Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D. C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

THE WAKE OF A SOURCE IN A TWO-LAYER FLUID

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

An interim report on a continuing problem.

5. AUTHOR(S) *(First name, middle initial, last name)*

J. M. Bergin

6. REPORT DATE May 20, 1971	7a. TOTAL NO. OF PAGES 48	7b. NO. OF REFS 8–9
8a. CONTRACT OR GRANT NO. NRL Problem G01-06 b. PROJECT NO. Project A37-370/F08-125-703 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7254 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Department of the Navy (Naval Air Systems Command) Washington, D.C. 20360

13. ABSTRACT

Abstract: The purpose of this report is to provide an analytical basis for evaluation of the internal wave system generated by a moving body in stratified fluids which can be represented by a two-layer structure. In view of the standard representation of rigid bodies by a distribution of hydrodynamic sources, we achieve our objective by developing analytical expressions for the disturbance due to a single source.

Expressions are derived for the velocity field produced by the source, which may be moving at any depth in the fluid; these expressions require for their validity a disturbance of small amplitude. The method of stationary phase, when applied to these expressions, yields a simplified description of the velocity field far behind the source and indicates that the internal wave system possesses significant amplitudes in a region limited to the vicinity of the track of the source. An additional quantity, the displacement function, is introduced in order to describe the amount by which fluid particles are displaced (in the vertical direction) from their equilibrium position. We also discuss the physical significance of the form of the displacement function and the manner in which various parameters affect its behavior.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Internal wave						
Ship wake problem						
Two-layer fluid						
Source in a two-layer fluid						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Security Classification

CONTENTS

Abstract ii
Problem Status ii
Authorization ii

1. INTRODUCTION 1

2. THE WAVE SYSTEM CAUSED BY A SOURCE IN THE TOP LAYER 2
   - Formulation and Formal Solution of the Problem 2
   - Asymptotic Form of the Velocity Potential 13
   - The Nature of the $\gamma(Z)$ Relationship 21
   - Evaluation and Discussion of the Displacement Function 26

3. THE WAVE SYSTEM CAUSED BY A SOURCE IN THE BOTTOM LAYER 33
   - The Formal Solution of the Problem 33
   - Asymptotic Form of the Velocity Potential 37
   - The Displacement Function 40

4. CONCLUSION 43

REFERENCES 44

ABSTRACT

The purpose of this report is to provide an analytical basis for evaluation of the internal wave system generated by a moving body in stratified fluids which can be represented by a two-layer structure. In view of the standard representation of rigid bodies by a distribution of hydrodynamic sources, we achieve our objective by developing analytical expressions for the disturbance due to a single source.

Expressions are derived for the velocity field produced by the source, which may be moving at any depth in the fluid; these expressions require for their validity a disturbance of small amplitude. The method of stationary phase, when applied to these expressions, yields a simplified description of the velocity field far behind the source and indicates that the internal wave system possesses significant amplitudes in a region limited to the vicinity of the track of the source. An additional quantity, the displacement function, is introduced in order to describe the amount by which fluid particles are displaced (in the vertical direction) from their equilibrium position. We also discuss the physical significance of the form of the displacement function and the manner in which various parameters affect its behavior.

PROBLEM STATUS

This is an interim report; work on this problem is continuing.

AUTHORIZATION

NRL Problem G01-06
Project A37370/652/0F08-125-703

Manuscript submitted January 25, 1971.

THE WAKE OF A SOURCE IN A TWO-LAYER FLUID

1. INTRODUCTION

A rigid body moving in a stratified fluid which possesses a strong thermocline (a region in which the density changes rapidly from one essentially constant value to another constant value) generates a system of internal waves in addition to the usual surface wave system. This situation represents the operational environment of conventional surface vessels in many areas of the world. In particular, it applies to areas such as the Scandinavian fjords and the Canadian inlets where glacier runoff provides a layer of fresh water over a much deeper layer of salt water.

Our main objective in this report is to develop an analytical basis for the interpretation of the internal (dead water) waves which occur in such areas. We achieve a certain analytical simplification by replacing the typical environmental stratification with two homogeneous fluids having constant densities. The effects of a rigid body moving through such a fluid are approximately equivalent to those of a source, associated with the bow, and a sink, associated with the stern. Since the source and sink are equivalent except for sign, the analytical determination of the disturbance created requires only knowledge of the disturbance produced by a source moving in a two-layer fluid.

The present work treats the problem of the uniform motion of a two-layer fluid past a point source held fixed in the flow. Both the top layer, which has finite thickness, and the bottom layer of infinite depth are regarded as incompressible, inviscid fluids that have constant, but different, densities. As usual, the inertia of the atmosphere is assumed to be negligible and this yields the condition of constant pressure there. A further assumption made is that we may linearize the governing equations. An equivalence exists between this problem and that of the source moving at constant horizontal velocity in the fluid which is undisturbed far ahead of the source as long as the initial transient features of the starting process are ignored.

This case of fluid flow provides a basis for the treatment of the wave system generated by a rigid body, such as a ship, moving in an ocean possessing two fluid layers of different densities. It is a standard technique to replace the body by an appropriate distribution of sources and the addition of the effects of each source then yields the wave motion accompanying the given body. As for the actual density structure of the ocean, it changes continuously with depth; but, often an adequate approximation to the continuous stratification is provided by two layers of different densities. This results in considerable simplification of the analysis and proves instructive for more complicated situations.

Hudimac (1) dealt with the two-layer system but considered only the case where the effective depth of the source was less than the thickness of the upper layer; he found a solution for the velocity potential in this layer and the displacement of the interface. However, his analysis has certain deficiencies. Most troubling are the numerous misprints which occur and render the analysis unfit for use in numerical studies of actual motions. Also, he based some of his statements about the general nature of the asymptotic solution on the results of numerical investigation of a few particular cases. In several respects, Hudimac's treatment must be considered incomplete. It appears that the problem of the source in a two-layer fluid requires further development in order to achieve the desired completeness.

Problems related to the above involving more complicated examples of density stratification have been considered by other investigators such as Grosch (2), who provides further references. Of more interest to the present work is the problem treated by Crapper (3) of a pressure point moving over the surface of a two-layer fluid in which attention is focused on general features of the motion.

The main purpose of this paper is to correct Hudimac's analysis and to extend it to allow treatment of the wave pattern due to a source located at any depth. The analytical method employed and much of the notation follow Hudimac rather closely, thus making for ease of comparison with his results. Our concern lies chiefly with the internal wave system, since it has not received the attention given the surface wave system; however, we do present some analysis and remarks about the surface waves in order to clarify their role in the present problem.

In the next section, an analysis is given for the case treated by Hudimac of the source in the top layer. A similar development in the third section for the source in the bottom layer rests upon the earlier work and results in a briefer exposition. We determine the velocity potential throughout the fluid in both cases and then obtain asymptotic representations for the region far behind the source. The displacement function in the far field then follows directly from the form of the velocity potential. Careful consideration of the development of the analysis provides a more acceptable basis for certain statements made by Hudimac. Moreover, some of the results are new and provide a better understanding of the nature of the wave pattern.

2. THE WAVE SYSTEM CAUSED BY A SOURCE IN THE TOP LAYER

Formulation and Formal Solution of the Problem

We consider in this section the uniform flow of an inviscid, two-layer fluid past a source held fixed in the top layer. The top layer has the constant density ρ and thickness h, while the bottom layer is infinite in depth and has a density ρ'. We regard the atmosphere above the top layer as possessing negligible inertia so that the pressure there must be a constant, which can be taken equal to zero. The x-z plane gives the location of the undisturbed interface and the y axis points vertically upward with the source positioned on it a distance f above the undisturbed interface. Far ahead of the source the flow consists of a uniform motion of speed c directed toward the source. The geometry of the flow situation is shown in Fig. 1.

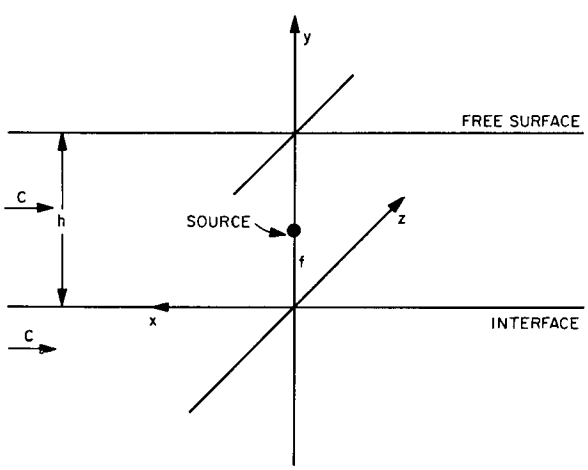

Fig. 1 - Geometry of the flow

Since each fluid is assumed inviscid, the problem involves the determination of potentials Φ, Φ' describing the steady irratational motion in the top and bottom layers, respectively. We note here the use of a prime on a variable to designate that it refers to the bottom layer, while the unprimed variable refers to the top layer. The boundary conditions are that at the free surface described by

$$y = h + \eta_s\,(x,z), \tag{2.1}$$

we must have

$$\frac{Dy}{Dt} = v = \frac{D\eta_s}{Dt} \tag{2.2}$$

and the pressure must satisfy

$$p = 0 \tag{2.3}$$

while at the interface, given by

$$y = \eta_i\,(x,z), \tag{2.4}$$

we must have

$$v = v' \tag{2.5}$$

and

$$p = p'. \tag{2.6}$$

We also require that the disturbance to the uniform prescribed state vanish at great depths as well as far upstream of the source.

The pressure in the top layer is given in terms of the flow by Bernoulli's integral

$$p + \frac{1}{2}\rho\,|\bar{v}|^2 + \rho g y = \text{constant}, \tag{2.7}$$

where g denotes the acceleration of gravity, and an analogous equation holds for the bottom layer. Employing the boundary conditions, we find that Eq. (2.7) takes the form

$$p + \frac{1}{2}\rho(|\bar{v}|^2 - c^2) + \rho g(y - h) = 0. \tag{2.8}$$

In the bottom layer, we have

$$p' + \frac{1}{2}\rho'\,(|\bar{v}'|^2 - c^2) + \rho' g y - \rho g h = 0. \tag{2.9}$$

A quantity useful for describing the disturbance produced by the source is the displacement function $\eta(x,y,z)$ which gives the elevation above its initial level of the streamline which when undisturbed was at height y. This function is defined throughout the flow and satisfies

$$\frac{D\eta}{Dt} = v; \tag{2.10}$$

we can also write

$$\eta_s(x,z) = \eta(x,h,z) \tag{2.11}$$

and

$$\eta_i(x,z) = \eta(x,0,z). \tag{2.12}$$

The above equations specify the exact boundary value problem, which is nonlinear owing to the dependence of the pressure on the square of the velocity shown in Eqs. (2.8) and (2.9). We shall assume that the source is sufficiently removed from the interface and the free surface to provide at these surfaces only a small departure from the uniform flow. Then, with

$$\Phi = cx + \varphi, \tag{2.13}$$

$$\Phi' = cx + \varphi',$$

the potentials φ, φ' are regarded as small quantities in order to permit the linearization of the boundary conditions. Carrying out this process, we find that the kinematic condition of Eq. (2.2) at the free surface is

$$\varphi_y = c\eta_x \text{ on } y = h, \tag{2.14}$$

and the pressure condition (Eq. (2.3)), through use of Eq. (2.8), is

$$c\varphi_x + g\eta = 0 \text{ on } y = h. \tag{2.15}$$

At the interface, the kinematic condition Eq. (2.5) becomes

$$\varphi_y = \varphi'_y \text{ on } y = 0, \tag{2.16}$$

and the pressure condition Eq. (2.6) takes the form

$$\rho c\varphi_x + \rho g\eta = \rho' c\varphi'_x + \rho' g\eta \text{ on } y = 0. \tag{2.17}$$

If the displacement is eliminated between Eqs. (2.14) and (2.15), the result is

$$c^2\varphi_{xx} + g\varphi_y = 0 \text{ on } y = h. \tag{2.18}$$

Likewise, at the interface,

$$\rho c^2\varphi_{xx} + \rho g\varphi_y = \rho' c^2\varphi'_{xx} + \rho' g\varphi'_y \text{ on } y = 0 \tag{2.19}$$

and

$$\varphi_y = \varphi'_y \text{ on } y = 0 \tag{2.20}$$

as before. The disturbance to the uniform flow must vanish at great depths, and this provides

$$\lim_{y \to -\infty} \varphi' = 0. \tag{2.21}$$

In addition, far ahead of the source, a radiation condition is imposed which requires that

$$\lim_{x \to \infty} x^{1/2} \text{ grad } \varphi = 0 \tag{2.22}$$

and

$$\lim_{x \to \infty} x^{1/2} \text{ grad } \varphi' = 0.$$

These results provide the boundary conditions needed to determine the potentials describing the disturbance to the uniform flow.

Use of the Fourier integral representation will enable us to find the solution to the above boundary value problem. Transforms in x and z will be employed with the transform of a function v (x, y, z) denoted by

$$\tilde{v}\ (\xi, \zeta, y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} v \exp\ [-\ i\ (\xi x + \zeta z)]\ dx\, dz \tag{2.23}$$

and its inverse as

$$v\ (x, y, z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \tilde{v} \exp\ [\ i\ (\xi x + \zeta z)]\ d\xi\, d\zeta. \tag{2.24}$$

To account explicitly for the presence of the source, we write

$$\begin{aligned} \varphi &= \frac{m}{r_1} + \varphi_b, \\ r_1 &= \left[x^2 + (y - f)^2 + z^2\right]^{1/2}, \end{aligned} \tag{2.25}$$

where m is the source strength and φ_b is a regular harmonic function; the source lies in the upper layer at (0, f, 0). The differential equations to be solved are

$$\nabla^2 \varphi_b = 0$$

and

$$\nabla^2 \varphi' = 0.$$

Taking the transforms, we obtain the equations

$$\tilde{\varphi}_{byy} - \tilde{\omega}^2 \tilde{\varphi}_b = 0 \tag{2.26}$$

and

$$\tilde{\varphi}'_{yy} - \tilde{\omega}^2 \tilde{\varphi}' = 0, \tag{2.27}$$

where

$$\tilde{\omega}^2 = \xi^2 + \zeta^2. \tag{2.28}$$

The solution to Eq. (2.26) is

$$\tilde{\varphi}_b = A\, e^{\tilde{\omega} y} + B\, e^{-\tilde{\omega} y} \tag{2.29}$$

and that of Eq. (2.27) is

$$\tilde{\varphi}' = A'\, e^{\tilde{\omega} y}, \tag{2.30}$$

where the boundary condition Eq. (2.21) was used in Eq. (2.30).
The coefficients A, B, A′ depend on the transform variables and are determined by use of the transformed boundary conditions.

Before transforming the boundary conditions, we must substitute for φ the expression of Eq. (2.25). The source term is then dealt with by use of the representation

$$\frac{1}{[x^2 + (y-f)^2 + z^2]^{1/2}} = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\exp\,[\pm\, \tilde{\omega}\,(y-f)]}{\tilde{\omega}} \exp\,[i(\xi x + \zeta z)]\, d\xi\, d\zeta \tag{2.31}$$

with the positive sign employed if $y < f$ and the negative sign if $y > f$. This result is not restricted by the sign of f and will be used again in the next section, which treats the case of the source situated in the bottom layer.

The transform of the free surface condition, Eq. (2.18), gives

$$g\tilde{\varphi}_{by} - c^2 \xi^2 \tilde{\varphi}_b = mg \exp\,[-\tilde{\omega}\,(h-f)] + \frac{mc^2 \xi^2}{\tilde{\omega}} \exp\,[-\tilde{\omega}\,(h-f)] \tag{2.32}$$

for $y = h$. At the interface, Eq. (2.20) provides

$$\tilde{\varphi}_{by} + m\, e^{-\tilde{\omega} f} = \tilde{\varphi}'_y \qquad \text{with } y = 0, \tag{2.33}$$

and Eq. (2.19) gives

$$\rho c^2 \xi^2 \tilde{\varphi}_b + \rho c^2 \xi^2 m \frac{e^{-\tilde{\omega} f}}{\tilde{\omega}} = \rho' c^2 \xi^2 \tilde{\varphi}' - (\rho' - \rho)\, g \tilde{\varphi}'_y \tag{2.34}$$

with $y = 0$.

Substitution of Eqs. (2.29) and (2.30) into the transformed boundary conditions gives

$$\tilde{\omega} A - \tilde{\omega} B - \tilde{\omega} A' = - m e^{-\tilde{\omega} f} \tag{2.35}$$

from Eq. (2.33),

$$- \rho c^2 \xi^2 (A + B) + [\rho' c^2 \xi^2 - (\rho' - \rho) g \tilde{\omega}] A' = \rho c^2 \xi^2 m \frac{e^{-\tilde{\omega} f}}{\tilde{\omega}} \tag{2.36}$$

from Eq. (2.34), and

$$(g\tilde{\omega} - c^2\xi^2) e^{\tilde{\omega} h} A - (g\tilde{\omega} + c^2\xi^2) e^{-\tilde{\omega} h} B = \frac{m}{\tilde{\omega}} (g\tilde{\omega} + c^2\xi^2) e^{-\tilde{\omega}(h - f)} \tag{2.37}$$

from Eq. (2.32). These three equations serve to determine A, B, and A′ by use of Cramer's rule. The determinant Q of the coefficients can be written as

$$Q = \tilde{\omega} (c^2\xi^2 - g\tilde{\omega}) \left\{ [(\rho' + \rho) c^2\xi^2 - (\rho' - \rho) g\tilde{\omega}] e^{\tilde{\omega} h} + (\rho' - \rho)(g\tilde{\omega} + c^2\xi^2) e^{-\tilde{\omega} h} \right\}; \tag{2.38}$$

and, with the solutions for the three functions, we eventually find

$$\tilde{\varphi}_b = \frac{mP}{Q} , \tag{2.39}$$

where

$$\begin{aligned} P = {} & (c^2\xi^2 + g\tilde{\omega}) [- (\rho + \rho') c^2\xi^2 + (\rho' - \rho) g\tilde{\omega}] \exp [-\tilde{\omega} (h - f - y)] \\ & - (c^2\xi^2 + g\tilde{\omega}) (c^2\xi^2 - g\tilde{\omega}) (\rho' - \rho) \left\{ \exp [-\tilde{\omega} (h - f - y)] + \exp [-\tilde{\omega} (h - f - y)] \right. \\ & + (c^2\xi^2 - g\tilde{\omega})^2 (\rho' - \rho) \exp [\tilde{\omega} (h - f - y)] \end{aligned} \tag{2.40}$$

and

$$\tilde{\varphi}' = 2\rho m c^2 \xi^2 \frac{R}{Q} e^{\tilde{\omega} y} , \tag{2.41}$$

in which

$$R = (c^2\xi^2 - g\tilde{\omega}) \exp [\tilde{\omega} (h - f)] - (c^2\xi^2 + g\tilde{\omega}) \exp [-\tilde{\omega} (h - f)]. \tag{2.42}$$

These results provide the transformed potentials.

The above transforms lead via the inversion formula to the potentials

$$\varphi_0 = \frac{m}{r_1} + \frac{m}{2\pi} \iint_{-\infty}^{\infty} \frac{P \exp [i (\xi x + \zeta z)]}{Q} d\xi \, d\zeta \tag{2.43}$$

and

$$\varphi_0' = \frac{\rho m}{2\pi} \iint_{-\infty}^{\infty} \frac{2c^2\xi^2 R e^{\tilde{\omega} y} \exp[i(\xi x + \zeta z)]}{Q} \, d\xi \, d\zeta, \tag{2.44}$$

where P, Q, and R are as defined above. We use the subscripts to indicate that these potentials are not those desired since we have yet to satisfy the radiation condition. The correct potentials, however, follow from the above after further analysis.

The expressions for φ_0 and φ_0' can be simplified by noting that P, Q, and R depend only on the squares of the transform parameters. Thus, if we write

$$\exp[i(\xi x + \zeta z)] = \cos(\xi x + \zeta z) + i \sin(\xi x + \zeta z)$$

in each integral, it is clear that the sine term does not contribute to the value of the integral due to the symmetries involved. Likewise, if we write

$$\cos(\xi x + \zeta z) = \cos \xi x \cos \zeta z - \sin \xi x \sin \zeta z,$$

the second term does not contribute to the value of the integral for the same reason. Furthermore, the cosine term leads to an equal contribution in each quadrant so that the integration can be taken over the first quadrant. If polar coordinates are used to describe the quadrant through

$$\begin{aligned} \xi &= \tilde{\omega} \cos \theta \\ \zeta &= \tilde{\omega} \sin \theta \end{aligned} \tag{2.45}$$

then the potentials take the form

$$\varphi_0 = \frac{m}{r_1} + \frac{m}{\pi c^2} \int_0^{\pi/2} \int_0^{\infty} \frac{M(\theta, \tilde{\omega}) \cos(\tilde{\omega} x \cos\theta) \cos(\tilde{\omega} z \sin\theta)}{(\tilde{\omega} - \tilde{\omega}_0) g(\theta, \tilde{\omega})} \, d\tilde{\omega} \, d\theta \tag{2.46}$$

and

$$\varphi_0' = \frac{2\rho m}{\pi} \int_0^{\pi/2} \int_0^{\infty} \frac{R e^{\tilde{\omega} y} \cos(\tilde{\omega} x \cos\theta) \cos(\tilde{\omega} z \sin\theta)}{(\tilde{\omega} - \tilde{\omega}_0) g(\theta, \tilde{\omega})} \, d\tilde{\omega} \, d\theta. \tag{2.47}$$

In these expressions, we have

$$\tilde{\omega}_0 = g c^{-2} \sec^2 \theta, \tag{2.48}$$

$$g(\theta, \tilde{\omega}) = \cos^2\theta \, (\rho' c^2 \tilde{\omega} \cosh h\tilde{\omega} + \rho c^2 \tilde{\omega} \sinh h\tilde{\omega}) - (\rho' - \rho) g \sinh h\tilde{\omega}, \tag{2.49}$$

$$M(\theta,\tilde{\omega}) = \sec^2\theta\Big\{(c^2\tilde{\omega}\cos^2\theta + g)\,[-(\rho+\rho')\,c^2\tilde{\omega}\cos^2\theta + (\rho'-\rho)g]\,\exp\,[-\tilde{\omega}(h-f-y)]$$

$$-(c^2\tilde{\omega}\cos^2\theta + g)\,(c^2\tilde{\omega}\cos^2\theta - g)\,(\rho'-\rho)\quad\exp\,[-\tilde{\omega}\,(h+f-y)] + \exp\,[-\tilde{\omega}\,(h-f-y)]$$

$$+(c^2\tilde{\omega}\cos^2\theta - g)^2\,(\rho'-\rho)\exp\,[\tilde{\omega}\,(h-f-y)]\Big\}\,, \tag{2.50}$$

and

$$R(\theta,\tilde{\omega}) = \tilde{\omega}\,[(c^2\tilde{\omega}\cos^2\theta - g)\,e^{\tilde{\omega}(h-f)} - (c^2\tilde{\omega}\cos^2\theta + g)\exp\,[-\tilde{\omega}\,(h-f)]\,. \tag{2.51}$$

The singularities of the integrands in Eqs. (2.46) and (2.47), which are associated with the non-uniqueness of the solution, are of crucial importance in determining the asymptotic behavior of the potentials. They are given by

$$\tilde{\omega} = \tilde{\omega}_0 = g c^{-2}\sec^2\theta \tag{2.52}$$

and

$$g\,(\theta,\tilde{\omega}_1) = 0. \tag{2.53}$$

These both specify curves in the transform plane and, from Eq. (2.49), we find that

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_0} = \frac{\rho'-\rho}{\rho'}\;\frac{\sinh\,\tilde{\omega}_1 h}{\cosh\,\tilde{\omega}_1 h + \dfrac{\rho}{\rho'}\,\sinh\,\tilde{\omega}_1 h}\,,$$

which implies that

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_0} \leqslant \frac{\rho'-\rho}{\rho'}\,\tanh\,\tilde{\omega}_1 h < 1.$$

This indicates that the curves do not intersect for $0 \leqslant \theta < \pi/2$. The implicit relation of Eq. (2.53) may be written as

$$h\tilde{\omega} = \frac{\sec^2\theta}{F^2}\;\frac{\sinh\,\tilde{\omega}h}{\cosh\,\tilde{\omega}h + \dfrac{\rho}{\rho'}\,\sinh\,\tilde{\omega}h}\,, \tag{2.54}$$

where the subscript has been omitted and in which

$$F^2 = \frac{\rho' c^2}{\Delta\rho g h} \tag{2.55}$$

and

$$\Delta\rho = \rho' - \rho. \tag{2.56}$$

We shall find that the Froude number F controls the general nature of the solution. For the case of the source advancing into the fluid which is otherwise at rest, the Froude number gives the ratio of the source speed to the maximum velocity at which internal plane waves can propagate in the two-layer system. Those motions for which the Froude number is less than one are termed subcritical, while motions with a Froude number greater than one are called supercritical.

Some consequences of Eq. (2.54) follow from writing it as

$$\tilde{\omega}_1 h \left(\coth \tilde{\omega}_1 h + \frac{\rho}{\rho'} \right) = F^{-2} \sec^2 \theta, \tag{2.57}$$

where the function on the left is an increasing function of $\tilde{\omega}_1 h$ which is equal to one at $\tilde{\omega}_1 = 0$. Thus, if $F < 1$ with θ in the first quadrant, a solution $\tilde{\omega}_1$ exists. However, if $F > 1$, a solution exists only when

$$F^2 \cos^2 \theta \leqslant 1,$$

which yields the condition

$$\theta \geqslant \theta_c = \cos^{-1} \left(\frac{1}{F} \right). \tag{2.58}$$

The general behavior of the singular curves is indicated in Fig. 2.

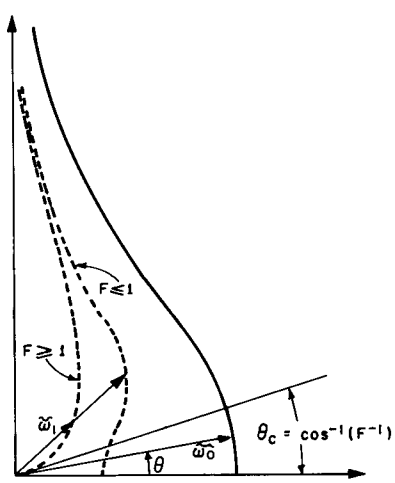

Fig. 2 - The singular curves in the transform plane (based on Hudimac (1)).

We now examine the radiation condition in regard to the potentials found above. In considering the behavior of the potentials far ahead of the source, we shall use the result from Fourier analysis that

$$\lim_{x \to \infty} \int_a^b f(s) \frac{\sin [x(s - s_0)]}{s - s_0} \, ds = \frac{\pi}{2} [f(s_0 +) + f(s_0 -)] \tag{2.59}$$

if $a < s_0 < b$. The result is zero if the cosine appears instead of the sine.

The potentials can be written as

$$\varphi_0 = \frac{m}{r_1} + I_1 + I_0 \tag{2.60}$$

and

$$\varphi_0' = I_1' + I_0', \tag{2.61}$$

with

$$I_1 = \frac{m}{\pi c^2} \int_0^{\pi/2} \int_0^{\tilde{\omega}_2} \frac{M \cos(\tilde{\omega}\, x \cos\theta) \cos(\tilde{\omega}\, z \sin\theta)}{(\tilde{\omega} - \omega_0)\, g(\theta, \tilde{\omega})}\, d\tilde{\omega} d\theta, \tag{2.62}$$

and

$$I_0 = \frac{m}{\pi c^2} \int_0^{\pi/2} \int_{\tilde{\omega}_2}^{\infty} \frac{M \cos(\tilde{\omega}\, x \cos\theta) \cos(\tilde{\omega}\, z \sin\theta)}{(\tilde{\omega} - \tilde{\omega}_0)\, g(\theta, \tilde{\omega})}\, d\tilde{\omega} d\theta, \tag{2.63}$$

in which $\tilde{\omega}_2$ is chosen so that $\tilde{\omega}_1 < \tilde{\omega}_2 < \tilde{\omega}_0$; a similar decomposition applies to φ_0'. The subscripts denote the singular curves involved in the integration. This decomposition enables the singularities to be treated separately.

Our interest lies in the asymptotic forms of the above integrals for large x, and we shall use standard limit notation to designate these asymptotic values. Consider, then,

$$\lim_{x\to\infty} I_0 = \frac{m}{\pi c^2} \lim_{x\to\infty} \int_0^{\pi/2} \int_{\tilde{\omega}_2}^{\infty} \frac{M \cos(\tilde{\omega}\, x \cos\theta) \cos(\tilde{\omega}\, z \sin\theta)}{(\tilde{\omega} - \tilde{\omega}_0)\, g(\theta, \tilde{\omega})}\, d\tilde{\omega} d\theta. \tag{2.64}$$

We write

$$\begin{aligned} \cos(\tilde{\omega}\, x \cos\theta) = {} & \cos[(\tilde{\omega} - \tilde{\omega}_0)\, x \cos\theta] \cos(\tilde{\omega}_0\, x \cos\theta) \\ & - \sin(\tilde{\omega}_0\, x \cos\theta) \sin[(\tilde{\omega} - \tilde{\omega}_0)\, x \cos\theta]. \end{aligned} \tag{2.65}$$

If this is substituted in Eq. (2.64), the cosine term does not contribute, so that

$$\lim_{x\to\infty} I_0 = -\frac{m}{\pi c^2} \lim_{x\to\infty} \int_0^{\pi/2} \int_{\tilde{\omega}_2}^{\infty} \frac{M \cos(\tilde{\omega}\, z \sin\theta) \sin(\tilde{\omega}_0\, x \cos\theta)}{g(\theta, \tilde{\omega})} \frac{\sin[(\tilde{\omega} - \tilde{\omega}_0)\, x \cos\theta]}{(\tilde{\omega} - \tilde{\omega}_0)}\, d\tilde{\omega} d\theta,$$

which gives by Eq. (2.59),

$$\lim_{x\to\infty} I_0 = -\frac{m}{c^2}\int_0^{\pi/2} \frac{M(\theta,\tilde{\omega}_0)\cos(\tilde{\omega}_0 z\sin\theta)\sin(\tilde{\omega}_0 x\cos\theta)}{g(\theta,\tilde{\omega}_0)}\,d\theta$$

This can be written as

$$\lim_{x\to\infty} I_0 = \frac{4pmg}{c^2}\int_0^{\pi/2} \frac{\sec^2\theta\exp[-\tilde{\omega}_0(h-f-y)]\cos(\tilde{\omega}_0 z\sin\theta)\sin(\tilde{\omega}_0 x\cos\theta)}{\rho e^{\tilde{\omega}_0 h}+(\rho'-\rho)e^{-\tilde{\omega}_0 h}}\,d\theta. \quad (2.66)$$

We note that this function can be interpreted throughout the top layer and, as can easily be shown, is a harmonic function there.

The integral I_1 is dealt with in a similar manner. First, we write

$$I_1 = \frac{m}{\pi c^2}\int_0^{\pi/2}\int_0^{\tilde{\omega}_2} \frac{M\cos(\tilde{\omega} z\sin\theta)\cos(\tilde{\omega} x\cos\theta)}{(\tilde{\omega}-\tilde{\omega}_0)\,\frac{g(\theta,\tilde{\omega})}{\tilde{\omega}-\tilde{\omega}_1}(\tilde{\omega}-\tilde{\omega}_1)}\,d\tilde{\omega}\,d\theta \quad (2.67)$$

and then use an expansion as in Eq. (2.65), with $\tilde{\omega}_1$ instead of $\tilde{\omega}_0$, to get

$$\lim_{x\to\infty} I_1 = -\frac{m}{c^2}\int_a^{\pi/2} \frac{M(\theta,\tilde{\omega}_1)\cos(\tilde{\omega}_1 z\sin\theta)\sin(\tilde{\omega}_1 x\cos\theta)}{(\tilde{\omega}_1-\tilde{\omega}_0)\,G(\theta,\tilde{\omega}_1)}\,d\theta \quad (2.68)$$

with $a = 0$ for $F < 1$, $a = \theta_c$ for $F > 1$, and

$$\begin{aligned} G(\theta,\tilde{\omega}_1) &= \left(\frac{\partial g}{\partial\tilde{\omega}}\right)_{\tilde{\omega}_1} \\ &= \rho' c^2\cos^2\theta\cosh\tilde{\omega}_1 h + \rho' h c^2\cos^2\theta\,\tilde{\omega}_1\sinh\tilde{\omega}_1 h \\ &\quad + \rho c^2\cos^2\theta\sinh\tilde{\omega}_1 h + \rho h c^2\cos^2\theta\,\tilde{\omega}_1\cosh\tilde{\omega}_1 h \\ &\quad - (\rho'-\rho)\,gh\cosh\tilde{\omega}_1 h. \end{aligned} \quad (2.69)$$

As before, the function represented by the integral in Eq. (2.68) can be interpreted throughout the top layer and is harmonic there. This completes the treatment of the integrals appearing in φ_0.

When the same process is applied to the integrals in φ_0', the results are

$$\lim_{x\to\infty} I_0' = \frac{4\rho mg}{c^2} \int_0^{\pi/2} \frac{\sec^2\theta \exp\left[-\tilde{\omega}_0(h-f-y)\right] \cos(\tilde{\omega}_0 z \sin\theta) \sin(\tilde{\omega}_0 x \cos\theta)}{\rho e^{\tilde{\omega}_0 h} + (\rho' - \rho) e^{-\tilde{\omega}_0 h}} d\theta \quad (2.70)$$

and

$$\lim_{x\to\infty} I_1' = -2\rho m \int_a^{\pi/2} \frac{R(\theta, \tilde{\omega}_1) e^{\tilde{\omega}_1 y} \cos(\tilde{\omega}_1 z \sin\theta) \sin(\tilde{\omega}_1 x \cos\theta)}{(\tilde{\omega}_1 - \tilde{\omega}_0) G(\theta, \tilde{\omega}_1)} d\theta, \quad (2.71)$$

with a and G defined as before. Both of these functions can be interpreted throughout the bottom layer and are harmonic there.

The above results yield the asymptotic nature of the potentials φ_0 and φ_0'. Further consideration of the integrals would indicate that they represent waves far ahead of the source in violation of the radiation condition. In order to satisfy the radiation condition, the asymptotic forms are subtracted from the potentials to get

$$\varphi = \varphi_0 - \lim_{x\to\infty} I_0 - \lim_{x\to\infty} I_1 \quad (2.72)$$

and

$$\varphi' = \varphi_0' - \lim_{x\to\infty} I_0' - \lim_{x\to\infty} I_1' . \quad (2.73)$$

That these functions represent the desired solution is clear from their harmonic nature and their satisfaction of the radiation condition; the remaining boundary conditions, which are satisfied by φ_0 and φ_0', continue to be satisfied by the potentials φ and φ' because the functions defined by the single integrals represent a system of waves which satisfy the same conditions as φ_0 and φ_0'. We note that the potentials φ and φ' are symmetric in z as expected.

Asymptotic Form of the Velocity Potential

The solutions φ, φ' for the velocity potential are not in a very useful form; however, it is possible to get an explicit form for these quantities by considering the region far behind the source. That is, we want to evaluate

$$\lim_{x\to-\infty} \varphi$$

and

$$\lim_{x\to-\infty} \varphi',$$

where the limits are again to be understood in an asymptotic sense.

First consider the potential for the top layer. The double integral appearing in φ_0 is reduced to a single integral by the same process employed earlier. Thus,

$$\lim_{x \to -\infty} \varphi_0 = \lim_{x \to -\infty} (I_0 + I_1),$$

with I_0 and I_1 defined as before. It is easily verified that

$$\lim_{x \to -\infty} I_0 = - \lim_{x \to \infty} I_0$$

and (2.74)

$$\lim_{x \to -\infty} I_1 = - \lim_{x \to \infty} I_1 .$$

These lead directly to the result that far behind the source

$$\varphi \approx - 2 \lim_{x \to \infty} I_0 - 2 \lim_{x \to \infty} I_1 ; \tag{2.75}$$

and, similarly,

$$\varphi' \approx - 2 \lim_{x \to \infty} I'_0 - 2 \lim_{x \to \infty} I'_1 . \tag{2.76}$$

In these expressions, we note that the functions indicated by the limit notation are defined for all x but here are to be evaluated far behind the source. They may be written in explicit terms as

$$\begin{aligned} \varphi \approx & - \frac{8 \rho m g}{c^2} \int_0^{\pi/2} \frac{\sec^2 \theta \exp [-\tilde{\omega}_0 (h-f-y)] \cos (\tilde{\omega}_0 z \sin \theta) \sin (\tilde{\omega}_0 x \cos \theta)}{\rho e^{\tilde{\omega}_0 h} + (\rho' - \rho) e^{-\tilde{\omega}_0 h}} d\theta \\ & + \frac{2m}{c^2} \int_a^{\pi/2} \frac{M (\theta, \tilde{\omega}_1) \cos (\tilde{\omega}_1 z \sin \theta) \sin (\tilde{\omega}_1 x \cos \theta)}{(\tilde{\omega}_1 - \tilde{\omega}_0) G (\theta, \tilde{\omega}_1)} d\theta \end{aligned} \tag{2.77}$$

and

$$\begin{aligned} \varphi' \approx & - \frac{8 \rho m g}{c^2} \int_0^{\pi/2} \frac{\sec^2 \theta \exp [-\tilde{\omega}_0 (h-f-y)] \cos (\tilde{\omega}_0 z \sin \theta) \sin (\tilde{\omega}_0 x \cos \theta)}{\rho e^{\tilde{\omega}_0 h} + (\rho' - \rho) e^{-\tilde{\omega}_0 h}} d\theta \\ & + 4 \rho m \int_a^{\pi/2} \frac{R (\theta, \tilde{\omega}_1) e^{\tilde{\omega}_1 y} \cos (\tilde{\omega}_1 z \sin \theta) \sin (\tilde{\omega}_1 x \cos \theta)}{(\tilde{\omega}_1 - \tilde{\omega}_0) G (\theta, \tilde{\omega}_1)} d\theta \end{aligned} \tag{2.78}$$

with $a = 0$, $F < 1$; $a = \theta_c$, $F > 1$.

The first integrals in Eqs. (2.77) and (2.78) are only slightly influenced by the density stratification as seen by the form of the denominators in the integrands. It is also clear that one is the analytical continuation of the other. In addition, the depth of the interface h has little influence on these integrals as long as $h-f$ (the depth of the source beneath the free surface) and $h-y$ (the depth of the observation point beneath the free surface) are fixed. The first integrals describe the usual surface wave system which, apart

from the slight influence of the stratification, the source would generate in a homogeneous ocean of infinite depth. For further remarks on the effect of the stratification in contrast to the homogeneous case but for the problem of the traveling pressure point on the surface of a two-layer system, we refer to Crapper (3). The surface wave system is not dealt with further since it differs little from the homogeneous case, which has been discussed by several authors (4).

The second integrals in Eqs. (2.77) and (2.78) describe the internal wave system generated by the uniform flow past the source; we designate them by I_2 and I_2', respectively, and drop the subscript on $\tilde{\omega}_1$ as it is no longer needed. A simplified form of these integrals is obtained by use of the identity

$$\sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$$

and by the introduction of polar coordinates (R, γ) to describe the (x, z) plane through

$$\begin{aligned} x &= R \cos \gamma \\ z &= R \sin \gamma. \end{aligned} \tag{2.79}$$

We find that

$$\sin (\tilde{\omega} x \cos \theta) \cos (\tilde{\omega} z \sin \theta) = \frac{1}{2} \operatorname{Im} \left\{ \exp [iR\tilde{\omega} \cos (\theta - \gamma)] + \exp [iR\tilde{\omega} \cos (\theta - \gamma)] \right\}. \tag{2.80}$$

Also, since $\tilde{\omega}$ is known in terms of θ through Eq. (2.54), the integration variable in both integrals can be changed to $\tilde{\omega}$. The integrals become

$$I_2 = \operatorname{Im} \int_b^\infty \frac{m}{c^2} \frac{M \frac{d\theta}{d\tilde{\omega}}}{(\tilde{\omega} - \tilde{\omega}_0) G (\theta, \tilde{\omega})} \left\{ \exp [iR\tilde{\omega} \cos (\theta - \gamma)] + \exp [iR\tilde{\omega} \cos (\theta + \gamma)] \right\} d\tilde{\omega} \tag{2.81}$$

and

$$I_2' = \operatorname{Im} \int_b^\infty 2\rho m \frac{R (\theta, \tilde{\omega}) e^{\tilde{\omega} y} \frac{d\theta}{d\tilde{\omega}}}{(\tilde{\omega} - \tilde{\omega}_0) G (\theta, \tilde{\omega})} \left\{ \exp [iR\tilde{\omega} \cos (\theta - \gamma)] + \exp [iR\tilde{\omega} \cos (\theta + \gamma)] \right\} d\tilde{\omega} \tag{2.82}$$

with

$$b = \tilde{\omega} (0), \ F < 1; \ b = 0, \ F > 1.$$

If we further introduce the dimensionless variable

$$Z = h\tilde{\omega} \tag{2.83}$$

and the quantities

$$h_+ (Z) = Z \cos (\theta - \gamma) \tag{2.84}$$

and

$$h_-(Z) = Z\cos(\theta + \gamma), \tag{2.85}$$

then the integrals may be written as

$$I_2 = \mathrm{Im}\int_{Z^*}^{\infty} T(Z)\left\{\exp\left[i\,\frac{Rh_+(Z)}{h}\right] + \exp\left[i\,\frac{Rh_-(Z)}{h}\right]\right\}dZ \tag{2.86}$$

and

$$I_2' = \mathrm{Im}\int_{Z^*}^{\infty} B(Z)\left\{\exp\left[i\,\frac{Rh_+(Z)}{h}\right] + \exp\left[i\,\frac{Rh_-(Z)}{h}\right]\right\}dZ, \tag{2.87}$$

where

$$Z^* = h\tilde{\omega}(0) \text{ for } F < 1,$$

$$Z^* = 0 \text{ for } F > 1,$$

$$T(Z) = \frac{m}{c^2}\,\frac{M(\theta,\tilde{\omega})\,\dfrac{d\theta}{dZ}}{(\tilde{\omega}-\tilde{\omega}_0)\,G(\theta,\tilde{\omega})}, \tag{2.88}$$

and

$$B(Z) = 2\rho m\,\frac{R(\theta,\tilde{\omega})\,e^{\tilde{\omega}y}\,\dfrac{d\theta}{dZ}}{(\tilde{\omega}-\tilde{\omega}_0)\,G(\theta,\tilde{\omega})}. \tag{2.89}$$

The integrals are now in a form appropriate for the method of stationary phase. Because of the symmetry of the disturbance with respect to the x axis, we can restrict γ to the interval

$$\frac{\pi}{2} < \gamma \leqslant \pi.$$

According to the method of stationary phase (5), the main contributions to the two integrals for large R arise from the stationary points given by

$$h_+'(Z) = 0, \tag{2.90}$$

$$h_-'(Z) = 0, \tag{2.91}$$

where the prime indicates differentiation with respect to Z. The general form of the main contribution to the value of I_2 made by a stationary point (with a similar form applying to I_2') is

$$\mathrm{Im}\left|\frac{2\pi h}{R h''(Z)}\right|^{1/2} T(Z)\exp\left[i\frac{Rh(Z)}{h} + \frac{i\pi}{4}\,\mathrm{sgn}\, h''(Z)\right] \tag{2.92}$$

where h(Z) designates the function of either Eq. (2.84) or (2.85). The stationary points depend on γ, and, if for a given γ there is no stationary point, the integral is of the order R^{-1} at most.

The condition of Eq. (2.90) can be written as

$$\tan(\theta - \gamma) = \frac{1}{Z}\frac{dZ}{d\theta} = a, \tag{2.93}$$

which gives

$$-\tan\gamma = \frac{a - \tan\theta}{1 + a\tan\theta} \tag{2.94}$$

or

$$-\tan\gamma = \frac{A\sinh Z + Z}{2F^2 Z A^2 - A\sinh Z - Z}\left(\frac{F^2 Z A - \sinh Z}{\sinh Z}\right)^{1/2} \tag{2.95}$$

with

$$A = \cosh Z + \frac{\rho}{\rho'}\sinh Z. \tag{2.96}$$

For each Z in the permitted range, the right side of Eq. (2.95) is nonnegative and we find a γ which must lie in the second quadrant. Thus, the terms in the integrals involving $h_+(Z)$ make a contribution to the asymptotic values for γ in the second quadrant.

As for the stationary points associated with $h_-(Z)$, they are determined by Eq. (2.95) with the sign of the left side made positive. We find that $h_-(Z)$ has stationary points only for γ in the third quadrant. The terms involving $h_-(Z)$ need not be considered further since they lead to a contribution to the integrals such as to insure symmetry with respect to the x axis. This permits us to limit our attention solely to the stationary points of $h_+(Z)$.

We note that the form of Eq. (2.92) requires the function $h''_+(Z)$ for which a simple result can be found. The second derivative evaluated at a stationary point is

$$h''_+(Z) = -\cos(\theta - \gamma)\left[\sec^2(\theta - \gamma) Z\left(\frac{d\theta}{dZ}\right)^2 + \tan(\theta - \gamma)\frac{d}{dZ}\left(Z\frac{d\theta}{dZ}\right)\right].$$

But Eq. (2.93) gives

$$Z\frac{d\theta}{dZ} = \frac{1}{\tan(\theta - \gamma)},$$

where, here, $\gamma = \gamma(Z)$. Thus,

$$\frac{d}{dZ}\left(Z\frac{d\theta}{dZ}\right) = -\frac{1}{\sin^2(\theta - \gamma)}\left(\frac{d\theta}{dZ} - \frac{d\gamma}{dZ}\right),$$

and we finally obtain, at a stationary point,

$$h''_+(Z) = -\frac{1}{\sin(\theta-\gamma)}\frac{d\gamma}{dZ}. \tag{2.97}$$

This simple result will prove useful later, in the discussion of the nature of the wave system.

Returning to Eq. (2.95) for the stationary points of $h_+(Z)$, we see that it defines a relation $\gamma(Z)$. The behavior of this function is difficult to ascertain; however, there is a simple consequence for the supercritical case. We must have in this case

$$\theta_c \leqslant \theta \leqslant \frac{\pi}{2}. \tag{2.98}$$

Write

$$\gamma = \pi - t,$$

where

$$0 \leqslant t \leqslant \frac{\pi}{2};$$

then

$$\tan(\theta-\gamma) = \tan(\theta+t),$$

which must be nonnegative by Eq. (2.93). This, together with Eq. (2.98), gives

$$\theta_c + t \leqslant \theta + t \leqslant \frac{\pi}{2},$$

so that

$$\theta_c \leqslant \frac{\pi}{2} - t$$

or

$$\cos\theta_c = \frac{1}{F} \geqslant \sin t.$$

Hence,

$$t \leqslant \sin^{-1}\left(\frac{1}{F}\right),$$

or, equivalently,

$$\gamma \geqslant \gamma_c = \pi - \sin^{-1}\left(\frac{1}{F}\right). \tag{2.99}$$

This result indicates that in the supercritical case there are no stationary points for angles less than the critical angle γ_c; consequently, the main disturbance is confined to an angular sector symmetric to the x axis with limiting angles given by $\pm\gamma_c$.

We shall now make the following assumptions about the nature of the $\gamma(Z)$ relationship:

For the supercritical case, we assume that as Z increases from its minimum value of zero γ increases from its minimum value γ_c and approaches the value π asymptotically. Figure 3 gives a sketch of the assumed behavior.

For the subcritical case, we assume that as Z increases from its minimum value γ decreases from the value π to a minimum value, which determines the limiting angle γ_c of the wake region, and then increases ultimately to make an asymptotic approach to the value π as Z tends to infinity. Figure 4 gives a sketch of the assumed behavior.

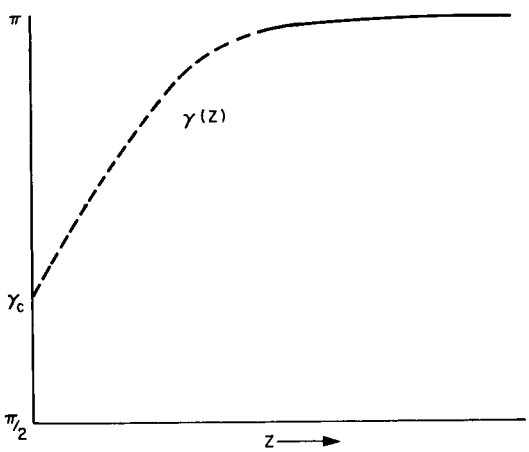

Fig. 3 - The behavior of the $\gamma(Z)$ curve for the supercritical case. The dashed portion of the curve represents the conjectural nature of the curve in the interval of Z where neither the small nor large Z behavior applies.

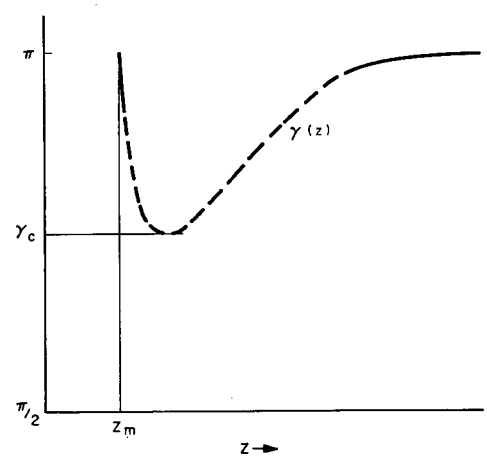

Fig. 4 - The behavior of the $\gamma(Z)$ curve for the subcritical case. The dashed portion of the curve represents the conjectured form for intermediate values of Z between Z_m and moderately large Z.

A complete analytical justification of these assumptions does not appear possible; and, to avoid their use in any given situation, a numerical evaluation of $\gamma(Z)$ could be performed. Hudimac made similar assumptions based on the numerical treatment of a few cases. In the next section, the $\gamma(Z)$ relationship will be considered in some detail and support will be given for the assumed behavior.

On the basis of these assumptions, we can give the main contributions to the values of I_2 and I_2' as provided by the method of stationary phase. For the subcritical case with

$$\gamma_c < \gamma \leqslant \pi,$$

there are two stationary points Z^* and Z^{**} which coalesce at $\gamma = \gamma_c$, and

$$I_2 \approx \operatorname{Im}\left\{\left|\frac{2\pi h}{Rh_+''(Z^*)}\right|^{1/2} T(Z^*)\exp\left[i\,\frac{Rh_+(Z^*)}{h} - \frac{i\pi}{4}\right] + \left|\frac{2\pi h}{Rh_+''(Z^{**})}\right|^{1/2} T(Z^{**})\exp\left[i\,\frac{Rh_+(Z^{**})}{h} + \frac{i\pi}{4}\right]\right\}, \tag{2.100}$$

where $Z^* \leqslant Z^{**}$. A similar result exists for I_2' with the function T replaced by the function B. For angles less than the critical angle, there are no stationary points and the asymptotic values of the integrals are of a smaller order than the above.

For the supercritical case with

$$\gamma_c \leqslant \gamma \leqslant \pi,$$

there is only one stationary point Z, and

$$I_2 \approx \operatorname{Im}\left\{\left|\frac{2\pi h}{Rh_+''(Z)}\right|^{1/2} T(Z) \exp\left[i\,\frac{Rh_+(Z)}{h} + \frac{i\pi}{4}\right]\right\}, \tag{2.101}$$

with a similar expression for I_2' in which T(Z) is replaced by B(Z). Again we find that for angles less than the critical angle the integrals are of a smaller order than the above.

The quantities T(Z) and B(Z) appearing in the above results were given earlier; but, a simplification of their form is possible which we now obtain. From Eqs. (2.88) and (2.89), we have

$$T(Z) = \frac{mh}{c^2}\,\frac{M(\theta,\tilde{\omega})\,\dfrac{d\theta}{dZ}}{(Z - h\tilde{\omega}_0)\,G(\theta,\tilde{\omega})}$$

and

$$B(Z) = 2\rho mh\,\frac{R(\theta,\tilde{\omega})\,e^{\tilde{\omega}y}\,\dfrac{d\theta}{dZ}}{(Z - h\tilde{\omega}_0)\,G(\theta,\tilde{\omega})}.$$

We can write

$$Z - h\tilde{\omega}_0 = \frac{\sec^2\theta}{c^2}\,D, \tag{2.102}$$

where

$$D = c^2 Z \cos^2\theta - gh. \tag{2.103}$$

After some simplification, the result that

$$\frac{\dfrac{d\theta}{dZ}}{(Z - h\tilde{\omega}_0)\,G(\theta,\tilde{\omega})} = \frac{1}{2\rho'\,DZA\tan\theta} \tag{2.104}$$

is obtained in which A was defined earlier by Eq. (2.96). It is useful to introduce the quantity

$$C = c^2 Z \cos^2\theta + gh. \tag{2.105}$$

These expressions enable us to write

$$T(Z) = -\frac{m}{c^2 h}\frac{1}{2\rho' DZA \sin\theta \cos\theta}\Bigg(C(\rho C + \rho' D)\exp\left[-Z\left(1 - \frac{f}{h} - \frac{y}{h}\right)\right]$$

$$+ (\rho' - \rho) CD \left\{\exp\left[-Z\left(1 + \frac{f}{h} - \frac{y}{h}\right)\right] + \exp\left[-Z\left(1 - \frac{f}{h} + \frac{y}{h}\right)\right]\right\} \tag{2.106}$$

$$- (\rho' - \rho) D^2 \exp\left[Z\left(1 - \frac{f}{h} - \frac{y}{h}\right)\right]\Bigg)$$

and

$$B(Z) = \frac{\rho m}{h}\frac{1}{\rho' DA \tan\theta}\left\{D\exp\left[Z\left(1 - \frac{f}{h}\right)\right] - C\exp\left[-Z\left(1 - \frac{f}{h}\right)\right]\right\} e^{Z\frac{y}{h}} . \tag{2.107}$$

These equations, with Eqs. (2.100) and (2.101), give the asymptotic form for the velocity potential associated with the internal wave system which is confined to an angular region symmetric with the x axis, outside of which the disturbance is of a smaller order. The nature of the wave system is discussed later in terms of the displacement function, which provides a more direct description of the disturbance than does the velocity potential.

The Nature of the $\gamma(Z)$ Relationship

We have found the asymptotic form of the velocity potential associated with the internal wave system on the basis of specific assumptions concerning the nature of the $\gamma(Z)$ relationship. In this section, we examine the continuous function $\gamma(Z)$ with a view to giving support to the assumptions made. The behavior of this function depends on the Froude number, and we consider the supercritical and subcritical cases separately.

For the supercritical case, we found that stationary points only occur for

$$\gamma_c \leqslant \gamma \leqslant \pi,$$

where γ_c is given by Eq. (2.99). The dependence of γ_c on the Froude number is shown in Fig. 5. It is interesting to note that the critical angle for the supercritical case is not influenced by the density ratio but is solely determined by the Froude number. We have then

$$\gamma(Z) \geqslant \gamma_c,$$

where the range of Z comprises

$$0 \leqslant Z < \infty.$$

It is easy to show by Eq. (2.95) that

$$\gamma(0) = \gamma_c \tag{2.108}$$

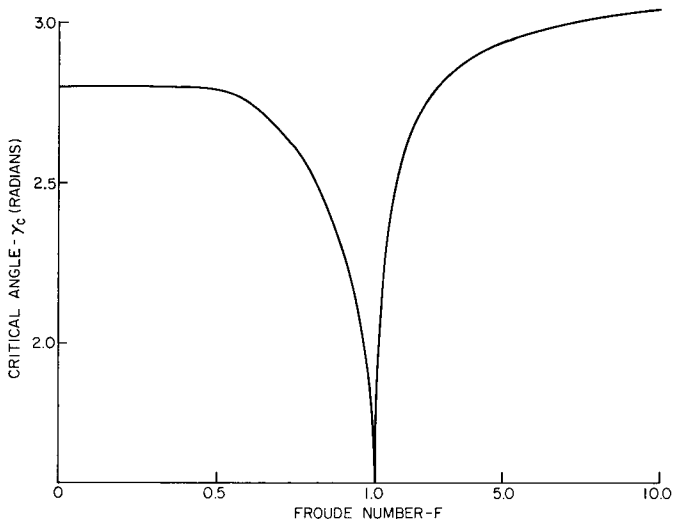

Fig. 5 - The dependence of the critical angle on the Froude number with density ratio unity. The value of the density ratio affects the dependence only in the subcritical region; in setting the ratio equal to unity, we consider a limiting situation.

and

$$\left(\frac{d\gamma}{dZ}\right)_{Z=0} = \frac{\rho}{\rho'} (F^2 - 1)^{-1/2} , \tag{2.109}$$

which indicate that $\gamma(Z)$ is an increasing function of Z near $Z = 0$ with a slope there dependent on density ratio as well as Froude number. At large Z, Eq. (2.95) yields the result that

$$\pi - \gamma \approx \frac{1}{2}\left[F^2 Z \left(1 + \frac{\rho}{\rho'}\right)\right]^{-1/2} , \tag{2.110}$$

which gives a fair approximation for moderately large Z as well. This relationship indicates that γ asymptotically approaches the value π as Z tends to infinity. We also have the expression

$$\frac{d\gamma}{dZ} \approx \frac{1}{4Z^{3/2}}\left[F^2 \left(1 + \frac{\rho}{\rho'}\right)\right]^{-1/2} \tag{2.111}$$

for the slope in the region of large Z.

The above results essentially determine the nature of the $\gamma(Z)$ curve for either very small or moderately large values of Z. A sketch of the typical behavior is given in Fig. 3. For those Z which do not lie in these limiting regions, further analysis is necessary to determine the behavior of the function. This can be accomplished by numerical calculation for any given case as was done by Hudimac. For our purposes, we assume that the forms for small and large Z join smoothly in the manner indicated.

The subcritical case is more complicated. We have in this case

$$0 \leqslant \theta \leqslant \frac{\pi}{2}$$

and

$$0 < Z_m \leqslant Z < \infty,$$

where the minimum Z_m occurs at $\theta = 0$ and is found from

$$\frac{AZ}{\sinh Z} = F^{-2} . \tag{2.112}$$

In terms of γ, this minimum value of Z occurs at $\gamma = \pi$ and, as Z increases from Z_m, γ must decrease from π. Consideration of the derivative indicates that as

$$Z \to Z_m$$

$$\frac{d\gamma}{dZ} \approx - \frac{\text{constant}}{\sin\theta} , \tag{2.113}$$

which tends to $-\infty$. For large Z, Eq. (2.95) yields

$$\pi - \gamma \approx \frac{1}{2}\left[F^2 Z \left(1 + \frac{\rho}{\rho'}\right)\right]^{-1/2} \tag{2.114}$$

as in the supercritical case. This demonstrates the asymptotic approach of γ to π as Z tends to infinity. Between Z_m and infinity there must be at least one minimum value of $\gamma(Z)$; and, we assume here that there is only one minimum, which is denoted by γ_c. This angle determines the limiting angle of the wake since for $\gamma < \gamma_c$ there will be no stationary points. Fig. 4 presents a sketch of the general nature of $\gamma(Z)$ for the subcritical case. Again it seems that the simplest means for examining the behavior of γ for intermediate values of Z is by numerical evaluation of specific cases. Hudimac gives a calculated $\gamma(Z)$ which agrees with the general form we indicate.

The conjectured form of $\gamma(Z)$ can be verified for a restricted range of Froude numbers near zero. In this case, Eq. (2.112) indicates that Z_m is large and is given approximately by

$$Z_m = \frac{1}{\left(1 + \frac{\rho}{\rho'}\right) F^2} .$$

This result can be expected to hold up to about F = 1/2. Since $Z \geqslant Z_m \gg 1$, Eq. (2.95) can be approximated by

$$-\tan\gamma = \frac{Z_m^{1/2}}{2Z - Z_m} \left(Z - Z_m\right)^{1/2} \tag{2.115}$$

for all Z in the appropriate range. This equation indicates that at $Z = Z_m$ and as $Z \to \infty$ we must have $\gamma = \pi$. It yields the result that

$$\frac{d\gamma}{dZ} = Z_m^{1/2} \cos^2 \gamma \frac{Z - \frac{3}{2} Z_m}{(2Z - Z_m)^2 (Z - Z_m)^{1/2}}. \tag{2.116}$$

The implications of this result are

$$\frac{d\gamma}{dZ} < 0 \qquad \text{for } Z < \frac{3}{2} Z_m,$$

$$\frac{d\gamma}{dZ} > 0 \qquad \text{for } Z > Z_m,$$

and the derivative vanishes in the finite range only at

$$Z = \frac{3}{2} Z_m. \tag{2.117}$$

This agrees with our conjectured behavior for $\gamma(Z)$ in the subcritical case. It follows from Eqs. (2.117) and (2.115) that the critical angle is found from

$$-\tan \gamma_c = \frac{1}{2\sqrt{2}},$$

which gives

$$\pi - \gamma_c = 19^\circ\, 28'. \tag{2.118}$$

This is exactly the same as the classical value for the Kelvin wake.

The general condition which determines the critical angle for the subcritical case is

$$\frac{d\gamma}{dZ} = 0,$$

where the derivative is evaluated by use of Eq. (2.95). It is clear that the critical angle depends on Froude number and density ratio; thus

$$\gamma_c = \gamma_c \left(F, \frac{\rho}{\rho'}\right).$$

This result is in contrast to the supercritical case where the Froude number alone determines the critical angle.

From our previous consideration of the range of small Froude number, we can state that the critical angle is essentially constant over the first half of the subcritical regime. It is also possible to find an

approximate expression for the critical angle when the Froude number is near one. For this case, Z_m is near zero and we assume that the minimum value of $\gamma(Z)$ occurs and for Z near zero—an assumption whose consistency will be demonstrated. We are thus mainly interested in $\gamma(Z)$ for the small Z range. Since $Z = Z(\theta)$, we have

$$\gamma = \gamma(\theta), \qquad \text{with } 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

The condition for the critical angle now becomes

$$\frac{d\gamma}{d\theta} = 0$$

with γ given by Eq. (2.94), where

$$a = 2 \tan\theta \frac{A \sinh Z}{A \sinh Z - Z}. \tag{2.119}$$

We find that

$$-\tan\gamma = \frac{Za - Z\tan\theta}{Z + Za\tan\theta}. \tag{2.120}$$

From Eq. (2.120), the condition for the critical angle becomes

$$Z^2 - Z(Za)' + 2(Za)^2 = 0, \tag{2.121}$$

where the prime indicates differentiation with respect to θ. Expansions for small θ can be used to solve Eq. (2.121). From Eqs. (2.54) and (2.119), it follows that

$$Z = \frac{\rho'}{\rho}(\epsilon + \theta^2 + \ldots) \tag{2.122}$$

and

$$Za = 2\frac{\rho'}{\rho}\theta + \ldots, \tag{2.123}$$

where

$$\epsilon = 1 - F^2. \tag{2.124}$$

Substitution of Eqs. (2.122) and (2.123) into Eq. (2.121) yields

$$\theta = \left(\frac{\epsilon}{3}\right)^{1/2},$$

so that at the critical angle we have

$$Z = \frac{4}{3}\frac{\rho'}{\rho}\epsilon,$$

$$Za = 2\frac{\rho'}{\rho}\left(\frac{\epsilon}{3}\right)^{1/2}.$$

The consistency of the earlier assumption about the location of the value of Z for the critical point is demonstrated by these results. They provide, through Eq. (2.120),

$$\pi - \gamma_c = \tan^{-1}(3\epsilon)^{-1/2}, \tag{2.125}$$

$$\epsilon = 1 - F^2$$

for the approximate value of the critical angle when the Froude number is near one.

We now have approximations for the critical angle valid for small F and for F near one. In both cases the critical angle depends solely on the Froude number. The behavior of the critical angle for intermediate values of the Froude number can be determined numerically; and, in Fig. 5, the results of such a calculation for density ratio equal to one are given. We note that the critical angle is constant up to about 0.5 in agreement with an earlier conclusion. Also, our approximate result for the critical angle when F is near one fairs in well with the exact numerical results. Calculations of the critical angle were made for

$$0.97 \leqslant \frac{\rho}{\rho'} \leqslant 1.00,$$

which should include the majority of cases relevant to the oceans. The effect of the density ratio is extremely small. For example, a 1% change in the density ratio produces at most a 1/20% change in the critical angle for F = 0.75, with this change decreasing to zero as the Froude number is made larger or smaller than 0.75.

This completes our consideration of the behavior of $\gamma(Z)$. An analytical basis has now been given for the earlier assumptions about the nature of this function which enabled us to find the asymptotic form of the velocity potential. While not complete in form, it does offer more insight than the mere numerical calculation of different cases used by Hudimac to justify his solution for the velocity potential. The dependence of the critical angle on the Froude number and the density ratio also was investigated with simplified forms found near the endpoints of the interval of subcritical Froude number. An interesting result is the slight effect of the density ratio on the critical angle for values of the density ratio appropriate for the ocean environment. Further remarks related to the behavior of $\gamma(Z)$ appear in the following section on the displacement function.

Evaluation and Discussion of the Displacement Function

Using the asymptotic form of the velocity potential found earlier, we can proceed to evaluate the displacement function, which is defined in general by Eq. (2.10). Our interest is confined to the region inside the wake angle which lies far behind the source. The displacement associated with the internal wave system in this region is found from

$$\eta_x \approx \frac{1}{c}\frac{\partial}{\partial y} I_2 \tag{2.126}$$

and

$$\eta'_x \approx \frac{1}{c}\frac{\partial}{\partial y} I'_2 . \tag{2.127}$$

We shall consider first the supercritical case and then the subcritical case.

The above expressions, together with the results found earlier for I_2 and I'_2, yield for the supercritical case

$$\begin{aligned}\eta_x \approx -\frac{m}{c^3 h}\left|\frac{2\pi h}{R h''_+(Z)}\right|^{1/2} \frac{1}{2\rho' h\, DA \sin\theta \cos\theta}\Bigg(&C(\rho C + \rho' D)\exp\left[-Z\left(1-\frac{f}{h}-\frac{y}{h}\right)\right] \\ &+ (\rho'-\rho) C D \left\{\exp\left[-Z\left(1+\frac{f}{h}-\frac{y}{h}\right)\right] - \exp\left[-Z\left(1-\frac{f}{h}+\frac{y}{h}\right)\right]\right\} \\ &+ (\rho'-\rho) D^2 \exp\left[Z\left(1-\frac{f}{h}-\frac{y}{h}\right)\right]\Bigg)\sin\left[\frac{RZ\cos(\theta-\gamma)}{h}+\frac{\pi}{4}\right]\end{aligned} \tag{2.128}$$

and

$$\begin{aligned}\eta'_x \approx \frac{\rho m}{ch}\left|\frac{2\pi h}{R h''_+(Z)}\right|^{1/2} \frac{Z}{\rho' h\, DA \tan\theta}&\left\{D\exp\left[Z\left(1-\frac{f}{h}\right)\right] - C\exp\left[-Z\left(1-\frac{f}{h}\right)\right]\right\} \\ &e^{Z\frac{y}{h}} \sin\left[\frac{RZ\cos(\theta-\gamma)}{h}+\frac{\pi}{4}\right].\end{aligned} \tag{2.129}$$

We consider these as differential equations for the displacement function, and from their form it seems reasonable to assume solutions having the form

$$\eta \approx \frac{m}{cR^{1/2}h^{1/2}}\,\eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right)\cos\left[\frac{RZ\cos(\theta-\gamma)}{h}+\frac{\pi}{4}\right] \tag{2.130}$$

and

$$\eta' \approx \frac{m}{cR^{1/2}h^{1/2}}\,\eta'\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right)\cos\left[\frac{RZ\cos(\theta-\gamma)}{h}+\frac{\pi}{4}\right]. \tag{2.131}$$

In these expressions, the factor

$$\eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right)$$

and the analogous term for the bottom layer are called the dimensionless amplitude functions; the trigonometric factor in each expression is referred to as the phase term.

It is necessary to find the dimensionless amplitude terms. This is done by evaluating the derivatives of Eqs. (2.130) and (2.131) with respect to x and requiring the results to be identical with the right sides of Eqs. (2.128) and (2.129). It is convenient to use

$$\eta_x = \eta_R \cos\gamma - \eta_\gamma \frac{\sin\gamma}{R} \tag{2.132}$$

with a similar expression for η'_x. The derivatives are to be evaluated to the lowest order, and Eq. (2.132) indicates that the differentiations need be performed only on the phase terms. Carrying out this process, we arrive at the results that

$$\begin{aligned}\eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = \left|\frac{2\pi}{h''_+(Z)}\right|^{1/2} \frac{1}{2\rho' c^2 DZA \sin\theta \cos^2\theta} \Bigg(& C(\rho C + \rho' D) \exp\left[-Z\left(1 - \frac{f}{h} - \frac{y}{h}\right)\right] \\ &+ (\rho' - \rho) CD \left\{\exp\left[-Z\left(1 + \frac{f}{h} - \frac{y}{h}\right)\right] - \exp\left[-Z\left(1 - \frac{f}{h} + \frac{y}{h}\right)\right]\right\} \\ &+ (\rho' - \rho) D^2 \exp\left[Z\left(1 - \frac{f}{h} - \frac{y}{h}\right)\right]\Bigg)\end{aligned} \tag{2.133}$$

and

$$\begin{aligned}\eta'\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = -\left|\frac{2\pi}{h''_+(Z)}\right|^{1/2} \frac{\rho}{\rho' DA \sin\theta} \\ \left\{D \exp\left[Z\left(1 - \frac{f}{h}\right)\right] - C \exp\left[-Z\left(1 - \frac{f}{h}\right)\right]\right\} e^{Z\frac{y}{h}}.\end{aligned} \tag{2.134}$$

These provide the dimensionless amplitude functions and, by Eqs. (2.130) and (2.131), the displacement function at all depths for the supercritical case.

The displacement function for the subcritical case also satisfies Eqs. (2.126) and (2.127), where there are now two terms in the asymptotic approximation of each integral due to the existence of the two stationary points Z^* and Z^{**} ($Z^* < Z^{**}$) for each $\gamma > \gamma_c$. Employing the same process as in the supercritical case, we find that

$$\begin{aligned}\eta \approx \frac{m}{c R^{1/2} h^{1/2}} \eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) \cos\left[\frac{RZ\cos(\theta - \gamma)}{h} - \frac{\pi}{4}\right]\Bigg|_{Z = Z^*} \\ + \frac{m}{c R^{1/2} h^{1/2}} \eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) \cos\left[\frac{RZ\cos(\theta - \gamma)}{h} + \frac{\pi}{4}\right]\Bigg|_{Z = Z^{**}}\end{aligned} \tag{2.135}$$

with an analogous expression for the displacement in the bottom layer in which the dimensionless amplitude functions above are replaced by those appropriate for the bottom layer. The dimensionless amplitude functions are given by Eqs. (2.133) and (2.134).

We now have the asymptotic form of the displacement function for both regimes of Froude number. Hudimac (1) evaluated the dimensionless amplitude at the interface for specific values of the parameters and gives drawings of the locations of the constant-phase lines. He also presents illustrations of the behavior of the displacement of each wave system along a constant phase line. In the supercritical case, the wave system consists of diverging waves which are convex to the axis of the wake (the negative x axis). For the subcritical case, there are a transverse wave system represented by the first term in Eq. (2.135) and a divergent system given by the second term in Eq. (2.135). The transverse wave system has its crests normal to the wake axis, whereas the crests of the divergent system are convex to the wake axis and intersect the boundary line of the wake determined by the critical angle.

The general situation is essentially the same as that for an ocean of finite depth (6). That this should be so can be seen by the following argument. Since only the properties of the internal wave system are of interest, the fact that the internal wave amplitude is negligible at the free surface (38) can be used to replace the free surface by a rigid horizontal plane. If we then imagine the entire configuration inverted, we have the gravitational force directed upward with the heavier fluid resting on top of a lighter fluid of finite depth with a rigid boundary below. The direction of the gravitational force, however, is mainly important for determining the static equilibrium configuration. Thus, we expect that the internal wave problem associated with the source should correspond to the wave problem of a source in an ocean of finite depth with an atmosphere possessing inertia. The inertia of the atmosphere, while modifying certain properties of the system such as the wave speed and amplitude, is not of major significance; it is, rather, the finite depth of the ocean which must be considered of primary importance in determining the properties of the wave system. This argument indicates that there should be a general qualitative similarity between our internal wave problem and the problem of a source in a finite-depth ocean with an atmosphere of negligible inertia. It also lends further support to the earlier assumptions about the general nature of the $\gamma(Z)$ relationship on which our solution is based.

We now proceed to examine the properties of the solutions for the displacement. The structure of the solution is the same for the two cases of subcritical and supercritical flow, with the essential difference being that in the supercritical case there is one wave system while in the subcritical case there are two wave systems. Each wave system is described by the product of three factors, each of which can be associated with specific physical aspects of the displacement. The first factor represents a simple radial attenuation and gives the only dependence of the amplitude of the wave system on radial distance from the source. The dimensionless amplitude function describes the effects of angular location, depth of the observation point, depth of the source, Froude number, and density ratio on the amplitude of the wave system. Next is the phase term which defines the actual pattern of the wave system. It depends solely on the dimensionless polar coordinates R/h and γ, Froude number, and density ratio; consequently, the pattern of the wave system is the same at all depths.

The behavior of the dimensionless amplitude function is of interest, especially near the edge of the wake and along the wake axis. We treat the supercritical case first. As $\gamma \to \pi$, we have $Z \to \infty$ and

$$h_+''(Z) = O(Z^{-3/2}), \tag{2.136}$$

so that

$$\eta'\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = O\left[Z^{3/4} \exp\left(Z\,\frac{y-f}{h}\right)\right], \tag{2.137}$$

with a similar result for η (F, ρ/ρ', f/h, y/h, γ). Hence, the dimensionless amplitude tends to zero at the wake axis. As $\gamma \to \gamma_c$, $Z \to 0$, and from Eq. (2.109) we find

$$\lim_{Z\to 0} \eta'(\ldots) = -\left[8\pi \frac{\rho}{\rho'} F^2 (F^2-1)^{-1/2}\right]^{1/2}. \tag{2.138}$$

We also have

$$\lim_{Z\to 0} \eta(\ldots) = \lim_{Z\to 0} \eta'(\ldots) \left(1 - \frac{\rho'}{\rho}\frac{y}{h}\right). \tag{2.139}$$

These results indicate that the dimensionless amplitude approaches a finite value at the critical angle. A peculiar behavior of the limit for the bottom layer exists in the fact that it does not change with depth. Further study is necessary in order to describe the manner in which the wave system interior to the critical angle joins with that outside the critical angle; this should resolve the above difficulty. We conclude that in the supercritical case the amplitude of the wave system far behind the source remains finite throughout the wake region.

In the subcritical case, there are two wave systems to be considered. At the critical angle,

$$h_+'' = 0$$

for both the transverse and diverging wave systems. This implies that the displacement becomes infinite as the critical angle is approached, whatever the depth may be. The singularity is due to a failure of the method of stationary phase in the form employed above. A similar difficulty arises in the classical ship wave problem (5). To find the correct behavior in the vicinity of the critical angle, a more careful consideration is required; it appears that the result will be that, on the critical line itself, the displacement decays as $1/\sqrt[3]{R}$ with a smooth transition from the solution interior to the wake angle to that in the exterior region.

The behavior of the wave systems is different as the wake axis is approached. For the diverging wave system, we have $Z \to \infty$ as $\gamma \to \pi$, and we can demonstrate in a manner similar to the supercritical case that

$$\eta(\ldots) \to 0$$

and

$$\eta'(\ldots) \to 0.$$

Thus, the amplitude of the diverging wave system vanishes at the wake axis. For the transverse wave system, $Z \to Z_m$ and $\theta \to 0$ as $\gamma \to \pi$. It is found that

$$\frac{d\gamma}{dZ} = O\left(\frac{1}{\sin\theta}\right)$$

and

$$h''_+(Z) = O\left(\frac{1}{\sin^2\theta}\right)$$

in this limit. This singularity in the derivative is overcome by the other factor in the dimensionless amplitude function, and the result is a finite amplitude for the transverse wave system at the wake axis.

We now proceed to discuss certain consequences of the phase term. As noted above, the phase term defines the actual pattern of the wave system, which is the same at all depths. The general nature of the divergent and transverse waves has been mentioned earlier with reference made to Hudimac for detailed calculations. It is clear from the solutions obtained that for each γ lying in the wake region there is a constant wavelength defined along this ray for both the divergent and the transverse waves. In the supercritical case, the wavelength λ is given by

$$-\frac{Z\cos(\theta-\gamma)}{h} = \frac{2\pi}{\lambda(Z)}; \tag{2.140}$$

it attains an infinite value at the critical angle but decreases to zero at the wake axis. The wavelengths of the divergent and transverse wave systems in the subcritical case are found by evaluating Eq. (2.140) for the appropriate values of Z. For the transverse wave system, the wavelength has a finite, nonzero value at the wake axis and does not change appreciably as the critical line is approached. On the other hand, the wavelength of the divergent system vanishes at the wake axis and increases to attain the value of the transverse wavelength at the critical line.

The wavelengths defined by Eq. (2.140) for the various cases are the natural ones to use in describing the wave pattern because they are constant along a prescribed ray. However, at least two other definitions of wavelength are useful in describing the divergent wave system; the simplicity of the transverse system with its crests normal to the wake axis and changing only slightly as the critical line is approached does not require further treatment. We introduce, then, the wavelength λ^* of the divergent system parallel to the wake axis and the wavelength λ^{**} of the system normal to the wake axis. Both of these wavelengths depend on position but describe interesting features of the pattern.

In order to determine λ^* and λ^{**}, it is necessary to consider the expression for the phase given by

$$-\frac{RZ\cos(\theta-\gamma)}{h} - \frac{\pi}{4}, \tag{2.141}$$

where the minus sign is used for convenience. The phase is formally the same for the divergent system in both the subcritical and supercritical case and this permits a unified treatment. We shall only be interested in finding expressions for λ^* and λ^{**} in a restricted region of the wake where some simplification can be carried out. In particular, we consider that region of the wake where Z is large. This region lies adjacent to the wake axis and extends out to an angle γ for which Z is sufficiently large. It is, of course, necessary to be at a sufficient distance behind the source for the asymptotic expressions of the wave system to have validity.

For this region of large Z, we can find a simplified form of the phase. It is clear from Eq. (2.93) that for large Z

$$\tan(\theta-\gamma) \approx 2\tan\theta$$

and, consequently,

$$\cos(\theta - \gamma) \approx -\frac{1}{2\tan\theta}.$$

This leads to the approximate result that

$$Z\cos(\theta - \gamma) \approx -\frac{Z}{2\tan\theta}.$$

We want this as a function of γ, and Eq. (2.95) yields

$$-\tan\gamma \approx \frac{\tan\theta}{2F^2 Z\left(1 + \frac{\rho}{\rho'}\right)},$$

which gives

$$Z\cos(\theta - \gamma) \approx \frac{1}{4F^2\left(1 + \frac{\rho}{\rho'}\right)\tan\gamma}.$$

This enables the phase to be written approximately as

$$-\frac{R}{4hF^2\left(1 + \frac{\rho}{\rho'}\right)\tan\gamma} - \frac{\pi}{4}, \tag{2.142}$$

in which the dependence on R and γ is now explicit.

An immediate consequence of Eq. (2.142) is the approximate form of the phase lines in the region. We have

$$\tan\gamma = \frac{z}{x}$$

and sufficiently near the wake axis

$$R \approx -x,$$

which enables Eq. (2.142) to be written as

$$\frac{x^2}{4hF^2\left(1 + \frac{\rho}{\rho'}\right)z} - \frac{\pi}{4}. \tag{2.143}$$

This expression implies that the lines of constant phase are approximately parabolic in the region of large Z. We note that the phase increases with distance behind the source.

The wavelength λ^* is found from Eq. (2.143) as the distance parallel to the x axis over which the phase changes by 2π. Thus, we have

$$\frac{(x-\lambda^*)^2}{4hF^2\left(1+\frac{\rho}{\rho'}\right)z} - \frac{x^2}{4hF^2\left(1+\frac{\rho}{\rho'}\right)z} = 2\pi,$$

which yields

$$\lambda^* \sim 4\pi hF^2\left(1+\frac{\rho}{\rho'}\right)\frac{z}{|x|}. \tag{2.144}$$

This gives the wavelength parallel to the wake axis as inversely proportional to the distance behind the source. Also, at a fixed distance behind the source, the wavelength increases with distance from the wake axis.

For the wavelength λ^{**}, we consider the distance along a line perpendicular to the wake axis over which the phase changes by 2π. We then have by Eq. (2.143)

$$\frac{x^2}{4hF^2\left(1+\frac{\rho}{\rho'}\right)(z-\lambda^{**})} - \frac{x^2}{4hF^2\left(1+\frac{\rho}{\rho'}\right)z} = 2\pi,$$

which provides

$$\lambda^{**} \approx 8\pi hF^2\left(1+\frac{\rho}{\rho'}\right)\left(\frac{z}{x}\right)^2, \tag{2.145}$$

upon using the fact that $\lambda^{**} \ll z$ near the wake axis. The wavelength perpendicular to the wake axis tends to zero as z^2 and is inversely proportional to the square of the distance behind the source.

This completes our discussion of the properties of the wave system described by the displacement function. The main question not treated here is the manner in which the solution interior to the wake region joins with that exterior to the wake. Consideration of this matter would modify the present solution in the neighborhood of the critical line, and reference can be made to the literature for the treatment of similar difficulties in related wave problems (5,6). However, the solution obtained inside the critical angle remains valid and permits evaluation of the internal wave system generated by a source in the top layer. It represents a correction and an extension of the analysis given by Hudimac.

3. THE WAVE SYSTEM CAUSED BY A SOURCE IN THE BOTTOM LAYER

The Formal Solution of the Problem

This chapter extends the analysis already given by dealing with the case of a source located in the bottom layer. Hudimac (1) did not deal with this case at all; however, only slight changes are needed in the earlier analysis in order to find the disturbance associated with a source in the bottom layer. The similarities involved permit a condensed treatment with the same notation used for the various quantities as before.

The formulation of the problem for the source in the bottom layer leads to the Eqs. (2.18) through (2.22) for the determination of the potentials describing the disturbance of the uniform flow. To account for the source in the bottom layer, we write the potential there as

$$\varphi' = \varphi'_b + \frac{m}{r_1}, \tag{3.1}$$

where φ'_b is regular and

$$r_1 = \left[x^2 + (y-f)^2 + z^2\right]^{1/2}. \tag{3.2}$$

We note that here and in the following analysis f is a negative number since the source is located below the interface.

For the determination of the potentials, we have

$$\nabla^2 \varphi = 0 \tag{3.3}$$

and

$$\nabla^2 \varphi'_b = 0 \tag{3.4}$$

in the appropriate regions. The Fourier transform representation and the boundary condition at infinite depth lead to the solutions for the transforms of the potentials in Eqs. (3.3) and (3.4) given by

$$\tilde{\varphi} = A e^{\tilde{\omega} y} + B e^{-\tilde{\omega} y} \tag{3.5}$$

and

$$\tilde{\varphi}'_b = A' e^{\tilde{\omega} y}. \tag{3.6}$$

It remains to find the coefficients A, B, A′, which depend on the transform variables, through use of the transformed boundary conditions.

From Eqs. (2.18) through (2.22) and (3.1), the transformed boundary conditions are found to be

$$\tilde{\varphi}_y = \tilde{\varphi}'_{by} - m \exp(\tilde{\omega} f) \quad \text{on} \quad y = 0, \tag{3.7}$$

$$\rho c^2 \xi^2 \tilde{\varphi} + (\rho' - \rho) g \tilde{\varphi}_y = \rho' c^2 \xi^2 \tilde{\varphi}'_b + m \rho' c^2 \xi^2 \frac{\exp(\tilde{\omega} f)}{\tilde{\omega}} \quad \text{on} \quad y = 0 \tag{3.8}$$

and

$$-c^2 \xi^2 \tilde{\varphi} + g \tilde{\varphi}_y = 0 \quad \text{on} \quad y = h. \tag{3.9}$$

Substitution of Eqs. (3.5) and (3.6) into the above leads to the system of equations

$$\tilde{\omega} A - \tilde{\omega} B - \tilde{\omega} A' = -m \exp(\tilde{\omega} f), \tag{3.10}$$

$$-\rho c^2 \xi^2 (A + B) + \left[\rho' c^2 \xi^2 - (\rho' - \rho) g\tilde{\omega}\right] A' = -m \left[\rho' c^2 \xi^2 + (\rho' - \rho) g\tilde{\omega}\right] \frac{\exp(\tilde{\omega} f)}{\tilde{\omega}}, \tag{3.11}$$

$$(g\tilde{\omega} - c^2 \xi^2) e^{\tilde{\omega} h} A - (g\tilde{\omega} + c^2 \xi^2) e^{-\tilde{\omega} h} B = 0. \tag{3.12}$$

The left sides of these equations are identical to those in Eqs. (2.35) through (2.37), but the right sides now reflect the change in source location. Equations (3.10) through (3.12) can be solved by Cramer's rule. The determinant Q of the coefficients is precisely as in Eq. (2.38) and, solving for the three functions, we find by Eqs. (3.5) and (3.6) that

$$\tilde{\varphi} = 2\rho' m c^2 \xi^2 \frac{S}{Q} \tag{3.13}$$

with

$$S = (c^2 \xi^2 - g\tilde{\omega}) \exp[\tilde{\omega}(h + f - y)] - (c^2 \xi^2 + g\tilde{\omega}) \exp[\tilde{\omega}(f - h + y)] \tag{3.14}$$

and

$$\tilde{\varphi}_b' = m \frac{W}{Q} e^{\tilde{\omega} y} \tag{3.15}$$

with

$$W = -(c^2 \xi^2 + g\tilde{\omega}) \left\{(\rho' - \rho)(c^2 \xi^2 - g\tilde{\omega}) \exp[\tilde{\omega}(f + h)] \right. \tag{3.16}$$

$$\left. + \left[(\rho + \rho') c^2 \xi^2 + (\rho' - \rho) g\tilde{\omega}\right] \exp[\tilde{\omega}(f - h)]\right\}.$$

The inversion formula provides the two potentials

$$\varphi_0 = \frac{m}{2\pi} \iint_{-\infty}^{\infty} \frac{2\rho' c^2 \xi^2 S}{Q} \exp[i(\xi x + \zeta z)] \, d\xi \, d\zeta \tag{3.17}$$

and

$$\varphi_0' = \frac{m}{r_1} + \frac{m}{2\pi} \iint_{-\infty}^{\infty} \frac{W e^{\tilde{\omega} y}}{Q} \exp[i(\xi x + \zeta z)] \, d\xi \, d\zeta, \tag{3.18}$$

where the zero subscripts indicate that these potentials are not those desired, since we have yet to satisfy the radiation condition. By use of the symmetries involved in the integrands of Eqs. (3.17) and (3.18) and introducing polar coordinates $(\tilde{\omega}, \theta)$ to describe the transform plane, we eventually find that

$$\varphi_0 = \frac{2\rho' m}{\pi} \int_0^{\pi/2} \int_0^{\infty} \frac{S(\theta, \tilde{\omega}) \cos(\tilde{\omega} x \cos\theta) \cos(\tilde{\omega} z \sin\theta)}{(\tilde{\omega} - \tilde{\omega}_0)\, g(\theta, \tilde{\omega})} \, d\tilde{\omega}\, d\theta \tag{3.19}$$

and

$$\varphi_0' = \frac{m}{r_1} + \frac{m}{\pi} \int_0^{\pi/2} \int_0^{\infty} \frac{N(\theta, \tilde{\omega})\, e^{\tilde{\omega} y} \cos(\tilde{\omega} x \cos\theta) \cos(\tilde{\omega} z \sin\theta)}{(\tilde{\omega} - \tilde{\omega}_0)\, g(\theta, \tilde{\omega})} \, d\tilde{\omega}\, d\theta. \tag{3.20}$$

The definitions of $\tilde{\omega}_0$ and $g(\theta, \tilde{\omega})$ are given by Eqs. (2.48) and (2.49), and we have

$$S(\theta, \tilde{\omega}) = \tilde{\omega} \left\{ (c^2 \tilde{\omega} \cos^2\theta - g) \exp[\tilde{\omega}(-y + f + h)] - (c^2 \tilde{\omega} \cos^2\theta + g) \exp[\tilde{\omega}(y + f - h)] \right\} \tag{3.21}$$

and

$$N(\theta, \tilde{\omega}) = -(\tilde{\omega} + \tilde{\omega}_0) \left\{ (\rho' - \rho)(c^2 \tilde{\omega} \cos^2\theta - g)\, e^{\tilde{\omega}(f+h)} \right. \tag{3.22}$$

$$\left. + [(\rho + \rho') c^2 \tilde{\omega} \cos^2\theta + (\rho' - \rho) g]\, e^{\tilde{\omega}(f-h)} \right\}.$$

In order to satisfy the radiation condition, we proceed to examine the asymptotic behavior of Eqs. (3.19) and (3.20) far ahead of the source, for which the singularities of the integrands have paramount importance. The singularities are exactly as in the earlier case (see Fig. 2). A process exactly as employed in the previous section can be used to find the asymptotic forms of the potentials. The inner integration is split up through introduction of a curve $\tilde{\omega}_2$ such that

$$\tilde{\omega}_1 < \tilde{\omega}_2 < \tilde{\omega}_0,$$

and this enables us to write

$$\varphi_0 = I_1 + I_0, \tag{3.23}$$

$$\varphi_0' = \frac{m}{r_1} + I_1' + I_0', \tag{3.24}$$

where the subscripts indicate the singularity involved in the integrals. We find that

$$\lim_{x \to \infty} I_0 = \frac{4\rho' mg}{c^2} \int_0^{\pi/2} \frac{\sec^2\theta \exp[\tilde{\omega}_0 (y + f - h)] \cos(\tilde{\omega}_0 z \sin\theta) \sin(\tilde{\omega}_0 x \cos\theta)}{\rho e^{\tilde{\omega}_0 h} + (\rho' - \rho) e^{-\tilde{\omega}_0 h}} \, d\theta, \tag{3.25}$$

$$\lim_{x\to\infty} I_1 = -2\rho' m \int_a^{\pi/2} \frac{S(\theta,\tilde{\omega}_1)\cos(\tilde{\omega}_1 z\sin\theta)\sin(\tilde{\omega}_1 x\cos\theta)}{(\tilde{\omega}_1-\tilde{\omega}_0)\,G(\theta,\tilde{\omega}_1)}\,d\theta, \tag{3.26}$$

$$\lim_{x\to\infty} I_0' = \frac{4\rho' mg}{c^2}\int_0^{\pi/2} \frac{\sec^2\theta\exp[\tilde{\omega}_0(y+f-h)]\cos(\tilde{\omega}_0 z\sin\theta)\sin(\tilde{\omega}_0 x\cos\theta)}{\rho e^{\tilde{\omega}_0 h}+(\rho'-\rho)e^{-\tilde{\omega}_0 h}}\,d\theta, \tag{3.27}$$

and

$$\lim_{x\to\infty} I_1' = -m\int_a^{\pi/2} \frac{N(\theta,\tilde{\omega}_1)\exp(\tilde{\omega}_1 y)\cos(\tilde{\omega}_1 z\sin\theta)\sin(\tilde{\omega}_1 x\cos\theta)}{(\tilde{\omega}_1-\tilde{\omega}_0)\,G(\theta,\tilde{\omega}_1)}\,d\theta. \tag{3.28}$$

In these expressions, the limit notation indicates the asymptotic values of the integrals involved and

$$G(\theta,\tilde{\omega}_1) = \left(\frac{\partial g}{\partial\tilde{\omega}}\right)_{\tilde{\omega}_1}$$

as in Eq. (2.69), with $a = 0$ for $F < 1$, $a = \theta_c$ for $F > 1$. The angle θ_c is again given by Eq. (2.58).

We now define

$$\varphi = \varphi_0 - \lim_{x\to\infty} I_0 - \lim_{x\to\infty} I_1 \tag{3.29}$$

and

$$\varphi' = \varphi_0' - \lim_{x\to\infty} I_0' - \lim_{x\to\infty} I_1'. \tag{3.30}$$

Here, as before, the quantities indicated by the limit notation represent functions which can be interpreted throughout the top or bottom layer, whichever is appropriate. The functions defined by Eqs. (3.29) and (3.30) are harmonic and satisfy the boundary conditions as well as the radiation condition; thus, they represent the potential in the top and bottom layers associated with the disturbance to the uniform flow produced by the source in the bottom layer.

Asymptotic Form of the Velocity Potential

We now proceed to determine the state of the flow far behind the source where Eqs. (3.29) and (3.30) may be greatly simplified. First, the potentials in Eqs. (3.23) and (3.24) are simplified by noting that

$$\lim_{x\to-\infty} I_0 = -\lim_{x\to\infty} I_0$$

and

$$\lim_{x\to-\infty} I_1 = -\lim_{x\to\infty} I_1,$$

with analogous results for the primed quantities. These results provide

$$\lim_{x\to-\infty} \varphi = -2 \lim_{x\to\infty} (I_0 + I_1)$$

$$= -\frac{8\rho' mg}{c^2} \int_0^{\pi/2} \frac{\sec^2\theta \exp[\tilde{\omega}_0 (y + f - h)] \cos(\tilde{\omega}_0 z \sin\theta) \sin(\tilde{\omega}_0 x \cos\theta)}{\rho e^{\tilde{\omega}_0 h} + (\rho' - \rho) e^{-\tilde{\omega}_0 h}} d\theta$$

$$+ 4\rho' m \int_a^{\pi/2} \frac{S(\theta, \tilde{\omega}_1) \cos(\tilde{\omega}_1 z \sin\theta) \sin(\tilde{\omega}_1 x \cos\theta)}{(\tilde{\omega}_1 - \tilde{\omega}_0) G(\theta, \tilde{\omega}_1)} d\theta \tag{3.31}$$

with $a = 0$ for $F < 1$, $a = \theta_c$ for $F > 1$. Similarly,

$$\lim_{x\to-\infty} \varphi' = -\frac{8\rho' mg}{c^2} \int_0^{\pi/2} \frac{\sec^2\theta \exp[\tilde{\omega}_0 (y + f - h)] \cos(\tilde{\omega}_0 z \sin\theta) \sin(\tilde{\omega}_0 x \cos\theta)}{\rho e^{\tilde{\omega}_0 h} + (\rho' - \rho) e^{-\tilde{\omega}_0 h}} d\theta$$

$$+ 2m \int_a^{\pi/2} \frac{N(\theta, \tilde{\omega}_1) \exp(\tilde{\omega}_1, y) \cos(\tilde{\omega}_1 z \sin\theta) \sin(\tilde{\omega}_1 x \cos\theta)}{(\tilde{\omega}_1 - \tilde{\omega}_0) G(\theta, \tilde{\omega}_1)} d\theta \tag{3.32}$$

with a given above. Comparing Eqs. (3.31) and (3.32) with the results obtained for the source above the interface, we identify the first terms in each equation with the usual surface wave system, slightly modified by the stratification.

Only the internal wave system will receive further treatment; it is described by the second integrals in Eqs. (3.31) and (3.32), and these integrals will be designated by I_2 and I_2', respectively. A form of these integrals appropriate for the method of stationary phase results upon introducing the polar coordinates (R, γ) to describe the x, z plane and by changing the variable of integration to $\tilde{\omega}_1$, which can henceforth be written without the subscript. Thus, with

$$Z = h\tilde{\omega},$$

we have

$$I_2 = \mathrm{Im} \int_{Z^*}^{\infty} T(Z) \left\{ \exp\left[i \frac{R}{h} h_+(Z) \right] + \exp\left[i \frac{R}{h} h_-(Z) \right] \right\} dZ \tag{3.33}$$

and

$$I_2' = \mathrm{Im} \int_{Z^*}^{\infty} B(Z) \left\{ \exp\left[i \frac{R}{h} h_+(Z) \right] + \exp\left[i \frac{R}{h} h_-(Z) \right] \right\} dZ \tag{3.34}$$

where

$$Z^* = h\,\tilde{\omega}(0) \text{ for } F < 1,$$

$$Z^* = 0 \text{ for } F > 1,$$

$$h_+(Z) \text{ and } h_-(Z)$$

are given by Eqs. (2.84) and (2.85),

$$T(Z) = \frac{2\rho' m\, S \dfrac{d\theta}{dZ}}{(\tilde{\omega} - \tilde{\omega}_0)\, G(\theta, \tilde{\omega})}, \tag{3.35}$$

and

$$B(Z) = \frac{m\, N \quad \exp\left(Z \dfrac{y}{h}\right) \dfrac{d\theta}{dZ}}{(\tilde{\omega} - \tilde{\omega}_0)\, G(\theta, \tilde{\omega})}. \tag{3.36}$$

Equations (3.33) and (3.34) are formally the same as Eqs. (2.86) and (2.87) with the difference between them given by the new expressions for T(Z) and B(Z) appropriate for the new location of the source.

The approximate evaluation of Eqs. (3.33) and (3.34) for large R can now be carried out by the method of stationary phase according to which the main contributions to the integrals arise from the stationary points given by

$$h'_+(Z) = 0$$

and

$$h'_-(Z) = 0.$$

These are the same conditions as those that arose for the case of the source in the top layer and have already been treated in the previous section. The symmetry involved allows us to restrict γ to the second quadrant, and this implies that only the stationary points of $h_+(Z)$ need be considered. Thus, the stationary points are determined by Eq. (2.95), which represents a relationship $\gamma(Z)$. In the previous section, the nature of this relationship received detailed treatment; and, it was found that in both the subcritical and supercritical regimes of the Froude number there is a critical angle γ_c such that for $\gamma < \gamma_c$ no stationary points exist and the disturbance is of a smaller order than that interior to the wake region.

In the supercritical case for $\gamma_c \leqslant \gamma \leqslant \pi$, there is for each γ only one stationary point, and the integrals take the approximate form

$$I_2 \approx \mathrm{Im} \left| \frac{2\pi h}{R h''_+(Z)} \right|^{1/2} T(Z) \quad \exp\left[i \frac{R h_+(Z)}{h} + \frac{i\pi}{4} \right] \tag{3.37}$$

and

$$I_2' \approx \mathrm{Im}\left|\frac{2\pi h}{Rh_+''(Z)}\right|^{1/2} B(Z)\ \exp\left[i\ \frac{Rh_+(Z)}{h} + \frac{i\pi}{4}\right], \tag{3.38}$$

where Z denotes the stationary point. We note that by use of Eqs. (2.102) through (2.105), T(Z) in Eq. (3.35) can be written as

$$T(Z) = \frac{m}{h}\ \frac{D\ \exp\left[Z\left(-\frac{y}{h} + \frac{f}{h} + 1\right)\right] - C\ \exp\left[Z\left(\frac{y}{h} + \frac{f}{h} - 1\right)\right]}{DA\tan\theta} \tag{3.39}$$

and B(Z) in Eq. (3.36) as

$$B(Z) = -\frac{m}{2\rho' c^2 h}\ \frac{C\left\{(\rho'-\rho)\,D\ \exp\left[Z\left(\frac{f}{h}+1\right)\right] + (\rho' C + \rho D)\ \exp\left[Z\left(\frac{f}{h}-1\right)\right]\right\}\exp\left(Z\frac{y}{h}\right)}{DZA\sin\theta\cos\theta}. \tag{3.40}$$

Equations (3.37) through (3.40) specify the asymptotic form in the supercritical case of the velocity potential associated with the internal wave system in the wake region.

The subcritical case for $\gamma_c < \gamma \leqslant \pi$ involves two stationary points Z* and Z** (Z* < Z**), which coalesce at γ_c. We have in this case

$$\begin{aligned} I_2 \approx\ & \mathrm{Im}\left|\frac{2\pi h}{Rh_+''(Z^*)}\right|^{1/2} T(Z^*)\ \exp\left[i\ \frac{Rh_+(Z^*)}{h} - \frac{i\pi}{4}\right] \\ & + \mathrm{Im}\left|\frac{2\pi h}{Rh_+''(Z^{**})}\right|^{1/2} T(Z^{**})\ \exp\left[i\ \frac{Rh_+(Z^{**})}{h} + \frac{i\pi}{4}\right], \end{aligned} \tag{3.41}$$

where the function T(Z) is given by Eq. (3.39). I_2' has a form similar to the above with T(Z) replaced by B(Z), where B(Z) is given by Eq. (3.40). This completes our evaluation of the asymptotic form of the velocity potential associated with the internal wave system.

The Displacement Function

The asymptotic form of the velocity potential provides a means for the determination of the displacement far behind the source through the equations

$$\eta_x \approx \frac{1}{c}\frac{\partial}{\partial y} I_2 \tag{3.42}$$

and

$$\eta_x' \approx \frac{1}{c}\frac{\partial}{\partial y} I_2'. \tag{3.43}$$

This displacement, of course, describes only the disturbance produced by the internal wave system.

We first consider the supercritical case, for which the right sides of Eqs. (3.42) and (3.43) may be evaluated by use of Eqs. (3.37) and (3.38). As in the second section, the solution for the displacement function can be immediately obtained in the form

$$\eta \approx \frac{m}{cR^{1/2}h^{1/2}}\, \eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) \cos\left[\frac{RZ\cos(\theta-\gamma)}{h} + \frac{\pi}{4}\right]. \tag{3.44}$$

and

$$\eta' \approx \frac{m}{cR^{1/2}h^{1/2}}\, \eta'\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) \cos\left[\frac{RZ\cos(\theta-\gamma)}{h} + \frac{\pi}{4}\right], \tag{3.45}$$

where the dimensionless amplitude functions are given by

$$\eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = \left|\frac{2\pi}{h_+''(Z)}\right|^{1/2} \frac{D \exp\left[Z\left(-\frac{y}{h}+\frac{f}{h}+1\right)\right] + C \exp\left[Z\left(\frac{y}{h}+\frac{f}{h}-1\right)\right]}{DA\sin\theta} \tag{3.46}$$

and

$$\eta'\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = \left|\frac{2\pi}{h_+''(Z)}\right|^{1/2} \frac{C\left\{(\rho'-\rho)D \exp\left[Z\left(\frac{f}{h}+1\right)\right] + (\rho' C + \rho D) \exp\left[Z\left(\frac{f}{h}-1\right)\right]\right\} e^{Z\frac{y}{h}}}{2\rho' c^2\, DZA \sin\theta \cos^2\theta}. \tag{3.47}$$

We note that Eqs. (3.44) and (3.45) are formally the same as Eqs. (2.130) and (2.131); the only difference between them occurs in the new forms of the dimensionless amplitude functions.

For the subcritical case, Eq. (3.41) and an analogous expression for the bottom layer are used in Eqs. (3.42) and (3.43) to find

$$\eta \approx \frac{m}{cR^{1/2}h^{1/2}}\,\eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right)\cos\left[\frac{RZ\cos(\theta-\gamma)}{h} - \frac{\pi}{4}\right]\Bigg|_{Z=Z^*}$$
$$+ \frac{m}{cR^{1/2}h^{1/2}}\,\eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right)\cos\left[\frac{RZ\cos(\theta-\gamma)}{h} + \frac{\pi}{4}\right]\Bigg|_{Z=Z^{**}} \tag{3.48}$$

and a similar expression for the displacement η' in which the dimensionless amplitude function $\eta'(F, \rho/\rho', f/h, y/h, \gamma)$ appears in place of $\eta(F, \rho/\rho', f/h, y/h, \gamma)$. In these expressions, the dimensionless amplitude functions are defined by Eqs. (3.46) and (3.47). We again note the similarity with the result in Eq. (2.135) for the case considered in the second section.

The similarities of these expressions and those obtained in the previous chapter permit a brief discussion of the properties of the displacement for the present case of the source in the bottom layer. We have, as before, the amplitude of the displacement exhibiting a radial decay of $1/\sqrt{R}$. Also, the phase terms are the same as in the previous case, and the discussion given there applies as well. The most important implication of the phase term lies in the fact that the wave pattern does not change with depth. Thus, we see that the location of the source only influences the dimensionless amplitude functions and, through them, the magnitude of the displacement.

A matter of some interest is the limiting behavior of the displacement at the wake boundary $\gamma = \gamma_c$ and at the wake axis $\gamma = \pi$. For the subcritical case, we again find that there is a singularity in the displacement at the boundary of the wake on account of the vanishing there of $h''_+(Z)$; however, this is associated with a failure of the method of stationary phase in the form employed above, and further analysis would correct this difficulty. At the wake axis, we find that the amplitude of the divergent wave system vanishes while the amplitude of the transverse wave system approaches a nonzero limit, exactly as in the earlier situation.

In the supercritical case, the amplitude of the divergent system tends to zero at the wake axis as before. However, at the boundary of the wake, we find that

$$\lim_{Z\to 0}\ \eta\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = 0 \tag{3.49}$$

and

$$\lim_{Z\to 0}\ \eta'\left(F, \frac{\rho}{\rho'}, \frac{f}{h}, \frac{y}{h}, \gamma\right) = 0, \tag{3.50}$$

which imply that the amplitude of the divergent wave tends to zero at the wake boundary regardless of the depth. The results in Eqs. (3.49) and (3.50) differ from Eqs. (2.138) and (2.139) and indicate a peculiar dependence of the dimensionless amplitude functions on the source location. The question of the behavior of the displacement in the vicinity of the wake boundary and the manner in which the solution interior to the wake joins with that in the exterior region requires a further development of the analysis given above and is not dealt with any further in this work.

4. CONCLUSION

The objective of furnishing an analytical basis for the interpretation of "dead water" waves has been accomplished through the analysis developed in this report. Although we have dealt with a source moving in a two-layer fluid, the disturbance generated by a rigid body regarded as a source and a sink can be determined immediately from the results obtained in this report. Detailed calculations indicating specific properties of the internal wave system associated with a dipole will be given elsewhere (9). We now proceed to summarize the specific results developed in this report concerning the disturbance caused by a point source.

We have obtained expressions for the velocity potential which describes the disturbance of a uniform flow in a two-layer fluid produced by a source located in either the top or bottom layer. As noted in the introductory remarks to this report, the situation of uniform flow past a fixed source is equivalent to the case of a source moving at a constant rate in the fluid which is at rest far ahead of the source. The expressions found for the potential require that the waves have a small amplitude in order to permit linearization of the boundary conditions. A simplification of the solution resulted by considering the region far behind the source in which the method of stationary phase yields an explicit, approximate expression for the internal wave system accompanying the source. The Froude number F, which gives the ratio of source speed to the maximum phase velocity, significantly influences the nature of the disturbance and should be regarded as the main parameter for describing the internal waves. There is also a surface wave system present; however, it differs only slightly from that found in a homogeneous fluid of infinite depth and no detailed treatment of it was necessary.

The most significant result of the approximate solution is the limitation of the main disturbance to an angular sector symmetric to the track of the source. Within this region, the magnitude of the disturbance decays as $1/\sqrt{R}$, with R denoting the distance from the source measured in a horizontal plane. Calculations of the angle defining the wake boundary in the subcritical regime ($F < 1$) indicated an exceedingly small dependence on the density ratio when this ratio takes values near one; in the supercritical regime ($F > 1$), the Froude number alone defines the angular extent of the wake. Thus, the Froude number essentially determines the wake angle in all cases.

By use of the approximate expression for the velocity potential, the displacement function was determined for the internal wave system. The displacement function specifies the vertical displacement of streamlines from their undisturbed position and provides a direct description of the properties of the internal wave system. Interpretation of the form of the solution for the displacement function revealed a situation qualitatively similar to wave disturbances produced in an ocean of finite depth; and, an argument was presented as to why this similarity exists. In the subcritical regime, we have a converging wave system and a diverging wave system in the wake region; but, in the supercritical regime, only a diverging wave system exists with crests that ultimately become parallel to the wake boundary in the far field. Two wavelengths, each differing from the wavelength normally used to describe the wake, were introduced in the discussion of the diverging wave system. One wavelength is measured parallel to the wake axis, whereas the other is measured normal to the wake axis. Simple expressions for these wavelengths were found for a region adjacent to the wake axis where they provide interesting information about the nature of the diverging wave pattern.

Certain aspects of the wave pattern have not been treated adequately, neither here nor elsewhere in the literature. The behavior of the displacement in the immediate vicinity of the wake boundary represents a primary question deserving further treatment. We have previously noted that for the subcritical case reference can be made to other works which deal with this question in related wave problems. On the other hand, it appears that the question has yet to receive attention in the supercritical case. One aspect of the wave pattern omitted from this report, but of limited interest, is the behavior of the wave pattern near the source—that is, the near-field behavior. However, both the vicinity of the wake boundary and the

near-field region have a limited spatial extent which restricts their importance for most investigations concerned with the wave pattern of a source.

It has already been mentioned that the problem treated in this work was studied by Hudimac (1). However, he considered only the case of the source located in the top layer and evaluated the displacement for the interface alone. Quite apart from correcting many misprints in his paper, we have also carried out the analysis for the situation in which the source is located in the bottom layer; and, for each case of source location, the displacement associated with the internal wave system has been evaluated for all depths. Moreover, our expressions for the displacement far behind the source have a simpler form than those given previously.

REFERENCES

1. Hudimac, A.A., "Ship Waves in a Stratified Ocean," J. Fluid Mech. 11:229-243 (1961)

2. Grosch, C.E., "Internal Wave Field of a Dipole in a Fluid with an Exponentially Varying Density, I," Stevens Institute of Technology, Davidson Laboratory Report 1042 (1964)

3. Crapper, G.D., "Ship Waves in a Stratified Ocean," J. Fluid Mech. 29:667-672 (1967)

4. Wehausen, J.V., and Laitone, E.V., "Surface Waves," in "Encyclopedia of Physics" vol. IX Fluid Dynamics III, edited by S. Flügge, Berlin:Springer-Verlag, p. 446, 1960

5. Stoker, J.J., "Water Waves," New York:Interscience, 1957

6. Crapper, G.D., "Surface Waves Generated by a Travelling Pressure Point," Proc. Roy. Soc. A282:547-558 (1964)

7. Lamb, H., "Hydrodynamics," 6th ed., New York:Dover Publications, Inc., 1945, Art. 231

8. Yih, C.-S., "Gravity Waves in a Stratified Fluid," J. Fluid Mech. 8:481-508 (1960)

9. Bergin, J.W., "On the Internal Wave System Created by a Dipole Moving in a Two-Layer Fluid," NRL Report 7289 (Confidential Report, Unclassified Title), in publication