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# Sufficiency Theorem for Discontinuous Optimal Cost Surfaces

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A $V$-type function is introduced for the purpose of modeling the optimal cost surfaces (OCS) of a very general class of optimal control processes, especially state-constrained processes. This classification of the OCS builds from the assumption that the OCS is the countable union of disjoint submanifolds. The $V$-type function is particularly suitable for handling OCSs that are discontinuous at submanifolds of codimension 1 of the state space. A discontinuous

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nontransversality condition is presented which keeps trajectories from entering any submanifold of the OCS from a discontinuous edge. A sufficiency theorem utilizing the $V$-type function is given. This theorem states that if certain conditions are met by a control policy, than it is optimal. The defining of these conditions, together with the sufficiency theorem, provides a classification of the OCS for general optimal control processes with state contraints. An example is provided to demonstrate the implementation of the results.

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# SUFFICIENCY THEOREM FOR DISCONTINUOUS OPTIMAL COST SURFACES*

## INTRODUCTION

The optimal cost surface (OCS) is the plot of the values of the optimal value function above the state space. This surface is obtained as part of the solution of an optimal control problem. Even though the properties of this surface are dependent entirely on the given elements of the problem, researchers have found it profitable to hypothesize about its properties in deriving results in optimal control theory. For instance, current field-type sufficiency theorems require that the OCS be continuous [1-8]. The derivations of the fundamental partial differential equation of dynamic programming in Ref. 9 are based on the OCS having bounded second partial derivatives. These properties, however, are not always met, even in rather simple control problems [10]. Recent Lotka-Volterra models of prey-predator control systems provide simple examples where the OCS is discontinuous and has unbounded first partial derivatives [11,12]. A discontinuous OCS arises particularly in state-constrained optimal control processes [13].

An assumption which has survived all known examples is that the OCS is the countable union of disjoint submanifolds. In this report we examine the problem of arranging these submanifolds in such a way that the properties of the resulting OCS can be utilized in establishing optimality over a very general class of optimal control processes, especially state-constrained processes. In particular, we model the optimal value function with a V-type function. This function may be locally discontinuous on submanifolds of codimension 1. A nontransversality condition and a discontinuous nontransversality condition are presented. The latter condition keeps a trajectory of the control process from entering any submanifold of the OCS from a discontinuous edge. These properties and conditions are formulated in a sufficiency theorem which states that, if certain conditions are met by a control policy, then it is optimal. The defining of these properties and conditions, together with the sufficiency theorem, provides a classification of the OCS for general optimal control processes with state constraints. An example is given to demonstrate the implementation of the results.

## DEFINITION OF OPTIMAL CONTROL PROCESS

The optimal control process under investigation has its dynamical behavior governed by a system of ordinary differential equations and has its evolution of state described by the motion of a point in $n$-dimensional Euclidean $E^n$. The seven basic elements needed in defining the optimal control process are four functions ($f$, $U$, $f_0$, and $g_0$), two sets ($X$ and $\Theta$), and a function space $\Omega$. These elements are described subsequently.

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The dynamical behavior of the optimal control process is modeled by the state velocity function $f$ in the state equation

$$\dot{x} = f(x, v), \; x \in E^n, \; v \in E^m, \tag{1}$$

where $f$ is a function with domain $E^n \times E^m$. We let $\varphi(\cdot)$, an absolutely continuous function, represent a solution of Eq. (1) when controlled by a control policy $u(\cdot)$, a Lebesgue measurable function of time. With initial time $t_0$ the initial state satisfies

$$\varphi(t_0) = x_0 . \tag{2}$$

The state space $X$ is a subset of $E^n$. It is considered to be a topological space possessing the induced topology from $E^n$. The terminal set $\Theta$ is a closed subset of $X$.

The controller of the process is equipped with the elements $\Omega$ and $U$. The control function space $\Omega$ is the space of all Lebesgue measurable functions of time defined on bounded intervals with range in $E^m$. Constraints on the control functions in $\Omega$ are given implicitly by the set-valued function

$$U : X \rightarrow \text{power set of } E^m . \tag{3}$$

Given $x \in X$, the set $U(x)$ is a set of control values available to the controller at the state $x$. $U(x)$ is nonempty for all $x \in X$.

A solution of Eq. (1) for some control $u \in \Omega$, $u : [t_0, t_f] \rightarrow E^m$, given initial conditions, is called a trajectory. A trajectory $\varphi : [t_0, t_f] \rightarrow E^n$ is said to be admissible if it lies entirely in the state space $X$ for all times $t$ contained in $[t_0, t_f]$. An admissible trajectory is said to be terminating if $\varphi(t_f)$ is contained in $\Theta$. The time $t_f$ is called the terminating or final time for a terminating admissible trajectory. The time $t_f$ belongs to the interval $[t_0, \infty)$; $t_f$ does not have to be the same terminating time for distinct trajectories unless it is constrained to be fixed by the terminal set $\Theta$. For nonautonomous systems (that is, $f$ an explicit function of $t$), one component of $\varphi$ is the time $t$ itself.

A control $u \in \Omega$, $u : [t_0, t_f] \rightarrow E^m$ is said to be admissible if it has at least one corresponding admissible trajectory $\varphi : [t_0, t_f] \rightarrow X$ such that

$$u(t) \in U[\varphi(t)] \qquad \forall\, t \in [t_0, t_f] . \tag{4}$$

Here, the trajectory $\varphi$ corresponds to the control $u$ if

$$\varphi(t) - \varphi(t_0) = \int_{t_0}^{t} f[\varphi(\tau), u(\tau)] \, d\tau \tag{5}$$

for all $t \in [t_0, t_f]$.

Let $x_0$ be contained in $X$. Let $C(x_0)$ denote the set of all admissible controls having at least one terminating admissible trajectory emanating from $x_0$. For $u \in C(x_0)$, let $T(x_0; u)$ denote the set of all terminating admissible trajectories $\varphi$ emanating from $x_0$,

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corresponding to the control $u$, and satisfying Eq. (4). The domains of $u$ and $\varphi$ coincide for all $\varphi$ contained in $T(x_0; u)$.

*Definition 2.1.* A denumerable decomposition $D$ of a set $X \subset E^n$ is defined to be a denumerable collection of pairwise disjoint subsets whose union is $X$. This is written as $D = \{X_j : j \in J\}$, where $J$ is a denumerable index set of the disjoint subsets.

Let $B$ be a subset of $E^n$. A mapping $F : B \to R$ is said to be differentiable locally Lipschitzian if and only if there is an open set $W$ containing $B$ such that $F$ may be extended to a function which is differentiable and locally Lipschitzian on $W$.

*Definition 2.2.* Let $X$ be a subset of $E^n$ and $D$ a denumerable decomposition of $X$. A real-valued function $V : X \to E^1$ is said to be piecewise differentiable locally Lipschitzian with respect to $D$ if, for $j \in J$, the restriction $V|X_j : X_j \to E^1$ is differentiable and locally Lipschitzian; that is, there exists a collection $\{(W_j, V_j) : j \in J\}$ such that $W_j$ is an open set containing $X_j$, $V_j : W_j \to E^1$ which is differentiable locally Lipschitzian, and $V_j(x) = V(x)$ for $x \in X_j$. We say the collection $\{(W_j, V_j) : j \in J\}$ is associated with $V$ and $D$.

If the process is in the state $x_0 \in X$, then it is to be controlled during a transfer of the process to the terminal set $\Theta$ so as to render the criterion

$$J(x_0, \varphi, u) \triangleq g_0[\varphi(t_f)] + \int_{t_0}^{t_f} f_0[\varphi(\tau), u(\tau)]\, d\tau \tag{6}$$

a minimum value, where $g_0$, a real-valued function, is continuous and piecewise differentiable locally Lipschitzian with respect to a decomposition over a neighborhood of the terminal set $\Theta$; $f_0$ is a real-valued bounded Borel-measurable function with domain $E^n \times E^m$; $u \in C(x_0)$; and $\varphi \in T(x_0; u)$.

In summary, the control process is represented by the septuple $(f, U, f_0, g_0, X, \Theta, \Omega)$ where $f$ is a function, $U$ is a set-valued map, $f_0$ is bounded Borel-measurable, $g_0$ is continuous and piecewise differentiable locally Lipschitzian, $X$ is a subset of $E^n$, $\Theta$ is closed in $X$, and $\Omega$ is the space of Lebesgue measurable controls. In particular, $C(x_0)$ represents the set of admissible controls at $x_0 \in X$, and $T(x_0; u)$ denotes the set of all terminating admissible trajectories emanating from $x_0$ due to the control $u \in C(x_0)$. We will let $\Gamma$ denote this control process.

*Definition 2.3.* Optimality. Let $x_0 \in X$, $u^* \in C(x_0)$, and $\varphi^* \in T(x_0; u^*)$. The pair $(u^*, \varphi^*)$ is said to be optimal at $x_0$ if and only if, for all $u \in C(x_0)$ and for all $\varphi \in T(x_0; u)$, the following inequality is satisfied:

$$J(x_0, \varphi^*, u^*) \leqslant J(x_0, \varphi, u). \tag{7}$$

If the pair $(u^*, \varphi^*)$ is optimal at $x_0$, then the value $J(x_0, \varphi^*, u^*)$ is arbitrarily defined to be $V(x_0)$. If an optimal pair $(u^*, \varphi^*)$ exists for every $x_0 \in X$, then we have a real-valued function defined on $X$:

$$V : X \to E^1. \tag{8}$$

It is for this function—the optimal value function (OVF)—that we want to prescribe very general properties, those that will hold over a large class of optimal control processes. We do this in the next section.

## FORMULATION OF DISCONTINUOUS OPTIMAL VALUE FUNCTION

This section is devoted to setting up some very general properties of the optimal value function that are compatible with the dynamics of optimal control processes which admit continuous or discontinuous optimal value functions.

**Lemma 3.1.** A monotonicity lemma for a discontinuous optimal value function. *Let $D = \{X_j : j \in J\}$ be a denumerable decomposition of $X$. Let $\varphi : [t_0, t_f] \to X$ be absolutely continuous and $h_0 : [t_0, t_f] \to E^1$ be integrable. Let $V : X \to E^1$ be piecewise differentiable locally Lipschitzian with respect to $D$. Let $\{W_j, V_j) : j \in J\}$ be a collection associated with $V$ and $D$. Let $T_j = \{t \in [t_0, t_f] : \varphi(t) \in X_j\}$ for $j \in J$. Suppose that*

(i) *For each $j \in J$*

$$h_0(t) + \frac{d}{dt}(V_j \circ \varphi)(t) \geqslant 0 \qquad \text{almost everywhere in } T_j.$$

(ii) *There exists a countable compact subset $T$ of $[t_0, t_f]$ such that $V \circ \varphi$ is continuous on the open set $\theta = (t_0, t_f) \sim T$.*

(iii) *If $t$ is a point of discontinuity of $V \circ \varphi$, then $\inf\{\sup\{(V \circ \varphi)(\tau) : 0 < t - \tau < r, \tau \in [t_0, t_f]\} : r > 0\} \leqslant (V \circ \varphi)(t) \leqslant \sup\{\inf\{(V \circ \varphi)(\tau) : 0 < \tau - t < r, \tau \in [t_0, t_f]\} : r > 0\}$. For the case $t = t_0$ or $t = t_f$ only one inequality is used.*
*Then the function*

$$g(t) = \int_{t_0}^{t} h_0(\tau)\, d\tau + (V \circ \varphi)(t)$$

*defined for $t \in [t_0, t_f]$ is monotone nondecreasing.*

*Proof.* The set $\theta$, being open, is the denumerable union of disjoint open intervals

$$\theta = \cup \{\theta_i : i \in I\}$$

where $I$ is the index set of these disjoint open intervals. For each $i \in I$ let the interval $(a_i, b_i)$ designate $\theta_i$. Let $[d_i, e_i] \subset (a_i, b_i)$. Since the function $V \circ \varphi$ is continuous on $[d_i, e_i]$, it follows from Theorem 3.1 of Ref. 6 that $g$ is monotone nondecreasing on $[d_i, e_i]$; an hypothesis of Theorem 3.1 requires $V$ to be continuous over $X$, but its proof only uses the continuity of $V \circ \varphi$.

Consequently, it follows that $g$ is monotone nondecreasing over $(a_i, b_i)$. From condition (iii) it follows that $g$ is monotone nondecreasing over $[a_i, b_i]$.

Suppose that the intervals $(a_i, b_i)$ and $(b_i, b_{i+1})$ are such that $b_i = a_{i+1}$. Thus, $b_i$ is an isolated point in $T$. From the above analysis we know that $g$ is monotone

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nondecreasing over $[a_i, b_i]$ and $[b_i, b_{i+1}]$. Thus $g$ is monotone nondecreasing over $[a_i, b_{i+1}]$. In this way, we can remove all isolated points of $T$. Every countable compact subset of the real line contains an isolated point [14]. Let $I_0$ be the set of all isolated points of $T$. Let $T_1 = T \sim I_0$. Let $\theta_1 = (t_0, t_f) \sim T_1$. The set $T_1$ is countable compact and $\theta_1$ is open. Let $I_1$ be the isolated points of $T_1$.

Let

$$\theta_1 = \bigcup_{k \in K} (a_{1k}, b_{1k}).$$

From the above analysis we have established that $g$ is monotone nondecreasing over $(a_{1k}, b_{1k}), k \in K$. If $b_{1k} \in I_1$, then by evoking condition (iii), we have $g$ monotone nondecreasing over $(a_{1k}, b_{1k+1})$ where $b_{1k} = a_{1k+1}$. By repeated use of condition (iii), we establish that the monotonicity of $g$ is extendable through all the isolated points of $T_1$. Let $T_2 = T_1 \sim I_1$ and $\theta_2 = (t_0, t_f) \sim T_2$. Let $I_2$ be the isolated points of $T_2$.

Using the process of transfinite induction as developed in Ref. 14, we extend the monotonicity of $g$ to the entire interval $[t_0, t_f]$ by removing, in the above fashion, the isolated points $I_k$ from $T_k$. If $T_{k+1}$ is nonempty then it contains isolated points. Since points of $T$ are being removed at each stage and $T$ is countable, the entire set $T$ is evacuated by transfinite induction. This completes the proof of the lemma.

In this section we speak of a manifold $M$ to mean an embedded continuously differential $n$-1 dimensional submanifold with boundary (the boundary may be empty) of $E^n$ such that there exists a manifold $\widetilde{M}$ containing $M$ where $\widetilde{M}$ is an embedded, continuously differentiable, ($n$-1)-dimensional submanifold without boundary of $E^n$. That is, $M \subset \widetilde{M}$ and the boundary of $\widetilde{M}$, $\partial \widetilde{M}$, is empty. For the case that the boundary of $M$, $\partial M$, is empty we take $\widetilde{M} = M$. The formal definitions of manifolds can be found in Ref. 15.

All neighborhoods are considered open.

*Definition 3.1.* ($\theta^+$ and $\theta^-$). Let $x \in M$ and let $\theta$ be a sufficiently small neighborhood of $x$ in $E^n$ such that $\widetilde{M}$ slices $\theta$ into parts $\theta^+$, $\theta^-$, and $\widetilde{M} \cap \theta$. The open set $\theta^+$ is that part of $\theta$ belonging to one side of $\widetilde{M}$, and the open set $\theta^-$ is that part belonging to the other side of $\widetilde{M}$. In particular, the intersections $\theta^- \cap \theta^+$, $\theta^+ \cap (\widetilde{M} \cap \theta)$, and, $\theta^- \cap (\widetilde{M} \cap \theta)$ are empty and $\theta = \theta^+ \cup (\widetilde{M} \cap \theta) \cup \theta^-$.

We say that two sets are Hausdorff separated if and only if they are contained in disjoint open sets.

In the next two definitions, let $X \subset E^n$ and let $V^*$ denote a function $V^*: X \to E^1$.

*Definition 3.2.* Locally Discontinuous Side. A manifold $M \subset X$ has at $x \in M$ a locally discontinuous side with respect to $V^*$ if and only if there exists a neighborhood $\theta$ of $x$ in $E^n$ such that the set $\{(y, V^*(y)) : y \in \widetilde{M} \cap \theta \cap X\}$ is Hausdorff separated from either the set $\{(y, V^*(y)) : y \in \theta^+ \cap X\}$ or the set $\{(y, V^*(y)) : y \in \theta^- \cap X\}$. Moreover, we say that $M$ has at $x$ two locally discontinuous sides with respect to $V^*$ if and only if the

set is Hausdorff separated from both the latter two sets*. We call $\theta^+$ the positive side and $\theta^-$ the negative side of $M$ at $x$.

*Definition 3.3.* Locally Continuous Side. A manifold $M$ has at $x \in M$ a locally continuous side with respect to $V^*$ if and only if there exists a neighborhood $\theta$ of $x$ in $E^n$ such that $V^*$ is continuous on either $(\theta^+ \cap X) \cup (\widetilde{M} \cap \theta \cap X)$ or $(\theta^- \cap X) \cup (\widetilde{M} \cap \theta \cap X)$. That is, $V^*$ is continuous on either the plus side or the minus side of $M$ at $x$.

Let $V^*: X \to E^1$ be a function and let $B \subset X$. The function $V^*$ when restricted to the subset $B$ is denoted as $V^*|_B$; that is,

$$V^*|_B : B \to E^1$$

and

$$V^*|_B(x) = V^*(x) \quad \forall x \in B.$$

*Definition 3.4.* $V$-type Function. A function $V^*: X \to E^1$ is called a $V$-type function if and only if there exist a denumerable decomposition $D = \{X_j : j \in J\}$ of $X$ and a denumerable collection of disjoint manifolds $\{M_i : i \in I\} \subset D$ such that

(i) $V^*$ is piecewise differentiable locally Lipschitzian with respect to $D$. (Let $\{(W_j, V_j^*) : j \in J\}$ be the associated collection.)

(ii) $V^*|_{X \sim \Theta \cup \bigcup_{i \in I} M_i}$ is continuous over $X \sim \Theta \cup \bigcup_{i \in I} M_i$.

(iii) $V^*|_{\bigcup_{i \in I} M_i}$ is continuous over $\bigcup_{i \in I} M_i$.

(iv) $\Theta \cup \bigcup_{i \in I} M_i$ is closed in $X$.

(v) $M_i$, $i \in I$, has at each $x \in M_i$ either a locally continuous side and a locally discontinuous side or two locally discontinuous sides, all with respect to $V^*$.

The collections $\{X_j : j \in J\}$, $\{M_i : i \in I\}$, and $\{(W_j, V_j^*) : j \in J\}$ are said to be associated with the $V$-type function $V^*$.

Note that each manifold $M_i$, $i \in I$, is by definition a member of the decomposition $D$. Here, $I$ denotes the index set of the manifolds.

*Remark 3.1.* Property (iv) is needed in some of the proofs of the theorems and lemmas contained in this report. It is included so that the set $X \sim \Theta \cup \bigcup_{i \in I} M_i$ is open in the topology of $X$.

*$V^*$ is continuous over $\widetilde{M} \cap \theta \cap X$, restrictively.

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The next definition gives a condition for the impossibility of a trajectory entering a point on a manifold. This is established in the lemma following it.

*Definition 3.5.* Nontransversality Condition. A manifold $M$ satisfies at a point $x_0 \in M$ the nontransversality condition on the positive side (on the negative side) with respect to the control process $\Gamma$ if and only if the following condition holds:

*NT Condition.* There exist a manifold $\widetilde{M}$ and a neighborhood $\theta$ of $x_0$ in $E^n$ such that, for every $y \in \theta \cap \widetilde{M} \cap X$, the inequality

$$N^+(y) \cdot f(x, v) \geqslant 0 \quad \forall x \in \theta^+ \cap X, \quad \forall v \in U(x) \tag{9}$$

holds, where $N^+(y)$ is the unit normal to $M$ at $y$ that points into $\theta^+$. The "negative side" version of this definition replaces $\theta^+$ with $\theta^-$ and $N^+(y)$ with $N^-(y)$, the unit normal pointing into $\theta^-$. If $x_0 \in M \sim \partial M$ then $\widetilde{M}$ can be replaced with $M$.

**Lemma 3.2.** Nontransversality of Trajectory. *Let $x_0$ be contained in a manifold $M$ without boundary where $M \subset X$. Suppose the NT condition is satisfied, say, on the positive side of $M$ at $x_0$ with respect to the control process $\Gamma$. Then there exists no control policy $u : [t_1, t_2] \to E^m$ with corresponding solution $\varphi : [t_1, t_2] \to \theta^+ \cup (M \cap \theta)$ of (1) satisfying (4) such that*

$$\varphi(t_1) \in \theta^+ \tag{10}$$

$$\varphi(t_2) = x_0 . \tag{11}$$

*Proof.* Suppose, to the contrary, there exists such a control $u$ with corresponding trajectory $\varphi$. Let $t_3 \in (t_1, t_2)$ be the first time that $\varphi(t_3) \in M \cap \theta$. Denote $y_0 = \varphi(t_3)$. Thus, $\varphi(t) \in \theta^+$ for all $t \in (t_1, t_3)$. Note that the *NT* condition is satisfied on the positive side of $M$ at $y_0$ with respect to the control process $\Gamma$. It then suffices to show the impossibility of the control policy $u : [t_1, t_3] \to E^m$ with corresponding trajectory $\varphi : [t_1, t_3] \to \theta^+ \cup (M \cap \theta)$ of (1) satisfying (4) and

$$\varphi(t) \in \theta^+ \quad \forall t \in (t_1, t_3) \tag{12}$$

$$\varphi(t_3) = y_0 . \tag{13}$$

From (9) it follows that, for all $y \in \theta \cap M \cap X$,

$$N^+(y) \cdot \dot{\varphi}(t) \geqslant 0 \quad \text{almost everywhere} \quad t \in (t_1, t_3) . \tag{14}$$

Integrating (14) over the interval $(t_1, t_3)$, $t \in (t_1, t_3)$, we have, for all $y \in \theta \cap M \cap X$,

$$N^+(y) \cdot [y_0 - \varphi(t)] \geqslant 0 \quad \forall t \in (t_1, t_3) . \tag{15}$$

Note that, in particular, Eq. (15) implies that

$$N^+(y_0) \cdot [y_0 - \varphi(t)] \geqslant 0 \quad \forall t \in (t_1, t_3). \tag{16}$$

This is a statement that the trajectory $\varphi$ belongs to the closed (flat) half space below the tangent plane to $M$ at $y_0$. Denote this half space by $H(y_0, y_0)$.

For $y \in \theta \cap M \cap X$, let $T_y(M)$ represent the tangent plane to $M$ at $y$. Translate this tangent plane by the vector $y_0 - y$ to the point $y_0$ and let $H(y, y_0)$ denote the closed (flat) half space below the translated tangent plane. Equation (15) is a statement that the trajectory $\varphi$ belongs to $H(y, y_0)$ for all $y \in \theta \cap M \cap X$.

Define the cone $H$ as

$$H = \bigcap_{y \in \theta \cap M \cap X} H(y, y_0).$$

Thus,

$$\varphi(t) \in H \ \forall t \in (t_1, t_3). \tag{17}$$

This is a contradiction, since the tip of the cone $H$ (a sufficiently small neighborhood of $y_0$ intersected with $H$) is contained in $\theta^- \cup (\theta \cap M)$. That is, it lies on the negative side of $M$ at $y_0$ rather than on the positive side. This completes the proof of the lemma.

*Definition 3.6.* Discontinuous Nontransversality Condition. A $V$-type function $V^* : X \to E^1$ is said to satisfy the discontinuous nontransversality (DNT) condition with respect to the control process $\Gamma$ if and only if, for each $x_0 \in M_i$, $i \in I$, where $M_i$ has at $x_0$ a locally discontinuous side in the positive direction (in the negative direction) with respect to $V^*$, the manifold $M_i$ satisfies at $x_0$ the nontransversality condition on the positive side (on the negative side) with respect to the control process $\Gamma$.

**Lemma 3.3.** Countable Compactness of Discontinuities. *Let $x_0 \in X$. Let $u \in C(x_0)$, $\varphi \in T(x_0; u)$, and $t_f$ be the terminating time for the trajectory $\varphi$. Let $V^* : X \to E^1$ be a $V$-type function satisfying the discontinuous nontransversality condition with respect to the control process $\Gamma$. Let $S$ denote the set of all discontinuities of the function $V^* \circ \varphi : [t_0, t_f] \to E^1$. Let $T = \{t_0\} \cup \{t_f\} \cup S$. Then $T$ is a countable compact subset of $[t_0, t_f]$. Moreover, $V^* \circ \varphi$ is continuous from the left except possibly at $t = t_f$.*

*Proof.* It suffices to show that for each $t \in S \cap (t_0, t_f)$ there exists $\delta > 0$ such that $V^* \circ \varphi$ is continuous on $(t - \delta, t)$. For, in this case we can write

$$[t_0, t_f] = \{t_0\} \cup \{t_f\} \cup \{[t_{\alpha(k)}, t_k] : k \in K\}$$

where $K$ indexes the points of $S \cap (t_0, t_f)$ and where $t_{\alpha(k)}$ represents the last discontinuity before $t_k$; for $k = 0$ we take $t_{\alpha(0)} = t_0$ if $t_0 \notin S$ and for $k = 1$ we take $t_1 = t_f$ if $t_f \notin S$.

Since

$$\bigcup_{k \in K} (t_{\alpha(k)}, t_k)$$

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is an open subset of the real line, it is the countable union of disjoint open intervals. Since the intervals $(t_{\alpha(k)}, t_k)$ are already disjoint it follows that the index set $K$ is countable. Thus $S$ and, therefore, $T$ are countable. $T$ is compact since $T = [t_0, t_f] \sim \bigcup_{k \in K} (t_{\alpha(k)}, t_k)$.

Let $t \in (t_0, t_f) \cap S$. That $\varphi(t)$ belongs to $\bigcup_{i \in I} M_i$ follows from the properties (ii) and (iv) of Definition 3.4. Suppose $\varphi(t) \in M_i$. It follows from Lemma 3.2 that the trajectory $\varphi$ cannot enter $M_i$ at $\varphi(t)$ from a discontinuous side since $V^*$ satisfies the DNT Condition. If $M_i$ has at $\varphi(t)$ two locally discontinuous sides with respect to $V^*$, then there is $\delta > 0$ such that $\varphi(t_1) \in M_i$ for all $t_1 \in (t - \delta, t)$. Property (iii) of Definition 3.4 states that $V^*$ is continuous over $M_i$. Thus $V^* \circ \varphi$ is continuous over $(t - \delta, t)$.

If $M_i$ has at $\varphi(t)$ only one locally discontinuous side with respect to $V^*$, then, according to property (v) of Definition 3.4, $M_i$ has at $\varphi(t)$ a locally continuous side. From Definition 3.3 it follows that there exists a neighborhood $\theta$ of $\varphi(t)$ in $E^n$ such that $V^*$ is continuous on, say, $(\theta^+ \cap X) \cup (M \cap \theta \cap X)$. There exists a $\delta > 0$ such that $\varphi(t_1)$ belongs to the latter set for all $t_1 \in (t - \delta, t)$. Consequently, $V^* \circ \varphi$ is continuous over $(t - \delta, t)$. This shows that $V^* \circ \varphi$ is continuous from the left except possibly at $t = t_f$.

## SUFFICIENCY THEOREM FOR DISCONTINUOUS OPTIMAL VALUE FUNCTION

The sufficiency theorem given below is designed to establish the optimality of a pair $(u^*, \varphi^*)$ in a control process that may not have a continuous optimal value function. A discontinuous optimal value function arises particularly in state-constrained optimal control processes [13].

A function $V^*: X \to E^1$ is lower semicontinuous if and only if, for all $x_0 \in X$,

$$V(x_0) \leqslant \sup \{\inf \{V(x) : 0 < \|x - x_0\| < r, x \in X\} : r > 0\}.$$

For a $V$-type function, the $V^*$ lower semicontinuity, for each $x \in M_i$, $i \in I$, is equivalent to

$$V^*(x) \leqslant \operatorname*{limit}_{x_m \to x} \{\inf [V(x_k) : k \geqslant m]\} \tag{18}$$

for all sequences $(x_m) \subset X$ converging to $x$.

**Theorem 4.1.** Sufficiency for Discontinuous Optimal Value Function. *Let $x_0 \in X$. Let $u^* \in C(x_0)$ and $\varphi^* \in T(x_0; u^*)$. For the optimality in $X$ of the pair $(u^*, \varphi^*)$, it is sufficient that there exists a $V$-type function $V^*: X \to E^1$ with associated collections $\{X_j : j \in J\}$, $\{M_i : i \in I\}$, and $\{W_j, V_j^*) : j \in J\}$ such that the following conditions are satisfied:*

(i) *$V^*$ satisfies the discontinuous nontransversality condition with respect to the control process $\Gamma$.*

(ii) *$V^*$ is lower semicontinuous.*

(iii) *For each* $u \in C(x_0)$ *and each* $\varphi \in T(x_0; u)$ *there exists a convergent sequence* $(t_k)$ *contained in* $[t_0, t_f]$ *of* $\varphi$ *such that*

$$g_0[\varphi(t_f)] \geqslant \underset{t_k \to t_f}{\text{limit}} \; V^*[\varphi(t_k)] \,. \tag{19}$$

(iv)

$$V^*(x_0) = g_0[\varphi^*(t_f^*)] + \int_{t_0}^{t_f^*} f_0[\varphi^*(\tau), \quad u^*(\tau)] \, d\tau$$

*where* $t_f^*$ *is the terminating time for* $\varphi^*$.

(v) *For all* $x \in X_j$, $v \in U(x)$, $j \in J$

$$f_0(x, v) + \text{grad } V_j^*(x) \cdot f(x, v) \geqslant 0 \,. \tag{20}$$

*Proof.* Let $u \in C(x_0)$, $\varphi \in T(x_0; u)$, and $t_f$ be the terminating time for the trajectory $\varphi$. In view of condition (iv) we want to show that

$$V^*(x_0) \leqslant g_0[\varphi(t_f)] + \int_{t_0}^{t_f} f_0[\varphi(\tau), \quad u(\tau)] \, d\tau \,. \tag{21}$$

Let

$$g(t) = \int_{t_0}^{t} f_0[\varphi(\tau), \quad u(\tau)] \, d\tau + (V^* \circ \varphi)(t)$$

for all $t \in [t_0, t_f - \delta]$, where $\delta > 0$. Condition (v) implies that

$$f_0[\varphi(t), \quad u(t)] + \frac{d}{dt} (V_j^* \circ \varphi)(t) \geqslant 0 \text{ almost everywhere in } T_j, \; j \in J,$$

where $T_j = \{t \in [t_0, t_f] : \varphi(t) \in X_j\}$. Thus, condition (i) of Lemma 3.1 is satisfied.

As a result of Lemma 3.3, condition (i) implies that $T = \{t_0\} \cup \{t_f\} \cup S$ is countable compact where $S$ are the discontinuities of $V^* \circ \varphi$. Condition (ii) of Lemma 3.1 is, therefore, satisfied. Invoking Lemma 3.3, we have that $V^* \circ \varphi$ is continuous from the left except possibly at $t = t_f$. This satisfies the left inequality of condition (iii) of Lemma 3.1 for $t \in [t_0, t_f]$. Condition (ii) of Theorem 4.1 together with condition (ii) of Definition 3.4, implies that the right inequality of condition (iii) of Lemma 3.1 is satisfied for $t \in [t_0, t_f]$. With all conditions of Lemma 3.1 met, it follows that $g : [t_0, t_f - \delta] \to E^1$ is monotone nondecreasing for all $\delta > 0$.

Consequently, $V^*(x_0) \leqslant g(t)$ for all $t \in [t_0, t_f)$ and, therefore,

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$$V^*(x_0) \leqslant \int_{t_0}^{t} f_0[\varphi(\tau), \quad u(\tau)]\, d\tau + V^*[\varphi(t)] . \tag{22}$$

From condition (iii) of Theorem 4.1, we can write

$$\operatorname*{limit}_{t_m \to t_f} V^*[\varphi(t_m)] \leqslant g_0[\varphi(t_f)] \tag{23}$$

for some convergent sequence $(t_m)$.

Inequality (21) follows from (22) and (23). This completes the proof.

## CONTROL PROCESS WITH DISCONTINUOUS OPTIMAL VALUE FUNCTION

Consider the bilinear control process with state equations

$$\dot{x}_1 = -v + x_2(1-v), \qquad v \in [0,1] \tag{24a}$$

$$\dot{x}_2 = -x_1(1-v) \tag{24b}$$

with state space $X = E^2$ and terminal set $\Theta = \{(0,0)\}$. For a given initial state $x_0 = (x_1^0, x_2^0)$, we desire to minimize the transfer time to the origin:

$$J(x_0, \varphi, u) = \int_{t_0}^{t_f} d\tau . \tag{25}$$

Thus, $f_0 = 1$ and $g_0 = 0$. The set-valued function is given by

$$U(x) = [0,1] \qquad \forall x \in E^2 . \tag{26}$$

The control function space $\Omega$ is the space of all Lebesgue measurable functions of time defined on bounded intervals with range in $E^1$. An admissible control policy $u \in C(x_0)$, $x_0 \in E^2$, with $u : [t_0, t_f] \to E^1$, satisfies the relation

$$u(t) \in [0,1] \qquad \forall t \in [t_0, t_f] \tag{27}$$

and has a terminating trajectory $\varphi \in T(x_0; u)$.

Let $\Gamma_0$ represent the above control process.

We seek to find a pair $(u^*, \varphi^*)$, $u^* \in C(x_0)$, $\varphi^* \in T(x_0; u)$ for each $x_0 \in E^2$ and to discover a $V$-type function $V^* : E^2 \to E^1$ with associated collections

$\{X_j : j \in J\}$

$\{M_i : i \in I\}$

and

$\{(W_j, V_j^*) : j \in J\}$

such that conditions (i)- (v) of Theorem 4.1 are satisfied with respect to the control process $\Gamma_0$.

For this purpose we make the following definitions. Let $J = \{1, 2, \ldots, 5\}$ and $I = \{1\}$. Define $h : E^2 \to E^1$ with

$$h(x_1, x_2) = 1 + \frac{x_2}{x_1^2 + x_2^2} - \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

for all $(x_1, x_2) \in E^2 \sim \{(0,0)\}$ and with $h(0,0) = 0$. Write $p : \{x_2 : x_2 < 0 \text{ and } x_2 > -2\} \to E^1$ such that

$$h[p(x_2), x_2] = 0 \quad \forall x_2 < 0. \tag{28}$$

The solutions $p(x_2)$ of (28) for $x_2 < 0$ are unique, and $p$ is a smooth function. The function $p$ is plotted in Fig. 1 for $p(x_2) \leqslant 2$. Define

$$X_1 = \{(x_1, x_2) : x_1 > 0,\ x_2 = 0\}$$

$$X_2 = \{(x_1, x_2) : h(x_1, x_2) > 0\}$$

$$X_3 = \{(x_1, x_2) : h(x_1, x_2) = 0,\ x_2 < 0\}$$

$$X_4 = \{(x_1, x_2) : h(x_1, x_2) < 0\}$$

$$X_5 = \{(0,0)\}$$

$$M_1 = X_1.$$

Let $W_1 = W_5 = E^2$, $W_2 = X_2$, $W_4 = X_4$, and $W_3 = \{(x_1, x_2) : x_2 < 0\}$.

Define $A : E^2 \sim \{(0,0)\} \to E^1$ as the arctangent function such that

$$A(x_1, x_2) = \begin{cases} \tan^{-1}\left(\dfrac{x_2}{x_1}\right), & x_1 > 0, \quad x_2 \geqslant 0 \\[2ex] \left(\dfrac{\pi}{2}\right) + \tan^{-1}\left(\dfrac{-x_1}{x_2}\right), & x_1 \leqslant 0, \quad x_2 > 0 \end{cases}$$

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$$\begin{cases} \pi + \tan^{-1}\left(\dfrac{-x_2}{-x_1}\right), & x_1 < 0, \quad x_2 \leqslant 0 \\[2ex] \dfrac{3\pi}{2} + \tan^{-1}\left(\dfrac{x_1}{-x_2}\right), & x_1 \geqslant 0, \quad x_2 < 0. \end{cases}$$

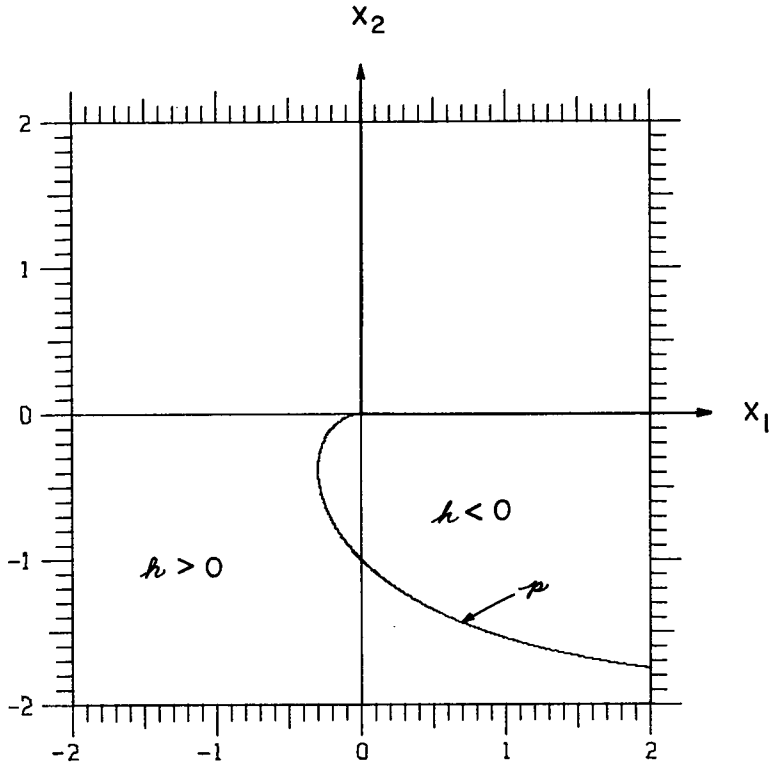


Fig. 1—The curve of $p$

Define $V_j^*$, $j \in J$, as follows:

$$V_1^*(x_1, x_2) = x_1 \quad \forall (x_1, x_2) \in E^2$$

$$V_2^*(x_1, x_2) = A(x_1, x_2) + \sqrt{x_1^2 + x_2^2} \quad \forall (x_1, x_2) \in X_2$$

$$V_3^*(x_1, x_2) = A(x_1, x_2) + \sqrt{x_1^2 + x_2^2} \quad \forall (x_1, x_2) \in W_3$$

$$V_4^*(x_1, x_2) = x_1 - p(x_2) + V_2^*[p(x_2), x_2] \quad \forall (x_1, x_2) \in X_4$$

$$V_5^*(x_1, x_2) = 0 \quad \forall (x_1, x_2) \in E^2 .$$

We define a $V$-type function candidate $V^* : E^2 \to E^1$ as

$$V^*(x) = V_j^*(x) \quad \forall x \in X_j, \quad j \in J.$$

This function is plotted in Fig. 2. Note the orientation of the axes. The curve defined by $[p(x_2), x_2, V^*(p(x_2), x_2)]$ is plotted in the surface; this is the intersection of $p$ with the optimal cost surface.

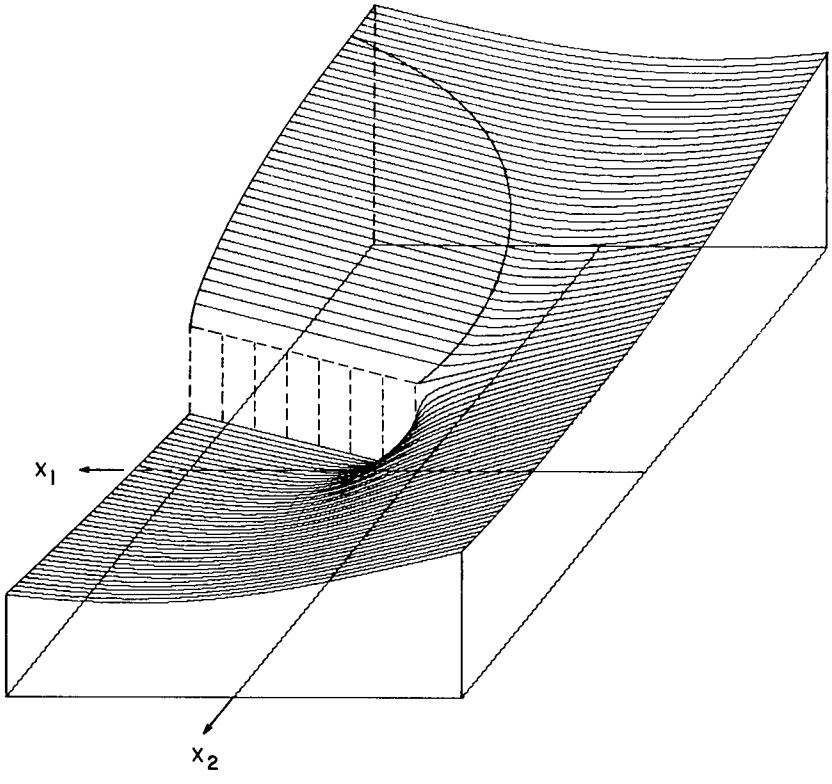


Fig. 2—Optimal value function $V^*$

It is easily checked that $V^*$ is piecewise differentiable, locally Lipschitzian with respect to $D$ since the collection $\{(W_j, V_j^*) : j \in J\}$ has the necessary properties satisfied; the $W_j$'s are open and the $V_j^*$'s are smooth. Note that the restrictions

$$V^*|_{E^2 \sim (M_1 \cup \Theta)} \quad \text{and} \quad V^*|_{M_1}$$

are continuous. The set $\Theta \cup M_1$ is closed in $E^2$. Let $N^+(x) = (0,1)$ and $N^-(x) = (0,-1)$ for all $x \in M_1$. It follows that $M_1$ has at each $x \in M_1$ a locally continuous side in the positive direction and a locally discontinuous side in the negative direction. Actually, these two directions have the values of $V^*(x)$ separated by the amount $\pi$. This completes the proof that $V^*$ is a $V$-type function.

$V^*$ satisfies the DNT condition with respect to $\Gamma_0$, provided $M_1$ satisfies at each $x_0 = (x_1^0, x_2^0) \in M_1$ the NT condition on the negative side with respect to $\Gamma_0$. Since $\partial M$, is empty, we take $\widetilde{M}_1 = M_1$. Let $\theta$ denote the open cube

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$$\left\{(x_1, x_2) : |x_1^0 - x_1| < \frac{x_1^0}{2} \quad \text{and} \quad |x_2^0 - x_2| < \frac{x_1^0}{2}\right\} .$$

Let $(y_1, y_2) \in \theta \cap M$. Let $(x_1, x_2) \in \theta^- \cap E^2$ and $v \in [0,1]$. Since $N^-(y_1, y_2) = (0,-1)$, the inequality (9) becomes

$$x_1(1 - v) \geqslant 0 . \tag{29}$$

This inequality holds since $x_1 > 0$ and $v \leqslant 1$. Consequently, condition (i) of Theorem 4.1 is satisfied.

Since $M_1$ has at each $x \in M_1$ a locally continuous side in the positive direction, the inequality (18) is satisfied for all sequences converging to $x$ from the positive side. Let $(x_m)$ be a sequence converging to $x = (x_1, 0)$ from the negative side. In this case we have $V^*(x_m) = V_4^*(x_{1m}, x_{2m})$ and, therefore,

$$V^*(x_m) = x_{1m} - p(x_{2m}) + V_2^*[p(x_{2m}), x_{2m}] ,$$

with

$$V_2^*[p(x_{2m}), x_{2m}] = \pi + \tan^{-1} \frac{-x_{2m}}{p(x_{2m})} + \sqrt{p(x_{2m})^2 + x_{2m}^2} .$$

Verify that

$$p(x_{2m}) \to 0 \quad \text{as} \quad x_{2m} \to 0$$

$$\tan^{-1} \frac{-x_{2m}}{p(x_{2m})} \to 0 \quad \text{as} \quad x_{2m} \to 0 .$$

Consequently,

$$\pi + x_1 = \operatorname*{limit}_{x_m \to x} V^*(x_m) . \tag{30}$$

Since $V^*(x) = x_1$, we see that condition (ii) is satisfied.

For condition (iii) it suffices to show, for $u \in C(x_0)$, $\varphi \in T(x_0; u)$, that

$$\operatorname*{limit}_{t_k \to t_f} V^*[\varphi(t_k)] = 0 \tag{31}$$

for some sequence $(t_k)$ converging to $t_f$ where $\varphi(t_f) = (0, 0)$. It follows from the dynamics (24) that the origin cannot be reached from the second, third, or fourth quadrants. This implies that $\varphi_1(t_k) > 0$ and $\varphi_2(t_k) > 0$ for all such sequences $(t_k)$. There is a $\delta > 0$ such that $\varphi(t)$ belongs to the first quadrant for all times $t \in [t_f - \delta, t_f)$. From (24) we have $\varphi_1(t)\dot{\varphi}_1(t) + \varphi_2(t)\dot{\varphi}_2(t) = -\varphi_1(t) - \dot{\varphi}_2(t)$ almost everywhere $t \in [t_0, t_f]$. Integrating this equation over $[t_k, t_f]$ gives

$$\varphi_1^2(t_k) + \varphi_2^2(t_k) + 2\varphi_2(t_k) = 2\int_{t_k}^{t_f} \varphi_1(\tau)\,d\tau .$$

There exists a sequence $(t_k)$ converging to $t_f$ such that

$$\int_{t_k}^{t_f} \varphi_1(\tau)\,d\tau \leqslant \varphi_1(t_k)\int_{t_k}^{t_f} d\tau ;$$

consequently, these two expressions imply that

$$\varphi_1(t_k) + \frac{\varphi_2(t_k)}{\varphi_1(t_k)}\,[\varphi_2(t_k) + 2] \leqslant 2(t_f - t_k) \tag{32}$$

for all $t_k$ belonging to the sequence $(t_k)$. This implies that

$$\operatorname*{limit}_{t_k \to t_f} \frac{\varphi_2(t_k)}{\varphi_1(t_k)} = 0 , \tag{33}$$

since the left-hand side of (32) is bounded above zero. In the first quadrant, we have

$$V^*[\varphi_1(t_k), \varphi_2(t_k)] \leqslant \tan^{-1}\left(\frac{\varphi_2(t_k)}{\varphi_1(t_k)}\right) + \sqrt{[\varphi_1(t_k)]^2 + [\varphi_2(t_k)]^2} .$$

This inequality, together with $V^* \geqslant 0$ and (33), implies (31).

We show that condition (v) is met and then define $(u^*, \varphi^*)$ and show that (iv) is satisfied.

Let $j = 1$, $x \in X_2$, and $v \in [0,1]$. In this case $x_2 = 0$ and grad $V_1^*(x) = (1, 0)$. The left-hand side of (20) reduces to $1 - v$. Thus (20) is met for $j = 1$.

Let $j = 2$, $x \in X_2$, and $v \in [0,1]$. In this case we have

$$\frac{\partial V_2^*(x)}{\partial x_1} = \frac{-x_2}{x_1^2 + x_2^2} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \tag{34a}$$

and

$$\frac{\partial V_2^*(x)}{\partial x_2} = \frac{x_1}{x_1^2 + x_2^2} + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} . \tag{34b}$$

The inequality (20) reduces to

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$$vh(x_1, x_2) \geqslant 0 , \tag{35}$$

which is satisfied for all $(x_1, x_2) \in X_2$.

Let $j = 3$, $x \in X_3$, and $v \in [0, 1]$. The partial derivatives of $V_3^*$ are given by (34) with $x_1 = p(x_2)$. The left-hand side of (20) reduces to the left-hand side of (35), with $h[p(x_2), x_2] = 0$.

Let $j = 4$, $x \in X_4$, and $v \in [0,1]$. Check that

$$\frac{\partial V_4^*}{\partial x_1} = 1$$

and

$$\frac{\partial V_4^*}{\partial x_2} = \frac{p(x_2)}{[p(x_2)]^2 + x_2^2} + \frac{x_2}{\sqrt{[p(x_2)]^2 + x_2^2}} .$$

The left-hand side of (20) reduces to

$$(1 - v)\left[1 + x_2 - \frac{x_1 p(x_2)}{[p(x_2)]^2 + x_2^2} - \frac{x_1 x_2}{\sqrt{[p(x_2)]^2 + x_2^2}}\right] .$$

After adding and subtracting the term

$$\frac{p(x_2)x_2}{\sqrt{[p(x_2)]^2 + x_2^2}}$$

inside the above brackets and making use of $h[p(x_2), x_2] = 0$, we obtain

$$(1 - v)[p(x_2) - x_1]\left[\frac{p(x_2)}{[p(x_2)]^2 + x_2^2} + \frac{x_2}{\sqrt{[p(x_2)]^2 + x_2^2}}\right] . \tag{36}$$

For $(x_1, x_2) \in X_4$, we have

$$p(x_2) < x_1 . \tag{37}$$

In order for (20) to hold, it sufficies then to have

$$\frac{p(x_2)}{\sqrt{[p(x_2)]^2 + x_2^2}} + x_2 < 0 . \tag{38}$$

For $x_2 \geqslant -1$, $p(x_2) \leqslant 0$, so that (38) holds. For $x_2 < -1$, (38) holds since it is always true that

$$\frac{p(x_2)}{\sqrt{[p(x_2)]^2 + x_2^2}} < 1 .$$

For $j = 5$, $x = (0, 0)$, $v \in [0,1]$ the inequality (20) holds since grad $V_5^*(0, 0) = (0, 0)$. This completes the proof that condition ($v$) is satisfied.

Define the closed-loop control policy $\sigma : E^2 \to [0,1]$ as

$$\sigma(x) = \begin{cases} 1 & \text{if} \quad x \in X_1 \cup X_4 \\ 0 & \text{if} \quad x \in X_2 \cup X_3 \cup X_5 . \end{cases}$$

The trajectories resulting from this control are plotted in Fig. 3.

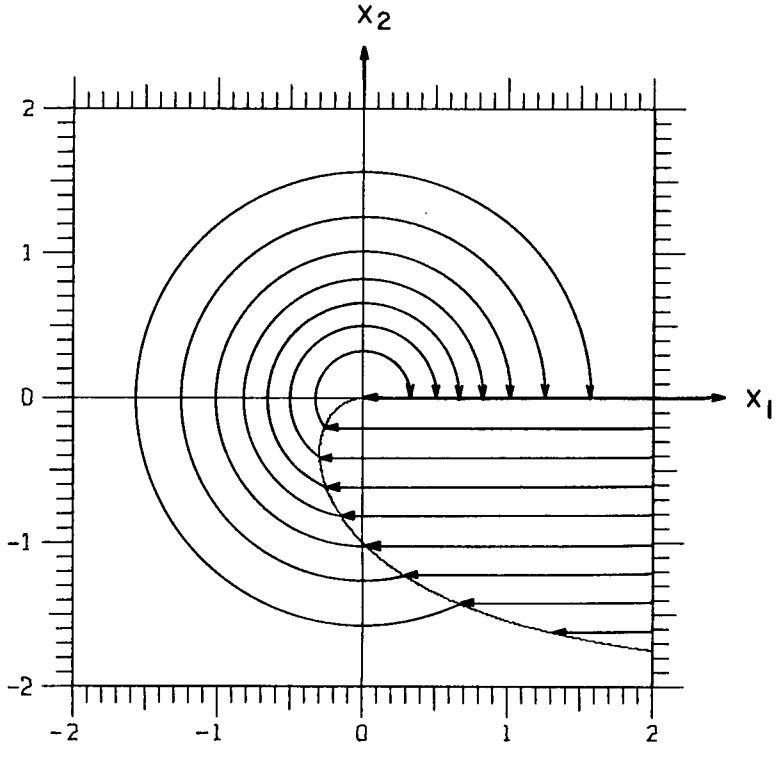


Fig. 3—Optimal trajectories

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Let $x_0 \in X_1$. Define the pair $(u^*, \varphi^*)$ as

$$u^*(t) = \begin{cases} 1 & \forall t \in [t_0, t_f] \\ 0 & t = t_f \end{cases}$$

$$\varphi_1^*(t) = x_1^0 - (t - t_0) \qquad \forall t \in [t_0, t_f]$$

$$\varphi_2^*(t) = 0 \qquad \forall t \in [t_0, t_f]$$

where

$$t_f - t_0 = x_1^0 . \tag{39}$$

Let $x_0 \in X_2 \cup X_3$. Define the pair $(u^*, \varphi^*)$ as

$$u^*(t) = \begin{cases} 0 & \text{if} \quad t \in [t_0, t_1) \cup \{t_f\} \\ 1 & \text{if} \quad t \in [t_1, t_f) \end{cases}$$

$$\varphi_1^*(t) = \begin{cases} \sqrt{(x_1^0)^2 + (x_2^0)^2} \cos(t_1 - t) & \forall t \in [t_0, t_1) \\ \sqrt{(x_1^0)^2 + (x_2^0)^2} - (t - t_1) & \forall t \in [t_1, t_f] \end{cases}$$

$$\varphi_2^*(t) = \begin{cases} \sqrt{(x_1^0)^2 + (x_2^0)^2} \sin(t_1 - t) & \forall t \in [t_0, t_1) \\ 0 \quad \forall t \in [t_1, t_f] \end{cases}$$

where $t_1 - t_0 \in [0, 2\pi)$ satisfies

$$\cos(t_1 - t_0) = \frac{x_1^0}{\sqrt{(x_1^0)^2 + (x_2^0)^2}}$$

$$\sin(t_1 - t_0) = \frac{x_2^0}{\sqrt{(x_1^0)^2 + (x_2^0)^2}}$$

and where

$$t_f - t_0 = A(x_1^0, x_2^0) + \sqrt{(x_1^0)^2 + (x_2^0)^2} . \tag{40}$$

Let $x_0 \in X_4$. Define $(u^*, \varphi^*)$ as

$$u^*(t) = \begin{cases} 1 & \text{if} \quad t \in [t_0, t_1) \cup [t_2, t_f) \\ 0 & \text{if} \quad t \in [t_1, t_2) \cup \{t_f\} \end{cases}$$

$$\varphi_1^*(t) = \begin{cases} x_1^0 - (t - t_0) & \forall\, t \in [t_0, t_1) \\ \sqrt{[p(x_2^0)]^2 + (x_2^0)^2} \cos(t_2 - t) & \forall\, t \in [t_1, t_2) \\ \sqrt{[p(x_2^0)]^2 + (x_2^0)^2} - (t - t_2) & \forall\, t \in [t_2, t_f] \end{cases}$$

$$\varphi_2^*(t) = \begin{cases} x_2^0 & \forall\, t \in [t_0, t_1) \\ \sqrt{[p(x_2^0)]^2 + (x_2^0)^2} \sin(t_2 - t) & \forall\, t \in [t_1, t_2) \\ 0 & \forall\, t \in [t_2, t_f] \end{cases}$$

where $t_1$ satisfies

$$t_1 = t_0 + x_1^0 - p(x_2^0)$$

and $t_2 - t_1 \in [0, 2\pi)$ satisfies

$$\cos(t_2 - t_1) = \frac{p(x_2^0)}{\sqrt{[p(x_2^0)]^2 + (x_2^0)^2}}$$

$$\sin(t_2 - t_1) = \frac{x_2^0}{\sqrt{[p(x_2^0)]^2 + (x_2^0)^2}},$$

with

$$t_f - t_0 = x_1^0 - p(x_2^0) + A[p(x_2^0), x_2^0] + \sqrt{[p(x_2^0)]^2 + (x_2^0)^2}\,. \tag{41}$$

Since $X_5$ is the terminal set, nothing needs to be undertaken there.

From Eqs. (39)—(41), the definitions of $V^*$, $u^*$, $\varphi^*$, it follows that condition (iv) is met. Since all conditions of Theorem 4.1 are met, the pairs $(u^*, \varphi^*)$ defined above are optimal. Equation (30) shows that the optimal value function $V^*$ is discontinuous along the positive $x_1$ axis.

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