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Criteria for Pareto-Optimality in Cooperative Differential Games

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ABSTRACT

Cooperative differential games having convex control sets are investigated. Two criteria are given for the utility of analyzing Pareto-optimal solutions of such games. The criteria are shown to determine whether a Pareto-optimal control policy belongs to the boundary of the control set or to the interior. A condition which is necessary for Pareto-optimality in such games is employed to obtain the results. An example is given to demonstrate that the results do not necessarily apply to games having nonconvex control sets.

CRITERIA FOR PARETO-OPTIMALITY IN COOPERATIVE DIFFERENTIAL GAMES

1. INTRODUCTION

Some background into cooperative differential games is given in Refs. 1 and 2. In obtaining optimal solutions in cooperative differential games, it can be quite advantageous to use necessary conditions of optimality which deal directly with properties pertaining to the control space. In Ref. 1 such conditions are derived for the cases in which solutions to cooperative differential games lie in the interior of the control set and the cases in which they lie on the boundary of the control set. The approach used in Ref. 1 is based on the dimension of the cost cone. The objective in this report is to obtain the bulk of the results of Ref. 1 by a simpler and more direct approach. This objective is accomplished by using two simple criteria.

2. PROBLEM STATEMENT

Consider a cooperative differential game involving p players, each desiring to minimize his cost criterion

$$J^i(u_1, \ldots, u_p) \triangleq g_0^i(x^f),\ i \in \{1, \ldots, p\} \tag{1}$$

through the selection of his control function $u_i : [t_0, t_f] \to U_i$, where the state governed by

$$\dot{x}(t) = f(x(t), u_1(t), \ldots, u_p(t), t),\ t \in [t_0, t_f] \tag{2}$$

evolves from the initial state

$$x(t_0) = x^0 \tag{3}$$

and terminates at some state $x(t_f) \triangleq x^f$.* Here, the control set U_i is a closed convex set in E^{r_i}, the playing space G is an open set in E^n, and the state $x(t)$ belongs to G for all t

*Even though each player has a cost criterion, he is not necessarily endowed with a control variable in Eq. (2). Rather, it may well happen that the ith player has only an indirect influence on the other players in their adopting of a control which is reasonably acceptable to himself. For instance, the influence may be of a nature similar to the undisputed bargaining position of the AFL-CIO president George Meany. Therefore, for the case that the ith player is not endowed with a control variable, the integer r_i is taken to be zero and the set U_i to be empty.

contained in $[t_0, t_f]$.* Also, the functions $g_0^i\colon G \to R$ and $f\colon G \times U \times [t_0, t_f] \to E^n$ are continuously differentiable, where the control set U denotes the cartesian product $U_1 \times \ldots \times U_p$. Thus the control set U is a convex subset of E^r, where

$$r = \sum_{i=1}^{p} r_i.$$

Definition. A p-tuple of control functions $(u_1^*, \ldots, u_p^*) \triangleq u^*$ is said to be Pareto-optimal if, and only if, for each p-tuple of control functions $(u_1, \ldots, u_p) \triangleq u$ we have either

$$J^i(u) = J^i(u^*), \text{ for all } i \in \{1, \ldots, p\}, \tag{4}$$

or there is at least one $j \in \{1, \ldots, p\}$ such that

$$J^i(u) > J^j(u^*). \tag{5}$$

Let $x^*\colon [t_0, t_f] \to G$ be the solution of Eqs. (2) and (3) for some control u^*. Consider the solutions $\lambda^i\colon [t_0, t_f] \to E^n$, $i \in \{1, \ldots, p\}$, of the adjoint equations

$$\dot{\lambda}^i(t) = -\lambda^i(t) \cdot \frac{\partial f}{\partial x}(x^*(t), u^*(t), t) \tag{6}$$

that satisfy the terminal conditions

$$\lambda^i(t_f) = \frac{\partial g_0^i}{\partial x}(x^f), \tag{7}$$

where $x^f = x^*(t_f)$. Furthermore, for each $i \in \{1, \ldots, p\}$ and each $t \in [t_0, t_f]$, let

$$C^i(t) \triangleq \lambda^i(t) \cdot \frac{\partial f}{\partial u}(x^*(t), u^*(t), t). \tag{8}$$

Note that $C^i(t)$ is an r-dimensional vector.

Necessary Condition. In order that some control u^* be Pareto-optimal, it is necessary that for each control value $v \in U$ and for almost each $t \in [t_0, t_f]$ there is at least one $j \in \{1, \ldots, p\}$, with j depending on v and t, yielding

$$C^j(t) \cdot (v - u^*(t)) = 0 \tag{9}$$

or

$$C^j(t) \cdot (v - u^*(t)) > 0. \tag{10}$$

*E^{r_i} denotes an r_i-dimensional Euclidean space.

The preceding necessary condition is derived in Ref. 1 by means of a classical variational approach. However, as is pointed out below in Section 4, this condition is not necessary when U is not convex.

3. CRITERIA FOR PARETO-OPTIMALITY

Throughout this section we will make use of an arbitrary but fixed control $u^*: [t_0, t_f] \to U$ with its corresponding solution of Eqs. (2) and (3), $x^*: [t_0, t_f] \to G$.

Following the terminology of Ref. 1, we define for each $t \in [t_0, t_f]$

$$C(t) = \left\{ C \in E^r: C = \sum_{i=1}^{p} \alpha_i C^i(t), \forall \alpha_i \geqslant 0, i = 1, ..., p \right\} \tag{11}$$

and

$$C_D(t) = \{ v \in E^r: v \cdot C \leqslant 0 \text{ for all } C \in C(t) \}. \tag{12}$$

The closed convex set $C(t)$ is known as the cost cone at time t and $C_D(t)$ as the dual cone to $C(t)$. We have used the closed dual rather than the open dual.

Criterion I. At time $t_1 \in [t_0, t_f]$ there exists no nonzero $C^i(t_1)$, $i \in \{1, ..., p\}$, such that $-C^i(t_1) \in C(t_1)$.

Criterion II. At time $t_1 \in [t_0, t_f]$ there exists a nonzero $C^i(t_1)$, $i \in \{1, ..., p\}$, such that $-C^i(t_1) \in C(t_1)$.

Criterion I provides a means of testing for variations in a control policy that result in a lower cost to first order for all players. Criterion II provides a means of testing a control policy for the times when variation in control results in unchanging cost to first order for all players or a decrease in cost to first order for some players and an increase for other players.

Theorem 1. Suppose Criterion I is satisfied at time $t_1 \in [t_0, t_f]$. Then, whenever $u^*(t_1)$ lies in the interior of U, there exists a control point $v_1 \in U$, with v_1 arbitrarily close to $u^*(t_1)$, such that

$$C^j(t_1) \cdot (v_1 - u^*(t_1)) < 0 \textit{ for all } j \in \{1, ..., p\}. \tag{13}$$

In other words, neither condition (9) or (10) is not met at the time t_1.

Proof. Consider the dual cone $C_D(t_1)$. Since Criterion I is satisfied at time t_1, it follows that $C(t_1)$ is proper, that is, $C(t_1) \cap -C(t_1) = \{0\}$. Thus $C(t_1)$ has lineality zero.* For, if $C(t_1)$ has lineality greater than zero, then there exists a nonzero $C \in C(t_1)$ such that $-C \in C(t_1)$; this contradicts $C(t_1) \cap -C(t_1) = \{0\}$.

*The largest linear subspace, through the origin, contained in $C(t_1)$ is called the lineality space of $C(t_1)$, and its dimension is called the lineality of $C(t_1)$.

A closed convex cone $C(t_1)$ has lineality zero if, and only if, the dual cone $C_D(t_1)$ is r-dimensional; see Sandgren [3]. Since $C_D(t_1)$ is r-dimensional, it has a nonempty interior. Let W_1 belong to the interior of $C_D(t_1)$. Thus $W_1 \cdot C < 0$ for all $C \in C(t_1)$. Let $v_1(\delta) = u^*(t_1) + \delta W_1$, where $\delta > 0$. Since $u^*(t_1)$ is an interior point of U, it follows that $v_1(\delta)$ belongs to U for small enough δ and that condition (13) holds. This completes the proof.

From Theorem 1, together with the necessary condition of (9) and (10), we have the following corollary.

Corollary 1. A Pareto-optimal control must necessarily lie on the boundary of the control set at almost all times that Criterion I holds. Moreover, Criterion II must be satisfied at almost all times that a Pareto-optimal control lies in the interior of the control set.

Theorem 2. Suppose Criterion II is satisfied at time $t_1 \in [t_0, t_f]$. Then, for each control function v: $[t_0, t_f] \to U$, the condition

$$C^j(t_1) \cdot (v(t_1) - u^*(t_1)) \geqslant 0 \tag{14}$$

is met for at least one $j \in \{1, ..., p\}$.

Proof. Let $v_1 \in U$. By hypothesis, there exists a nonzero $C^i(t_1)$, $i \in \{1, ..., p\}$, such that $-C^i(t_1) \in C(t_1)$. Therefore,

$$-C^i(t_1) = \sum_{j=1}^{p} \alpha_j C^j(t_1) \tag{15}$$

for some $\alpha_j \geqslant 0$, $j = 1, ..., p$. Equation (15) is equivalent to

$$0 = \sum_{j=1}^{p} \beta_j C^j(t_1), \tag{16}$$

where $\beta_i = \alpha_i + 1$ and $\beta_j = \alpha_j$ for $j \neq i$. Note that

$$0 \cdot (v_1 - u^*(t_1)) = \sum_{j=1}^{p} \beta_j C^j(t_1) \cdot (v_1 - u^*(t_1)). \tag{17}$$

If $C^j(t_1) \cdot (v_1 - u^*(t_1)) < 0$ for all $j \in \{1, ..., p\}$, then Eq. (17) implies that $0 < 0$. Therefore Eq. (14) is met for at least one $j \in \{1, ..., p\}$. This concludes the proof.

The above theorem yields immediately the following corollary.

Corollary 2. If Criterion II is satisfied for almost all times t in $[t_0, t_f]$, then the necessary condition (9) and (10), for u^* to be Pareto-optimal, is met.

The preceding theorems and corollaries address Pareto-optimality with respect to the control set U or, more specifically, to its interior and boundary. Letting ∂U_i denote the boundary of U_i, note that the boundary of U is equal to the union

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$$\bigcup_{i=1}^{p} U_1 \times \ldots \times U_{i-1} \times (\partial U_i) \times U_{i+1} \times \ldots \times U_p.$$

For a given cooperative differential game, it may be known that a Pareto-optimal control must necessarily lie on the boundary of the control set U over some time interval. This knowledge is often not enough information to determine a Pareto-optimal solution to the game. That is, knowledge of a control $u^*(t) = (u_1^*(t), \ldots, u_p^*(t))$ belonging to ∂U does not determine whether the ith player's control $u_i^*(t)$ belongs to ∂U_i. It could indeed happen that $u_1^*(t)$ belongs to U_1 and $u_j^*(t)$ belongs to ∂U_j for $j = 2, \ldots, p$. It would be useful to know that $u^*(t)$ belongs to $\partial U_1 \times \partial U_2 \times \ldots \times \partial U_p$ whenever this is the case.

It is possible to obtain additional information about a Pareto-optimal control by holding one or more of the control variables fixed, say $u_2, \ldots, u_p$, and varying the remaining control variables, in this case u_1. Using the cost criteria of all players and fixing the control $(u_2, \ldots, u_p)$ of the players, we can apply the approach and results of this section to ascertain whether it is necessary, in the sense of Pareto-optimality, that u_1 belongs to the boundary of U_1. From such effort, additional knowledge can often be extracted which pertains to the control set of each individual player.

4. COUNTEREXAMPLE FOR A NONCONVEX CONTROL SET *U*

We now demonstrate by means of a counterexample that conditions (9) and (10) are not, in general, necessary for arbitrary U. Although this example involves only one player, it illustrates a method for constructing counterexamples involving multiple players. For simplicity the control values of this one player are denoted by $(v_1, v_2) \in E^2$.

Let $U = \{(v_1, v_2) \in E^2\colon (v_1)^2 + (v_2)^2 \geqslant 1\}$ and let $G = E^2$. Note that U is not convex. The state $x = (x_1, x_2)$ is governed by

$$\dot{x}_1 = (v_1)^2 \tag{18}$$

and

$$\dot{x}_2 = (v_2)^2 \tag{19}$$

with the initial condition

$$(x_1^0, x_2^0) = (-1, 0). \tag{20}$$

The time interval $[t_0, t_f]$ is taken to be $[0, 1]$, and the cost criterion to be minimized is given to be

$$g_0^1(x_1^f, x_2^f) = \frac{(x_1^f + 1)^2}{2} + \frac{(x_2^f + 1)^2}{2}, \tag{21}$$

where (x_1^f, x_2^f) is the terminal state at time $t = 1$.

Let (v_1, v_2) be a pair of control functions such that $(v_1(t))^2 + (v_2(t))^2 \geqslant 1$ for all $t \in [0, 1]$. The corresponding solutions (x_1, x_2) have as the terminal state

$$x_1^f \geqslant - \int_0^1 (v_2(t))^2 \, dt \tag{22}$$

and

$$x_2^f = \int_0^1 (v_2(t))^2 \, dt. \tag{23}$$

Equation (23), together with (21) and (22), yields

$$g_0^1(x_1^f, x_2^f) = 1 + \frac{(x_1^f)^2 + (x_2^f)^2}{2} + (x_1^f + x_2^f) \geqslant 1. \tag{24}$$

The optimal cost of $g_0^1(x_1^f, x_2^f) = 1$ is rendered by the pair (v_1^*, v_2^*), where $v_1^*(t) = 1$ and $v_2^*(t) = 0$ for all $t \in [0, 1]$.

However, the optimal pair (v_1^*, v_2^*) does not satisfy the necessary condition. To see this, note that $\lambda_1^1 \equiv 1$ and $\lambda_2^1 \equiv 1$. Thus conditions (9) and (10) reduce to

$$(\lambda_1^1, \lambda_2^1) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1(t) - 1 \\ v_2(t) \end{pmatrix} = v_1(t) - 1 \geqslant 0 \tag{25}$$

for all pairs (v_1, v_2) satisfying $(v_1)^2 + (v_2)^2 \geqslant 1$. Condition (25) is not satisfied for $v_1 < 1$; thus a violation of condition (25) results within every neighborhood of $v_1^*(t) = 1$, $t \in [0, 1]$. This concludes the demonstration of our counterexample.

Although condition (25) does not hold, note that

$$\lambda^1(t) \cdot f(x^*(t), v_1, v_2, t) \geqslant \lambda^1(t) \cdot f(x^*(t), v_1^*(t), v_2^*(t), t) \tag{26}$$

holds for all $t \in [0, 1]$ and all $(v_1, v_2) \in U$. Indeed, recalling the maximal principal of optimal control theory, we would expect condition (26) to be fulfilled.

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