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Successive-Approximations Method for Solutions of Nonlinear Differential Equations at an Irregular-Type Singular Point

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CONTENTS

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Abstract: Two fundamental existence theorems for the study of analytic solutions of nonlinear ordinary differential equations with an irregular-type singularity are proved. A method of successive approximations involving improper contour integrals and analyticity with respect to several complex variables is employed.

I. INTRODUCTION

1. The Problem

In the course of studying a nonlinear differential equation of a complex variable at an irregular-type singular point, one of the main problems is to find the analytic meaning of formal solutions. In the process of tackling this problem, one often encounters two types of existence theorems to be discussed in this paper. These two theorems have been proved recently by M. Iwano [1, 2] by means of Tychonoff-type fixed point theory, which was originally devised by M. Hukahara [3, 4]. The aim of this report is to modify the proof of these theorems by using a method considerably more constructive, namely successive approximations. While doing this, one can see that there are two difficulties which are not usually seen in the conventional successive-approximations method; one involves the improper contour integral, while the other requires the analyticity with respect to several complex variable.

In this chapter, we shall clarify some notations and definitions. Chapter II will state the assumptions and the main theorems. Since the proof of these theorems are alike, only the sketch of the first theorem will be given in Chapter III. However, the paths of integration will be fully explored there. The complete proof of the second theorem will be given in Chapters IV and V.

2. Notations

The quantity $\mathbf{1}_m$ is defined as the $m \times m$ unit matrix. For an m-column vector y with elements y_j, the expression $\mathbf{1}_m(y)$ denotes an $m \times m$ diagonal matrix with diagonal elements $\{ y_j \}$

If u is an m-column vector with elements $\{ u_j \}$, then $[u]$ denotes an m-column vector with elements $|u_j|$. Obviously, $[u]$ coincides with u when all the components u_j are nonnegative and real.

For m-column vectors u and $\widetilde{u}$ with elements $\{ u_j \}$ and $\{ \widetilde{u}_j \}$, respectively, a vectorial inequality $[u] \leqslant [\widetilde{u}]$ means that $|u_j| \leqslant |\widetilde{u}_j|$ for each index j.

The components of an m-row vectors $q = (q_1, \ldots, q_m)$ are all nonnegative integers and

$$|q| = q_1 + q_2 + \ldots q_m \, . \tag{2.1}$$

For an m-column vector y with elements y_j, the symbol y^q stands for the scalar quantity

$$y^q = {y_1}^{q_1} \, {y_2}^{q_2} \ldots {y_m}^{q_m} \tag{2.2}$$

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The norm of an m-vector y with elements $\{y_j\}$ is

$$\|y\| = \max_{j=1}^{m} |y_j| \cdot \tag{2.3}$$

To simplify the notion, for a scalar w and an m-row vector y with elements $\{y_j\}$,

$$w^y = (w^{y_1}, \ldots, w^{y_m}) \tag{2.4}$$

$$\exp y = (\exp y_1, \ldots, \exp y_m), \text{ or } e^y = (e^{y_1}, \ldots, e^{y_m}) \tag{2.5}$$

$$\operatorname{Re} y = (\operatorname{Re} y_1, \ldots, \operatorname{Re} y_m), \text{ and } \operatorname{Im} y = (\operatorname{Im} y_1, \ldots, \operatorname{Im} y_m). \tag{2.6}$$

If y is a column vector, w^y, $\exp y$, $\operatorname{Re} y$, and $\operatorname{Im} y$ are all column vectors.

For an m-column vector y with elements $\{y_j\}$ and an n-column vector function $f(x,y)$ with elements $\{f_j(x,y)\}$, the notation $f_y(x,y)$ denotes an $n \times m$ matrix given by

$$f_y(x,y) = \left(\frac{\partial}{\partial y_1} f(x,y), \ldots, \frac{\partial}{\partial y_m} f(x,y) \right) \quad . \tag{2.7}$$

3. Definitions

A function $f(x)$, which is holomorphic and bounded in x for

$$0 < |x| < \xi, \; \underline{\Theta} < \arg x < \overline{\Theta}, \tag{3.1}$$

where ξ, $\underline{\Theta}$, and $\overline{\Theta}$ are given constants, and which admits an asymptotic expansion in powers of x as x tends to 0 through (3.1), is said to *belong to class* $C(\underline{\Theta}, \overline{\Theta}, \xi)$.

A vector $f(x,y,z)$, which is a holomorphic function of (x,y,z) for

$$0 < |x| < \xi, \underline{\Theta} < \arg x < \overline{\Theta}, \; \|y\| < \delta, \; \|z\| < \delta, \tag{3.2}$$

is said to have Property-$\mathfrak{U}$ with respect to y and z in (3.2) if the components of $f(x, y, z)$ admit uniformly convergent expansions in powers of y and z for (3.2) and if the coefficients of these expansions belong to class $C(\underline{\Theta}, \overline{\Theta}, \xi)$.

Suppose a finite number of monomials of x^{-1} of the same degree, say σ, are given:

$$\Omega_j(x) = -\frac{\gamma_j}{\sigma x^\sigma} \quad (j = 1, 2, \ldots, M).$$

Then the sectors of the form

$$\frac{1}{\sigma}(\arg \gamma_j - \frac{\pi}{2} + 2\pi h_j) < \arg x < \frac{1}{\sigma}(\arg \gamma_j + \frac{\pi}{2} + 2\pi h_j) \tag{3.3}$$

and

$$\frac{1}{\sigma}(\arg \gamma_j + \frac{\pi}{2} + 2\pi h_j') < \arg x < \frac{1}{\sigma}(\arg \gamma_j + \frac{3\pi}{2} + 2\pi h_j'), \tag{3.4}$$

where h_j and h_j' are any integers, are said to be *a maximal negative region* of $\Omega_j(x)$ and *a maximal positive region* of $\Omega_j(x)$, respectively. The maximal negative (or positive) region means that if x tends to the origin through any subsector of (3.3) or (3.4)), the function exp $(\mathrm{Re}\Omega_j(x))$ tends to zero (or infinity) exponentially.

A sector $\underline{\Theta} < \arg x < \overline{\Theta}$ is said to have Property-$\mathfrak{J}$ with respect to monomials $\{\Omega_1(x), \ldots, \Omega_M(x)\}$ if this sector does not contain any maximal negative region of $\Omega_j(x)$ for each index j and if there exists in this sector a direction for each index j such that, x tends to the origin along this direction, $\exp(\mathrm{Re}\Omega_j(x))$ tends to infinity exponentially.

Remark: Since the sectors

$$S_j\colon \frac{1}{\sigma}\left(\arg\gamma_j - \frac{5\pi}{2} + 2\pi h_j\right) + \epsilon_1 < \arg x < \frac{1}{\sigma}\left(\arg\gamma_j + \frac{\pi}{2} + 2\pi h_j\right) - \epsilon_2$$

and

$$S_j'\colon \frac{1}{\sigma}\left(\arg\gamma_j - \frac{\pi}{2} + 2\pi h_j\right) + \epsilon_1 < \arg x < \frac{1}{\sigma}\left(\arg\gamma_j + \frac{5\pi}{2} + 2\pi h_j'\right) - \epsilon_2 ,$$

where ϵ_1 and ϵ_2 are constants satisfying the relations $0 < \epsilon_1, \epsilon_2 < (2\pi/\sigma)$, $\epsilon_1 + \epsilon_2 < (3\pi/\sigma)$, then both have property-$\mathfrak{J}$ with respect to $\{\Omega_j(x)\}$. If a direction $\arg x = \theta_0$ is given, we can choose h_j and h_j' properly such that $\theta_0 \in S_j$ and $\theta_0 \in S_j'$. Put

$$S = \bigcap_{j=1}^{M} S_j,\quad S' = \bigcap_{j=1}^{M} S_j'$$

Then, both S and S' are nonempty and have Property-$\mathfrak{J}$ with respect to $\{\Omega_1(x), \ldots, \Omega_M(x)\}$. As a matter of fact, since

$$\max_{j=1}^{M}\{2\pi h_j + \arg\gamma_j\} - \min_{j=1}^{M}\{2\pi h_j + \arg\gamma_j\} < 2\pi$$

and

$$\max_{j=1}^{M}\{2\pi h_j' + \arg\gamma_j\} - \min_{j=1}^{M}\{2\pi h_j' \div \arg\gamma_j\} < 2\pi ,$$

we can choose ϵ_1 and ϵ_2 so small that the sectors S and S' have central angle $> \pi/\sigma$.

II. MAIN THEOREMS

4. First Existence Theorem

Let there be given two systems of $\alpha + \beta$ nonlinear differential equations:

$$x^{\sigma+1}y' = f(x,y,z),\; xz' = g(x,y,z) \tag{E$_1$}$$

where $' = d/dx$. Here we assume that

(i) x is a complex independent variable and σ is a positive integer.

(ii) y and z are both a- and β-column vectors with components $\{ y_j \}$ and $\{ z_k \}$, respectively.

(iii) $f(x,y,z)$ and $g(x,y,z)$ are a- and β-column vectors, respectively, whose components have Property-$\mathfrak{U}$ with respect to y and z in the domain

$$0 < |x| < \xi, \Theta < \arg x < \overline{\Theta}, \ \|y\| < d, \ \|z\| < d, \tag{4.1}$$

where $\underline{\Theta}$, $\overline{\Theta}$, ξ, and d are constants, with ξ and d positive.

(iv) The matrices f_y and f_z satisfy

$$f_y(0,0,0) = \mathbf{1}_a(\gamma) + D, \det \mathbf{1}_a(\gamma) \neq 0, f_z(0,0,0) = 0, \tag{4.2}$$

where γ is an a-column vector with elements $\{ \gamma_j \}$ and D is an a x a nilpotent matrix with lower triangular form.

(v) Equations (E_1) possess a formal solution of the form

$$y \sim \sum_{\ell=0}^{\infty} x^{\ell} f_{\ell}, \ z \sim \sum_{\ell=0}^{\infty} x^{\ell} g_{\ell}, \tag{4.3}$$

where f_ℓ and g_ℓ are a- and β-column constant vectors, respectively, and in particular,

$$\| f_0 \| < d, \ \|g_0\| < d. \tag{4.4}$$

Let

$$\Omega_j(x) = -\frac{\gamma_j}{\sigma x^\sigma} \quad (j = 1, 2, \ldots, a); \tag{4.5}$$

the first existence theorem is stated as follows:

THEOREM A. *Assume that, in the sector* $\underline{\Theta} < \arg x < \overline{\Theta}$, *there exists a subsector* $\underline{\Theta}^* < \arg x < \overline{\Theta}^*$ *which has Property-*$\mathfrak{S}$ *with respect to* $\{ \Omega_1(x), \ldots, \Omega_a(x) \}$. *Then the equations* (E_1) *have a unique solution* $\{ F(x), G(x) \}$ *which is holomorphic and bounded in* x *for*

$$0 < |x| < \xi_0, \underline{\Theta}^* < \arg x < \overline{\Theta}^*, \tag{4.6}$$

where $0 < \xi_0 \leqslant \xi$, *and which admits asymptotic expansions of the form* (4.3) *as* x *tends to zero in the sector* (4.6).

The sketch of this theorem will be given in the next section.

5. Second Existence Theorem

Let μ be a given n-column vector with elements $\{ \mu_k \}$. The second existence theorem concerns a system of equations similar to (E_1), except that the vectorial functions f and g, besides x, y and z, depend on an arbitrary function of the form $V(x) = \mathbf{1}_n(x^\mu)C$, where C is an arbitrary n-column vector. This system is:

$$x^{\sigma+1} y' = f(x, V(x); y, z), \ \ xz' = g(x, V(x); y, z). \tag{E_2}$$

Here we assume that

(i) $f(x,v;y,z)$ and $g(x,v;y,z)$ are α- and β-column vectors, respectively, which admit uniformly convergent series in powers of y and z in the domain

$$0 < |x| < \xi, \underline{\Theta} < \arg x < \overline{\Theta}, \ \|v\| < \delta, \|y\| < d, \ \|z\| < d, \tag{5.1}$$

and whose coefficients are functions with Property-$\mathfrak{U}$ with respect to v in

$$0 < |x| < \xi, \underline{\Theta} < \arg x < \overline{\Theta}, \ \|v\| < \delta, \tag{5.2}$$

with δ a positive constant.

(ii) The matrices f_y and f_z satisfy

$$f_y(0,0;0,0) = \mathbf{1}_a(\gamma) + D, \det \mathbf{1}_a(\gamma) \neq 0, f_z(0,0;0,0) = 0. \tag{5.3}$$

(iii) Equations (E_2) have a formal solution of the form

$$y \sim \sum_{|q|=0}^{\infty} V(x)^q f_q(x), \ z \sim \sum_{|q|=0}^{\infty} V(x)^q g_q(x), \tag{5.4}$$

where $f_q(x)$ and $g_q(x)$ are α- and β-column vector functions, respectively, which belong to class $C(\underline{\Theta}, \overline{\Theta}, \xi)$, and in particular,

$$\| f_0(x)\| < d, \ \|g_0(x)\| < d. \tag{5.5}$$

Now, the second existence theorem is stated as following:

THEOREM B. *Assume that, in the sector* $\underline{\Theta} < \arg x < \overline{\Theta}$, *there exists a subsector* $\underline{\Theta}^* < \arg x \ \overline{\Theta}^*$ *which has Property-*$\mathfrak{J}$ *with respect to* $\{ \Omega_1(x), \ldots, \Omega_a(x) \}$. *Then the equations* ($E_2$) *have a solution of the form* $\{ F(x, V(x)), G(x, V(x)) \}$ *whenever x and* $V(x)$ *are in the domain*

$$0 < |x| < \xi^0, \underline{\Theta}^* < \arg x < \overline{\Theta}^*, \ \|v\| < \delta^0, \tag{5.6}$$

where $0 < \xi^0 \leqslant \xi, 0 < \delta^0 \leqslant \delta$. *Furthermore, this solution admits uniformly convergent expansions of the form* (5.4) *so that* $F(x, v)$ *and* $G(x, v)$ *are* α- *and* β-*column vector functions with Property-*$\mathfrak{U}$ *with respect to* v *in the domain* (5.6).

The complete proof of this theorem will be given in Chapters IV and V.

III. PATHS OF INTEGRATION AND PROOF OF THEOREM A

6. Reduction of Theorem A

In order to prove Theorem A, we consider for a positive integer N the following transformations to (E_1):

$$y = \sum_{\ell=0}^{N-1} x^{\ell} f_{\ell} + \eta_N, \ z = \sum_{\ell=0}^{N-1} x^{\ell} g_{\ell} + \zeta_N, \tag{6.1}$$

and

$$\eta_N = \mathbf{1}_a(e^{\Omega(x)})\ P_N,\ \zeta_N = Q_N, \tag{6.2}$$

where $\Omega(x)$ denotes the a-column vector with elements $\{\Omega_j(x)\}$. Then P_N and Q_N satisfy

$$\left.\begin{aligned} x^{\sigma+1} P_N' &= \mathbf{1}_a(e^{-\Omega(x)})\ \hat{f}(x, \mathbf{1}_a(e^{\Omega(x)})P_N, Q_N) \\ xQ_N' &= \hat{g}(x, \mathbf{1}_a(e^{\Omega(x)})P_N, Q_N) \end{aligned}\right\} \tag{6.3}$$

where $\hat{f}(x,\eta,\zeta)$ and $\hat{g}(x,\eta,\zeta)$ are a- and β-column vector functions, respectively, which have Property-$\mathfrak{U}$ with respect to η and ζ in

$$0 < |x| < \xi_N, \underline{\Theta} < \arg x < \overline{\Theta},\ \ \|\eta\| < d_N,\ \ \|\zeta\| < d_N, \tag{6.4}$$

where ξ_N and d_N are constants which depend on N, $\xi_N < \xi$, d_N depends on d and ξ_N, and

$$\left.\begin{aligned} \hat{f}_\eta(0,0,0) &= \mathrm{D},\ \hat{f}_\zeta(0,0,0) = 0, \\ \hat{f}(0,0,0) &= 0,\ \hat{g}(0,0,0) = 0 \end{aligned}\right\} . \tag{6.5}$$

Therefore, we have the inequalities

$$\left.\begin{aligned} \|\hat{f}(x,\eta,\zeta)\| &\leqslant H(\|\eta\| + \|\zeta\|) + B_N\ |x|^N \\ \|\hat{g}(x,\eta,\zeta)\| &\leqslant H'(\|\eta\| + \|\zeta\|) + B_N |x|^N \end{aligned}\right\}, \tag{6.6}$$

where H, H', and B_N are positive constants, and H and H' are independent of N, for (6.4). Moreover, $\hat{f}$ and $\hat{g}$ satisfy Lipschitz conditions with respect to (η, ζ), namely

$$\|\hat{f}(x,\eta^1,\zeta^1) - \hat{f}(x,\eta^2,\zeta^2)\| \leqslant H(\|\eta^1 - \eta^2\| + \|\zeta^1 - \zeta^2\|)$$

and

$$\|\hat{g}(x,\eta^1,\zeta^1) - \hat{g}(x,\eta^2,\zeta^2)\| \leqslant H'(\|\eta^1 - \eta^2\| + \|\zeta^1 - \zeta^2\|)$$

for (x,η^1,ζ^1) and (x,η^2,ζ^2) in (6.4). By the fact that D is nilpotent, we can assume without loss of generality that H satisfies

$$8H < \|\gamma\|' \sin 2\sigma\epsilon \tag{6.7}$$

where $\|\gamma\|' = \min\limits_{1 \leqslant j a} |\gamma_j|$. Also, we take N so large that

$$4H' < N. \tag{6.8}$$

Then, the proof of Theorem A is reduced to that of solving the following:

PROBLEM A. *If we have* (6.7) *and* (6.8), *then there exists a unique solution* $\{\phi_N(x), \psi_N(x)\}$ *of* (6.3) *such that for suitable chosen* ξ'_N *and* K_N

(i) $\phi_N(x)$ *and* $\psi_N(x)$ *are holomorphic and bounded* α- *and* β-*column vector functions, respectively, for*

$$0 < |x| < \xi'_N, \underline{\Theta}^* < \arg x < \bar{\Theta}^*, \tag{6.9}$$

(ii) $\phi_N(x)$ *and* $\psi_N(x)$ *satisfy the inequalities*

$$[\phi_N(x)] \leqslant K_N \; |x|^N \; [e^{-\mathrm{Re}\;\Omega(x)}], \;\; \| \psi_N(x) \| \leqslant K_N \; |x|^N \tag{6.10}$$

for x in (6.9).

Moreover, a solution of (6.3) *satisfying*

$$[P_N] \; = 0(|x|^N)[e^{-\mathrm{Re}\;\Omega(x)}], \; \|Q_N\| \leqslant 0(|x|^N) \tag{6.11}$$

is unique.

Theorem A can be derived from the solution of this problem by using an argument similar to that presented in Section 12.

7. A Fundamental Lemma

Let $A(\tau)$ be a scalar function of τ, defined in the interval

$$\underline{\Theta}^* < \tau < \bar{\Theta}^* \tag{7.1}$$

such that

$$2\sigma\epsilon \leqslant A(\tau) \leqslant \pi - 2\sigma\epsilon \tag{7.2}$$

for any preassigned ϵ; This function $A(\tau)$ will be given specifically in the next section. Define the function $\omega(\tau)$ by

$$\omega(\tau) \;=\; \exp \int_{\theta_0}^{\tau} \;\cot A(t)dt, \tag{7.3}$$

where θ_0 is a fixed angle in (7.1).

Instead of finding the solutions of Problem A, we shall prove the following:

LEMMA A. *There exists a continuous function* $A(\tau)$ *defined in* (7.1) *and positive constants* ξ''_N *and* K_N *such that* (6.3) *has a unique solution* $\{\tilde{\varphi}_N(x), \tilde{\psi}_N(x)\}$ *satisfying*

(i) $\tilde{\varphi}_N(x)$, *and* $\tilde{\psi}_N(x)$ *are holomorphic and bounded* α- *and* β-*column vector functions, respectively, for*

$$0 < |x| < \xi''_N \; \omega(\arg x), \underline{\Theta}^* < \arg x < \bar{\Theta}^*. \tag{7.4}$$

(ii) $\tilde{\varphi}_N(x)$ *and* $\tilde{\psi}_N(x)$ *satisfy the inequalities*

$$[\widetilde{\varphi}_N(x)] \leqslant K_N\,|x|^N\,[e^{-\mathrm{Re}\,\Omega(x)}],\quad \|\widetilde{\psi}_N(x)\| \leqslant K_N\,|x|^N \tag{7.5}$$

for x in (7.4).

Since $\omega(\tau)$ is positive and bounded, *the domains* (7.4) *and* (6.9) *are equivalent* in the sense that any point in (7.4) is contained in (6.9) if we choose ξ_N'' suitably, and vice versa. Thus Problem A is solved if Lemma A is proved.

8. Determination of $A(\tau)$

The directions $\arg x = \theta_j$ in the sector

$$\underline{\Theta}^* < \arg x < \overline{\Theta}^*, \tag{8.1}$$

such that

$$\mathrm{Re}\,\Omega_j(x) = 0 \text{ for } \arg x = \theta_j,$$

are called singular directions of $\Omega_j(x)$ and are given by

$$\frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h\right) \tag{8.2}$$

or

$$\frac{1}{\sigma}\left(\arg \gamma_j - \frac{\pi}{2} + 2\pi h'\right) \tag{8.3}$$

where h and h' are some integers. Singular directions of the form (8.2) are called *ascending singular directions of $\Omega_j(x)$*, and those of the form (8.3) are called *descending singular directions.* It is to be noticed that, when we consider $\mathrm{Re}\,\Omega_j(x)$ as a function of $\arg x = \theta$, $\mathrm{Re}\,\Omega_j(x)$ is a monotonic increasing (or decreasing) function of $\arg x$ in a small neighborhood of each singular direction of the form (8.2) (or the form (8.3));

For those j such that $\mathrm{Re}\,\Omega_j(x)$ change their signs in (8.1), we shoose $\arg \gamma_j$ so that at least one of the two singular directions

$$\theta_{j+} = \frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2}\right) \tag{8.4}$$

or

$$\theta_{j-} = \frac{1}{\sigma}\left(\arg \gamma_j + \frac{3\pi}{2}\right) \tag{8.5}$$

is contained in (8.1). By the assumption that (8.1) has Property-$\mathfrak{Z}$ with respect to $\{\Omega_1(x), \ldots, \Omega_a(x)\}$, we can classify the set $J = \{1, 2, \ldots, a\}$ of indices j into four classes:

$$J_0 = \{\, j; \mathrm{Re}\,\Omega_j(x) > 0 \text{ for } \underline{\Theta}^* \leqslant \arg x \leqslant \overline{\Theta}^* \,\},$$

$$J_1 = \{\, j; \Theta^* < \theta_{j+} < \theta_{j-} < \overline{\Theta}^* \,\},$$

$$J_2 = \left\{ j; \underline{\Theta}^* < \theta_{j+} < \overline{\Theta}^* < \theta_{j-} \right\},$$

$$J_3 = \left\{ j; \theta_{j+} < \underline{\Theta}^* < \theta_{j-} < \overline{\Theta}^* \right\}.$$

For $j \in J_2$, we define θ_{j-} by (8.5) and for $j \in J_3$, we define θ_{j+} by (8.4). Some of these four sets may be empty. Specially, either J_0 or J_1 must be empty, for $\overline{\Theta}^* - \underline{\Theta}^* < (\pi/\sigma)$ when J_0 is not empty and $\overline{\Theta}^* - \underline{\Theta}^* > (\pi/\sigma)$ when J_1 is not empty. Therefore $\{1, 2, \ldots, a\} = J_1 \cup J_2 \cup J_3$ or $\{1, 2, \ldots, a\} = J_0 \cup J_2 \cup J_3$.

Since the sector (8.1) has Property-$\mathfrak{Z}$ with respect to $\{\Omega_1(x), \ldots, \Omega_a(x)\}$, the angles $\underline{\Theta}^*$ and $\overline{\Theta}^*$ must satisfy the inequality, for sufficient small $\epsilon > 0$,

$$\max_{j=1}^{a} \theta_{j+} - \left(\frac{\pi}{\sigma} + 6\epsilon\right) \leqslant \underline{\Theta}^* < \overline{\Theta}^* \leqslant \min_{j=1}^{a} \theta_{j-} + \left(\frac{\pi}{\sigma} - 6\epsilon\right) \tag{8.6}$$

for all $j \in J_1 \cup J_2 \cup J_3$ or $j \in J_2 \cup J_3$. Put

$$\Theta_{k+} = \max_{j \in J_k} \theta_{j+}, \Theta_{k-} = \min_{j \in J_k} \theta_{j-} \tag{8.7}$$

where $k = 1, 2, 3$ or $k = 2, 3$. Then $A(\tau)$ is defined by

$$A(\tau) = \begin{cases} \sigma(\tau - \Theta_{3-} + 4\epsilon), \Theta_{3-} + \dfrac{\pi}{2\sigma} - 4\epsilon \leqslant \tau \leqslant \overline{\Theta}^* \\[2ex] \dfrac{\pi}{2}, \; \Theta_{2+} - \dfrac{\pi}{2\sigma} + 4\epsilon \leqslant \tau \leqslant \Theta_{3-} + \dfrac{\pi}{2\sigma} - 4\epsilon \\[2ex] \sigma(\tau - \Theta_{2+} - 4\epsilon) + \pi, \underline{\Theta}^* \leqslant \tau \leqslant \Theta_{2+} - \dfrac{\pi}{2\sigma} + 4\epsilon. \end{cases} \tag{8.8}$$

Noticing that

$$\left.\begin{aligned} \Theta_{2+} &= \max \theta_{j+} \; (j \in J_1 \cup J_2 \cup J_3 \text{ or } j \in J_2 \cup J_3) \\ \Theta_{3-} &= \min \theta_{j-} \; (j \in J_1 \cup J_2 \cup J_3 \text{ or } j \in J_2 \cup J_3) \end{aligned}\right\}, \tag{8.9}$$

we see that by (8.6) $A(\tau)$ satisfies

$$2\sigma\epsilon \leqslant A(\tau) \leqslant \pi - 2\sigma\epsilon \text{ for } \underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*. \tag{8.10}$$

9. A Fundamental Inequality

In order to prove Lemma A, we need an integral inequality stated as follows:

LEMMA 1. *Let x_1 be an arbitrary point in the domain (7.4). Then there exists an a-vector path Γ_{x_1} with elements $\{\Gamma_{jx_1}\}$ such that*

(i) *Each curve Γ_{jx_1} joins the point x_1 with the origin and is contained in the domain (7.4) except for the origin.*

(ii) *If ξ_N'' satisfies*

$$2N(\xi_N'' \max_{\underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*} \omega(\tau))^\sigma \leqslant \|\gamma\|' \sin 2\sigma\epsilon, \left(\|\gamma\|' = \min_{j=1}^{a} |\gamma| \right), \tag{9.1}$$

then

$$\int_0^{x_1} |x|^{N-\sigma-1} e^{-Re\,\Omega_j(x)}\,|dx| \leqslant \frac{2}{\|\gamma\|' \sin 2\sigma\epsilon}\,|x_1|^N e^{-Re\,\Omega_j(x_1)} \qquad (j = 1,2,\dots,a). \tag{9.2}$$

Here, the integration is carried along Γ_{jx_1}.

We shall define the path vector Γ_{x_1} and prove assertion (i) of Lemma 1 in this section. Assertion (ii) will be proved in the next section.

First, let us define an a-column vector function $a(\tau)$ with elements $\{a_j(\tau)\}$ in the interval (8.1).

If $j \in J_0$,

$$a_j(\tau) = \frac{\pi}{2},\ \underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*. \tag{9.3}$$

If $j \in J_1$,

$$a_j(\tau) = \begin{cases} \sigma(\tau - \theta_{j-} + 4\epsilon),\ \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \overline{\Theta}^*, \\ \frac{\pi}{2},\ \theta_{j+} + 2\epsilon < \tau < \theta_{j-} - 2\epsilon, \\ \sigma(\tau - \theta_{j+} - 4\epsilon) + \pi,\ \Theta^* \leqslant \tau \leqslant \theta_{j+} + 2\epsilon. \end{cases} \tag{9.4}$$

If $j \in J_2$,

$$a_j(\tau) = \begin{cases} \frac{\pi}{2},\ \theta_{j+} + 2\epsilon \leqslant \tau \leqslant \overline{\Theta}^*, \\ \sigma(\tau - \theta_{j+} - 4\epsilon) + \pi,\ \underline{\Theta}^* \leqslant \tau \leqslant \theta_{j+} + 2\epsilon. \end{cases} \tag{9.5}$$

If $j \in J_3$

$$a_j(\tau) = \begin{cases} \sigma(\tau - \theta_{j-} + 4\epsilon),\ \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \overline{\Theta}^*, \\ \frac{\pi}{2},\ \underline{\Theta}^* \leqslant \tau \leqslant \theta_{j-} - 2\epsilon. \end{cases} \tag{9.6}$$

It should be noticed that either J_0 or J_1 is empty. By virtue of (8.6), it is easily seen that

$$2\sigma\epsilon \leqslant a_j(\tau) \leqslant \pi - 2\sigma\epsilon \text{ for } \underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*. \tag{9.7}$$

Moreover, by virtue of (8.7),

$$\left.\begin{aligned} &a_j(\tau) \leqslant A(\tau),\ \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \overline{\Theta}^*\ (j \in J_1, J_3), \\ &a_j(\tau) \geqslant A(\tau),\ \underline{\Theta}^* \leqslant \tau \leqslant \theta_{j+} + 2\epsilon\ (j \in J_1, J_2) \end{aligned}\right\}. \tag{9.8}$$

Hence, we have

$$\int_{\theta}^{\tau} \cot a_j(t)\,dt \leqslant \int_{\theta}^{\tau} \cot A(t)\,dt \tag{9.9}$$

for $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon\ (j \in J_1, J_2)$ and for $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta\ (j \in J_1, J_3)$.

Let (r,θ) and (ρ,τ) be the polar coordinates of the point x_1 and of the variable point x on the curve Γ_{jx_1} respectively. Then the curve Γ_{jx_1} is defined as follows:

(i) If $\theta < \theta_{j+} + 2\epsilon$ or $\theta_{j-} - 2\epsilon < \theta$, the curve Γ_{jx_1} consists of a curvilinear part Γ_j'

$$\begin{gathered} \rho = r \exp \int_{\theta}^{\tau} \cot a_j(t)\,dt \\ \theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon \ \text{ or } \ \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta, \end{gathered} \tag{9.10}$$

and of a rectilinear part Γ_j''

$$\begin{gathered} 0 \leqslant \rho \leqslant r \exp \int_{\theta}^{\tau} \cot a_j(t)\,dt, \\ \tau = \theta_{j+} + 2\epsilon \ \text{ or } \ \theta_{j-} - 2\epsilon. \end{gathered} \tag{9.11}$$

(ii) If $\theta_{j+} + 2\epsilon \leqslant \theta \leqslant \theta_{j-} - 2\epsilon$, the curve Γ_{jx_1} consists of only a rectilinear part Γ_j''

$$0 \leqslant \rho \leqslant r,\ \tau = \theta. \tag{9.12}$$

By virtue of (9.9), the curves Γ_{jx_1} defined by (9.10), (9.11), or (9.12) are contained entirely in the domain (7.4), except for the origin. This proves assertion (i) of Lemma 1.

10. Completion of Proof of Lemma 1

In order to prove assertion (ii) of Lemma 1, we need the following differential inequalities.

LEMMA 2. *Let s_j be the arc length of the curve Γ_{jx_1} measured from the origin to the variable point x on this curve. Then,*

$$\frac{d}{ds_j} e^{-Re\,\Omega_j(x)} \geqslant |x|^{-\sigma-1} e^{-Re\,\Omega_j(x)} \|\gamma\|' \sin 2\sigma\epsilon \tag{10.1}$$

and

$$|x|^{-1} \frac{d|x|}{ds_j} \geqslant -|x|^{-1} \tag{10.2}$$

holds as x moves on Γ_{jx_1}.

In fact, if (10.1) and (10.2) hold, then

$$\frac{d}{ds_j}\left(|x|^N e^{-Re\ \Omega_j(x)}\right) \geqslant |x|^{N-\sigma-1} e^{-Re\ \Omega_j(x)}\left(\|\gamma\|'\, 2\sigma\epsilon - N|x|^{\sigma}\right). \tag{10.3}$$

If we choose ξ_N'' such that (9.1) is satisfied, then

$$2(\|\gamma\|' \sin 2\sigma\epsilon - N|x|^{\sigma}) \geqslant \|\gamma\|' \sin 2\sigma\epsilon$$

for x in (7.4). Thus

$$\frac{d}{ds_j}\left(|x|^N e^{-Re\ \Omega_j(x)}\right) \geqslant \frac{\|\gamma\|' \sin 2\sigma\epsilon}{2}\ |x|^{N-\sigma-1} e^{-Re\ \Omega_j(x)} \tag{10.4}$$

for x on Γ_{jx_1}. Then, assertion (ii) follows and Lemma 1 is completely proved.

In order to prove Lemma 2, put $x = \rho e^{i\tau}$. Notice that on the curvilinear part Γ_j', ρ is a function of τ given by (9.10). Since

$$\frac{d\rho}{d\tau} = \rho \cot a_j(\tau), \tag{10.5}$$

we have

$$\frac{ds_j}{d\tau} = \pm\left[\left\{\frac{d}{d\tau}(\rho\cos\tau)\right\}^2 + \left\{\frac{d}{d\tau}(\rho\sin\tau)\right\}^2\right]^{1/2} = \frac{\mp\rho}{\sin a_j(\tau)}, \tag{10.6}$$

where the "-" is for $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$ and the "+" is for $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$. Thus,

$$\frac{dx}{ds_j} = \mp e^{i\tau}(\cot a_j(\tau) + i)\sin a_j(\tau) = \mp e^{(a_j(\tau)+\tau)i} \tag{10.7}$$

according as $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$ or $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$. Hence, we have the equality

$$\frac{d}{ds_j}(-Re\ \Omega_j(x)) = \pm\rho^{-\sigma-1}|\gamma_j|\cos(a_j(\tau) - \sigma\tau + \arg\gamma_j) \tag{10.8}$$

where "+" is for $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$ and "-" is for $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$, and consequently,

$$\frac{d}{ds_j}\ e^{-Re\ \Omega_j(x)} = \pm\rho^{-\sigma-1}|\gamma_j| e^{-Re\ \Omega_j(x)}\cos(a_j(\tau) - \sigma\tau + \arg\gamma_j) \tag{10.9}$$

according as $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$ or $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$.

On the other hand, by the definitions of the functions $a_j(\tau)$ and of the angles θ_{j+}, θ_{j-}, we have

$$a_j(\tau) - \sigma\tau + \arg\gamma_j = \begin{cases} \dfrac{\pi}{2} + 4\sigma\epsilon \pmod{2\pi}, & \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta_{j-} + \dfrac{\pi}{\sigma} - 6\epsilon, \\[2ex] \dfrac{\pi}{2} - 4\sigma\epsilon \pmod{2\pi}, & \theta_{j+} - \dfrac{\pi}{\sigma} + 6\epsilon \leqslant \tau \leqslant \theta_{j+} + 2\epsilon. \end{cases} \tag{10.10}$$

Hence

$$\pm \cos (a_j(\tau) - \sigma\tau + \arg \gamma_j) = \sin 4\sigma\epsilon > \sin 2\sigma\epsilon.$$

This proves the inequality (10.1) for x on Γ_j'.

On the rectilinear part Γ_j'', $x = \rho e^{i\theta}$ and $s = \rho$. Thus,

$$\frac{d}{ds_j} e^{-Re\, \Omega_j(x)} = -e^{-Re\, \Omega_j(x)} \frac{d}{d\rho} Re\, \Omega_j(x) \tag{10.11}$$

$$= -e^{-Re\, \Omega_j(x)} \rho^{-\sigma-1} |\gamma_j| \cos (\arg \gamma_j - \sigma\theta) \geqslant e^{-Re\, \Omega_j(x)} \rho^{-\sigma-1} |\gamma_j| \sin 2\sigma\epsilon$$

because $\theta_{j+} + 2\epsilon \leqslant \theta \leqslant \theta_{j-} - 2\epsilon$. Therefore, (10.1) is true as x moves on Γ_{jx_1}.

In order to prove (10.2), we observe that s_j is real. Then

$$|x|^{-1} \frac{d|x|}{ds_j} = \frac{d}{ds_j} \log |x| = \frac{d}{ds_j} (Re \log x) = Re\, (\frac{d}{ds_j} \log x) = Re\, (x^{-1} \frac{dx}{ds_j}) \geqslant - |x|^{-1} \tag{10.12}$$

Here we use (10.7) when x in on Γ_j'. When x is on Γ_j'', this inequality follows immediately from the fact the $|x| = s_j$.

Thus, Lemma 2 is proved.

11. Solution of Problem A

Consider, for an arbitrary point x_1 in (7.4), the system of integral equations

$$\left.\begin{aligned} \varphi(x_1) &= \int_0^{x_1} x^{-\sigma-1} \mathbf{1}_a (e^{-\Omega(x)}) \hat{f} \left(x, \mathbf{1}_a (e^{\Omega(x)}) \varphi(x), \psi(x)\right) dx \\ \psi(x_1) &= \int_0^{x_1} x^{-1} \hat{g} \left(x, \mathbf{1}_a (e^{\Omega(x)}) \varphi(x), \psi(x)\right) dx \end{aligned}\right\} \tag{11.1}$$

where the integration of the jth component of the first equation of (11.1) is carried along the curve Γ_{jx_1} defined in Section 9 and the integrations of the second equation are carried along the segment $\overline{Ox_1}$ joining x_1 and the origin. Applying the integral inequality (9.2) and using successive approximations, a discussion analogous to that in Chapter V will prove Lemma A, and consequently give the solution of Problem A.

Thus Theorem A is proved.

IV. PROOF OF THEOREM B

12. Reduction of Theorem B

In order to prove Theorem B, we first consider, for a positive integer N, the following transformations to (E_2):

$$y = \sum_{|q|<N} V(x)^q f_q(x) + \eta_N, \quad z = \sum_{|q|<N} V(x)^q g_q(x) + \zeta_N. \tag{12.1}$$

Then the transformed equations can be written as

$$\left.\begin{aligned} x^{\sigma+1}\eta_N &= 1_a(\gamma)\eta_N + \hat{f}(x, V(x); \eta_N, \zeta_N) \\ x\zeta' &= \hat{g}(x, V(x); \eta_N, \zeta_N) \end{aligned}\right\} \tag{12.2}$$

where $\hat{f}(x, \nu; \eta, \zeta)$ and $\hat{g}(x, \nu; \eta, \zeta)$ are holomorphic and bounded vector functions of (x, ν, η, ζ) for

$$0 < |x| < \xi_N, \underline{\Theta}^* < \arg x < \overline{\Theta}^*, \ \|\nu\| < \delta_N, \ \|\eta\| < d_N, \ \|\zeta\| < d_N. \tag{12.3}$$

Here ξ_N, δ_N and d_N are constants which depend on $N, \xi_N < \xi, \delta_N < \delta$, and d_N depends on d, ξ_N, and δ_N. Further,

$$\left.\begin{aligned} &\hat{f}_\eta(0,0;0,0) = D, \hat{f}_\zeta(0,0;0,0) = 0, \\ &\hat{f}(0,0;0,0) = 0, \ \hat{g}(0,0;0,0) = 0. \end{aligned}\right\} \tag{12.4}$$

Therefore, we have the inequalities

$$\left.\begin{aligned} &\|\hat{f}(x, \nu; \eta, \zeta)\| \leq H(\|\eta\| + \|\zeta\|) + B_N \|\nu\|^N \\ &\|\hat{g}(x, \nu; \eta, \zeta\| \leq H'(\|\eta\| + \|\zeta\|) + B_N \|\nu\|^N \end{aligned}\right\} \tag{12.5}$$

for (12.3), where H, H', and B_N are positive constants and H and H' are independent on N. Moreover, $\hat{f}$ and $\hat{g}$ satisfy Lipschitz conditions with respect to (η,ζ) with Lipschitz constants H and H', respectively, in (12.3). Since D is a nilpotent matrix, we can assume without loss of generality that H satisfies

$$8H < \|\gamma\|' \sin 2\sigma\epsilon \tag{12.6}$$

for a preassigned number ϵ. Also, we take N so large that

$$4H' < N \|Re\,\mu\|', (\|Re\,\mu\|' = \min\{Re\,\mu_k\}). \tag{12.7}$$

Put

$$\eta_N = 1_a(e^{\Omega(x)})P_N, \zeta_N = Q_N. \tag{12.8}$$

Then, the equation (12.1) is reduced to

$$\left.\begin{aligned} P_N' &= x^{-\sigma-1} 1_a(e^{-\Omega(x)})\hat{f}(x, V(x); 1_a(e^{\Omega(x)})P_N, Q_N) \\ Q_N' &= x^{-1}\hat{g}(x, V(x); 1_a(e^{\Omega(x)})P_N, Q_N) \end{aligned}\right\}. \tag{12.9}$$

Thus, the proof of Theorem B is reduced to solving the following:

PROBLEM B. *If we have* (12.6) *and* (12.7), *then there exists a solution* $\{\varphi_N(x, V(x)), \psi_N(x, V(x))\}$ *of* (12.9) *such that for suitably chosen* ξ_N' δ_N' *and* K_N

(i) $\varphi_N(x,\nu)$ *and* $\psi_N(x,\nu)$ *are holomorphic and bounded* α*- and* β*-column vector functions, respectively, for*

$$0 < |x| < \xi'_N, \Theta^* < \arg x < \Theta^*, \ \|v\| < \delta'_N; \tag{12.10}$$

(ii) $\varphi_N(x,v)$ *and* $\psi_N(x,v)$ *satisfy the inequalities*

$$[\varphi_N(x,v)] \leqslant K_N \ \|v\|^N \ [e^{-\mathrm{Re}\,\Omega(x)}], \ \|\psi_N(x,v)\| \leqslant K_N \ \|v\|^N, \tag{12.11}$$

for (x, v) *in* (12.10).

Moreover, a solution of (12.9) *satisfying*

$$[P_N] = 0(\|V(x)\|^N) \ [e^{-\mathrm{Re}\,\Omega(x)}], \ \|Q_N\| = 0(\|V(x)\|^N) \tag{12.12}$$

is unique.

In fact, we can prove Theorem B from the solution of Problem B in the following manner. Owing to the transformations (12.1) and (12.8), the quantities

$$\left.\begin{array}{l} \displaystyle\sum_{|q|<N} V(x)^q f_q(x) + 1_a(e^{\Omega(x)})\, \varphi_N(x, V(x)) \\[2em] \displaystyle\sum_{|q|<N} V(x)^q g_q(x) + \psi_N(x, V(x)) \end{array}\right\} \tag{12.13}$$

are a solution of equations (E_2) provided that $(x, V(x))$ is in the domain (12.10). Let N' be an integer greater than N.

Then

$$\left.\begin{array}{l} \displaystyle 1_a(e^{-\Omega(x)}) \sum_{N \leqslant |q| < N'} V(x)^q f_q(x) + \varphi_{N'}(x, V(x)) \\[2em] \displaystyle\sum_{N \leqslant |q| < N'} V(x)^q g_q(x) + \psi_{N'}(x, V(x)) \end{array}\right\} \tag{12.14}$$

are a solution of equations (12.9), satisfying $(12.12)_N$ if $(x, V(x))$ belongs to the common part of the domains $(12.10)_N$ and $(12.10)_{N'}$. Hence, by the uniqueness of solution, the solution (12.14) must coincide with $\{\varphi_N(x, V(x)), \psi_N(x, V(x))\}$. From this, the solution of (E_2) expressed by (12.13) is independent of N provided that N satisfies (12.7). We denote this solution by $\{F(x, V(x)), G(x, V(x))\}$. Then by analytic continuation, the functions $F(x,v)$ and $G(x,v)$ are defined in the domain of the form (5.6) with $\xi^0 = \sup \xi'_N$, $\delta^0 = \sup \delta'_N$.

On the other hand, $v = 0$ is an interior point of the domain (5.6) in which the vector functions $F(x,v)$ and $G(x,v)$ are defined. Therefore, by Cauchy's theorem, $F(x, V(x))$ and $G(x, V(x))$ can be expanded into a uniformly convergent power series of $V(x)$ whenever $(x, V(x))$ is in the domain (5.6). Clearly, from Problem B, we know that $F(x, V(x))$ and $G(x, V(x))$ admit the asymptotic expansions (5.4). By the uniqueness of asymptotic expansions, these asymptotic expansions must coincide with the uniformly convergent expansions. This proves the uniform convergence of the formal solutions (5.4).

Thus Theorem B is proved.

13. A Fundamental Lemma for Problem B

In order to find the solution of Problem B, similar to that for Problem A, it is necessary to replace $(12.10)_N$ by *an equivalent domain* of the form

$$0<|x|<\xi_N'' \,\omega(\arg x),\ [v]<\delta_N''\,[\chi(\arg x)],\ \underline{\Theta}^*<\arg x<\overline{\Theta}^*. \tag{13.1}$$

Here $\omega(\tau)$ is the scalar function defined by (7.3) with $A(\tau)$ defined by (8.6) and $\chi(\tau)$ is an n-column vector function with elements $\{\chi_k(\tau)\}$ defined as

$$\chi_k(\tau) = \exp\left\{(\operatorname{Re}\mu_k)\int_{\theta_0}^{\tau} \cot A(t)\,dt + (\operatorname{Im}\mu_k)(\theta_0-\tau)\right\}, \tag{13.2}$$

where θ_0 is a fixed angle satisfying $\underline{\Theta}^* \leqslant \theta_0 \leqslant \overline{\Theta}^*$.

Instead of finding the solution of Problem B, we shall prove the following

LEMMA B. *There exists positive constants ξ_N'', δ_N'' and K_N such that* (12.9) *has a unique solution* $\{\tilde{\varphi}_N(x, V(x)), \tilde{\psi}_N(x, V(x))\}$ satisfying

(i) $\tilde{\varphi}_N(x,v)$ *and* $\tilde{\psi}_N(x,v)$ *are holomorphic and bounded α- and β-column vector functions, respectively, for* (x,v) *in* (13.1);

(ii) $\tilde{\varphi}_N(x,v)$ *and* $\tilde{\psi}_N(x,v)$ *satisfy the inequalities*

$$[\tilde{\varphi}_N(x,v)] \leqslant K_N \,\|v\|^N\,[e^{-\operatorname{Re}\Omega(x)}],\quad \|\tilde{\psi}_N(x,v)\| \leqslant K_N\,\|v\|^N \tag{13.3}$$

for (x,v) *in* (13.1).

This Lemma will be proved in Chapter V.

14. Fundamental Inequalities for Problem B

In order to prove Lemma B, we must prove fundamental inequalities stated in the following.

LEMMA 3. *Let (x_1, v^1) be an arbitrary point in a domain of the form*

$$0<|x|<\xi_N\,\omega(\arg x),\ [v]<\delta_N\,[\chi(\arg x)],\ \underline{\Theta}^*<\arg x<\overline{\Theta}^*. \tag{14.1}$$

Choose the n-column vector C so that $V(x_1)=v^1$, namely, $C=1_n(x_1^{\,-\mu})\,v^1$. Then there exists an α-vector path Γ_{x_1} with elements $\{\Gamma_{jx_1}\}$ such that

(i) *Each curve Γ_{jx_1} joins the point x_1 with the origin and is contained in the domain*

$$0<|x|<\xi_N\,\omega(\arg x),\ \underline{\Theta}^*<\arg x<\overline{\Theta}^* \tag{14.2}$$

except for the origin;

(ii) *As x moves on the curve Γ_{jx_1}, we have*

$$[V(x)]<\delta_N\,[\chi(\arg x)],\ \underline{\Theta}^*<\arg x<\overline{\Theta}^*; \tag{14.3}$$

(iii) *If ξ_N satisfies*

$$2N \, \| \mu \| \Big(\xi_N \max_{\underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*} \omega(\tau)\Big)^{\sigma} \leqslant \| \gamma \|' \sin 2\sigma\epsilon, \tag{14.4}$$

then

$$\int_0^{x_1} |x|^{-\sigma-1} \, \|V(x)\|^N \, e^{-\operatorname{Re}\Omega_j(x)} \, |dx| \leqslant \frac{2}{\|\gamma\|' \sin 2\sigma\epsilon} \, \|V(x_1)\|^N \, e^{-\operatorname{Re}\Omega_j(x_1)} \tag{14.5}$$

$$(j = 1, 2, \ldots, a).$$

Here the integration is carried along Γ_{jx_1}.

The curves Γ_{jx_1} are defined exactly in the same way as in Section 9 for the proof of Lemma 1. Then assertion (i) is evidently satisfied. Assertion (ii) will be proved in this section and the proof of assertion (iii) will be given in the next section.

Let $x_1 = re^{i\theta}$ and the variable point $x = \rho e^{i\tau}$. Let the components of the n-vectors $V(x)$, v^1, and μ be $\{ V_k(x) \}$, $\{ v_k^1 \}$, and $\{ \mu_k \}$, respectively. Then (14.3) is equivalent to n inequalities.

$$|V_k(x)| < \delta_N \exp \left\{ (\operatorname{Re}\mu_k) \int_{\theta_0}^{\tau} \cot A(t)dt + (\operatorname{Im}\mu_k)(\theta_0 - \tau) \right\} \tag{14.6.k}$$

as x moves on the curve Γ_{jx_1}. Observe that the curve Γ_{jx_1} consists of two parts Γ_j' and Γ_j'' in general, and we have $V_k(x) = v_k^1 (x/x_1)^{\mu_k}$. Thus

$$|V_k(x)| = |v_k^1| \left(\frac{\rho}{r}\right)^{\operatorname{Re}\mu_k} \exp \{ (\operatorname{Im}\mu_k)(\theta - \tau) \} . \tag{14.7.k}$$

For x on Γ_j', ρ is a function of τ given by (9.10) and we have

$$|V_k(x)| = |v_k^1| \exp \left\{ (\operatorname{Re}\mu_k) \int_{\theta}^{\tau} \cot a_j(t)dt + (\operatorname{Im}\mu_k)(\theta - \tau) \right\} ;$$

consequently, by (9.9)

$$|V_k(x)| \leqslant |v_k^1| \exp \left\{ (\operatorname{Re}\mu_k) \int_{\theta}^{\tau} \cot A(t)dt + (\operatorname{Im}\mu_k)(\theta - \tau) \right\} . \tag{14.8.k}$$

On the other hand, since $v^1 = V(x_1)$, v_k^1 must satisfy the inequality (14.6.k) with $\tau = \theta$. Namely,

$$|v_k^1| < \sigma_N \exp \left\{ (\operatorname{Re}\mu_k) \int_{\theta_0}^{\theta} \cot A(t)dt + (\operatorname{Im}\mu_k)(\theta_0 - \theta) \right\} . \tag{14.9.k}$$

Hence by (14.8.k) and (14.9.k), (14.3) holds for x on Γ_j'.

For x on Γ_j'', $\rho \leqslant r$ and τ is constant. Hence, by virtue of (14.7.k), $|V_k(x)| \leqslant |v_k^1|$. Thus, by (14.9.$k$), (14.3) holds for x on Γ_j''.

Thus, assertion (ii) of Lemma 3 is proved.

15. Completion of Proof of Lemma 3

In order to prove (iii) of Lemma 3, we need the following differential inequality.

LEMMA 4. *Let s_j be the arc length of the curve Γ_{jx_1} measured from the origin to the variable point x on this curve. Then*

$$\frac{d}{ds_j} \|V(x)\| \geqslant - |x|^{-1} \|\mu\| \, \|V(x)\| \tag{15.1}$$

holds as x moves on Γ_{jx_1}.

In fact, if (15.1) holds for x on Γ_{jx_1}, by virtue of (10.1) we have

$$\frac{d}{ds_j} \left(\|V(x)\|^N e^{-\mathrm{Re}\,\Omega_j(x)} \right) \geqslant |x|^{-\sigma-1} \|V(x)\|^N e^{-\mathrm{Re}\,\Omega_j(x)} \tag{15.2}$$

$$\cdot (\|\gamma\|' \sin 2\sigma\epsilon - N \|\mu\| \, |x|^{\sigma})$$

for x on Γ_{jx_1}. Thus, if ξ_N satisfies (14.4), then for $x \in \Gamma_{jx_1}$ we have

$$\frac{d}{ds_j} \left(\|V(x)\|^N e^{-\mathrm{Re}\,\Omega_j(x)} \right) \geqslant \frac{\|\gamma\|' \sin 2\sigma\epsilon}{2} |x|^{-\sigma-1} \|V(x)\|^N e^{-\mathrm{Re}\,\Omega_j(x)}.$$

Consequently, (14.5) follows immediately and the proof of Lemma 3 is completed.

To prove Lemma 4, notice, analogous to that for (10.2), that

$$|V_k(x)|^{-1} \frac{d}{ds_j} |V_k(x)| = \mathrm{Re} \left(V_k(x)^{-1} \frac{d}{ds_j} V_k(x) \right) = \mathrm{Re}\,\mu_k x^{-1} \frac{dx}{ds_j}. \tag{15.3}$$

Since $|dx/ds_j| = 1$ except for the joint of the curves Γ_j' and Γ_j'', it follows that

$$\frac{d}{ds_j} |V_k(x)| \geqslant - |\mu_k| \, |x|^{-1} |V_k(x)| \geqslant - \|\mu\| \, |x|^{-1} \|V(x)\|$$

for x on Γ_{jx_1} and for $k = 1, 2, \ldots, n$.

Thus Lemma 4 is proved.

V. PROOF OF LEMMA B

16. Successive Approximations

We shall prove Lemma B by means of successive approximations in this chapter.

Let (x_1, v^1) be an arbitrary point in the domain (13.1), where ξ_N'' and δ_N'' are to be specified in the next section. It is easy to see that the system of equations (12.9) is equivalent to the system of $\alpha + \beta$ integral equations

$$
\left.\begin{aligned}
\Phi(x_1, \nu^1) &= \int_0^{x_1} x^{-\sigma-1}\, 1_a(e^{-\Omega(x)}) \hat{f}(x, V(x); 1_a(e^{\Omega(x)})\, \Phi(x, V(x)), \Psi(x, V(x)))\,dx \\
\Psi(x_1, \nu^1) &= \int_0^{x_1} x^{-1}\, \hat{g}(x, V(x); 1_a(e^{\Omega(x)})\, \Phi(x, V(x)), \Psi(x, V(x)))\,dx
\end{aligned}\right\} \quad (16.1)
$$

where $V(x) = 1_n(x^{\mu})C$, with C chosen so that $V(x_1) = \nu^1$; namely, $C = 1_n(x_1^{-\mu})\,\nu^1$. For the first equation, the paths are taken along the a-vector path Γ_{x_1} defined in Section 9, and that for the second equation is taken along the straight line segment $\overline{0x_1}$.

The successive approximations for (16.1) are defined to be the sequence of functions $\{\Phi^{(m)}(x_1, \nu^1), \Psi^{(m)}(x_1, \nu^1)\}$ $(m = 0, 1, 2, \ldots)$ given recursively by the formulas

$$
\Phi^{(0)}(x_1, \nu^1) \equiv 0,\ \Psi^{(0)}(x_1, \nu^1) \equiv 0 \qquad (16.2)
$$

and

$$
\left.\begin{aligned}
\Phi^{(m+1)}(x_1, \nu^1) &= \int_0^{x_1} x^{-\sigma-1}\, 1_a(e^{-\Omega(x)}) \hat{f}(x, V(x); \\
&\qquad 1_a(e^{\Omega(x)})\, \Phi^{(m)}(x, V(x)), \Psi^{(m)}(x, V(x)))\,dx, \\
\Psi^{(m+1)}(x_1, \nu^1) &= \int_0^{x_1} x^{-1} \hat{g}(x, V(x); 1_a(e^{\Omega(x)}) \\
&\qquad \Phi^{(m)}(x, V(x)), \Psi^{(m)}(x, V(x)))\,dx \\
&\qquad (m = 0, 1, 2, \ldots)\ .
\end{aligned}\right\} \quad (16.3m)
$$

Here the paths for (16.3) are taken as those for (16.1).

We shall prove that a sequence defined as $\{\Phi^{(m)}(x_1, \nu^1), \Psi^{(m)}(x_1, \nu^1)\}$ $(m = 0, 1, 2, \ldots)$ converges to the desired solution of (16.1), or, equivalently, to that of (12.9) in the following steps:

(I) Each term of the sequence $\{\Phi^{(m)}(x_1, \nu^1), \Psi^{(m)}(x_1, \nu^1)\}$ given by (16.3) is well defined and holomorphic in (x_1, ν^1) for (13.1);

(II) The sequence $\{\Phi^{(m)}(x_1, \nu^1), \Psi^{(m)}(x_1, \nu^1)\}$ $(m = 0, 1, 2, \ldots)$ converges uniformly to $\{\varphi(x_1, \nu^1), \psi(x_1, \nu^1)\}$ in any compact subset of (13.1);

(III) The limit functions $\{\varphi(x_1, \nu^1), \psi(x_1, \nu^1)\}$ satisfy the integral equations (16.1);

(IV) The functions $\{\varphi(x_1, \nu^1), \psi(x_1, \nu^1),\}$ are a solution of the differential equation (12.9) satisfying the properties described in Lemma B;

(V) A solution of (12.9) satisfying (13.3) is unique.

Step (III) means the interchange of limiting process and contour integration. Also, due to the relationship between x_1 and ν^1 through the function $V(x)$, step (IV) is not a trivial consequence of step (III).

The constants ξ_N'', δ_N'' and K_N will be specified in step (I). If steps (I) through (V) are proved, the unique solution $\{\varphi(x, \nu), \psi(x, \nu)\}$ will be denoted by $\{\tilde{\varphi}_N(x, \nu), \tilde{\psi}_N(x, \nu)\}$. Thus, Lemma B is proved.

17. The Functions $\Phi^{(1)}(x, \nu)$ and $\Psi^{(1)}(x, \nu)$

We shall prove step (I) by means of mathematical induction.

Let (x_1, ν^1) be an arbitrary point in a domain of the form (13.1), $V(x)$ be a holomorphic solution of $x\nu' = 1_n(\mu)\nu$ such that $V(x_1) = \nu^1$. Then the vectorial functions $\Phi^{(1)}(x_1, \nu^1)$ and $\Psi^{(1)}(x_1, \nu^1)$ are given by

$$\left.\begin{aligned} \Phi^{(1)}(x_1, \nu^1) &= \int_0^{x_1} f^{(1)}(x, V(x))\,dx \\ \Psi^{(1)}(x_1, \nu^1) &= \int_0^{x_1} g^{(1)}(x, V(x))\,dx \end{aligned}\right\}, \tag{17.1}$$

where

$$f^{(1)}(x, \nu) = x^{-\sigma-1}\, 1_a(e^{-\Omega(x)})\hat{f}(x, \nu; 0,0)$$

$$g^{(1)}(x, \nu) = x^{-1}\,\hat{g}(x, \nu; 0,0)$$

and the first integration is taken along Γ_{x_1} while the second integration is taken along $\overline{Ox_1}$.

A. <u>Existence of Integrations</u>–Since the integrands of (17.1) are holomorphic on their respective paths, except at $x = 0$, we shall first prove the convergence of the integrals at $x = 0$.

For the first integration, let x^*_{1j} be the last point of Γ_{jx_1} on $\tau = \theta_{j+} + 2\epsilon$ or on $\tau = \theta_{j-} - 2\epsilon$, according as whether $\underline{\Theta}^* \leqslant \theta \leqslant \theta_{j+} + 1\epsilon$ or $\theta_{j-} - 2\epsilon \leqslant \theta \leqslant \overline{\Theta}^*$, when x moves from 0 to x_1 along Γ_{jx_1}. Let $r^*_j = |x^*_{1j}|$. Then, from the definition of Γ_{jx_1} and the formulas (9.4) to (9.6), it is readily seen that

$$r^*_j = r\left\{\frac{\sin 2\sigma\epsilon}{\sin a_j(\theta)}\right\}^{\frac{1}{\sigma}} \geqslant r(\sin 2\sigma\epsilon)^{\frac{1}{\sigma}} > 0. \tag{17.2}$$

This means that for each j, no matter where x_1 is located in (13.1), the path Γ_{jx_1} is always has rectilinear portion Γ''_j of *positive* length. Furthermore,

$$\operatorname{Re}\Omega_j(x) > 0 \text{ on } \Gamma''_j. \tag{17.3}$$

Denote the jth components of $f^{(1)}$ by $f^{(1)}_j$. Then, by (12.5), we have

$$|f^{(1)}_j(x, V(x))| = O\left(|x|^{-\sigma-1+N\|\operatorname{Re}\mu\|}e^{-\operatorname{Re}\Omega_j(x)}\right). \tag{17.4}$$

By (17.3), the right-hand side of (17.4) tends to zero exponentially as x approaches the origin along Γ''_j. Thus, the jth components of the first integral exists at $x = 0$.

Furthermore, the integrand is bounded on Γ_{jx_1}, whether there is a curvilinear portion Γ'_j or not, and the length of Γ_{jx_1} is finite. Thus, the jth component of the first integral exists.

These facts are true for every component. Hence, $\Phi^{(1)}(x_1, \nu^1)$ exists for arbitrary (x_1, ν^1) in (13.1).

For the second integral, notice that the paths are taken along the straight line $\overline{Ox_1}$ and, consequently, $x^{-1}dx = d\rho$. By the fact that each component of the integrand is bounded, thanks to (6.6), the existence of the second integral, namely $\Psi^{(1)}(x_1, v^1)$, follows immediately.

B. Upper Bounds–By (12.5), we have

$$[f^{(1)}(x, V(x))] \leqslant |x|^{-\sigma-1} B_N \|V(x)\|^N \left[e^{-\mathrm{Re}\,\Omega(x)}\right], \quad \|g^{(1)}(x, V(x))\| \leqslant B_N \|V(x)\|^N \qquad (17.5)$$

for

$$0 < |x| < \xi_N'' \omega(\arg x), \quad \underline{\Theta}^* < \arg x < \overline{\Theta}^*. \qquad (17.6)$$

Choose ξ_N'' so small that

$$2N\|\mu\| (\xi_N'' \max_{\underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*} \omega(\tau))^\sigma \leqslant \|\gamma\|' \sin 2\sigma\epsilon. \qquad (17.7)$$

Then, by (14.5), we get

$$\left[\Phi^{(1)}(x_1, v^1)\right] \leqslant \frac{2B_N}{\|\gamma\|' \sin 2\sigma\epsilon} \|v^1\| \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right] \qquad (17.8)$$

for (x_1, v^1) in (13.1).

To estimate the upper bound of $\Psi^{(1)}(x_1, v^1)$, notice that

$$|V_k(x)| = \left(\frac{x}{x_1}\right)^{\mathrm{Re}\,\mu_k} e^{(\mathrm{Im}\,\mu_k)}(\arg x_1 - \arg x).$$

Then

$$\frac{d}{d|x|}\|V(x)\| = \frac{d}{d|x|}|V_k(x)| = (\mathrm{Re}\,\mu_k)\,|x|^{-1}|V_k(x)|$$

$$= (\mathrm{Re}\,\mu_k)\,|x|^{-1}\|V(x)\| \geqslant \|\mathrm{Re}\,\mu\|'\,|x|^{-1}\|V(x)\|$$

for some k. Thus

$$\frac{d}{d|x|}\|V(x)\|^N \geqslant N\|\mathrm{Re}\,\mu\|'\,|x|^{-1}\|V(x)\|. \qquad (17.9)$$

Hence, by (12.5), we have

$$\|\Psi^{(1)}(x_1, v^1)\| \leqslant \frac{B_N}{N\|\mathrm{Re}\,\mu\|'}\|v^1\|. \qquad (17.10)$$

for (x_1, v^1) in (13.1).

Now, we can choose K_N and δ_N'' such that

$$K_N = \max\left\{\frac{4B_N}{\|\gamma\|' \sin 2\sigma\epsilon}, \frac{2B_N}{N\|\mathrm{Re}\,\mu\|'}\right\}, \qquad (17.11)$$

and δ_N'' satisfies

$$K_N \left\{ \delta_N'' \max_{\underline{\Theta}^* \leqslant \tau \leqslant \overline{\Theta}^*} \|\chi^{(\tau)}\| \right\}^N < d_N. \tag{17.12}$$

These inequalities are needed to define $\Phi^{(m)}$ and $\Psi^{(m)}$ by (16.3).

C. Analyticity–First of all, when x_1 is fixed, (17.8) and (17.10) imply that the integrals (17.1) converge uniformly with respect to ν^1. Thus $\Phi^{(1)}(x_1, \nu^1)$ and $\Psi^{(1)}(x_1, \nu^1)$ are holomorphic in ν^1 for $[\nu^1] \leqslant \delta_N'' [\chi(\arg x_1)]$ when x_1 is fixed.

Next, we shall prove that $\Phi^{(1)}(x_1, \nu^1)$ and $\Psi^{(1)}(x_1, \nu^1)$ are holomorphic in x_1 for (17.6) when ν^1 is fixed. Let x_0 be a point in (17.6) and sufficiently near x_1. Observe that

$$\left\{ \begin{aligned} \int_0^{x_1} f^{(1)}(x, V(x))\,dx &= \int_0^{x_0} f^{(1)}(x, V(x))\,dx + \int_{x_0}^{x_1} f^{(1)}(x, V(x))\,dx \\ \int_0^{x_1} g^{(1)}(x, V(x))\,dx &= \int_0^{x_0} g^{(1)}(x, V(x))\,dx + \int_{x_0}^{x_1} g^{(1)}(x, V(x))\,dx \end{aligned} \right\}. \tag{17.13}$$

Here, in the first equation, the jth component of the first integral is carried along the path Γ_{jx_0}, and that of the second is carried along $\overline{x_0 x_1}$. In the second equation, the first and the second integrals are carried along $\overline{0x_0}$ and $\overline{x_0 x_1}$, respectively.

For the proof of the first relation of (17.13), it is sufficient to prove that

$$\Phi_j^{(1)}(x_1, \nu^1) = \int_0^{x_0} f_j^{(1)}(x, V(x))\,dx + \int_{x_0}^{x_1} f_j^{(1)}(x, V(x))\,dx \tag{17.14.j}$$

for each index j, where $\Phi_j^{(1)}$ is the jth component of $\Phi^{(1)}$.

Let t_0 and t_1 be, respectively, the intersection points of the paths Γ_{jx_0} and Γ_{jx_1} with a circle $|x| = \ell$ of small radius. Since ν^1 is fixed and $f_j^{(1)}(x, V(x))$ is holomorphic in (17.6) by the use of Cauchy's theorem, (17.14.j) is an immediate consequence of

$$\left| \int_{t_0}^{t_1} f_j^{(1)}(x, V(x))\,dx \right| \to 0 \text{ as } \ell \to 0. \tag{17.15.j}$$

Here the path of integration is taken along the circular arc $|x| = \ell$ in (17.6). However, from the construction of Γ_{jx_0} and Γ_{jx_1}, we know that $\operatorname{Re} \Omega_j(x) > 0$ for x on $t_0 t_1$. Thus, the left-hand side of (17.15.j) tends to zero exponentially as ℓ tends to zero. This proves (17.14.j).

Similarly, the second relation of (17.13) can be proved from the fact that

$$\left\| \int_{t_0}^{t_1} g^{(1)}(x, V(x))\,dx \right\| \leqslant 2\pi B_N \ell^{\|\operatorname{Re} \mu\| + 1} \to 0 \text{ as } \ell \to 0.$$

Now, let $V(x)$ be specifically denoted by $W(x, x_1, \nu^1)$, namely $W(x_1, x_1, \nu^1) = \nu^1$. Let $\hat{\nu}^1 = W(x_1, x_1, \nu^1)$. Then,

$$\Phi^{(1)}(x_1,\nu^1) - \Phi^{(1)}(\hat{x}_1,\nu^1) = \{\Phi^{(1)}(x_1,\nu^1) - \Phi^{(1)}(\hat{x}_1,\hat{\nu}^1)\} + \{\Phi^{(1)}(\hat{x}_1,\hat{\nu}^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)\}$$

$$= \int_0^{x_1} f^{(1)}(x, W(x,x_1,\nu^1))dx - \int_0^{x_1} f^{(1)}(x, W(x,\hat{x}_1,\nu^1))dx$$

$$+ \{\Phi^{(1)}(\hat{x}_1,\hat{\nu}^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)\}$$

$$= \int_{x_1}^{\hat{x}_1} f^{(1)}(x, W(x,x_1,\nu^1))dx + \{\Phi^{(1)}(\hat{x}_1,\hat{\nu}^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)\} . \tag{17.16}$$

Here we use (17.13), and the paths of integration are taken accordingly. Thus, we have that

$$\lim_{\hat{x}_1 \to x_1} \frac{\Phi^{(1)}(x_1,\nu^1) - \Phi^{(1)}(\hat{x}_1,\nu_1)}{\hat{x}_1 - x_1} = -f^{(1)}(x_1,\nu^1) + \Phi^{(1)}_\nu \left.\frac{d\nu}{dx}\right|_{\substack{x=x_1 \\ \nu=\nu_1}} \tag{17.17}$$

exists, since we have just proved that the matrix $\Phi^{(1)}_\nu (x_1,\nu^1)$ is well defined. Therefore, $\Phi^{(1)}(x_1,\nu^1)$ is holomorphic with respect to x_1 for (17.6) when ν^1 is fixed.

Thus, by Hartog's theorem, $\Phi^{(1)}(x_1,\nu^1)$ is holomorphic in (x_1,ν^1) for (13.1).

In the same manner, $\Psi^{(1)}(x_1,\nu^1)$ is holomorphic for (13.1).

18. The Functions $\Phi^{(m)}(x,\nu)$ and $\Psi^{(m)}(x,\nu)$

We have seen in Section 17 that $\Phi^{(1)}(x_1,\nu^1)$ and $\Psi^{(1)}(x_1,\nu^1)$ are holomorphic in (x_1,ν^1) for (13.1) and satisfying

$$\left[\Phi^{(1)}(x_1,\nu^1)\right] \leqslant \frac{K_N}{2} \|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x)}\right] , \quad \|\Psi^{(1)}(x_1,\nu^1)\| \leqslant \frac{K_N}{2} \|\nu^1\|^N \tag{18.1}$$

for (x_1,ν^1) in (13.1) with ξ''_N, δ''_N, and K_N specified.

Let $\Pi(m)$ denote the following proposition.

$\Pi(m)$. (i) *The functions* $\Phi^{(m)}(x_1,\nu^1)$ *and* $\Psi^{(m)}(x_1,\nu^1)$ *are well defined and holomorphic for* (x_1,ν^1) *in* (13.1);

(ii) $\Phi^{(m)}(x_1,\nu^1)$ *and* $\Psi^{(m)}(x_1,\nu^1)$ *satisfy*

$$\left[\Phi^{(m)}(x_1,\nu^1) - \Phi^{(m-1)}(x_1,\nu^1)\right] \leqslant \frac{K_N}{2^m} \|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right] , \tag{18.2.m}$$

$$\left[\Phi^{(m)}(x_1,\nu^1)\right] \leqslant K_N\left(\frac{1}{2} + \cdots + \frac{1}{2^m}\right) \|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right] , \tag{18.3.m}$$

$$\|\Psi^{(m)}(x_1,\nu^1) - \Psi^{(m-1)}(x_1,\nu^1)\| \leqslant \frac{K_N}{2^m} \|\nu^1\|^N , \tag{18.4.m}$$

$$\|\Psi^{(m)}(x_1,\nu^1)\| \leqslant K_N\left(\frac{1}{2}+\cdots+\frac{1}{2^m}\right)\|\nu^1\|^N \tag{18.5.m}$$

for (x_1,ν^1) *in* (13.1).

We have seen that the proposition $\Pi(1)$ is true, due to (18.1) and (16.2).

Suppose that $\Pi(m)$ is true for $m = 1,2,\ldots,k$. We want to show that $\Pi(k+1)$ is true.

First of all, by (17.12), (18.3.k) and (18.5.k), the functions $\hat{f}(x, V(x);\ 1_a(e^{\Omega(x)})\,\Phi^{(k)}(x, V(x)),\ \Psi^{(k)}(x, V(x)))$ and $\hat{g}(x, V(x);\ 1_a(e^{\Omega(x)})\,\Phi^{(k)}(x, V(x)),\ \Psi^{(k)}(x, V(x)))$ are well defined and holomorphic in (17.6). Thus, $\Phi^{(k+1)}(x_1,\nu^1)$ and $\Psi^{(k+1)}(x_1,\nu^1)$ are given by (16.3.k+1). These integrals exist by the same reasoning as that in A of Section 17.

Since $\hat{f}(x,\nu;\eta,\zeta)$ and $\hat{g}(x,\nu;\eta,\zeta)$ satisfy Lipschitz conditions with respect to (η,ζ) with Lipschitz constants H and H', respectively, by using (18.2.k), (18.4.k), (14.5), and (17.9) we have

$$\left[\Phi^{(k+1)}(x_1,\nu^1)-\Phi^{(k)}(x_1,\nu^1)\right] \leqslant \frac{2}{\|\gamma\|'\sin 2\sigma\epsilon}\cdot 2H\cdot\frac{K_N}{2^k}\|\nu^1\|^N\left[e^{-\operatorname{Re}\Omega(x_1)}\right]$$

$$\leqslant \frac{K_N}{2^{k+1}}\|\nu^1\|^N\left[e^{-\operatorname{Re}\Omega(x_1)}\right] \tag{18.2.k+1}$$

and

$$\|\Psi^{(k+1)}(x_1,\nu^1)-\Psi^{(k)}(x_1,\nu^1)\| \leqslant \frac{1}{N\|\operatorname{Re}\mu\|'}\cdot 2H'\cdot\frac{K_N}{2^k}\|\nu^1\|^N \leqslant \frac{K_N}{2^{k+1}}\|\nu^1\|^N \tag{18.4.k+1}$$

for (x_1,ν^1) in (13.1), thanks to (12.6) and (12.7). Furthermore, by the use of (13.3.k), (18.5.k), (18.2.k+1), and (18.4.k+1), we have

$$\left[\Phi^{(k+1)}(x_1,\nu^1)\right] \leqslant K_N\left(\frac{1}{2}+\cdots+\frac{1}{2^{k+1}}\right)\|\nu^1\|^N\left[e^{-\operatorname{Re}\Omega(x_1)}\right] \tag{18.3.k+1}$$

$$\|\Psi^{(k+1)}(x_1,\nu^1)\| \leqslant K_N\left(\frac{1}{2}+\cdots+\frac{1}{2^{k+1}}\right)\|\nu^1\|^N \tag{18.5.k+1}$$

for (x_1,ν^1) in (13.1).

Now, by (18.3.k+1) and (18.5.k+1) and the same reasoning as that in C of Section 17, $\Phi^{(k+1)}(x_1,\nu^1)$ and $\Psi^{(k+1)}(x_1,\nu^1)$ are holomorphic in (x_1,ν^1) for (13.1).

Thus, the proposition $\Pi(k+1)$ is true

Therefore, by means of mathematical induction, $\Pi(m)$ is true for all positive integers m. Namely, step (I) is proved, and better yet, we have the inequalities (18.2.m) to (18.5.m).

19. Convergence of $\{\Phi^{(m)}(x,\nu),\ \Psi^{(m)}(x,\nu)\}$

Since

$$\Phi^{(m)}(x_1,\nu^1)=\Phi^{(0)}(x_1,\nu^1)+\sum_{k=0}^{m-1}\left\{\Phi^{(k+1)}(x_1,\nu^1)-\Phi^{(k)}(x_1,\nu^1)\right\} \tag{19.1}$$

$$\Psi^{(m)}(x_1,\nu^1) = \Psi^{(0)}(x_1,\nu^1) + \sum_{k=0}^{m-1} \left\{ \Phi^{(k+1)}(x_1,\nu^1) - \Psi^{(k)}(x_1,\nu^1) \right. \quad , \tag{19.2}$$

the sequence $\{ \Phi^{(m)}(x_1,\nu^1), \Psi^{(m)}(x_1,\nu^1) \}$ converges if and only if the series in the right-hand side of (19.1) and (19.2) converge. However, by (18.2.k) and (18.4.k), these series converge absolutely and uniformly in any compact subset of (13.1).

Since each term of the sequence $\{ \Phi^{(m)}(x_1,\nu^1)\, \Psi^{(m)}(x_1,\nu^1) \}$ is holomorphic in (x_1,ν^1) for (13.1), the limit, denoted by $\{ \varphi(x_1,\nu^1), \psi(x_1,\nu^1) \}$ is also holomorphic in (x_1,ν^1) for (13.1).

Moreover, due to (18.2.k) and (18.4.k), we have

$$\left[\varphi(x_1,\nu^1)\right] \leqslant K_N \,\|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right] \tag{19.3}$$

and

$$\|\psi(x_1,\nu^1)\| \leqslant K_N \,\|\nu^1\|^N \tag{19.4}$$

for (x_1,ν^1) in (13.1). Thus, step (II) is proved.

20. Integral Expression of $\{ \varphi(x,\nu), \psi(x,\nu) \}$

We shall prove that the limit function $\varphi(x_1,\nu^1)$ and $\psi(x_1,\nu^1)$ satisfy integral equations (16.1).

For the first equation, let φ_j and $\hat{f}_j$ denote the jth components of the vectors φ and $\hat{f}$, respectively. We want to show that, given $\epsilon > 0$, there exists an integer $M(\epsilon, x_1)$, depending on ϵ and x_1, such that

$$\left| \int_0^{x_1} x^{-\sigma-1} e^{-\Omega_j(x)} \left\{ \hat{f}_j \Big(x, V(x); \mathbf{1}_a (e^{\Omega(x)})\, \varphi(x,V(x)), \psi(x,V(x))\Big) \right.\right.$$
$$\left.\left. -\hat{f}_j \Big(x, V(x); \mathbf{1}_a (e^{\Omega(x)})\, \Phi^{(m)}(x,V(x)), \Psi^{(m)}(x,V(x))\Big) \right\} dx \right| < \epsilon \tag{20.1}$$

for $m \geqslant M(\epsilon, x_1)$.

From (18.3.m) and (18.5.m), we know that $\Phi^{(m)}$ and $\Psi^{(m)}$ satisfy the same inequalities as (19.3) and (19.4), namely

$$\left[\Phi^{(m)}(x_1,\nu^1)\right] \leqslant K_N \,\|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right] , \tag{20.2}$$

and

$$\|\Psi^{(m)}(x_1,\nu^1)\| \leqslant K_N \,\|\nu^1\|^N \tag{20.3}$$

for (x_1,ν^1) in (13.1), independent of m.

Since the vector $\hat{f}(x,\nu;\eta,\zeta)$ satisfies a Lipschitz condition in (η,ζ) with Lipschitz constant H, the left-hand side of (20.1) is dominated by

$$4HK_N \int_{\Gamma_{jx_1}} |x|^{-\sigma-1} e^{-\mathrm{Re}\,\Omega_j(x)} \,\|V(x)\|^N \,|dx| , \tag{20.4}$$

independent of m. By the same reason as we have seen in A of Section 17, the integral (20.4) exists. Hence, we can choose a point x_{j1}^0 on Γ_j'', independent of m, such that

$$\left| \int_0^{x_{j1}^0} x^{-\sigma-1} e^{-\Omega_j(x)} \left\{ \hat{f}_j \left(x, V(x); 1_a(e^{\Omega(x)}) \varphi(x, V(x)), \psi(x, V(x))\right) \right.\right.$$
$$\left.\left. -\hat{f}_j \left(x, V(x); 1_a(e^{\Omega(x)}) \Phi^{(m)}(x, V(x)), \Psi^{(m)}(x, V(x))\right) \right\} dx \right| < \frac{\epsilon}{2}. \tag{20.5}$$

On the other hand, since the arc of Γ_{jx_1} from x_{j1}^0 to x_1 has finite length, and $\Phi^{(m)}(x,\nu)$ and $\Psi^{(m)}(x,\nu)$ converge uniformly in any compact subset of (13.1), we can choose a compact subset of (13.1), containing the portion of Γ_{jx_1} from x_{j1}^0 to x_1, and an integer $M(\epsilon, x_1)$ such that

$$\left| \int_{x_{j1}^0}^{x_1} x^{-\sigma-1} e^{-\Omega_j(x)} \left\{ \hat{f}_j \left(x, V(x); 1_a(e^{\Omega(x)}) \varphi(x, V(x)), \psi(x, V(x))\right) \right.\right.$$
$$\left.\left. -\hat{f}_j \left(x, V(x); 1_a(e^{\Omega(x)} \Phi^{(m)}(x, V(x)), \Psi^{(m)}(x, V(x))\right) \right\} dx \right| < \frac{\epsilon}{2} \tag{20.6}$$

for $m \geqslant M(\epsilon, x_1)$.

By (20.5) and (20.6), (20.1) is proved. This is true for all components; thus, $\varphi(x_1, \nu^1)$ satisfies the first integral equation of (16.1).

Similarly, $\psi(x_1, \nu^1)$ satisfies the second integral equation of (16.1), and therefore step (III) is proved.

21. $\{\varphi(x, V(x)), \psi(x, V(x))\}$ as a Solution of (12.9)

We shall prove that $\{\varphi(x, V(x)), \psi(x, V(x))\}$ is a solution of (12.9) whenever $(x, V(x))$ belongs to (13.1) To prove this, rewrite the integral equations satisfied by $\varphi(x, \nu)$ and $\psi(x,\nu)$ as

$$\begin{aligned} \varphi(x_1, \nu^1) &= \int_0^{x_1} \hat{\Phi}(x, V(x))\, dx \\ \psi(x_1, \nu^1) &= \int_0^{x_1} \hat{\Psi}(x, V(x))\, dx \end{aligned} \tag{21.1}$$

where

$$\hat{\Phi}(x,\nu) = x^{-\sigma-1} 1_a(e^{-\Omega(x)}) \hat{f}(x, \nu; 1_a(e^{\Omega(x)}) \varphi(x,\nu), w(x,\nu)).$$

$$\hat{\Psi}(x,\nu) = x^{-1} \hat{g}(x, \nu; 1_a(e^{\Omega(x)}) \varphi(x,\nu), \psi(x,\nu)).$$

Also, write $V(x) = W(x, x_1, \nu^1)$. Then it is sufficient to prove that

$$\frac{d}{dx_0} \varphi(x_0, \nu^0) = \hat{\Phi}(x_0, \nu^0), \quad \frac{d}{dx_0} \psi(x_0, \nu^0) = \hat{\Psi}(x_0, \nu^0) \tag{21.2}$$

where ν^0 is a vector function of x_0 given by $W(x_0, x_1, \nu^1)$.

We shall prove the first equation of (21.2). Since $W(x, x_0, \nu^0) = W(x, x_1, \nu^1)$, the first equation of (21.1) can be written as

$$\varphi(x_0, \nu^0) = \int_0^{x_0} \Phi(x, W(x, x_0, \nu^0))\, dx. \tag{21.3}$$

Hence

$$\frac{d}{dx_0} \varphi(x_0, \nu^0) = \Phi(x_0, \nu^0)$$

$$+ \int_0^{x_0} \frac{\partial \hat{\Phi}(x,W)}{\partial W} \left\{ \frac{\partial W(x,x_0,\nu^0)}{\partial x_0} + \frac{\partial W(x,x_0,\nu^0)}{\partial \nu_0} \cdot \frac{\partial W(x_0,x_1,\nu^1)}{\partial x_0} \right\} dx. \tag{21.4}$$

However, for any constant $\hat{\xi}$, $\hat{\eta} = W(\hat{\xi}, x, \nu)$ is an integral of the equation $x\nu' = \mathbf{1}_n(\mu)\nu$. Thus $\hat{\eta} = W(\hat{\xi}, x_0, \nu^0) = W(\hat{\xi}, x_1, \nu^1)$, and

$$\frac{dW(\xi, x_0, \nu^0)}{dx_0} = 0.$$

Namely, the expression in the braces of the integrand in (21.4) vanishes identically. Therefore, the first equation of (21.2) is proved.

Similarly, we can prove the second equation of (21.2). Thus step (IV) is proved.

22. Uniqueness

To complete the proof of Lemma B it remains to prove step (V), namely, a solution of (12.9) satisfying (13.3) which is unique.

Suppose that there are two solutions satisfying (13.3). Let $\{P(x, V(x)), Q(x, V(x))\}$ be the difference of these two solutions. Then, there exists a positive constant K such that

$$P(x_1, \nu^1) \leqslant K \|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right], \; \|Q(x_1, \nu^1)\| \leqslant K \|\nu_1\|^N \tag{22.1}$$

for (x_1, ν_1^1) in (13.1). Since $\hat{f}(x, \nu; \eta, \zeta)$ and $\hat{g}(x, \nu; \eta, \zeta)$ satisfy Lipschitz conditions with respect to (η, ζ) with Lipschitz constants H and H', respectively, we have

$$\left[P(x_1, \nu^1)\right] \leqslant 2HK \int_{\Gamma_{x_1}} |x|^{-\sigma-1} \|V(x)\|^N \left[e^{-\mathrm{Re}\,\Omega(x)}\right] |dx|$$

$$\leqslant \frac{4HK}{\|\gamma\|' \sin 2\sigma\epsilon} \|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right]$$

$$\leqslant \frac{K}{2} \|\nu^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right] \tag{22.2}$$

and

$$\|Q(x_1, v^1)\| \leqslant 2H'K \int_0^{|x_1|} |x|^{-1} \|V(x)\|^N |dx|$$

$$\leqslant \frac{2H'K}{N\|\mathrm{Re}\,\mu\|'} \|v^1\|^N \leqslant \frac{K}{2} \|v^1\|^N, \tag{22.3}$$

Here, we use (14.5), (17.9), (12.6) and (12.7). Repeating this process, we have, for any positive integer p,

$$\left[P(x_1, v^1)\right] \leqslant \frac{K}{2^p} \|v^1\|^N \left[e^{-\mathrm{Re}\,\Omega(x_1)}\right], \|Q(x_1, v^1)\| \leqslant \frac{K}{2^p} \|v^1\|^N \tag{22.4}$$

for (x_1, v^1) in (13.1). Hence

$$P(x_1, v^1) \equiv 0, \ Q(x_1, v^1) \equiv 0, \tag{22.5}$$

for (x_1, v^1) in (13.1), and these prove the uniqueness.

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13. ABSTRACT

Two fundamental existence theorems for the study of analytic solutions of nonlinear ordinary differential equations with an irregular-type singularity are proved. A method of successive approximations involving improper contour integrals and analyticity with respect to several complex variables is employed.

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