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Some Properties of a Generalized Moving-Window Binary Detector

J. J. FREEMAN

Security Systems Branch
Tactical Electronic Warfare Division

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ABSTRACT

For a pulsed, binary detector system, when one is given the probability that a target register a "1" is p_k, where k indexes the relative position of the target with respect to the edge (or some arbitrary benchmark) of the antenna pattern, both the optimum linear detector and the likelihood ratio detector are shown to be weighted moving windows.

A rigorous formula for the probability of detection of a moving window is derived (for the case p_k is a constant) which consists of a finite series, of which the first few terms suffice for reasonable accuracy.

PROBLEM STATUS

This is an interim report; work on other phases is continuing.

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SOME PROPERTIES OF A GENERALIZED MOVING-WINDOW BINARY DETECTOR

INTRODUCTION

We consider an interrogation system consisting of a pulsed transmitter, a receiver, a binary detector, and an antenna which scans the region for the detection of targets. The transmitter periodically sends out a sequence of pulses or pulse patterns. If a target is present within the solid angle scanned by the antenna pattern, the target responds with an appropriate pulse pattern. These responses will be received by the interrogator, demodulated, and finally the binary detector will register a "1," indicating reception of a response appropriate to a true target. In the absence of such a response, the binary detector will register a "0."

Despite the presence of a target, the binary detector may sometimes falsely register a 0 because the signal level is too low, especially if the target is near the edge of the antenna pattern. Also, despite the absence of a target within the scanned solid angle, the interrogator may sometimes falsely register a 1 because of the response from unwanted targets through side lobes, or reflections, or noise. In Fig. 1 we indicate a typical antenna pattern and beneath it we show how the probability of registering a 1 varies with the relative position of a target with respect to the antenna pattern. In the absence of a target, i.e., if the target is outside the scanning angle, we will assume that the probability of the interrogator registering a 1 is constant and equal to p_0. When the target lies within the antenna pattern, the probability that its response registers a 1 will be indicated by p_k, where k indexes the relative position of the target with respect to the edge (some arbitrary benchmark) of the antenna pattern. Given these a priori probabilities, we derive an expression for (a) an optimum (with respect to signal/noise ratio) linear digital detector, (b) a maximum-likelihood digital detector, and (c) a closed explicit representation of the probability of detection for both (a) and (b) for the case where the antenna pattern can be approximated by a rectangle. For this case, the optimum detector becomes a moving window detector (1-4).

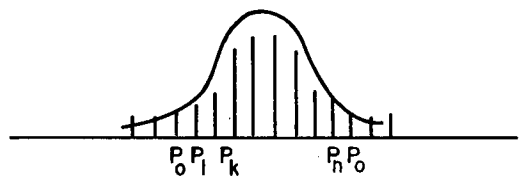

Fig. 1 - Probability that a target registers a 1 as a function of its angle (ordinate) from the edge of the antenna pattern

OPTIMUM LINEAR FILTER

At the time t, we have available past observations of the output of the binary detector, namely, Y_t, Y_{t-1}, ... Y_{t-j} ..., where the Y_{t-j} are independent binary variates which may take on the values 1 or 0 with varying probability, p_{k_j} or $1 - p_{k_j}$, respectively, depending upon the relative position of a target with respect to the edge of the scanning pattern, indexed by the subscript k_j.

We choose a linear combination of the elements of our time sequence as a statistic Z_t, and choose the coefficients so as to maximize its signal/noise ratio.

Thus

$$Z_t = \sum_{j=1}^{N} C_j \, Y_{t-j} \, , \tag{1}$$

where C_j are parameters to be optimized. The signal/noise ratio is

$$S/N = \left(\bar{Z}_{t,s} - \bar{Z}_{t,n} \right)^2 \Big/ \sigma_n^2 \, , \tag{2}$$

where

$$\bar{Z}_{t,s} = \left(\sum_{j=1}^{N} C_j \, \bar{Y}_{t-j} \right)_s = \sum_{j=1}^{N} C_j \, p_{k_j} \tag{3}$$

is the mean value of the statistic under the hypothesis that a signal exists, i.e., a target lies within the antenna patterns; and where

$$\bar{Z}_{t,n} = \left(\sum_{j=1}^{N} C_j \, \bar{Y}_{t-j} \right)_n = p_o \sum_{j=1}^{N} C_j \tag{4}$$

is the mean value under the hypothesis that no signal exists, i.e., no target lies within the antenna pattern, and where σ_n^2 is the variance of Z_t under the hypothesis of no signal:

$$\sigma_n^2 = \overline{Z_{t,n}^2} - \left(\bar{Z}_{t,n} \right)^2 = \left(p_o - p_o^2 \right) \sum_{i=1}^{N} C_i^2 \, . \tag{5}$$

Letting $p'_{k_j} = p_{k_j} - p_o$ and substituting Eqs. (3), (4), (5) into Eq. (2) yields

$$S/N = \left(\sum_i C_i \, p'_{k_j} \right)^2 \Big/ \left(p_o - p_o^2 \right) \sum_i C_i^2 \, . \tag{6}$$

Differentiating S/N with respect to C_j and setting the derivative equal to zero, one obtains

$$C_j = p'_{k_j} \frac{\Sigma C_i^2}{\Sigma C_i \, p'_{k_i}} = b \, p'_{k_j} \, , \tag{7}$$

where b is an arbitrary constant. Substituting Eq. (7) into Eq. (1) results in

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$$Z_t = \sum_{j=1}^{N} p_{k_j} Y_{t-j} , \tag{8}$$

which is the optimum detector whose S/N ratio is

$$(S/N)_{opt} = \frac{\sum_{j=1}^{N} p'^{2}_{k_j}}{p_o q_o} , \tag{9}$$

where

$$q_o = 1 - p_o .$$

Notice that if N, the number of samples collected, exceeds the number of attempted hits or trials per target, i.e., the maximum number of samples occurring within the scanning angle, then for $j > N$, $p_k = p_o$, and $p'_{k_j} - p_o = 0$. Subject to the above assumptions, the optimum detector collects N samples (where N is the maximum of hits that can occur on a target), weighs each binary response by p_k, and measures the weighted sum. At the instant the target is leaving the beam, at position N in Fig. 1, S/N assumes its maximum value.

LIKELIHOOD RATIO DETECTOR

To investigate the likelihood ratio detector (5), we assume that at each instant t, we have an N-termed subsequence

$$Y_t, Y_{t-1}, \ldots, Y_{t-N+1}$$

of an evolving time series. (In the time series, t is a variable, but within each subsequence t is fixed.) Elements of each subsequence may originate from a target, and we wish to test each running subsequence to determine the occurrence of a target using the likelihood ratio technique, and not limit ourselves to the constraint of a linear filter. In Fig. 2 is plotted a sequence of probabilities of 1 occurring in time.

Fig. 2 - Probability of 1 occurring in time

At each instant the entire infinite sequence may be considered to move one step forward, forming a new subsequence in the window which holds N terms. The probability that each element equals 1 is p_k, where again, k is the index describing the position of the target with respect to the edge of the scanning pattern at the time the target generated that particular response element.

Let

$$\mathbf{Y}_t = Y_t, Y_{t-1} \ldots Y_{t-N+1} .$$

Then the likelihood ratio is

$$L = P[Y_t|S]/P[Y_t|NS] \quad , \tag{10}$$

where $P[Y_t|S]$ is the probability of obtaining Y_t under the hypothesis that a target is within the scan, and $P[Y_t|NS]$ is the similar expression for the case of no target. It is convenient to reorder $Y_t, \ldots, Y_{t-N+1}$ as $Y_1, \ldots, Y_N$ and designate the generic term as Y_i, $i = 1, \ldots N$.

If the Y_i are statistically independent,

$$L = \prod_{i=1}^{N} \frac{P(Y_i|S)}{P(Y_i|NS)} = \prod_{i=1}^{N} \frac{p_{k_i}^{Y_i} \; q_{k_i}^{(1-Y_i)}}{p_o^{Y_i} \; q_o^{(1-Y_i)}} \quad . \tag{10a}$$

Since terms in the numerator and denominator of L cancel if the sample length exceeds the number of samples in a scan, we limit the subsequence to the number of samples in a scan, which we designate by N. Consider

$$\log L = \sum_{i=1}^{N} \left(Y_i \log \frac{p_{k_i} \; q_o}{q_{k_i} \; p_o} + \log \frac{q_{k_i}}{q_o} \right) . \tag{10b}$$

Just as the target is leaving the antenna pattern, when the subsequence is in the position indicated in Fig. 1, $\log L$ reaches its maximum value. At this time the position of the target causing each Y_i is known, and $p_{k_i} = p_i$, as shown in Fig. 1. The likelihood ratio statistic is

$$Z = \sum_{i=1}^{N} \log \left(\frac{p_i \; q_o}{p_o \; q_i} \right) Y_i \quad , \tag{11}$$

which also indicates a linear filter.

To compare the S/N ratios of the two filters, let

$$a_i = \log \frac{p_i \; q_o}{q_i \; p_o} \quad . \tag{12}$$

The signal in the S/N ratio is given by the mean difference between the indication in the presence of a signal and that in the absence of a signal, or

$$\overline{Z}_s - \overline{Z}_n = \sum_{i=1}^{N} a_i(p_i - p_o) = \sum_{i=1}^{N} a_i \; p_i' \tag{13}$$

$$\sigma_n^2 = \left(\sum_i a_i \; Y_i \right)^2 - \left(\sum_i a_i \; p_o \right)^2 = p_o q_o \sum a_i^2 \tag{14}$$

$$(S/N)_L = \frac{\left(\sum a_i \; p_i' \right)^2}{p_o \; q_o \sum a_i^2} \quad . \tag{15}$$

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The S/N ratio of the likelihood detector divided by that of the optimum linear detector is

$$\frac{(S/N)_{Lik}}{(S/N)_{cpt}} = \frac{\left(\sum a_i \, p_i'\right)^2}{\sum a_i^2 \sum p_i'^2} \leq 1 \ ,$$

which shows that the S/N ratio, albeit useful, is not an impeccable criterion of excellence, since the likelihood ratio detector will yield a higher probability of detection than one optimized for S/N ratio. For the case when the p_i distribution is rectangular, both filters become identical, namely,

$$Z = \sum_i Y_i \ ,$$

and their S/N ratio is given by

$$S/N = \frac{(p_s - p_o)^2 \, N}{p_o \, q_o} \ , \tag{16}$$

where p_s = probability of 1 if a target exists, and N is the number of trials per scan.

PROBABILITY OF DETECTION

Using an optimum filter and making a rectangular approximation to the antenna gain pattern, we derive an expression for the probability of detection. If N is the number of trials per scan, the optimum linear filter examines each successive subsequence of length N from the output of the binary detector, sums the N terms, and if the sum equals or exceeds a threshold T, registers the occurrence of a target. If N = 2, T = 2, and a target exists (Fig. 3), it will be registered provided the sequence sum equals or exceeds T in the window (or sequence) W_o, or W_{-1}, or W_1, i.e., provided the sequence symbolized by $[Y_i]$, i = 1, 4 contains at least one subsequence of length 2 for which the threshold is at least equaled. In general, for N trials per scan, the subsequence window sums N successive terms, and the probability of detection is the probability that (Fig. 4) there exists a subsequence of N terms whose sum will at least equal T. The sequence of Fig. 4 consists of a subsequence of N - 1 binary variables for which $p(1) = p_o$, followed by a subsequence of N terms for which $p(1) = p$, followed by a sequence of N - 1 terms for which $p(1) = p_o$.

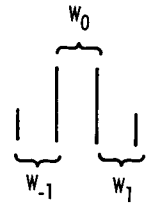

Fig. 3 - Probability of 1 occurring in the sequence of responses for which a target exists within a window of length 2

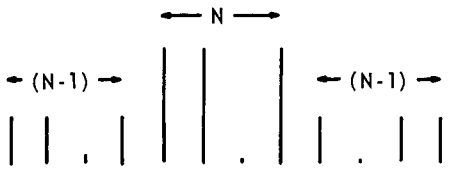

Fig. 4 - Probability of 1 occurring in the sequence of responses for which a target exists within a window of length N

Let E denote the event that a subsequence of length N containing at least T 1's occurs. The event E is a composite event consisting of a subset of the set of all possible sequences of length $(3N-2)$. In all, there are $2^{(3N-2)}$ of these sequences, and they are mutually exclusive. This set of all possible sequences may be partitioned into $T + 2$ nonoverlapping sets, namely $S(> T)$, the set of sequences containing more than T successes in the central window W_o, and $S(j)$, $j = 0, 1, \ldots, T$, the set of sequences containing precisely j successes in the central window. The event E certainly includes $S(>T)$ and $S(T)$, as well as certain elements of $S(j)$ for $j < T$. We proceed to enumerate those elements and compute their probability.

Consider the set of the sequences $S(T-1)$ for which there are $T-1$ successes occurring in the central window W_o. Of these, those for which the $(T-1)$ 1's occur in the first $(N-1)$ cells of W_o, and for which a 1 occurs immediately to the left of W_o qualify for the occurrence of event E (Fig. 5a), regardless of the composition of the sequences to the left, and regardless of the composition to the right. Since we are limiting ourselves to $(T-1)$ 1's in W_o, it is essential that a 0 occur in the cell to the extreme right of W_o. If $S_1(T-1)$ designates the set of these sequences,

$$\text{Prob}(S_1(T-1)) = 2\, p_o q\, P_{T-1}^{N-1}$$

is the probability of their occurrence, where P_{T-1}^{N-1} is the probability of obtaining $T-1$ successes in $(N-1)$ trials. The factor 2 occurs because of symmetry, interchanging right with left. Here,

$$P_{T-1}^{N-1} = \frac{(N-1)!}{(T-1)!\ (N-T)!}\, p^{T-1}\, q^{N-T} ,$$

where

$$q = 1 - p .$$

Consider now those sequences in $S(T-1)$ for which $(T-1)$ successes occur in the first $(N-2)$ cells, and 0's occur in the last two cells of W_o (Fig. 5b). If the subsequence (1, 0) immediately precedes W_o, then these sequences qualify for the event E. Furthermore, none of these sequences are contained in $S_1(T-1)$. If $S_2(T-1)$ designates the set of these sequences, then

$$\text{Prob}(S_2(T-1)) = 2\, p_o q_o\, q^2\, P_{T-1}^{N-2}$$

is the probability of $S_2(T-1)$. Similarly (Fig. 5c),

$$\text{Prob}(S_3(T-1)) = 2\, p_o q_o^2\, q^3\, P_{T-1}^{N-3}$$

is the probability of another subset of $S(T-1)$ which qualifies for E and does not overlap $S_2(T-1)$ or $S_1(T-1)$ and so on, until we exhaust cells to the left of W_o. The probability of those sequences of $S(T-1)$ which qualify for E is then

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		$\longleftarrow W_0 \longrightarrow$	
			T-1 in W_0
(a)	1	$\longleftarrow$ T-1 in W_0 -1 $\longrightarrow$ 0	$p_0 q \; P^{N-1}_{T-1}$
(b)	1 0	$\longleftarrow$ T-1 in W_0 -2 $\longrightarrow$ 0 0	$p_0 q_0 q^2 \; P^{N-2}_{T-1}$
(c)	1 0 0	$\longleftarrow$ T-1 in W_0 -3 $\longrightarrow$ 0 0 0	$p_0 q_0^2 q^3 \; P^{N-3}_{T-1}$
(d)	1 0 0 0	$\longleftarrow$ T-1 in W_0 -4 $\longrightarrow$ 0 0 0 0	$p_0 q_0^3 q^4 \; P^{N-4}_{T-1}$
			T-2 in W_0
(e)	1 1	$\longleftarrow$ T-2 in W_0 -2 $\longrightarrow$ 0 0	$(p_0 q)^2 \; P^{N-2}_{T-2}$
(f)	1 1 0	$\longleftarrow$ T-2 in W_0 -3 $\longrightarrow$ 0 0 0	$(p_0 q)^2 \; q_0 q \; P^{N-3}_{T-2}$
(g)	1 1 0 0	$\longleftarrow$ T-2 in W_0 -4 $\longrightarrow$ 0 0 0 0	$(p_0 q)^2 (q_0 q)^2 \; P^{N-4}_{T-2}$
			T-3 in W_0
(h)	1 1 1	$\longleftarrow$ T-3 in W_0 -3 $\longrightarrow$ 0 0 0	$(p_0 q)^3 \; P^{N-3}_{T-3}$
(i)	1 1 1 0	$\longleftarrow$ T-3 in W_0 -4 $\longrightarrow$ 0 0 0 0	$(p_0 q)^3 \; q_0 q \; P^{N-4}_{T-3}$

Fig. 5 - Table of possible configurations of binary digits qualifying for the event E, namely, that at least T 1's occur

$$\mathrm{Prob}(S(T-1)|E) = 2\, p_o q \sum_{j=0}^{N-1} (q_o\, q)^j \, P^{N-1-j}_{T-1} \; .$$

Since the number of successes cannot exceed the number of trials,

$$\max(j) \le N - T \; .$$

Consider now these sequences of S (T - 2) which qualify for E, diagrammed in Fig. 5e, f, and g. One such set of sequences (Fig. 5e) consists of (1,1) immediately to the left of W_o, followed by a sequence of N - 2 trials in which T - 2 1's occur, followed by (0,0). The probability of all sequences of this type is

$$\mathrm{Prob}(S_1(T-2)|E) = (p_o\, q)^2 \, P^{N-2}_{T-2} \; .$$

Similarly, one finds for the probability of all those sequences in S (T - 2) which qualify for E (Fig. 5e, f, g),

$$\mathrm{Prob}[S(T-2)|E] = 2(p_o\ q)^2 \sum_{j=0}^{N-T} (q_o\ q)^j\ P_{T-2}^{N-2-j} \quad .$$

Similarly, one finds for the probability of those sequences in $S(T-3)$ which qualify for E (Fig. 5h, i),

$$\mathrm{Prob}[S(T-3)|E] = 2(p_o\ q)^3 \sum_{j=0}^{N-T} (q_o\ q)^j\ P_{T-3}^{N-3-j} \quad .$$

Summing up,

$$\sum_{n=0}^{T} \mathrm{Prob}(S(T-n)|E) = 2 \sum_{j=0}^{N-T} \sum_{n=1}^{T} (p_o\ q)^n\ (q_o\ q)^j\ P_{T-n}^{N-j-n} \quad .$$

The probability of detection P_D, i.e., the probability of getting at least one window with at least T 1's is

$$P_D = \sum_{j=T}^{N} P_j^N + 2 \sum_{j=0}^{N-T} \sum_{n=1}^{T} (p_o\ q)^n\ (q_o\ q)^j\ P_{T-n}^{N-j-n} \quad .$$

On evaluating P_D numerically for several reasonable value of p_o and p, it was found that the second summation converged extremely fast and only the first few terms were required for reasonable accuracy.

CONCLUSIONS

1. For a pulsed, binary detector system, when one is given the probability that a target register a 1 is p_k, where k indexes the relative position of the target with respect to the edge (or some arbitrary benchmark) of the antenna pattern, the optimum linear detector is a weighted moving window; the likelihood ratio detector is also a weighted linear detector whose weighting coefficients differ slightly from those of the linear detector, and whose signal/noise ratio never exceeds that of the optimum linear detector. For the case when p_k is a constant for all targets within the antenna beam, both detectors become identical.

2. A formula was derived for the probability of detection of a moving window (for the case p_k is a constant) which consists of a finite series, of which the first few terms suffice for reasonable accuracy.

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REFERENCES

1. Dinneen, G.P. and Reed, I.S., "An Analysis of Signal Detection and Location by Digital Methods," IRE Trans. Inform. Theor. IT-2 (No. 1):29, (Mar. 1956)

2. Dillard, G.M., "A Moving-Window Detector for Binary Integration," IEEE Trans. Inform. Theor. IT-13 (No. 1):2, (Jan. 1967)

3. Hansen, V.G., "Performance of the Analog Moving Window Detector," IEEE Trans. Aerospace Electron. Syst. AES-6 (No. 2):173 (March 1970)

4. Capsers, J.W., "Automatic Detection for Radar," NELC Report 1680, (Secret Report, Unclassified Title) Oct. 20, 1969, Naval Electronics Laboratory Center, San Diego, Calif.

5. Walter, C.M., "A Quantitative Analysis of Automatic Target Detection-Position Estimation Schemes Observing Scintillating Targets in Noise," 1958 IRE Nat. Conv. Rec., Pt. 5, 107-119

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13. ABSTRACT

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