

# Optimal SAM Defense System

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## An Application of Optimal Control Concept to Operations Research

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Abstract: An operations research problem concerning the optimal SAM firing pattern to defend an aircraft carrier is solved via applications of the concept of closed-loop (feedback) and open-loop optimal control. The SAM defense problem is formulated as a Markov decision process with the number of SAMs in each salvo as the decision variable. Interesting cases, including the presence of imperfect sensor observation and a bound on the number of SAMs available, are considered. The principle of dynamic programming and the technique of nonlinear integer programming are applied to reach closed-loop and open-loop solutions. Numerical examples are given for illustration.

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### INTRODUCTION

There has been much recent successful cross-fertilization between the fields of optimal control and operations research (1). Modern control theory has found applications in solving economic (2), management science, and resource allocation problems (3). Pontryagin's Maximum Principle of control theory is generally the main technique used in this applications. The present report, however, emphasizes the concept of closed-loop (feedback) and open-loop optimal control in solving the surface-to-air-missile (SAM) defense problem for an aircraft carrier under various sensor conditions.

An air defense and offense game model was formulated by Brodheim and others (4). They considered the problem as a two-person zero-sum game. The problem treated in this report is different in many aspects. In particular: (a) Only defensive systems are of interest; the strategies of the offensive are not considered. (b) The objective of the defense is to protect a ship from enemy missiles with minimum expected cost and damage to the ship by the enemy missiles which survive interception by SAMs. (c) The sensor conditions are more involved; the observations concerning the number of enemy missiles in the attack are considered for the following cases: perfect observation, imperfect observation, and no observation. (d) The problem under consideration is simpler than that considered in Ref. 4, but this simple defense model permits much more extensive study and analysis.

The problem is formulated as a Markov decision process with the size of each salvo, the number of SAMs, as the decision variable (or "control variable"). Corresponding to different sensor conditions, the optimal decisions are found by applying the concepts of closed- and/or open-loop optimal control. We also consider the case where the number of SAMs onboard is limited. A Markov decision process of two-state variables is formed for this case where the states are arranged in matrix form. The principle of dynamic programming and the technique of nonlinear integer programming are used to solve problems of this type.

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NRL Problem B01-10; Project RR 003-02-41-6152. This is a final report on one phase of the problem; work is continuing on other phases. Manuscript submitted October 2, 1970.

## NOTATION

Small letters  $c$ ,  $g$ , *etc.*, represent vectors with elements denoted by  $c_j$ ,  $g_j$ , *etc.*, Capital letters  $F$ , *etc.*, represent matrixes with elements  $f_{jk}$ . Scalars are explicitly mentioned. The vector  $c$  and the matrix  $F$  at stage  $i$  are denoted by  $c(i)$  and  $F(i)$ , respectively. The transposition of  $F$  is denoted by  $F^T$ .

## MODEL OF SAM DEFENSE SYSTEM

It is assumed that a group of enemy missiles (EMs) is on its way to attack a ship which is defended by SAMs. From the observations and information concerning the speed and position of EMs the number of SAM salvos that can be launched in time to intercept the EMs before the time of final impact on the ship is determined at initial time. We denote the number by  $I$ . The problem is to choose the size of each SAM salvo such that an object function is minimized. Further assumptions concerning this model are listed as follows:

1. From radar output and other sources of information, it is assumed that the defense has an initially perfect knowledge of the number of EMs, which is denoted by  $j$ .

2. The EMs are assumed to arrive in a group and the SAM salvo is aimed at this group. It is further assumed that one SAM can destroy at most one EM. Therefore, for instance, if the probability of killing an EM is  $q$ , then the probability of killing two EMs from a group of three, when five SAMs are launched, is

$$\binom{5}{2} q^2 (1 - q)^3.$$

3. The objective function of  $n$  EMs with  $I$  SAM salvos available for launching can be expressed by

$$\left( \begin{array}{l} \text{The object} \\ \text{function} \end{array} \right) = \left( \begin{array}{l} \text{The expected cost} \\ \text{of total SAMs to be} \\ \text{launched} \end{array} \right) + \left( \begin{array}{l} \text{The expected cost of} \\ \text{damage caused by the} \\ \text{final impact with EMs} \end{array} \right).$$

### A Markov Decision Process

The problem described is formulated into a Markov decision process. Let the state variable of the process be the number of EMs which survive the SAMs' attack. The state number corresponds to the number of EMs surviving. For example, if 3 EMs remain at a certain time, the state of the system is state 3. After the next salvo of SAMs, the number of EMs surviving could be 0, 1, 2, or 3, which corresponds to states 0, 1, 2, or 3. In other words, the state of the process could be transferred to state 0, 1, 2, or 3. The probabilities associated with these transitions depend on the number of SAMs in the salvo, which is called the decision variable. The state transition diagram is shown in Fig. 1. As shown in Fig. 1, this decision process has the special property that there are no transitions from a given state to one of higher index. State 0 is the "terminal state," which means that a transition to state 0 implies that the process will remain in state 0. States 1, 2, ..., and  $n$  are "transient" states.  $l_j(i)$  as shown in Fig. 1 is the number of SAMs to be launched at  $i$ th state in state  $j$  (*i.e.*,  $j$  EMs remaining).

For convenience the index  $i$ , which denotes the  $i$ th transition stage, runs from  $-I$  to 0, where  $I$  is defined previously as the number of SAM salvos allowed. Therefore, the  $(-I)$ th stage is the beginning stage and the zeroth stage is the final stage. At the zeroth stage, all enemy missiles have reached the ship; no further defensive action can be taken. The equations which govern the probabilities of state transitions are

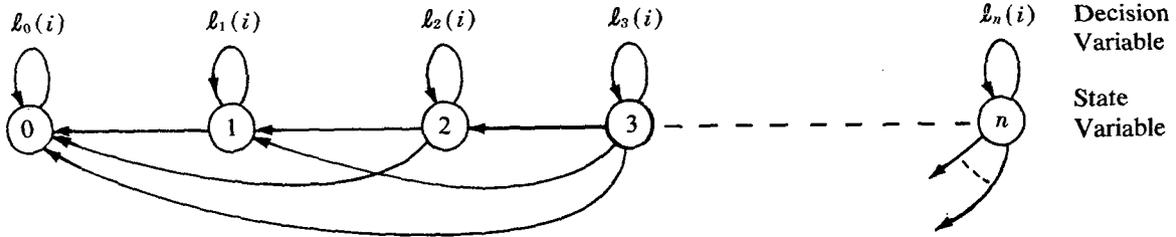


Fig. 1 - State transition diagram

$$\begin{bmatrix} p_0(i+1) \\ p_1(i+1) \\ p_2(i+1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_n(i+1) \end{bmatrix} = \begin{bmatrix} 1, f_{01}(i), f_{02}(i), \dots, f_{0n}(i) \\ 0, f_{11}(i), & \cdot \\ 0, 0, f_{22}(i), & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot \\ 0, \dots, \dots, 0, f_{nn}(i) \end{bmatrix} \begin{bmatrix} p_0(i) \\ p_1(i) \\ p_2(i) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_n(i) \end{bmatrix}, \quad (1)$$

with initial conditions

$$\begin{bmatrix} p_0(-I) \\ p_1(-I) \\ p_2(-I) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_n(-I) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix},$$

where  $p_j(i)$  is the probability of being in state  $j$  at the  $i$ th stage, and  $f_{jk}(i)$  is the conditional probability of transition from state  $k$  to state  $j$  at the  $i$ th stage. The values of  $f_{jk}(i)$  are calculated based on the special characteristics of the process and the preceding basic assumptions about the probability of interception;

$$f_{jk}(i) = \begin{cases} 0 & \text{if } j < k \\ \binom{\xi}{k-j} (1-q(i))^{\xi-k+j} q(i)^{k-j}, & \text{if } j \leq k \text{ and } j \neq 0 \\ 1 - \sum_{r=1}^n f_{rk}(i) & \text{if } j = 0 \end{cases} \quad (2)$$

with

$$\xi \equiv \ell_j(i),$$

where  $q(i)$  is a given probability of hitting an EM by SAM and  $\ell_j(i)$  represents the number of SAMs to be launched at  $i$ th stage when  $j$  EMs survive.

Equation (1) can be written in vector-matrix form:

$$p(i+1) = F(\ell(i)) p(i); \quad p(-I) = \text{given}, \quad (3)$$

where  $\ell(i)$  is the  $(n+1)$ -dimensional decision vector where elements are  $\ell_j(i)$ , for  $j=0, \dots, n$ .

### Markov Process with Cost

The cost function described previously can be associated with this Markov process. Define  $c_j(i)$  = the expected total cost from  $i$ th stage to the end of the process, if the system is now in state  $j$ , given  $\ell_j(m)$ , for  $m=i, \dots, -1$ . The expected cost includes the cost of SAMs launched and the cost of damage to the ship by surviving EMs.

Based on this definition,

$$c_j(i) = \left[ \ell_j(i) + \sum_{m=i+1}^{-1} \ell(m)^T p(m) \right] b + p(0)^T g; \quad \text{for } i = -I, \dots, -1, \quad (4)$$

where the first term represents the expected cost of the total number of SAMs launched, and  $b$ , a scalar, is the cost of each individual SAM. The second term represents the expected cost of the terminal damage on the ship and the vector  $g$  is an  $(n+1)$  vector whose element  $g_k$  gives the expected cost of damage to the ship, should  $k$  enemy missiles survive all SAMs' attack.

Using the state transition equation (Eq. (3)), a recurrence relation of vector  $c(i)$  is derived. After manipulations, we have

$$c(i) = b \ell(i) + F(\ell(i))^T c(i+1); \quad \text{for } i = -1, -2, \dots, -I, \quad (5)$$

with

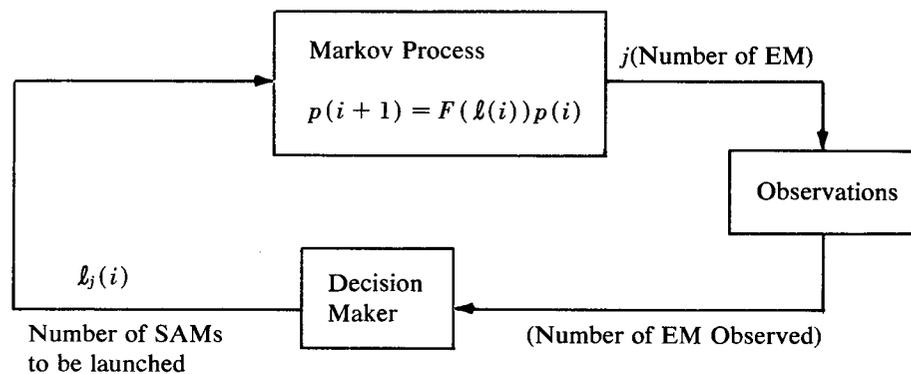
$$c(0) = g. \quad (6)$$

$F(\ell(i))$  is the transition matrix given in Eq. (1). Equation (5) is the key equation in this report.

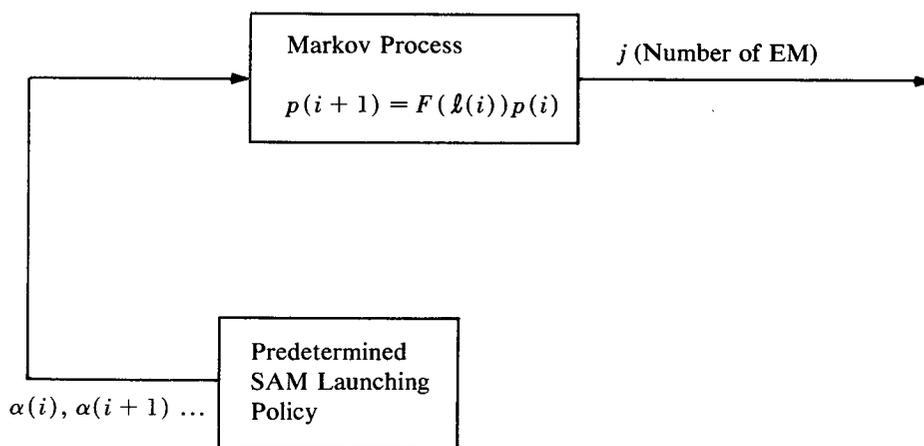
Howard (5) has formulated an economic decision process and has made a significant contribution in finding the optimal decisions in the steady state. The process treated in this report is different from Howard's in that the terminal cost contributes a great part of the total cost, and the special properties as indicated by Fig. 1 make the problem of finding the optimal  $\ell_j(i)$  computationally feasible even if the number of states  $n$  is large (say 50).

**Closed- and Open-Loop Controls**

In the following sections, the problems of finding optimal vector  $l(i)$  for  $i = -I, \dots, -1$  are considered under different sensor observations. The concept of open- and closed-loop (feedback) control is extensively applied to this problem. For clarity we briefly discuss the concept here. In the closed-loop control of Fig. 2, the output of the Markov process  $j$  is observed at  $i$ th stage. The observed results are used to make a decision on  $l_j(i)$  which in turn drives the Markov process. The whole thing forms a closed loop. In the open-loop control, the size of salvos, scalar  $\alpha(i)$  for  $i = -I, \dots, -1$ , is predetermined. No decision-making is involved. The open-loop control is applied when the observations are not available. The closed-loop control is a more sophisticated algorithm than the open-loop control, and it will yield the smallest possible expected cost for the decision process. In this report, an additional case which is called semiclosed loop is introduced. It is interesting to compare the numerical results of these three cases.



(a) Closed-loop control



(b) Open-loop control

Fig. 2 – Closed- and open-loop control

*Case I: Perfect Observation (Closed-Loop Control)\**

In this case the defense has perfect knowledge of the number of EMs remaining at every moment (*i.e.*, at all  $i$ ). In other words, the decision maker has perfect knowledge of what the state of the system is. From the principle of dynamic programming, the optimal  $l(i)$ , denoted by  $l^*(i)$ , can be found iteratively from

$$c^*(i) = \min_{l(i)} \{bl(i) + F(l(i))^T c^*(i+1)\}; \quad c^*(0) = g \quad (7)$$

$$\text{for } i = -1, -2, \dots, -I,$$

where Eq. (7) is obtained directly from Eq. (5) and  $c^*(i)$  is the optimal cost function (or the optimal return function in control theory). The optimization of Eq. (7) is carried out backwards, stage by stage and state by state. In more detail, at each stage, there are  $n$  optimal  $l_j(i)$  to be chosen to minimize the corresponding  $n$  cost functions  $c_j(i)$ . Since  $l_j(i)$  has to be a positive integer and  $F(l(i))$  is a nonlinear function of  $l_j(i)$ , this constitutes a nonlinear integer programming problem. Numerical results of  $l^*(i)$  are found by assuming that  $c_j(i)$  has a single relative minimum. This assumption is intuitively reasonable since the cost function shows a tradeoff between the cost of SAMs and the cost of damage suffered by the ship. If this assumption is not true, a process of choosing the absolute minimum out of a finite set of relative minima has to be taken.

It should be noted that the concept of closed-loop control (feedback control) has been applied to the problem. The optimal closed-loop decisions, where the state is assumed known when the decision is made, yield the smallest possible expected cost for this Markov decision process. This smallest expected cost serves as an upper bound for all the cases to be discussed in the subsequent sections.

A numerical example is given below to illustrate how the optimal decisions are chosen at each stage. The ship has perfect observations at all times. The important data concerning this example are given as follows:

The probability of destroying a missile,  $q(i) = 0.632$ , for all  $i$ .

The cost of a single SAM,  $b = 1$  unit.

The cost of ship damage by  $j$  EMs,  $g_j = j \times 100$  units for  $j = 0, \dots, n$ .

The number of enemy missiles in the raid,  $n = 10$ .

The maximum number of SAM salvos that can be launched,  $I = 8$ .

A list of optimal  $l_j^*(i)$  and  $c_j^*(i)$  for  $j = 1, \dots, 10$  and  $i = -8, -7, \dots, 0$ , are shown in Table 1. In real-time applications,  $l_j^*(i)$  is chosen at every stage by the state of the system. For example, if  $i = -3$ , and  $j = 9$ , from the table,

$$l_9^*(-3) = 12,$$

$$c_9^*(-3) = 15.662.$$

This means that 12 SAMs is the optimal decision at  $i = 3$  when 9 EMs remain and the associated optimal cost from  $i = -3$  to the end is 15.662. At  $i = 0$ , since there is no time for further defense, the cost is  $g_j$  if  $j$  enemy missiles are left. Once the process reaches the state 0 (no EM left), the process is terminated; no SAM is to be fired and the expected cost is 0.

*Remarks.* 1. The optimal closed-loop decision  $l_j^*(i)$  does not determine the state of the system at  $(i+1)$  stage, but it does determine the probability of the state occurring at  $(i+1)$  stage.

2. Table 1 can be applied to the situation where  $n \leq 10$  and  $I \leq 8$  because of the special characteristics of this decision process.

\*The solution to this problem for the special case of one EM has been obtained independently by D. Kaplan (6) using a different method.

TABLE 1  
Optimal Closed-Loop Solution

State of EMs (j)	Stage (i)									
	-8	-7	-6	-5	-4	-3	-2	-1	0	
1	1 1.5852	1 1.5903	1 1.6040	1 1.6410	1 1.7429	1 2.0188	2 2.7685	5 5.6749	100	$\dots \ell_j^*(i)$ $\dots c_j^*(i)$
2	2 3.1704	2 3.1805	2 3.2078	2 3.2811	2 3.4736	2 3.9469	4 4.8668	7 8.2618	200	
3	3 4.7557	3 4.7708	3 4.8114	3 4.9188	3 5.1844	4 5.6364	5 6.6820	9 10.734	300	
4	4 6.3409	4 6.3610	4 6.4117	4 6.5583	5 6.8520	5 7.3602	7 8.5181	12 12.968	400	
5	5 7.9261	5 7.9511	5 8.0177	5 8.1854	6 8.5772	7 9.0403	8 10.316	14 15.111	500	
6	6 9.5112	6 9.5412	6 9.6203	7 9.7967	7 10.116	8 10.702	10 12.050	16 17.210	600	
7	7 11.096	7 11.131	7 11.222	8 11.400	9 11.751	10 12.378	12 13.829	18 19.273	700	
8	8 12.682	8 12.721	9 12.816	9 13.010	10 13.359	11 14.007	13 15.513	20 21.307	800	
9	9 14.267	9 14.311	10 14.408	11 14.620	11 14.979	12 15.662	15 17.243	22 23.318	900	
10	10 15.852	11 15.898	11 16.003	12 16.214	13 16.596	14 17.289	16 18.930	24 25.311	1000	

3. It is interesting to note in Table 1 that  $\ell_j^*(i) \rightarrow j$  as  $i$  becomes a negative large number. This is essentially a steady-state optimal decision when the process is at a stage which is very far away from the end. The proof is straightforward (5) and is not given here.

4. The computer time consumed in computing Table 1 is 30 sec on the CDC 3800. It takes about 5 min to compute a similar table for  $n = 50$  and  $I = 20$ .

#### Case II. Imperfect Observation (Semiclosed-Loop Control)

The closed-loop solution as discussed in Case I assumes perfect observations. In the case of imperfect observations on the state, the closed-loop solution cannot be applied since the state of the system is not completely known. In this section, it is assumed that the sensor can only determine whether there are some EMs remaining or there is no EM at all (i.e., all EMs have been intercepted). In other words, the outputs from the sensor are either zero or not zero. However, the assumption of perfect knowledge on the number of EMs in the raid at the initial time,  $i = -I$ , is still sustained. The interpretation of this situation is that at the initial time, the defense has observations from all information sources which would provide a sufficient amount of data for a best estimate to the number of EMs, but during the combat time, the only observations

available are given by the radar, which can only detect whether there are EMs or not. This is a practical assumption since if the enemy missiles are coming in a group, they cluster so that the radar source cannot determine how many missiles are in the raid.

The approach used in solving this optimization problem resembles that of calculus of variations. The optimization cannot be carried out stage by stage, state by state. Instead the optimal decisions for all stages have to be optimally chosen simultaneously to minimize the cost function. The cost function as given by Eqs. (5) and (6) is still sustained except that

$$l_j(i) = \alpha(i) \quad \text{for all } j \neq 0, \quad (8)$$

and

$$l_0(i) = 0,$$

where  $\alpha(i)$  is a scalar. The interpretation of Eq. (8) is that at the  $i$ th stage the number of SAMs to be launched is independent of the state  $j$  for  $j \neq 0$  since the defense does not know the state of the system.  $l_0(i) = 0$  corresponds to the case where the observation shows no EMs surviving.

The technique used in finding the optimal set of  $\alpha(i)$  for  $i = -I, \dots, -1$  is the iterative descent search method under the assumption that the cost function  $c_n(-I)$  has a single relative minimum. An example with the same numerical data as given in Case I is carried out for  $I=5$  and  $n=5$ . The optimal solution is listed in Table 2. As shown in Table 2, at the  $(-5)$ th stage, the defense should fire 6 SAMs and then observe the result. If the observation shows that there are still some EMs left, the defense should launch two SAMs at the  $(-4)$ th stage. It is interesting to compare the optimal semiclosed-loop cost and the closed-loop cost. The optimal cost for the closed-loop policy is 8.1854 as given by Table 1. The semiclosed-loop policy provides an optimal cost of 8.5798. In other words, the expected cost would be 4.8% more if one applied the optimal semiclosed-loop policy instead of the closed-loop.

TABLE 2  
Semiclosed Loop  
with  $I/5$  and  $n=5$

Variable	Optimal Solution
$c^*(-5)$	8.5798
$\alpha^*(-5)$	6
$\alpha^*(-4)$	2
$\alpha^*(-3)$	2
$\alpha^*(-2)$	3
$\alpha^*(-1)$	6

*Remarks.* 1. The results in Table 2 can be interpreted physically with the assistance of Table 1. At the  $(-5)$ th stage,  $\alpha^*(-5)$  is close to  $l_5^*(-5) = 5$ , which is the optimal solution in the steady state. After  $\alpha^*(-5)$  has been launched, the probability of two and three EMs surviving is higher than other states if the system is not in zero state. Hence  $\alpha^*(-4)$  is 2. Finally, at the  $(-1)$  stage, if there are still some enemy missiles left, intuitively the probability of one EM being left is much higher than two or three. Therefore,  $\alpha^*(-1) = 6$ , which is one more SAM than  $l_1^*(-1) = 5$  in Table 1.

2. The assumption of a single relative minimum of the cost function can be loosely justified numerically. The optimal solution listed in Table 2 is obtained by using a descent search algorithm iteratively from an arbitrary initial guess of  $\alpha(i)$ , for  $i = -5, \dots, -1$ . Three different sets of initial guesses of  $\alpha(i)$  have been tested, and all three sets converge to the optimal solution of Table 2. This means that there is a single relative minimum in the domain which is defined by these sets of initial guesses.

### Case III. No Observation (Open-Loop Control)

This case is a slight extension of Case II. The defense has no observations at all except that the initial knowledge of the number of EMs is assumed. This situation may happen when the radar fails to detect the enemy missiles or when the ship commander distrusts the radar observations. The optimal launching policy for this case is easier to apply than that of the closed-loop case, but at the expense of higher expected cost.

Consider the same numerical example as in Case II. Since the probability of hitting an EM,  $q(i)$ , is assumed independent of  $i$ , there is no difference in firing a SAM salvo earlier or later. The only thing of importance is the total number of SAMs launched. Based on this argument, it is immediately found from the (-1)st stage of Table 1 that for  $n = 5$ , the total number of SAMs to be launched is 14, and the optimal expected cost is 15.111, which is much higher than the cost of semiclosed loop, 8.5798. Distributing these 14 SAMs into any combination of salvos does not affect the expected cost. If  $q(i)$  is a function of  $i$ , the solution is also easily obtained from the closed-loop program, since intuitively, the optimal launching policy is to fire all SAMs concentrated at a single stage where the probability of hitting  $q(i)$  is the highest of all.

#### Case IV. Limited Number of SAMs

In the preceding three cases, the process of optimization is carried out without any limitation on the number of SAMs available, contrary to the practical situation. The optimization problem with this limitation can be viewed both as an optimal control problem with inequality control constraints and as a resource allocation problem with the resource being the total number of SAMs available. However, in the case of perfect observations, a different concept can be adopted to reach the optimal solution. Let the total number of SAMs remaining be another state variable denoted by  $m$ . Then, the optimal closed-loop solution denoted by  $l_{jm}^*(i)$  will be a function of both state  $j$ , the number of EMs, and state  $m$ , the number of SAMs left. The transition of state  $m$  at any stage is "deterministically" determined by  $l_{jm}^*(i)$ . For example, if  $l_{jm}^*(i)$  is launched, then the system will shift to state  $[m - l_{jm}^*(i)]$  from state  $m$ . The transition of state  $j$  is still probabilistically determined by  $l_{jm}^*(i)$ .

The same preceding numerical example is calculated for  $n = 3$ ,  $J = 3$ , and  $m = 8$ . The results of  $l_{jm}^*(i)$  and  $c_{jm}^*(i)$  are given in Table 3. For instance, we launch four SAMs at the (-3)rd stage when three EMs are remaining and when eight SAMs are available. At (-2)nd stage, the total of SAMs available reduces to four. From the observations, if one EM is left, we should launch one SAM. The optimal process is carried on in this way.

*Remarks.* 1. If the total number of SAMs available at  $j = 3$  and  $i = -3$  is more than 18, which is obtained from Table 1 by adding 9, 5, and 4, the constraint on the total number of SAMs does not exist. The optimal launching policy follows the Table 1 of the closed-loop case.

2. The numerical results of Table 3 can be interpreted physically. For example, the reason why  $l_{23}^*(-2)$  is 2 instead of 1 is that if we launch two SAMs at (-2)nd stage, there is a possibility that both of these EMs are intercepted. Therefore, at the (-1)st stage, we have the chance to save the remaining one SAM. However, if we launch one instead of two, the number of EMs left at (-1)st stage would be at least one; therefore, the remaining two SAMs are to be fired. In other words, there is no possibility of saving a SAM if we fire one instead of two.

3. The case of imperfect observation with limited SAMs can be calculated by applying the techniques of dynamic programming (7). The case of no observation with limited SAMs is trivial, following the argument of the last section.

## CONCLUSION

This report demonstrates that some concepts from control theory can be employed to solve certain operations research problems. The author believes that the model and technique used in this report can be applied to problems of various areas, such as economic decision processes and inventory control. Other areas of control theory, such as optimal estimation, stochastic control, and differential games, should find many applications in operations research. For example, arguments in the present report can be extended to a problem in finite state stochastic games (8), when the enemy has the option of sending more missiles.

TABLE 3  
Closed-Loop Control with Limited SAMs

Number of EMs ( $j$ )	Number of SAMs Left ( $m$ )							
	1	2	3	4	5	6	7	8
THE (-3)RD STAGE								
1	(0,1) 37.800	(0,1) 14.910	1 6.487	1 3.472	1 2.449	1 2.133	1 2.026	1 2.018
2	(0,1) 137.80	(0,1,2) 75.600	(0,1,2) 38.244	2 19.173	2 10.304	2 6.468	2 4.890	2 4.242
3	(0,1) 237.80	(0,1,2) 175.60	(0,1,2,3) 113.40	(0,1,2,3) 66.901	3 37.737	3 21.434	3 13.06	4 8.957
THE (-2)ND STAGE								
1	(0,1) 37.800	1 14.910	1 6.719	1 3.938	2 3.081	2 2.790	2 2.768	2 2.768
2	(0,1) 137.800	(0,1,2) 75.600	2 38.244	2 19.467	3 10.758	3 6.975	3 5.508	3 5.062
3	(0,1) 237.800	(0,1,2) 175.60	(0,1,2,3) 113.40	3 66.901	4 37.989	4 21.789	4 13.539	5 9.594
THE (-1)ST STAGE								
1	1 37.800	2 15.542	3 7.983	4 5.834	5 5.674	5 5.674	5 5.674	5 5.674
2	1 137.800	2 75.600	3 38.643	4 20.266	5 12.145	6 9.055	7 8.286	7 8.286
3	1 237.800	2 175.60	3 113.400	4 67.154	5 38.521	6 22.851	7 15.132	8 11.802

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<b>13. ABSTRACT</b> <p>An operations research concerning the optimal SAM firing pattern to defend an aircraft carrier is solved via applications of the concept of closed-loop (feedback) and open-loop optimal control. The SAM defense problem is formulated as a Markov decision process with the number of SAMs in each salvo as the decision variable. Interesting cases, including the presence of imperfect sensor observation and a bound on the number of SAMs available, are considered. The principle of dynamic programming and the technique of nonlinear integer programming are applied to reach closed-loop and open-loop solutions. Numerical examples are given for illustration.</p>			

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