggests, the derivation consists of transforming Eq. (2) into a suitable form by a change in the independent parameter. Finally, we mention that the variational principle discussed here seems to be related to the principles described by Birkhoff [4, II.3].

NRL Problem B01-11; Project RR 003-02-41-6153. This is a final report on one phase of the problem; work is continuing on other phases. Manuscript submitted January 31, 1972.

2. VARIATIONAL PRINCIPLES

A. Let Γ be a class of paths in R^N which consists of either (a) paths with two fixed, given end-points, or (b) closed paths. For every path f in Γ given in euclidean coordinates by $x = x(t)$, let

$$Q^*(f) = \int Q(x(t), \dot{x}(t))\, dt,$$

where Q is some smooth function on R^{2N}. Then, as is well known, if Γ is a reasonable class, the condition that f be an extremal for Q^* is that $\nabla^o Q^*(f) = 0$, where ∇^o is the Euler-Lagrange operator

$$-\nabla^o Q^* = \frac{d}{dt}\left(\frac{\partial Q}{\partial \dot{x}}\right) - \frac{\partial Q}{\partial x}.$$

The operator $\nabla^o Q^*$ can be rigorously defined as a distribution of the same class as the Dirac δ function, and when multiplied by a certain Green's operator, it becomes a "true" gradient vector field in an appropriate Sobolev space of paths (cf. [5,6]).

Now let M be a riemannian manifold with metric g_{ij}, and consider the conservative dynamical system corresponding to the (velocity independent) potential V on M. Jacobi's Variational Principle asserts that every dynamical trajectory with total energy h is a reparametrized geodesic corresponding to the "Jacobi metric" $(h - V)g_{ij}$. But geodesics are extremals for either of the following two functionals: $f \to \int |\dot{x}| dt$ or $f \to \int |\dot{x}|^2\, dt$. This latter variational principle is the most convenient one. In fact, if the Euler-Lagrange operator is applied to the former functional, one does *not* obtain the usual equations for a geodesic. In the older texts this difficulty is handled by allowing as admissible paths only those that are parametrized by arc length. We shall examine this phenomenon more closely later when we consider how certain second order differential equations (sprays) transform under a change of parameter.

B. To express Jacobi's Principle in a more convenient form, we introduce the following functionals:

$$E(f) = \int |\dot{x}(t)|^2 V(x(t))\, dt;$$

$$J(f) = \frac{1}{2}\int |\dot{x}(t)|^2\, dt.$$

Then Jacobi's Principle can be written $\nabla^o(E - 2hJ) = 0$, or

$$\nabla^o E(f) = 2h \nabla^o J(f). \tag{4}$$

That is every trajectory with total energy h satisfies Eq. (3), modulo a change in parameter. The parameter in Eq. (5) is not real physical time but is the arc length associated with the Jacobi metric. For future reference we note that

$$\nabla^o E(f) = -2V(x)\ddot{x} - 2(\dot{x} \cdot \nabla V(x))\dot{x} + |\dot{x}|^2 \nabla V(x) \tag{5}$$

$$\nabla^o J(f) = -\ddot{x}. \tag{6}$$

UNCLASSIFIED

Warning. It is *not* true that every solution of Eq. (4) is a reparametrized solution of the dynamical equation (1). In fact, each solution of Eq. (4) is a reparametrized solution of one of the following:

$$\ddot{x} + \nabla V = 0 \tag{7a}$$

$$\nabla V = 0 \tag{7b}$$

$$\ddot{x} - \nabla V = 0 \tag{7c}$$

From Eqs. (4-6) it is easy to see how solutions of Eq. (7b) are obtained: they are curves $V =$ constant. To see how solutions of Eq. (7c) are obtained from Eq. (4) suppose for simplicity that V is a convex function which assumes an absolute minimum value at $x = p$, and let $V(p) = 0$. That is, we suppose that p is an attractor. The total energy of any nonequilibrium solution of Eq. (1) is positive. Suppose we set $h < 0$ in Eq. (4). An inspection of Eqs. (4-6) shows that changing the sign of h is equivalent to changing the sign of V the attractor becomes a repulsor). Hence for $h < 0$, we obtain a solution of Eq. (8c).

To obtain a formal proof that these three cases exhaust all the possibilities, we first note that $(h - V)|\dot{x}|^2$ is an integral (= constant of motion) of Eq. (4). The three cases Eqs. (7a-7c) correspond to the sign of $(h - V)$, (positive, zero, negative). In the first case, Eq. (4) is reduced to Eq. (1) through the parameter transformation $dt = d\tau/(h - V)$, where τ is the parameter in Eq. (4). The third case reduces to the first case by changing the signs of h and V.

Finally, we mention that when we use variational methods to obtain periodic solutions to Eq. (4) for a dynamical system consisting of a number of attractors, it is easy to exclude the case of Eq. (7c), i.e., periodic solutions in a neighborhood of repulsors.

C. We now state a variational principle for Eq. (2).

PROPOSITION. *Let* $K(f) = \int x \cdot B\dot{x}\, dt$, *so that* $\nabla^o K = 2B\dot{x}$. *Then every solution of Eq.* (2) *is a reparametrized solution of*

$$\nabla^o(E + K)\ (f) = 2h\nabla^o J(f). \tag{8}$$

Moreover, every solution to Eq. (9) *is a solution of*

$$\ddot{x} + 2B\dot{x} + \nabla V(x) = 0, \text{ or} \tag{2}$$

$$\ddot{x} - 2B\dot{x} - \nabla V(x) = 0. \tag{2*}$$

The proof makes use of a simple lemma. Let F and G be two Lipshitz continuous maps from $R^N \times R^N$ to R^N, and consider the two second order systems

$$\ddot{x} = F(x, \dot{x}) \tag{9}$$

$$\ddot{x} = G(x, \dot{x}). \tag{10}$$

We want to know when two such systems are equivalent modulo a change in parameter. The answer is easy in the case when one of the maps F or G is homogeneous of degree 2 in $\dot{x}$. (In the language of differential geometry, such a system is called a *spray*.)

LEMMA. *Suppose that G is homogeneous of degree 2 in $\dot{x}$ and that*

$$F(x, \dot{x}) = G(x, \dot{x}) + \phi(x, \dot{x})\dot{x}, \tag{11}$$

where ϕ is a smooth scalar-valued function on $R^N \times R^N$. Then every solution of Eq. (9) *is a reparametrized solution of Eq.* (10) *and every solution of Eq.* (10) *is a reparametrized solution of Eq.* (9). *In particular, if $x = x(t)$ is a solution of Eq.* (9) *with initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, then there exists a solution $y = y(s)$ of Eq.* (10) *such that $y(0) = x_0$, $\dot{y}(0) = \dot{x}_0$ and such that $x(t) = y(s(t))$, where the function $s = s(t)$ is obtained by integrating*

$$\frac{\ddot{s}}{\dot{s}} = \phi(x(t), \dot{x}(t)),\ s(0) = 0,\ \dot{s}(0) = 1.$$

The proof is elementary and involves a straightforward transformation of parameters. The homogeneity condition on G is essential; as is well known, it implies that if $x = x(t)$ is a solution of Eq. (10) then so is $x = x(kt)$, where k is any real number.

D. We now sketch a proof of the proposition. The proof will also serve as a derivation of Jacobi's Principle. (For the latter case, set $B = 0$.) The essential point of the proof is that the system of Eq. (2) has a first integral, viz., the well-known Jacobi constant

$$h = \frac{1}{2}|\dot{x}|^2 + V(x). \tag{12}$$

Rewriting Eq. (2) in terms of the parameter $s =$ arc length and using Eq. (12), one obtains

$$2(h - V)\ddot{x} - (\dot{x} \cdot \nabla V)\dot{x} + [2\sqrt{2(h - V)}]B\dot{x} + \nabla V = 0,$$

where now denotes d/ds. Hence $|\dot{x}|^2 = 1$, so that this last equation is equivalent to

$$2(h - V)\ddot{x} - (\dot{x} \cdot \nabla V)\dot{x} + [2\sqrt{2(h - V)}]|\dot{x}B\dot{x} + |\dot{x}|^2\nabla V = 0.$$

Then, provided that $V \neq h$, this system is a second order system homogeneous in $\dot{x}$. Applying the lemma with $\phi(x, \dot{x}) = -\dot{x} \cdot \nabla V$, we obtain the equivalent equation

$$2(h - V)\ddot{x} - 2(\dot{x} \cdot \nabla V)\dot{x} + [2\sqrt{2(h - V)}]|\dot{x}|B\dot{x} + |\dot{x}|^2\nabla V = 0.$$

This system has the first integral $|\dot{x}|^2(h - V) =$ constant. Setting this constant equal to 1/2, we obtain

$$2(h - V)\ddot{x} - 2(\dot{x} \cdot \nabla V)\dot{x} + 2B\dot{x} + |\dot{x}|^2\nabla V = 0,$$

which is (6).

This completes the proof.

UNCLASSIFIED

REFERENCE

1. Melvyn S. Berger, "On periodic solutions of second order Hamiltonian systems (I)," J. Math. Anal. *29* (1970), pp. 512-522.
2. Melvyn S. Berger, "Multiple solutions of nonlinear operator equations arising from the calculus of variations," in Proceedings of the Symposia in Pure Mathematics, Vol. XVIII, Part 1, Am. Math. Soc., (1970), pp. 10-27.
3. Melvyn S. Berger, "Periodic solutions of second order dynamical systems and isoperimetric variational problems," Am. J. Math., to appear.
4. G.D. Birkhoff, *Dynamical Systems,* American Mathematical Society Colloquium Publications Vol. 9 (1927).
5. W.B. Gordon, "On the Riemannian structure of certain function space manifolds, J. Differential Geometry *4* (1970), pp. 499-508.
6. W.B. Gordon, "A theorem on the existence of periodic solutions to Hamiltonian systems with convex potential," J. Differential Equations *10* (1971), pp. 324-335.
7. W.B. Gordon, "Periodic solutions to Hamiltonian systems whose potentials have infinitely deep wells," *Proceedings of the NRL Mathematics Research Center Conference on Ordinary Differential Equations,* Academic Press, New York, N.Y., 1972.

Security Classification

UNCLASSIFIED

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified	
	2b. GROUP	
3. REPORT TITLE JACOBIAN VARIATIONAL PRINCIPLES AND THE EQUIVALENCE OF SECOND ORDER SYSTEMS		
4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* This is a final report on one phase of the problem. Work is continuing on other phases.		
5. AUTHOR(S) *(First name, middle initial, last name)* William B. Gordon		
6. REPORT DATE April 5, 1972	7a. TOTAL NO. OF PAGES 8	7b. NO. OF REFS 7
8a. CONTRACT OR GRANT NO. NRL Problem B01-11 b. PROJECT NO. RR 003-02-41-6153 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7397 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Department of the Navy Office of Naval Research Arlington, Virginia 22217	

13. ABSTRACT

Two systems of differential equations are said to be equivalent if the trajectories of one coincide with the trajectories of the other modulo a transformation of independent (time) variable. The equivalence of second order systems is discussed, and the results obtained are used to derive a variational principle for the plane restricted three-body problem

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Variational principles Restricted three-body problem Equivalence of second order systems Sprays						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Security Classification