Naval Research Laboratory
Washington, DC 20375-5000

NRL/FR/5341--92-9367

Non-Gaussian Noise Models and Coherent Detection of Radar Targets

KEVIN J. SANGSTON AND KARL R. GERLACH

Target Characteristics Branch
Radar Division

November 5, 1992

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave Blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	November 5, 1992	Interim

4. TITLE AND SUBTITLE

Non-Gaussian Noise Models and Coherent Detection of Radar Targets

5. FUNDING NUMBERS

PE - 61153N
PR - 021-05-43
WU - DN480-006

6. AUTHOR(S)

Kevin J. Sangston and Karl R. Gerlach

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Research Laboratory
Washington, DC 20375-5320

8. PERFORMING ORGANIZATION REPORT NUMBER

NRL/FR/5341—92-9367

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Chief of Naval Research
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

The problem of detecting radar targets against a background of coherent, correlated non-Gaussian clutter is studied with a two-step procedure. In the first step, the structure of the multidimensional probability density function (PDF) describing the statistical properties of the clutter is derived. The starting point for this derivation is the basic scattering problem, and the statistics are determined from an extension of the Central Limit Theorem that is formulated and proved herein. Some basic mathematical properties of the resulting distributions are studied, and the problem of fitting data to such distributions is also discussed. The end product of the first step is a multidimensional PDF, which is then used in step 2 in the derivation of both the optimal and a suboptimal detection structure for detecting radar targets in this type of clutter. Performance results for the new detection processor are also given.

14. SUBJECT TERMS

Detectors	Radar
Non-Gaussian noise	Clutter

15. NUMBER OF PAGES
61

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std 239-18
298-102

CONTENTS

NON-GAUSSIAN NOISE MODELS AND COHERENT DETECTION OF RADAR TARGETS

1. INTRODUCTION

The detection of radar targets against a background of unwanted clutter due to echoes from the sea, land, or weather is a problem of fundamental interest in the radar community. Attacking the problem generally requires an understanding of the statistics of clutter echoes. Insight into the possible structure of the optimal (or quasi-optimal) detector can be gained only after the statistics are understood. Hence, the task of formulating optimal detection processors separates naturally into two fundamental steps: 1) formulation of a statistical model for the noise from which the desired target is to be discriminated, and 2) application of the results of detection theory to determine the structure of the detector for the type of noise modelled in step 1. (Although noise often refers to the ubiquitous thermal noise in the receiving equipment, here it refers to any unwanted background disturbance, including clutter echoes.)

For many years, radar systems had relatively low resolution capabilities such that clutter echoes were thought to comprise a sum of responses from a very large number of elementary scatterers. From the Central Limit Theorem (CLT) of probability theory, workers in the field were led to conclude that the appropriate statistical model for clutter was the Gaussian model. With this model in hand, researchers were able to attack and solve the detection problem. The optimal detection structure was discovered to be the now well-known matched filter. The importance of this result for the radar detection problem should not be underestimated. The structure of the matched filter may be viewed as that of a whitening filter followed by a correlator that maximizes the output signal-to-noise ratio. The structure of the frequently used coherent detection processor, which is composed of a moving target indication (MTI) filter followed by a bank of Doppler filters, is essentially a suboptimal implementation of a bank of matched filters. The MTI filter fulfills the role of the whitening filter, whereas the bank of Doppler filters plays the role of a bank of correlators. In a sense, this coherent detection processor may be said to have been derived according to the two-step approach described earlier.

In the quest for better performance, the resolution capabilities of radar systems have steadily improved. For detection performance, the belief originally was that a higher resolution radar system would intercept less clutter than a lower resolution system, thereby increasing detection performance. However, as resolution has increased, the statistics of the noise have no longer been observed to be Gaussian, and the detection performance of the traditional detection processor has not necessarily improved. The radar system is now plagued by target-like "spikes" that give rise to non-Gaussian observations. These spikes are passed by the detection processor as targets at a much higher false alarm rate than the system is designed to tolerate. The reason for the poor performance can be traced to the fact that the traditional radar detection processor is designed to operate against Gaussian noise. New detection processors are required to reduce the effects of the spikes and to improve detection

Manuscript approved Oct. 28, 1991.

performance. According to the two-step paradigm described above, the detection problem cannot be addressed, though, until the statistical nature of the non-Gaussian statistics is understood. These problems, i.e., the development of a model for the non-Gaussian statistics and the subsequent development of a detection processor, are the focus of this report.

Although at first glance the approach to solving these problems appears to be straightforward, the solution to the problem in step 1, i.e., determining an appropriate statistical model, is quite difficult. This problem is complicated by the fact that it is not sufficiently well-defined to ensure a unique solution. As stated, the goal is to determine a model for non-Gaussian statistics, yet this problem undoubtedly has an infinity of solutions. Which of these solutions is proper for the radar problem? To answer this question, one could attempt to measure the statistics by fitting empirical models to data collected with high-resolution radar systems. This approach is actually quite reasonable for determining first-order statistics of the non-Gaussian clutter, i.e., the amplitude probability density function (PDF), and much work has been devoted to this effort. However, two problems arise. First, if one group of experimentalists measures Weibull statistics, and a second group measures K statistics, which results should be accepted as correct? Second, the pulse-to-pulse clutter returns are usually correlated with one another and, hence, require a multidimensional model for the proper description of their statistics. Measurement of multidimensional statistics is prohibitively difficult. Moreover, even if the measurement of multidimensional statistics were feasible, very few closed-form expressions for non-Gaussian multidimensional PDF's of correlated random variables are available to describe the measurements. For those models that are available, no useful criteria for deciding among them are known. Note that this latter problem also arises for Gaussian statistics. Even if the first-order statistics are Gaussian, modelling the multidimensional statistics by the multivariate Gaussian PDF must be justified in any particular problem. Fortunately, as indicated earlier, the choice of the multivariate Gaussian PDF in the radar problem appears to be justified by the multidimensional form of the CLT. An analogous theorem is needed in the non-Gaussian case.

The appropriate multidimensional non-Gaussian model for use in radar detection studies must thus incorporate the following features: a) it must account for the measured first-order statistics; b) it must incorporate pulse-to-pulse correlation between data samples, and c) it must be chosen according to some criterion that clearly distinguishes it from the multitude of multidimensional non-Gaussian models satisfying a) and b). This problem forms the focus of Sections 2 through 5 of this report.

The resolution of the problems listed above is found in an extension of the CLT to incorporate number fluctuations, i.e., fluctuations in the instantaneous number of elementary scatterers contributing to the scattered echo. This result gives rise to a class of first-order statistics sufficiently general to encompass the Rayleigh, K, and Weibull distributions, as well as mixtures of these distributions. These distributions are among the most prevalent distributions used in empirical studies. The result also allows for pulse-to-pulse correlation between the samples. Most importantly, since the result arises from a reexamination of the scattering problem itself, it gives a natural criterion for selecting the appropriate multidimensional model.

The availability of a multidimensional model for the non-Gaussian noise allows one to attack the problem of detecting radar targets against such noise. This problem is the subject of Section 6. The goal of this aspect of the problem is an understanding of the structure of the optimal (and possibly suboptimal) processor for detecting radar targets. Although the structure is described by the likelihood ratio of classical detection theory, one seeks greater insight into this structure, with the ultimate goal of deriving a detection processor that is implementable yet close to optimal in the same sense that the current processor based on the MTI filter followed by the Doppler filter bank is an implementable but suboptimal version of a bank of matched filters.

Section 2 presents a general limit theorem for a sum of a random number of random vectors. This theorem represents an extension of the classical CLT and as such may have application in areas other than the problem of interest here. An essential feature of this new result is that it incorporates fluctuations in the number of random vectors being summed; this feature gives rise to limiting statistics that may be non-Gaussian in nature. Along with this theorem, the notion of a canonical model for the probability mass function (PMF) describing the number fluctuations is also introduced.

Section 3 then applies the new theorem to the study of scattering of electromagnetic waves. This work leads to the idea of modelling the radar clutter amplitude statistics by the class of Rayleigh mixture distributions. Previous investigators have modelled amplitude statistics by special cases of Rayleigh mixtures, and a brief review of their work is presented. Because identifying a given distribution as a Rayleigh mixture distribution is a problem of interest, necessary and sufficient conditions for a given cumulative distribution function (CDF) to be a Rayleigh mixture are also presented in this section. These conditions are then used to show several examples of Rayleigh mixture distributions. These examples include the Weibull distribution for the range of its parameters that are generally of interest in the radar problem. This demonstration is important because it offers a physical interpretation for the appearance of certain distributions, such as the Weibull distribution, in radar clutter studies. These distributions appear to be justifiable otherwise only on empirical grounds. It also unifies the approach to modelling amplitude statistics and clarifies the distinction between modelling clutter by the K distribution or the Weibull distribution; both models arise from the same underlying physical picture with the difference between them occurring because they have different underlying number fluctuation models.

The Rayleigh mixture distribution is defined by a certain CDF, $F_\tau(\tau)$, $\tau > 0$, and the determination of this CDF both analytically and empirically is important for the successful application of Rayleigh mixture distributions to radar studies. Section 4 presents results on both of these problems. A particularly interesting result is that the K distribution, which has been applied to empirical fits of radar clutter data with success, appears as the first term in a particular type of infinite series expansion of the general Rayleigh mixture distribution. This result would seem both to explain the appearance of the K distribution in empirical studies of scattering problems and to offer an approach to achieving a data fit better than the fit of the K distribution by including more terms in the series. Several examples illustrating this latter idea are presented.

Because many radar scenarios involve making detection decisions based on multiple pulse data that is correlated from pulse to pulse, the development of a theory of target detection for these types of problems is necessary. The development of such a theory requires models for the multidimensional statistics of correlated, non-Gaussian clutter; the formulation of such a multidimensional model is the focus of Section 5. Based on a physical interpretation of the theorem presented in Section 2, a multidimensional model in the form of a Gaussian mixture is presented. This model can incorporate first-order statistical information about the amplitude of the clutter as well as information about the correlation between pulses. It also has the advantage of being motivated by a physical picture and, hence, is not an arbitrary construction of a multidimensional model. However, the model is based on an idealized physical picture, and the problem of incorporating a more realistic physical picture is briefly discussed. This particular problem is quite difficult and will form the focus of future research.

Based on the model for the multidimensional statistics of the correlated clutter, Section 6 explores the problem of detecting a signal against a background of such clutter. Previous work on this problem by the present authors is briefly reviewed. The problem of detecting a signal of known Doppler structure but unknown complex amplitude is then attacked using the generalized likelihood methodology.

The structure of the detector is shown to be that of a square law detector whose output is processed through a nonlinearity. The structure may also be interpreted as a matched filter compared to a data-dependent threshold. Section 6 closes by presenting performance results for several cases of interest.

Although the report is long and heavily oriented towards mathematics and statistics, the results are nonetheless useful to the average radar engineer who may not have an interest in the mathematics but who is interested in either the modelling of non-Gaussian statistics or the detection of targets in non-Gaussian noise. For the worker interested in applying these results but not interested in the theory, Section 4.2 presents a methodology for fitting data to a Rayleigh mixture distribution, whereas Section 6.4 provides both closed form expressions and curves for the detection performance of the detection processor derived in this report. For the worker interested solely in the problem of modelling non-Gaussian statistics in scattering problems, Section 2 is particularly pertinent. The results of Section 2 also suggest that a possible physical mechanism for non-Gaussian statistics is a fluctuation in the number of scatterers contributing to the scattered field. This observation in turn suggests that dynamical models that could give rise to these fluctuations be studied further. Furthermore, as discussed in Section 6, the availabilty of such dynamical models would engender further results in the multidimensional modelling problem. For the worker interested in the structure of the detection processor derived herein, both Sections 6.1 and 6.3 give simple interpretations of this detection structure; these interpretations could form the starting points for deriving implementable structures that would be advances over the current MTI/Doppler filter bank processors.

2. A PHENOMENOLOGICAL MODEL FOR SCATTERING

In many propagation and scattering problems, the electric field at a given point may be represented by the phasor relationship

$$E(\bar{r}, t) = e^{j\omega t} \sum_{i=1}^{N} a_i(\bar{r}, t) e^{j\phi_i(\bar{r}, t)}, \tag{1}$$

where ω is the carrier frequency of the incident radiation, $a_i(\bar{r}, t)$ is a form factor that determines the angular distribution of the radiation from the ith scatterer and $\phi(\bar{r}, t)$ is a phase factor for the ith scatterer. Generally, the individual contributions to the sum are assumed to be statistically independent. For example, this kind of expression forms the starting point for Chapter 7 of Beckmann and Spizzichino (1963) on the probability distribution function of a wave scattered from a rough surface. This type of expression is also implied in the work of Dashen (1979) and Flatte et al. (1987) in which propagation through random media is treated via the path integral formalism; Eq. (1) would be valid in the so-called saturation regime. If the physics of the problem indicates that N is "large," then often the CLT is invoked to conclude that E has Gaussian statistics. However, in many cases, the physics of the problem also indicates that N fluctuates, a case to which the CLT does not apply. For example, in the work cited above, Dashen states that in the saturation regime, N fluctuates. As the notion of large N is meaningless if N is a random variable, a parameter that may become large is the mean of N, denoted $\bar{N}$. However, as indicated in the work of Jakeman and Pusey (1978), the statistics of E as $\bar{N} \rightarrow \infty$ need not be Gaussian. They studied a modification of Eq. (1), namely

$$E'(\bar{r}, t) = \frac{e^{j\omega t}}{\sqrt{\bar{N}}} \sum_{i=1}^{N} a_i(\bar{r}, t) e^{j\phi_i(\bar{r}, t)} \tag{2}$$

in which N has a negative binomial distribution. (In the standard development of the CLT, Eq. (1) is normalized by $1/\sqrt{N}$. The normalizations there and in Eq. (2) are necessary to ensure that the mean intensity remains finite.) Under the assumption that the a_i's and ϕ_i's are mutually independent, the a_i's independent and identically distributed (i.i.d.), and the ϕ_i's i.i.d. and uniformly distributed on $[0, 2\pi]$, they showed that the distribution of $A = |E'|$ as $\bar{N} \rightarrow \infty$, is given by the K distribution, not the Rayleigh distribution predicted by the CLT. On the other hand, they also performed this calculation with N being a Poisson random variable instead of a negative binomial random variable and showed that the asymptotic statistics are Rayleigh. Thus, inclusion of number fluctuations enriches the problem considerably, and a general limit theorem appropriate to this problem is needed. A first step towards such a theorem is given below.

A conclusion that may be drawn from the theorem given here is that the distribution of amplitude statistics predicted by this type of random walk model is a member of the general class of Rayleigh mixture distributions. The K distribution studied by Jakeman and Pusey (1978) is a member of this class. It is also shown below that the Weibull, gamma, and Nakagami-m distributions are also members of this class for suitably restricted domains of their respective parameters. These results suggest that the general Rayleigh mixture distribution be considered a model for amplitude statistics of scattering problems and that the role of number fluctuations in these problems be further studied.

2.1 A Limit Theorem for a Random Sum of Random Vectors

As is well-known, if $X_1, X_2, \ldots, X_N$ are i.i.d. random variables with zero mean and finite variance σ^2, then the random variable

$$S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \tag{3}$$

converges in distribution, as $N \rightarrow \infty$, to a Gaussian random variable Y with zero mean and variance σ^2. If, however, N is also considered to be a random variable, the classical results are no longer applicable. Theorems for random N in which convergence to a Gaussian random variable occurs (hence, these theorems are called CLT's) are presented by Chow and Teicher (1978) and the references therein. In this report, the problem is defined in a natural way, and it is shown that convergence to a non-Gaussian random variable can occur.

Let N be a random variable defined on the nonnegative integers, and let $(p(n; \bar{N}), \bar{N} > 0)$ be a collection of PMF's associated with N such that $E[N] = \bar{N}$, where $E[N]$ is the expected value of N. Let $k \geq 1$ be an integer and let $X_1, X_2, \ldots$ be a sequence of i.i.d. random vectors defined in R^k with zero mean and finite second-order moments given by the matrix $E[X_i X_i^T] = Q$ for all i (where T denotes transpose.) For each $\bar{N} > 0$, define a random vector by

$$S_{\bar{N}} = \frac{1}{\sqrt{\bar{N}}} \sum_{i=1}^{N} X_i, \tag{4}$$

where an empty sum (i.e., $N = 0$) is set equal to zero. Finally, for each $\bar{N} > 0$, define a function by

$$G(z; \bar{N}) = \sum_{n=0}^{\infty} (1 - \frac{z}{\bar{N}})^n p(n; \bar{N}), \quad 0 \le z \le \bar{N}. \tag{5}$$

The following theorem may now be stated:

THEOREM: If $G(z) = \lim_{\bar{N} \to \infty} G(z; \bar{N})$ converges for $z \in [0, \infty)$, then $S_{\bar{N}}$ converges in distribution as $\bar{N} \to \infty$ to a random vector Y with characteristic function given by

$$C_Y(u) = \int_0^{\infty} e^{-\frac{u^T Q u}{2} t} dF_t(t), \tag{6}$$

where $F_t(t)$ is a CDF of a positive random variable. Moreover, $F_t(t)$ may be obtained by Laplace inversion of $G(z)$, i.e.,

$$G(z) = \int_0^{\infty} e^{-zt} dF_t(t). \tag{7}$$

PROOF: The proof is presented in two parts. First, under the conditions of the theorem, the characteristic function of Y is shown to be $G(u^T Qu \,/\, 2)$. $G(z)$ is then shown to have the representation given in Eq. (7), which will complete the proof.

Conditioned on $N = n$, the characteristic function of $S_{\bar{N}}$ is given by

$$C_{\bar{N}}(u \mid N = n) = E[\exp(jS_{\bar{N}} u) \mid N = n]$$

$$= (C_X(\frac{u}{\sqrt{\bar{N}}}))^n, \tag{8}$$

where the i.i.d. property of the X_i's has been used and $C_X(u)$ is the common characteristic function of the X_i's. From the following asymptotic expansion as $|u| \to 0$,

$$C_X(u) = 1 - \frac{u^T Qu}{2} + o(u^T u), \tag{9}$$

(cf Cramer 1970), the unconditional characteristic function of $S_{\bar{N}}$ becomes

$$C_{\bar{N}}(u) = \sum_{n=0}^{\infty} [1 - \frac{u^T Qu}{2\bar{N}} + o(\frac{u^T u}{\bar{N}})]^n \, p(n; \bar{N}), \quad \text{as } \frac{|u|}{\sqrt{\bar{N}}} \to 0,$$

$$= G(\frac{u^T Qu}{2} - \bar{N} o(\frac{u^T u}{\bar{N}}) \, ; \bar{N}) \quad \text{if } 0 \le \frac{u^T Qu}{2} - o(\frac{u^T u}{\bar{N}}) \le \bar{N}. \tag{10}$$

From the definition of $G(z; \bar{N})$ it follows that for $0 \le z_1 \le z_2 \le \bar{N}$, then $G(z_2; \bar{N}) \le G(z_1; \bar{N})$. Fix u. Since $\bar{N} o(u^T u \,/\, \bar{N}) \to 0$ as $\bar{N} \to \infty$, then given $\varepsilon > 0$, a number $K \,\varepsilon\, R$ exists such that

$$\frac{u^T Qu}{2} - \varepsilon < \frac{u^T Qu}{2} - \bar{N} o(\frac{u^T u}{\bar{N}}) < \frac{u^T Qu}{2} + \varepsilon \tag{11}$$

for all $\bar{N} \ge K$, from which it follows

$$G(\frac{u^T Qu}{2} + \varepsilon \, ; \bar{N}) < G(\frac{u^T Qu}{2} - \bar{N} o(\frac{u^T u}{\bar{N}}) \, ; \bar{N}) < G(\frac{u^T Qu}{2} - \varepsilon \, ; \bar{N}). \tag{12}$$

Taking the limit as $\bar{N} \to \infty$ yields

$$G(\frac{u^T Qu}{2} + \varepsilon) < \lim_{\bar{N} \to \infty} G(\frac{u^T Qu}{2} - \bar{N} o(\frac{u^T u}{\bar{N}}) \, ; \bar{N}) < G(\frac{u^T Qu}{2} - \varepsilon). \tag{13}$$

Finally, since ε was arbitrary, then by Eqs. (10) and (13),

$$C_Y(u) = \lim_{\bar{N} \to \infty} G(\frac{u^T Qu}{2} - \bar{N} o(\frac{u^T u}{\bar{N}}) \, ; \bar{N})$$

$$= G(\frac{u^T Qu}{2}), \quad \frac{u^T Qu}{2} < \infty. \tag{14}$$

This completes the first part of the proof.

The second part of the proof follows from theorem 3 of Schoenberg (1938), which states that $C(u) = g(|u|^2)$ is the characteristic function of a radially symmetric random vector in R^k for all integers $k \ge 1$ if and only if $g(t)$ is a completely monotonic (c.m.) function for $t \ge 0$. A function $g(t)$ is said to be c.m. for $t > 0$ if $(-1)^l g^{(l)}(t) \ge 0$, $l = 0, 1, 2, \ldots$. The function $g(t)$ is furthermore said to be c.m. for $t \ge 0$ if it is c.m. for $t > 0$ and $g(0) = g(0^+) < \infty$. To apply this result here, let $Q = 2I$ in Eq. (14). This choice of Q has no effect on the structure of the function $G(z)$, which depends only on the collection $(p(n; \bar{N}), \bar{N} > 0)$. With this choice, Eq. (14) becomes

$$C(u) = G(|u|^2), \tag{15}$$

which shows that $C(u)$ is a radially symmetric characteristic function in R^k, $k \geq 1$. Hence, by Schoenberg's theorem, $G(z)$ is a c.m. function for $t \geq 0$. It follows immediately now from a theorem of Bernstein (1929) and Widder (1931) (cf Tamarkin 1931) that

$$G(z) = \int_0^\infty e^{-zt}\, dF_t(t) \tag{16}$$

where $F_t(t)$ is a bounded nondecreasing function. Without loss of generality, we may assume $F_t(0^-) = 0$, (if not, simply add a constant, $-F_t(0^-)$). Since $G(0; \bar{N}) = 1$, $\bar{N} > 0$, it follows immediately that $F_t(\infty) = 1$. Finally, since e^{-zt} is continuous, the value of $F_t(t)$ at any discontinuities, of which there can be at most a countable number (Kolmogorov and Fomin 1970), may be chosen so as to make $F_t(t)$ right-continuous without affecting the value of $G(z)$. Hence, $F_t(t)$ is a CDF of a positive random variable, and the theorem is proved.

2.2 Canonical Point Mass Functions

The limit theorem given above is predicated on the convergence of the functions $G(z; \bar{N})$ to $G(z)$. Therefore, it is of interest to study under what conditions convergence occurs. To investigate the structure of the collections of PMF's, $(p(n; \bar{N}), \bar{N} > 0)$, that yield convergence in the above development, the following corollary to a theorem of Boas (1939) appears to be useful.

COROLLARY: Let $\sum_{n=0}^{\infty} a_n$ be a convergent series. Then there exists a function $\beta: [0, \infty] \rightarrow R$ of bounded variation such that

$$a_n = \int_0^\infty \frac{t^n}{n!} e^{-t}\, d\beta(t) \quad n = 0, 1, 2, \ldots . \tag{17}$$

PROOF: By Boas' theorem, it is permissible to write

$$a_n = \int_0^\infty \frac{t^n}{n!}\, dF(t) \quad n = 0, 1, 2, \ldots , \tag{18}$$

where $F(t)$ is a function of bounded variation. Define $f_N(t) = \sum_{n=0}^{N} \frac{t^n}{n!}$, $t \in [0, \infty]$. Then $f_N(t)$ increases with N to e^t and by the monotone convergence theorem

$$\sum_{n=0}^{\infty} a_n = \int_0^\infty e^t\, dF(t) = C, \tag{19}$$

which is finite by hypothesis.

Define

$$\beta(t) = \int_0^t e^{\alpha}\, dF(\alpha)$$

$$= \int_0^t e^{\alpha}\, dF_1(\alpha) - \int_0^t e^{\alpha}\, dF_2(\alpha)$$

$$= \beta_1(t) - \beta_2(t), \tag{20}$$

where $F_1(t)$ and $F_2(t)$ are monotonic nondecreasing functions of t. It follows immediately that $\beta_1(t)$ and $\beta_2(t)$ are monotonic nondecreasing functions of t, and since by Eq. (19)

$$\lim_{t \to \infty} \beta(t) = C < \infty, \tag{21}$$

it follows that $\beta_1(t)$ and $\beta_2(t)$ are bounded; hence $\beta(t)$ is a function of bounded variation.

Now examine

$$\beta(t) = \int_0^t e^{\alpha}\, dF(\alpha), \tag{22}$$

where without loss of generality we may assume that $F(t)$ is a monotonic nondecreasing function of t. Also let $\beta(t) = 0,\ t \leq 0$. Let μ_β and μ_F be the Lebesgue-Stieltjes measures associated with β and F, respectively. Finally, let F^0 be the field consisting of finite disjoint unions of left-open, right-closed intervals, $(a, b]$, $a \leq b$, $a, b \in R$. By straightforward computation, one may easily show that

$$\mu_\beta(A) = \int_A e^{\alpha}\, d\mu_F(\alpha) \tag{23}$$

for all sets $A \in F^0$. Since each side of Eq. (23) is a measure on F^0, and each of these measures has a unique extension to the minimal σ-field over F^0, which in this case consists of the Borel sets, it follows that Eq. (23) holds for all Borel sets. By the Radon-Nikodym theorem,

$$\mu_F(A) = \int_A e^{-\alpha}\, d\mu_\beta(\alpha), \tag{24}$$

and hence from Eq. (18)

$$a_n = \int_0^\infty \frac{t^n}{n!} e^{-t} \, d\beta(t) \quad n = 0, 1, 2, \ldots . \tag{25}$$

This completes the proof of the corollary.

In particular, the corollary states that any PMF, $p(n; \bar{N})$, $n = 0, 1, 2, \ldots$, has the representation given by Eq. (17), i.e., any random variable with such a PMF is a "mixture" of Poisson random variables. Use of this representation in the definition of $G(z)$ and $G(z; \bar{N})$ now yields in a straightforward manner

$$G(z) = \lim_{\bar{N} \to \infty} \int_0^\infty e^{-zt} \, d\beta_{\bar{N}}(\bar{N}t), \tag{26}$$

where $\beta_{\bar{N}}(t)$ is a function of bounded variation associated with the representation of $p(n; \bar{N})$. Little may be said about the existence of $G(z)$ as defined by Eq. (26) without better knowledge of $\beta_{\bar{N}}(t)$. However, it appears possible to define a set of parameterized PMF's that yield convergence of $G(z)$ and that are *canonical* in the sense that if all collections of PMF's that yield convergence to the same function are identified in an equivalence class, then this canonical set of parameterized PMF's contains one representative of each possible equivalence class. In that sense, the canonical set represents all collections of PMF's that yield convergence.

In the theorem, let $k = 1$ and let σ^2 be the variance of the X_i's. The result above becomes $C_Y(u) = G(\sigma^2 u^2 / 2)$. From the relation $\sigma^2 = -C_Y{}^{(2)}(0)$ is easily found the relation $G^{(1)}(0) = -1$, which implies

$$\int_0^\infty t \, dF_t(t) = 1. \tag{27}$$

Thus, any CDF satisfying Eq. (27) yields, through Eq. (16), an admissible $G(z)$ for this problem. On the other hand, if $F_t(t)$ does not satisfy Eq. (27), the resulting $G(z)$ obtained by Eq. (16) is not admissible for this problem. For each CDF F_t such that $F_t(-\infty) = F_t(0^-) = 0$, $F_t(\infty) = 1$, and F_t satisfies Eq. (27), define a collection of PMF's parameterized by $\bar{N} > 0$ by

$$p_c(n; \bar{N}) = \int_0^\infty \frac{t^n}{n!} e^{-t} \, dF_t(t/\bar{N}) \quad n = 0, 1, 2, \ldots . \tag{28}$$

The canonical set $PM(\bar{N})$ (for Poisson mixture with mean $\bar{N}$) is defined to be the set of all such parameterized PMF's as defined by Eq. (28). It follows from Eq. (26) that if $p_c \,\varepsilon\, PM(\bar{N})$, then $G(z)$ associated with p_c is given by Eq. (16).

2.2.1 Examples

Let N be a Poisson random variable with PMF

$$p(n;\bar{N}) = \frac{(\bar{N})^n}{n!}e^{-\bar{N}},\ \bar{N} > 0. \tag{29}$$

It is immediately evident that this parameterized PMF is a member of the canonical set with $F_t(t) = U(t-1)$, where $U(t)$ is the Heaviside step function. Hence, $G(z) = e^{-z}$, and $S_{\bar{N}}$ converges to a normal random variable. To exhibit other members of the equivalence class represented by the Poisson distribution, examine the degenerate case, with PMF given by

$$p(n;\bar{N}) = 1,\ \ n = \bar{N} > 0, \tag{30}$$

and the binomial distribution, with PMF given by

$$p(n;\bar{N}) = \binom{N}{n} p^n q^{N-n}, \quad n = 0, 1, 2, \ldots, N, \tag{31}$$

where $0 < p = 1 - q < 1$, and $\bar{N} = Np$. That the degenerate case is a member of this equivalence class is merely a statement of the classical form of the CLT. It is straightforward to show that $G(z) = e^{-z}$ in this case. To see that the binomial distribution is also in this equivalence class, examine $G(z;\bar{N})$. After a simple computation, one finds

$$G(z;\bar{N}) = (1 - \frac{pz}{\bar{N}})^N. \tag{32}$$

For a fixed $p > 0$, $\bar{N} \to \infty$ if and only if $N \to \infty$. It follows then that

$$\lim_{\bar{N}\to\infty} G(z;\bar{N}) = \lim_{N\to\infty} (1 - \frac{z}{N})^N$$

$$= e^{-z}, \tag{33}$$

which shows that the binomial distribution is also in this class.

As a second example, let N be a negative binomial random variable with PMF

$$p(n;\bar{N}) = \frac{\Gamma(n+\nu)}{\Gamma(n+1)\Gamma(\nu)} \frac{r^n}{(1+r)^{n+\nu}}, \quad \nu > 0, \tag{34}$$

where $r = \bar{N}/\nu$. This PMF is also a member of the canonical set with

$$F_t(t) = \int_0^t \frac{\nu^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\nu x}\, dx, \tag{35}$$

as may be verified directly. Hence $G(z)$ converges for the negative binomial distribution and is given by

$$G(z) = \left[\frac{\nu}{\nu+z}\right]^\nu . \tag{36}$$

The attributes of the canonical PMF's that are of interest in this problem are: a) they form a set of parameterized PMF's for which convergence always occurs, b) they represent all collections of PMF's for which convergence occurs and c) they are related to the limiting CDF's $F_t(t)$ in a straightforward way.

3. CLUTTER MODELLING: RAYLEIGH MIXTURE DISTRIBUTIONS

If the random vectors in the theorem are defined in R^2, the limiting random variable denoted as $Y = (Y_1 \;\; Y_2)$ and the matrix $Q = \sigma^2 I$ where $\sigma^2 > 0$ is a constant and I is the 2×2 identity matrix, then the problem being discussed here takes the form of a spherically symmetric random walk in the plane with a random number of steps. Typically in applications involving narrowband signals (or noise), the two components Y_1 and Y_2 are interpreted as the inphase and quadrature components of a complex random variable. The relationship with the phasor approach used in formulating Eqs. (1) and (2) is established by transforming Y_1 and Y_2 to polar coordinates, say R and ϕ. The assumption that $Q = \sigma^2 I$ is tantamount in this problem to assuming that R and ϕ are independent and that ϕ is uniformly distributed on $[0, 2\pi]$. If the resultant (or amplitude), R, of the limiting case of this random walk as $\overline{N} \to \infty$ is defined by $R^2 = {Y_1}^2 + {Y_2}^2$, then inversion of the characteristic function yields

$$f_R(x) = \int_0^\infty \frac{x}{\sigma^2 t} e^{-\frac{x^2}{2\sigma^2 t}}\, dF_t(t) \tag{37}$$

where $F_t(t)$ is defined in the theorem. The choice of $F_t(t)$ obtained from the negative binomial distribution yields the result given by Jakeman and Pusey (1978). Evidently, the only permissible distributions for this resultant are mixtures of Rayleigh distributions. Use of this type of model for the amplitude statistics of clutter has been proposed by previous investigators, whose work is discussed below. The derivation of this model that is presented here, however, is new.

After studying the problem of sea clutter modelling, Trunk (1972; 1976) concluded that some types of non-Rayleigh sea clutter may be modelled as a locally homogeneous Rayleigh process whose Rayleigh parameter (which represents clutter power in this case) is modulated by the radar's large scale spatial sampling of the environment. The amplitude PDF of such a model is given by Eq. (37). In the remainder of this report, the model described by Eq. (37) is referred to as a Rayleigh mixture model. Prior to this work, Valenzuela and Laing (1971) had argued that the scattering of a radar pulse from the surface of the sea modelled by the so-called composite surface model yields amplitude statistics whose PDF, although complicated in form, may be described as a Rayleigh mixture model. Jakeman and Pusey (1976) used the phenomenological model described by Eq. (1) to study radar sea clutter modelling. Specifically, they assumed the a_i's to be i.i.d. according to a K distribution, the ϕ_i's to be i.i.d. according to the uniform distribution on $[0, 2\pi]$, and the a_i's to be mutually independent of the ϕ_i's. From these assumptions, they showed that the amplitude of the scattered field E is statistically described by a K distribution, which is also an example of a Rayleigh mixture model. Note that in the 1976 work, the number of scatterers N was a deterministic, nonfluctuating quantity; hence, the need for a specific model of the statistics of the individual scatterers a_i arose. In the 1978 work cited above, they presented an alternative derivation of K distributed amplitude statistics by introducing number fluctuations into the model and relaxing the requirement that the individual a_i's have K distributed statistics. Since the introduction of the K distribution, it has been used with qualitative success in empirical fits to clutter data (Ward 1981, 1982).

In addition to the models described above, the Weibull distribution has also yielded qualitatively accurate empirical fits to clutter data (Skolnik 1980; Olin 1982; Trizna 1988, 1989) and has emerged as a candidate model for sea clutter statistics. Its effectiveness as a model appears to have been strictly empirical; it does not appear to have the physical appeal of the K distribution or other Rayleigh mixture models. Conte and Longo (1987) addressed this issue through an analysis that strongly suggests that the Weibull distribution is also a Rayleigh mixture distribution and, hence, would have the same physical interpretation as the K or other Rayleigh mixture models. Unfortunately, they were not able to reach a definitive conclusion. A demonstration based on an early result of Bochner (1937a) is presented below and shows rigorously that the Weibull distribution is in fact a Rayleigh mixture distribution for a limited range of the shape parameter (i.e., $b \leq 2$; see Eq. (47).)

Thus, as first suggested by Conte and Longo, almost all of the major statistical models that have been proposed for the amplitude statistics of sea clutter (the log-normal being an exception) may be represented as a Rayleigh mixture model. This observation coupled with the result that physical/phenomenological models that have been proposed to model the scattering process also lead to the Rayleigh mixture model suggests that the model described by Eq. (37) be considered a unified model for sea clutter amplitude statistics. A study of the properties of such distributions, especially to determine if any of the previously proposed models for amplitude statistics of scattering problems are examples of Rayleigh mixture distributions, is therefore warranted and is given below.

3.1 Conditions for Rayleigh Mixture Distributions

A CDF defined on the positive real line will be said to be the distribution of a Rayleigh mixture if this CDF, $F_x(x)$, may be represented in the form

$$F_x(x) = \int_0^\infty (1 - e^{-\frac{x^2}{2\tau}})\, dF_\tau(\tau), \tag{38}$$

where τ (i.e., the Rayleigh parameter) is a random variable defined on the positive real line, and $F_\tau(\tau)$ is the CDF of τ. The problem of interest here is to determine necessary and sufficient conditions on $F_x(x)$ to ensure that it has the representation given in Eq. (38) for some CDF, $F_\tau(\tau)$. These conditions follow as a corollary of the Bernstein-Widder theorem for c.m. functions that was referred to earlier in the proof of the theorem presented herein.

COROLLARY: A CDF $F_x(x)$ defined on the positive real line is a Rayleigh mixture distribution if and only if the associated function $K(y) = 1 - F_x(\sqrt{y})$ is c.m. in the interval $0 \leq y < \infty$.

PROOF: Assume $K(y)$ is c.m. in the interval $0 \leq y < \infty$. Then by the Bernstein-Widder theorem,

$$K(y) = \int_0^\infty e^{-y\alpha} \, d\beta(\alpha), \tag{39}$$

where $\beta(\alpha)$ is nondecreasing. Without loss of generality, $\beta(0)$ may be chosen to be equal to 0. This choice has no effect on the value of $K(y)$. Since $K(0) = 1$, it follows that $\beta(\infty) - \beta(0) = \beta(\infty) = 1$. $\beta(\alpha)$ is seen to be bounded and, therefore, has at most countably many points of discontinuity. Since $e^{-y\alpha}$ is continuous, the value of $\beta(\alpha)$ at each point of discontinuity may be assigned to be an arbitrary finite value without affecting the value of $K(y)$. Therefore, the values of $\beta(\alpha)$ at its points of discontinuity can be chosen so as to make $\beta(\alpha)$ right-continuous.

Clearly, $y = x^2$ yields

$$F_x(x) = \int_0^\infty (1 - e^{-\alpha x^2}) \, d\beta(\alpha). \tag{40}$$

The substitutions $\alpha = 1/2\tau$ and $F_\tau(\tau) = 1 - \beta(1/2\tau)$ yield

$$F_x(x) = \int_0^\infty (1 - e^{-\frac{x^2}{2\tau}}) \, dF_\tau(\tau). \tag{41}$$

It remains only to show that $F_\tau(\tau)$ is a CDF. To this end,

1. $F_\tau(\tau)$ is nondecreasing, since $\beta(\alpha)$ is nondecreasing.
2. $\lim_{\tau \to -\infty} F_\tau(\tau) = \lim_{\tau \to 0} F_\tau(\tau) = 0$ since $\lim_{\alpha \to \infty} \beta(\alpha) = 1$.
   Similarly, $\lim_{\tau \to \infty} F_\tau(\tau) = 1$.
3. $F_\tau(\tau)$ is right-continuous, since $\beta(\alpha)$ is right-continuous.

Therefore, $F_\tau(\tau)$ is a CDF, and the proof in one direction is complete.

Conversely, assume $F_x(x)$ is a Rayleigh mixture distribution. Then, from the substitutions

$$\alpha = 1/2\tau \tag{42}$$

and

$$\beta(\alpha) = 1 - F_\tau(1/2\alpha), \tag{43}$$

it follows that

$$K(y) = \int_0^\infty e^{-y\alpha}\, d\beta(\alpha) \tag{44}$$

where $\beta(\alpha)$ is nondecreasing. Therefore, by the Bernstein-Widder theorem, $K(y)$ is c.m. in the interval $0 < y < \infty$. Because $K(0) = K(0^+) = 1$, $K(y)$ is also c.m. in the interval $0 \leq y < \infty$. The proof is complete.

This corollary fully characterizes Rayleigh mixture distributions and establishes a one-to-one relationship between Rayleigh mixture CDF's and functions, $K(y)$, that are c.m. in the interval $0 \leq y < \infty$ and normalized to $K(0) = 1$. Given a CDF F_x, one may establish that it represents a Rayleigh mixture distribution by establishing that the associated function K is c.m.. Although this task requires examination of an infinite number of derivatives, some techniques are presented below that make this problem tractable in some instances. Further conditions that ensure $dF_\tau(\tau) = f_\tau(\tau)\, d\tau$ follow from the work of Widder (1931) and Hille and Tamarkin (1933, 1934). The interested reader is referred to the references for details.

3.2 Examples of Rayleigh Mixture Distributions

To assess if a given distribution is a Rayleigh mixture, two previously obtained results relating to the class of c.m. functions are useful. First, let C be the class of functions that are c.m. on $0 < y < \infty$. Bochner (1937a) shows that

1. If $f_1, f_2, \varepsilon\, C$, then $af_1 + bf_2 \,\varepsilon\, C$ for $a, b > 0$.

2. If $f_1, f_2, \varepsilon\, C$, then $f_1 f_2 \,\varepsilon\, C$.

The second result is a lemma also due to Bochner (1937a, pp. 498-499):

LEMMA: If $\psi(\rho)$ is c.m. on $0 \leq \rho < \infty$ and $\chi(\rho)$ is such that $\chi(0) = 0$, and $\chi^{(1)}(\rho)$ is c.m. on $0 < \rho < \infty$, then the composition $\psi(\chi(\rho))$ is c.m. on $0 \leq \rho < \infty$.

As stated earlier, a well-known example of a Rayleigh mixture distribution is the K distribution with PDF defined by

$$f_x(x) = \frac{(\sqrt{2\nu/\eta})^{\nu+1}}{2^{\nu-1}\Gamma(\nu)} x^\nu K_{\nu-1}(\sqrt{2\nu/\eta}\, x) \qquad x, \nu, \eta > 0, \tag{45}$$

from which the following representation may be obtained (Watson 1944, p.183)

$$f_x(x) = \int_0^\infty \frac{x}{\tau} e^{-\frac{x^2}{2\tau}} \frac{(\nu/\eta)^\nu}{\Gamma(\nu)} \tau^{\nu-1} e^{-\frac{\nu}{\eta}\tau}\, d\tau. \tag{46}$$

A second example is given by the Weibull distribution with PDF

$$f_x(x) = abx^{b-1}e^{-ax^b} \qquad x, a, b > 0. \tag{47}$$

If the parameter b is restricted to the range $0 < b \leq 2$, Bochner's lemma may be applied to show that this PDF is a mixture of Rayleigh PDF's (Bochner 1937b). To see this result, examine

$$K(y) = 1 - G(\sqrt{y})$$

$$= e^{-ay^{b/2}}. \tag{48}$$

In Bochner's lemma, let $\psi(\rho) = e^{-a\rho}$, which is easily seen to be c.m. on $0 \leq \rho < \infty$ for any $a > 0$, and let $\chi(y) = y^{b/2}$. Clearly, $\chi(0) = 0$ and $\chi^{(1)}(y) = (b/2)y^{b/2-1}$, which for $b < 2$ may be written

$$\frac{b}{2}y^{b/2-1} = \frac{b/2}{\Gamma(1 - b/2)}\int_0^\infty e^{-yt}t^{-b/2}\,dt. \tag{49}$$

By the Bernstein-Widder theorem, Eq. (49) shows that $(b/2)y^{b/2-1}$ is c.m. on $0 < y < \infty$ for $0 < b < 2$. By Bochner's lemma, then, $K(y)$ is c.m. on $0 \leq y < \infty$. It follows that the Weibull distribution is a Rayleigh mixture distribution for the range $0 < b \leq 2$. (The case $b = 2$ is obvious.)

To apply the results presented in this report to a lesser known case, consider as a third example the gamma distribution with PDF

$$f_x(x) = \frac{a^b}{\Gamma(b)}x^{b-1}e^{-ax} \qquad x, a, b > 0. \tag{50}$$

Clearly, the following relations hold:

$$K(y) = 1 - F_x(\sqrt{y}) \geq 0, \tag{51}$$

$$-K^{(1)}(y) = \frac{f_x(\sqrt{y})}{2\sqrt{y}} \geq 0. \tag{52}$$

Also, since

$$(-1)^n K^{(n)}(y) = (-1)^{n-1}\frac{d^{n-1}}{dy^{n-1}}[-K^{(1)}(y)]$$

$$= (-1)^{n-1}\frac{d^{n-1}}{dy^{n-1}}\left[\frac{f_x(\sqrt{y})}{2\sqrt{y}}\right], \tag{53}$$

to show $(-1)^n K^{(n)}(y) \geq 0$, it suffices to show

$$(-1)^{n-1}\frac{d^{n-1}}{dy^{n-1}}\left[\frac{f_x(\sqrt{y}\,)}{2\sqrt{y}}\right] \geq 0 \quad n \geq 1, \tag{54}$$

i.e., that the function $f_x(\sqrt{y}\,)\,/\,2\sqrt{y}$ is c.m. on $0 < y < \infty$. To show the complete monotonicity of this function, examine

$$\frac{f_x(\sqrt{y}\,)}{2\sqrt{y}} = \frac{a^b}{b\Gamma(b)}\frac{by^{b/2-1}}{2}e^{-a\sqrt{y}}. \tag{55}$$

It suffices to show that the functions $e^{-a\sqrt{y}}$ and $(b/2)y^{b/2-1}$ are each c.m. and to use the closure properties of c.m. functions. But these functions have already been shown above to be c.m. for $0 < b < 2$. If $b = 2$, then $(b/2)y^{b/2-1} = 1$, which is c.m. because all of its derivatives are equal to 0 (indeed, any positive constant is c.m. for the same reason). Therefore, by the closure property of c.m. functions, the gamma PDF is also a mixture of Rayleigh PDF's for the case $0 < b \leq 2$.

As a final example, consider the Nakagami-m distribution with PDF given by

$$f_x(x) = 2(\frac{m}{\eta})^m \frac{x^{2m-1}}{\Gamma(m)}e^{-\frac{m}{\eta}x^2}, \qquad x, m, \eta > 0. \tag{56}$$

A computation similar to the one above for the gamma distribution shows that this distribution is a Rayleigh mixture distribution for $m \leq 1$.

3.3 Exponential Mixture Distributions

As it is sometimes of interest to study the distribution of intensity rather than the distribution of amplitude, let $I = R^2$ in Eq. (37). In radar applications, I represents the radar cross-section. The CDF of intensity becomes

$$F_I(y) = \int_0^\infty (1-e^{-\frac{y}{2\tau}})\, dF_\tau(\tau), \tag{57}$$

where $\tau = \sigma^2 t$ and $F_\tau(\tau) = F_t(\tau/\sigma^2)$. Distributions of this form will be called exponential mixture distributions.

In problems of propagation through random media, previous investigators have sometimes studied the moments of intensity rather than its distribution. From Eq. (57), the moments of intensity predicted by this model are given by

$$\frac{<I^n>}{<I>^n} = 2^n n!E[\tau^n]. \tag{58}$$

This model is consistent with models presented by previous investigators (Dashen 1984). From Eq.

(58), one immediately obtains for the scintillation index S,

$$S = \frac{<I^2>}{<I>^2} - 1 \geq 1, \tag{59}$$

with equality if and only if $F_\tau(\tau)$ is a unit step function. For application to problems of propagation through continuous random media, this model is thus only applicable in the saturation regime. But this region is precisely the region in which Dashen (1979) indicates that the phasor interpretation described by Eq. (2) is applicable.

4. METHODOLOGIES FOR DETERMINING $F_\tau(\tau)$

4.1 Theoretical Inversion of a Rayleigh Mixture Distribution

Thus far the results concerning Rayleigh mixtures have ascertained only the existence of the CDF $F_\tau(\tau)$. Once the Rayleigh mixture property is established for a given $f_x(x)$, the natural question to ask is what is the form of the associated $F_\tau(\tau)$? The general problem appears to be quite difficult. As a result, the discussion in this section is limited to suggestions for attacking the problem.

A general result for the inversion of Eq. (39), i.e., obtaining $\beta(\alpha)$ for a given $K(y)$, is presented by Widder (1934). Note that $F_\tau(\tau)$ may be obtained directly from $\beta(\alpha)$ by a simple change of variable. For the problem of interest here, the result becomes

$$\beta(\alpha) = \lim_{n \to \infty} (-1)^n \int_{\frac{n}{\alpha}}^{\infty} \frac{u^n}{n!} K^{(n)}(u)\, du. \tag{60}$$

Although general, the result requires knowledge of the derivatives of all orders of $K(y)$ in a neighborhood as $y \to \infty$. Since the actual computation of these derivatives can be difficult for some choices of $K(y)$, this method can be intractable. The interested reader is referred to the reference for details.

A second, quite general result for the inversion of Eq. (39) is also given by Widder (1935). This method requires knowledge of the derivatives of all orders of $K(y)$ at the point $y = 1$ and yields the following series solution for $\beta(\alpha)$ in terms of Laguerre polynomials:

$$\beta(\alpha) = \sum_{n=0}^{\infty} \int_0^{\alpha} L_n(y)\, dy \sum_{k=0}^{n} \binom{n}{k} \frac{K^{(k)}(1)}{k!}. \tag{61}$$

This approach may be illustrated by an application to determine the inversion for the Weibull distribution. In this case, $K(y) = e^{-ay^c}$ where $c = b/2$. To apply Widder's result, knowledge of $K^{(n)}(1)$ is necessary. These derivatives do not appear to be obtainable in closed form. However,

$$K^{(n)}(y) = \frac{d^{n-1}}{dy^{n-1}}[K^{(1)}(y)], \tag{62}$$

and

$$K^{(1)}(y) = -acy^{c-1}e^{-ay^c}. \tag{63}$$

Application of the Leibniz rule for the differentiation of a product then yields

$$K^{(n)}(y) = -acy^{c-n}\sum_{k=0}^{n-2}\begin{bmatrix} n-1 \\ k \end{bmatrix}K^{(k)}(y)c_k y^k - acy^{c-1}K^{(n-1)}(y),\ n \geq 1, \tag{64}$$

where $c_k = (c-1)(c-2) \cdots (c-n+k+1)$. This computation is a recursive computation for the derivatives of $K(y)$. It therefore may be used in conjunction with Widder's result to give a recursive scheme for computing $\beta(\alpha)$ and, thus, $F_\tau(\tau)$. The drawback, of course, is that this approach does not result in a closed form solution for $F_\tau(\tau)$.

For many cases of interest, τ will be a continuous random variable and $dF_\tau(\tau) = f_\tau(\tau)\, d\tau$ will hold. In cases for which this relationship does hold, Eq. (39) is an ordinary Laplace transform, and $f_\tau(\tau)$ may be obtained (at least conceptually) from the inverse Laplace transform of Eq. (39). This inversion result is given by

$$f_\tau(\tau) = \frac{1}{\tau}\frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty}\frac{f_x(\sqrt{s})}{2\sqrt{s}}e^{\frac{s}{2\tau}}\, ds \qquad \sigma > 0. \tag{65}$$

However, in some cases inversion via the standard contour integral approach may be difficult. For example, for the Weibull distribution, Eq. (65) becomes

$$f_\tau(\tau) = \frac{1}{2\pi j\tau}\int_{\sigma - j\infty}^{\sigma + j\infty}\frac{ab}{2}s^{c-1}e^{-as^c}e^{\frac{s}{2\tau}}\, ds \qquad \sigma > 0, \tag{66}$$

where $c = b/2$. The multivaluedness of the integrand renders this approach quite difficult (see, however, Pollard (1946)).

Under the assumption that $f_\tau(\tau)$ is piecewise continuous, an alternate approach for computing $f_\tau(\tau)$ based on the Mellin transform (Courant and Hilbert 1953) is as follows.

Consider the function

$$F_{x,M}(2p+1) = \int_0^\infty x^{2p} f_x(x)\, dx. \tag{67}$$

If the integral converges for $Re(p) > \sigma$, then

$$F_{x,M}(2p+1) = \int_0^\infty x^{2p}\int_0^\infty \frac{x}{\tau}e^{-\frac{x^2}{2\tau}}f_\tau(\tau)\, d\tau dx \qquad \mathrm{Re}(p) > \sigma. \tag{68}$$

The integrals on the right-hand side may be interchanged to obtain

$$F_{x,M}(2p + 1) = 2^p \Gamma(p + 1) \int_0^{\infty} \tau^p f_\tau(\tau)\, d\tau$$

$$= 2^p \Gamma(p + 1) F_{\tau,M}(p + 1) \tag{69}$$

where $F_{\tau,M}(p)$ is the Mellin transform of $f_\tau(\tau)$ and is convergent for $Re(p) > \sigma$. Since $f_\tau(\tau)$ is piecewise smooth by assumption, it follows that

$$f_\tau(\tau) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F_{\tau,M}(p + 1)\tau^{-p-1}\, dp. \tag{70}$$

Substitution for $F_{\tau,M}(p + 1)$ yields

$$f_\tau(\tau) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \frac{F_{x,M}(2p + 1)\tau^{-p-1}}{2^p \Gamma(p + 1)}\, dp. \tag{71}$$

This result indicates that under the appropriate conditions (i.e., the piecewise continuity of $f_\tau(\tau)$ and the convergence of $F_{x,M}(2p+1)$), $f_\tau(\tau)$ as defined by Eq. (71) is an inversion integral for this problem. This inversion integral may also be intractable, but it does provide an alternative approach to the inversion integral of Eq. (65).

For an application of this approach, let $f_x(x)$ be the PDF of the K distribution. For this case, τ is known to be a continuous random variable, and

$$F_{x,M}(2p + 1) = \left(\frac{2}{\sqrt{2\nu/\eta}}\right)^{2p} \frac{\Gamma(\nu + p)\Gamma(p + 1)}{\Gamma(\nu)}. \tag{72}$$

This integral converges for $Re(p) > (1/2)(|Re(\nu - 1)| - \nu - 1)$. Inserting this expression into Eq. (71) and letting $z = \nu + p$ yields

$$f_\tau(\tau) = \frac{(\nu/\eta)^\nu}{\Gamma(\nu)} \tau^{\nu-1} \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \Gamma(z)(\nu\tau/\eta)^{-z}\, dz \qquad \sigma > (1/2)(|Re(\nu - 1)| + \nu - 1). \tag{73}$$

Note that the contour lies in the right half-plane for any choice of $\nu > 0$. The singularities of the integrand are simple poles due to the presence of $\Gamma(z)$ and occur at $z = -n$, $n = 0, 1, 2, \ldots$. The residues of the gamma function at these points are equal to $(-1)^n/n!$. Closing the contour to the left by a semicircle of radius R_N that encloses N poles and taking the limit as $N \to \infty$ yields

$$f_\tau(\tau) = \frac{(\frac{\nu}{\eta})^\nu}{\Gamma(\nu)}\tau^{\nu-1}\sum_{i=0}^{\infty}\frac{(-1)^i(\frac{\nu}{\eta}\tau)^i}{i!}$$

$$= \frac{(\nu/\eta)^\nu}{\Gamma(\nu)}\tau^{\nu-1}e^{-\frac{\nu}{\eta}\tau}, \tag{74}$$

which is known to be correct.

An application of this same method to the case in which $f_x(x)$ is the PDF of the Weibull distribution yields

$$F_{x,M}(2p+1) = \frac{\Gamma(\frac{2p}{b}+1)}{a^{\frac{2p}{b}}}. \tag{75}$$

The computation proceeds similarly to the one just given and yields

$$f_\tau(\tau) = ab\,(2\tau)^{\frac{b}{2}-1}\sum_{i=0}^{\infty}\frac{(-a(\sqrt{2\tau})^b)^i}{i!\Gamma(1-\frac{b}{2}(i+1))} \quad 0 < b < 2. \tag{76}$$

This result agrees with a result obtained independently by Pollard (1946) and by Grosjean et al. (1989), and, in contrast to the result obtained earlier herein, is a closed-form solution. (The issue as to whether this result is more desirable from a computational point of view than the previous result is not addressed here.)

Formal application of this approach to the case in which $f_x(x)$ is the gamma PDF yields

$$F_{x,M}(2p+1) = \frac{\Gamma(2p+b)}{a^{2p}\Gamma(b)}. \tag{77}$$

Again, the computation proceeds similarly to the one given above and yields

$$f_\tau(\tau) = \frac{a^b}{\Gamma(b)}(2\tau)^{\frac{b}{2}-1}\sum_{i=0}^{\infty}\frac{(-a\sqrt{2\tau})^i}{i!\Gamma(1-\frac{1}{2}(i+b))}, \qquad 0 < b \le 2. \tag{78}$$

For $b = 1$ and $b = 2$, this series sums to

$$f_\tau(\tau) = \frac{a}{\sqrt{\pi}\sqrt{2\tau}}e^{-\frac{a^2}{2}\tau}, \tag{79}$$

and

$$f_\tau(\tau) = \frac{a^3\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{a^2}{2}\tau}, \tag{80}$$

respectively (Prudnikov et al. 1986, p. 707). The results in Eqs. (79) and (80) may be ascertained to be correct inversions by direct computation via Eq. (65), or by inserting them in Eq. (37) with $\tau = \sigma^2 t$ and performing the integration.

Finally, formal application of this approach to the Nakagami-m distribution proceeds analogously to the above derivations and yields

$$f_\tau(\tau) = 2(\frac{m}{\eta})^m \frac{(2\tau)^{m-1}}{\Gamma(m)} \sum_{i=0}^{\infty} \frac{(-2\tau m/\eta)^i}{i!\Gamma(1-(m+i))}, \quad m < 1. \tag{81}$$

4.2 Fitting a Rayleigh Mixture to Data

If the Rayleigh mixture model is to used to model data, the problem of empirically fitting data to a distribution becomes a problem of finding the $F_\tau(\tau)$ that yields the closest fit to the data. Moreover, since in this case an entire function rather than a limited number of parameters (usually two) is chosen during the fitting process, this model should give closer empirical fits to the data. For this approach to be effective, however, a methodology for obtaining $F_\tau(\tau)$ from a given set of data is required. This section of the report presents such a methodology. If an estimate for $F_\tau(\tau)$ can be obtained from a given set of data, then the PDF of the clutter amplitude can be estimated from Eq. (37) with $\tau = \sigma^2 t$. For $f_R(x)$ described by Eq. (37) it can be shown that

$$\int_0^\infty x^{2p} f_R(x)\, dx = 2^p \Gamma(p+1) \int_0^\infty \tau^p \, dF_\tau(\tau), \tag{82}$$

where $p = \sigma + j\omega$ and $\Gamma(p)$ is the gamma function, provided the left-hand side exists. For the problem of interest here, let $p = n$. Then Eq. (82) yields

$$\mu_{x,2n} = 2^n n! \, \mu_{\tau,n} \tag{83}$$

where

$$\mu_{x,2n} = 2n\text{th moment of } x,$$

$$\mu_{\tau,n} = n\text{th moment of } \tau.$$

As the moments of the amplitude x can be determined directly from experimental data, the moments of τ can be calculated from Eq. (83).

To illustrate this idea, consider in place of real data a model distribution for which the moments can be written in closed form. For $f_R(x)$ modelled by the K distribution, the moments are given by

$$\mu_{x,2n} = (\frac{2\eta}{\nu})^n \frac{\Gamma(\nu + n)n!}{\Gamma(\nu)}, \tag{84}$$

which yields for the moments of τ

$$\mu_{\tau,n} = (\frac{\eta}{\nu})^n \frac{\Gamma(\nu + n)}{\Gamma(\nu)}. \tag{85}$$

These moments are the moments of the gamma distribution, which is known to be the correct $F_\tau(\tau)$ for $g(x)$ given by the K distribution.

The problem, therefore, is to find a distribution $F_\tau(\tau)$ that is consistent with a set of moments $\mu_{\tau,n}$ obtained as described above. This problem is of classical origin, and many approaches are available. One approach, based on an orthogonal polynomial expansion, will be examined here. In this approach, it is assumed that the distribution of τ admits a probability density function $f_\tau(\tau)$. Although this assumption seems rather arbitrary, the empirical results obtained in the past, namely the success of the K and Weibull distributions, suggest that this assumption is valid for modelling sea clutter. With this assumption, Eq. (37) (with $\tau = \sigma^2 t$) becomes

$$f_R(x) = \int_0^\infty \frac{x}{\tau} e^{-\frac{x^2}{2\tau}} f_\tau(\tau)\, d\tau, \tag{86}$$

and the task is to determine $f_\tau(\tau)$ from the moments $\mu_{\tau,n}$, $n = 0, 1, \ldots, N$.

To determine this PDF, it is now assumed that $f_\tau(\tau)$ may be expanded in an infinite series of Laguerre polynomials

$$f(\tau) = \sum_{n=0}^{\infty} c_n e^{-\frac{\tau}{b}} (\frac{\tau}{b})^a L_n{}^a(\frac{\tau}{b}) \quad a > -1, b > 0, \tag{87}$$

where $L_n{}^a(x)$ is the generalized Laguerre polynomial of order n and a and b are constants to be determined later. Since these polynomials are orthogonal on the interval $[0, \infty)$ (which is the interval of interest for this problem) with respect to the weight $x^a e^{-x}$, they are an attractive choice as basis functions for the expansion of the unknown $f_\tau(\tau)$. This orthogonality condition implies

$$c_n = \frac{n!}{\Gamma(a + 1 + n)} \int_0^\infty L_n{}^a(\frac{\tau}{b}) \frac{f(\tau)}{b}\, d\tau, \tag{88}$$

which, upon substitution of the formula for the Laguerre polynomial, becomes

$$c_n = \frac{n!}{\Gamma(a+1+n)} \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n+a}{n-m} \frac{\mu_{\tau,m}}{b^{m+1}}. \tag{89}$$

The coefficients in the expansion are seen from Eq. (89) to be directly obtainable from knowledge of the moments of τ. Once this expansion has been obtained, it may be substituted into Eq. (86) to obtain

$$f_R(x) = 2x \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n+a}{n-m} \left(\frac{x}{2}\sqrt{2/b}\,\right)^{a+m} K_{a+m}(x\sqrt{2/b}\,). \tag{90}$$

The parameters a and b are arbitrary parameters that define the Laguerre polynomials used in the expansion and may be advantageously chosen as follows. Knowledge of the moments up through the second order implies that the expansion has three terms with the coefficients c_0 through c_2 depending on these three moments (note that $\mu_0 = 1$ is considered a moment). Because a and b are arbitrary, it is possible to choose these parameters so that $c_1 = c_2 = 0$. This choice is

$$a = \frac{\mu_{\tau,1}{}^2}{\mu_{\tau,2} - \mu_{\tau,1}{}^2} - 1, \tag{91}$$

$$b = \frac{\mu_{\tau,2} - \mu_{\tau,1}{}^2}{\mu_{\tau,1}}. \tag{92}$$

Note that in Eq. (91) the denominator is the variance of τ, whereas in Eq. (92) the numerator is this variance. Denote this variance by $\sigma_\tau{}^2$ and let

$$\frac{\mu_{\tau,1}{}^2}{\sigma_\tau{}^2} = \nu, \tag{93}$$

and

$$\mu_{\tau,1} = \eta. \tag{94}$$

With these choices, the expression for the expansion coefficients becomes

$$c_n = \frac{\nu}{\eta} \sum_{m=0}^{n} \frac{\left(-\frac{\nu}{\eta}\right)^m}{\Gamma(\nu+m)} \binom{n}{m} \mu_{\tau,m}. \tag{95}$$

4.2.1 Examples

As real clutter data was not available for this study, the application of the methodology developed herein will be illustrated with models for which closed form expressions for the moments are available. First consider clutter for which the amplitude is K distributed. In this case, the moments of τ are given by Eq. (85), and substitution in Eq. (95) yields

$$c_n = \frac{(\frac{\nu}{\eta})}{\Gamma(\nu)} \sum_{m=0}^{n} (-1)^m \binom{n}{m}. \tag{96}$$

The sum in Eq. (96) equals 1 for $n = 0$ and equals 0 for $n > 0$. Thus, based on knowledge of moments of the K distribution up through at least second order, the approach described herein yields

$$f_R(x) = \frac{2\nu/\eta}{\Gamma(\nu)} (\frac{\sqrt{2\nu/\eta}}{2})^{\nu-1} x^\nu K_{\nu-1}(\sqrt{2\nu/\eta}\, x), \tag{97}$$

which is the correct PDF for the K distribution.

Consider now a case in which the clutter is not exactly K distributed but instead is distributed according to the Weibull distribution. Specifically, for the PDF

$$g(x) = abx^{b-1}e^{-ax^b}, \quad a, b > 0, \tag{98}$$

the related moments of τ for use in Eq. (95) are

$$\mu_{\tau,n} = \frac{\Gamma(\frac{2n}{b} + 1)}{\Gamma(n + 1)} \frac{1}{(2a^{2/b})^n}. \tag{99}$$

Figures 1 through 3 compare the exact distributions with the distributions obtained from the expansion method described herein. In all cases, the series approximation converges to the actual distribution function with only a small number of terms.

As a second example, consider the case in which the true distribution of the clutter is a mixture of Weibull distributions of the form

$$G(x) = (1 - \rho)W(x; a_1, b_1) + \rho W(x; a_2, b_2), \tag{100}$$

where $W(x; a, b)$ represents a Weibull distribution with parameters a and b. For concreteness, let $a_1 = a_2 = 1$, $b_1 = 1$, $b_2 = 2$, and $\rho = 0.7$. Physically, this distribution corresponds to Rayleigh clutter ($b = 2$) that is corrupted by spikes ($b = 1$) 30% of the time. Figure 4 compares the exact mixture distribution with the distributions obtained by the expansion method.

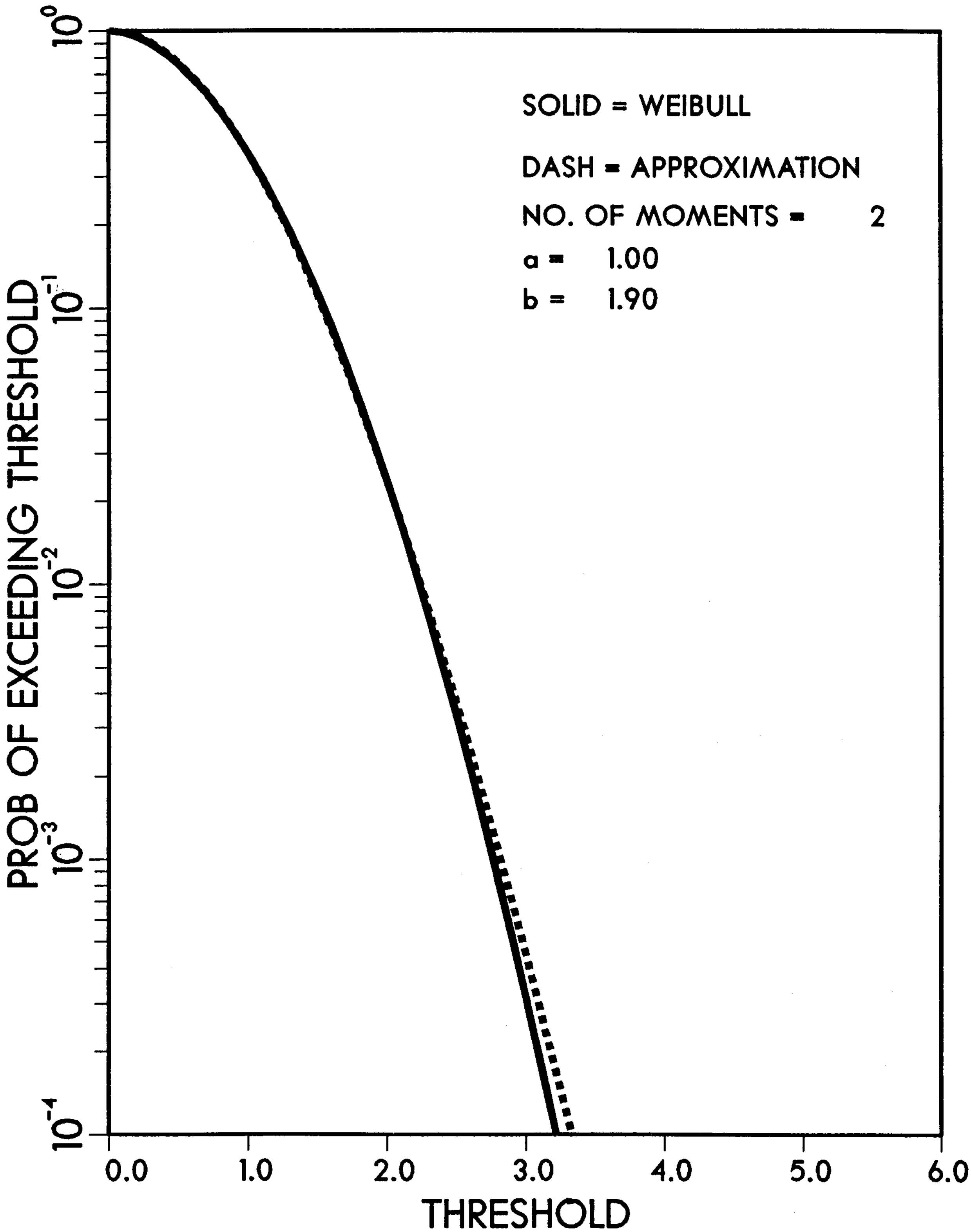

Fig. 1(a) — Approximation of Weibull ($a = 1$, $b = 1.9$) with 2 moments

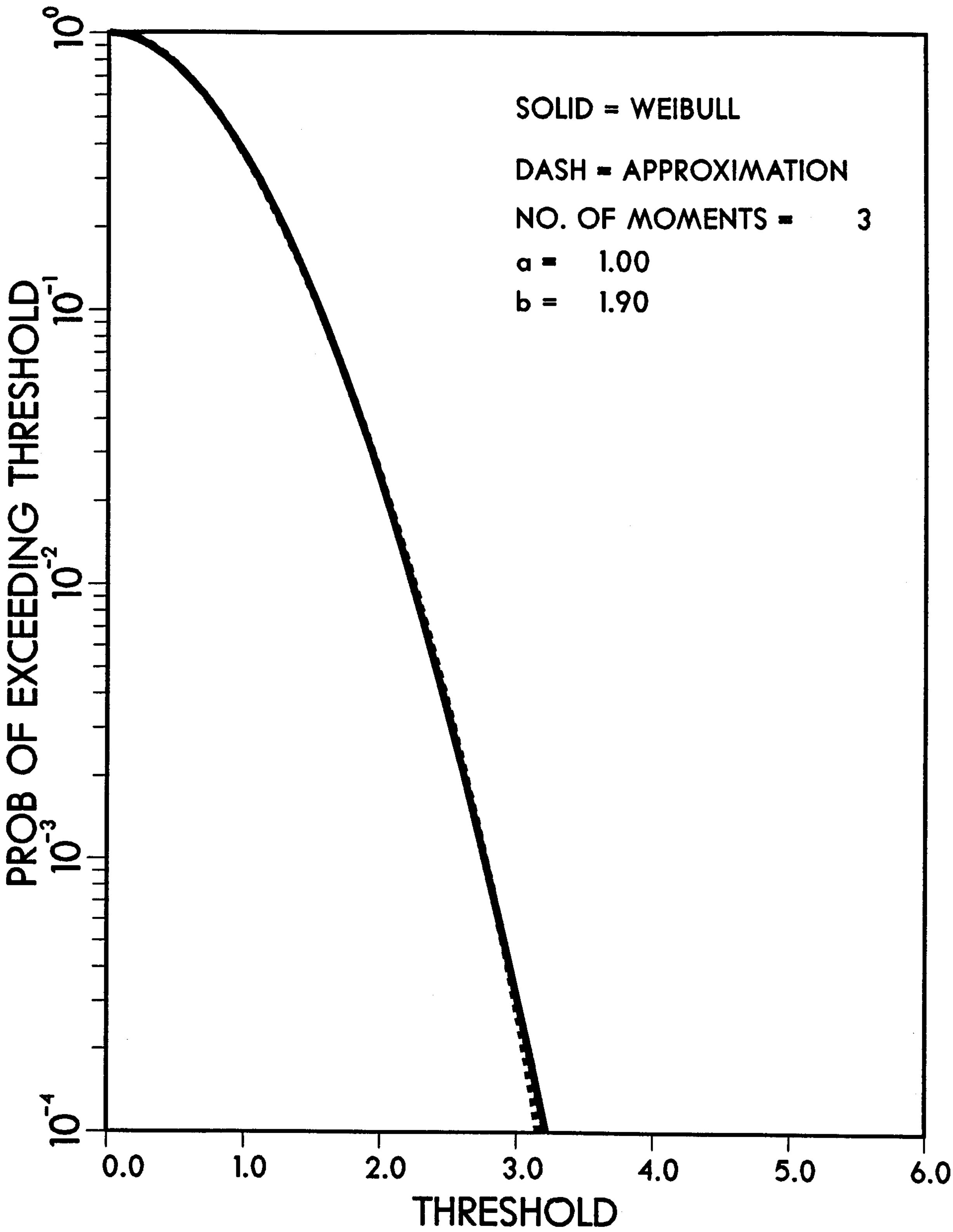

Fig. 1(b) — Approximation of Weibull (a = 1, b = 1.9) with 3 moments

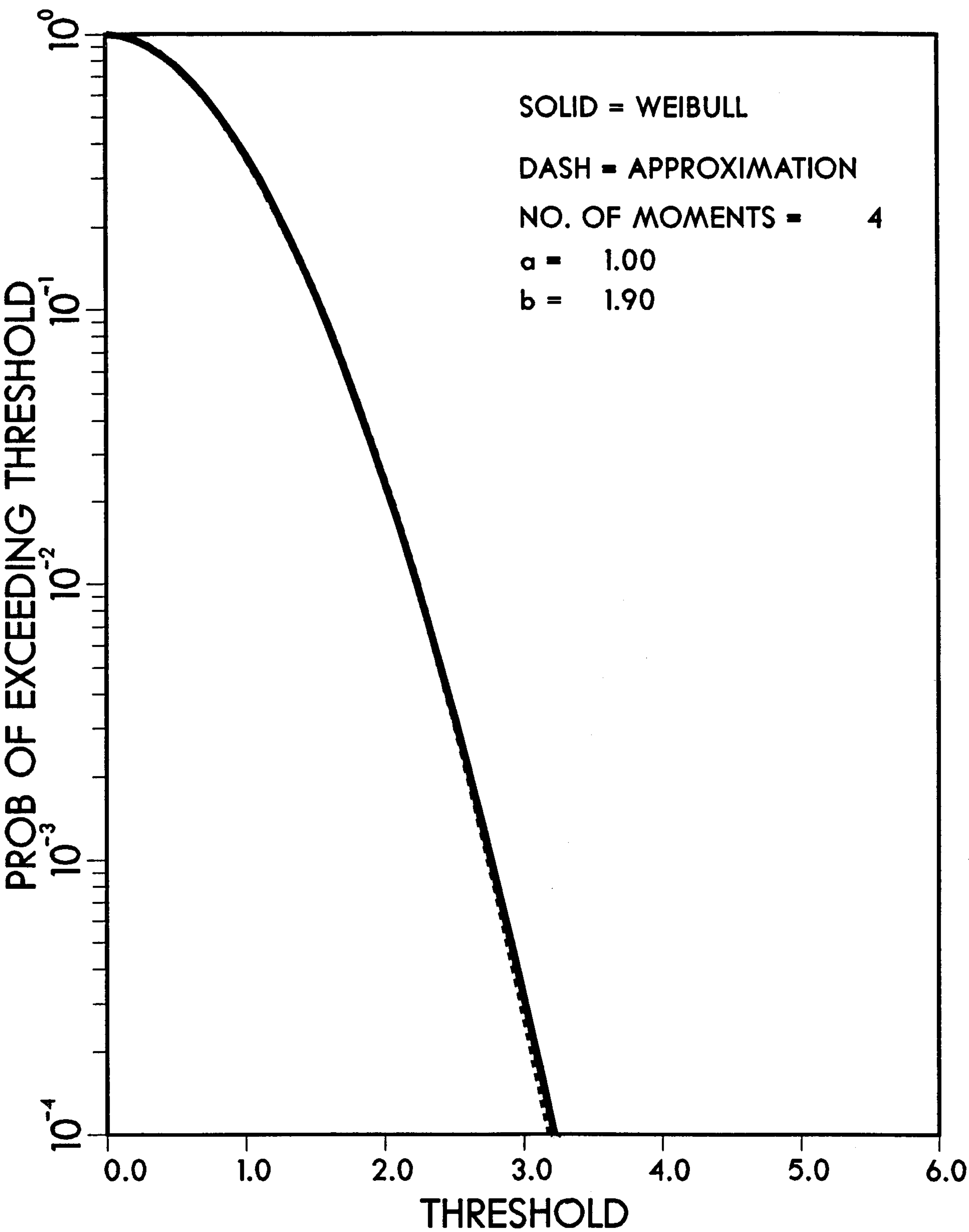

Fig. 1(c) — Approximation of Weibull (a = 1, b = 1.9) with 4 moments

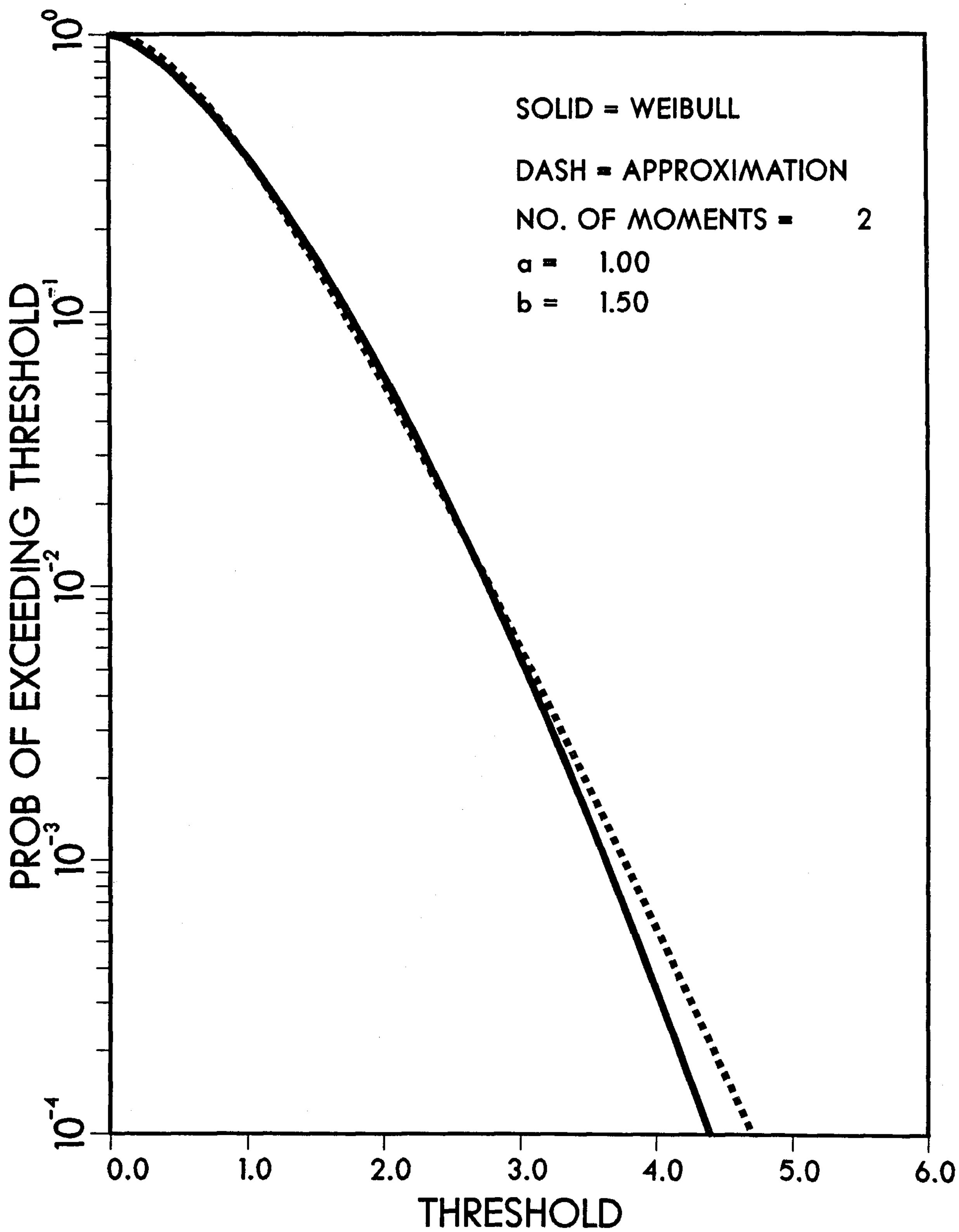

Fig. 2(a) — Approximation of Weibull ($a = 1$, $b = 1.5$) with 2 moments

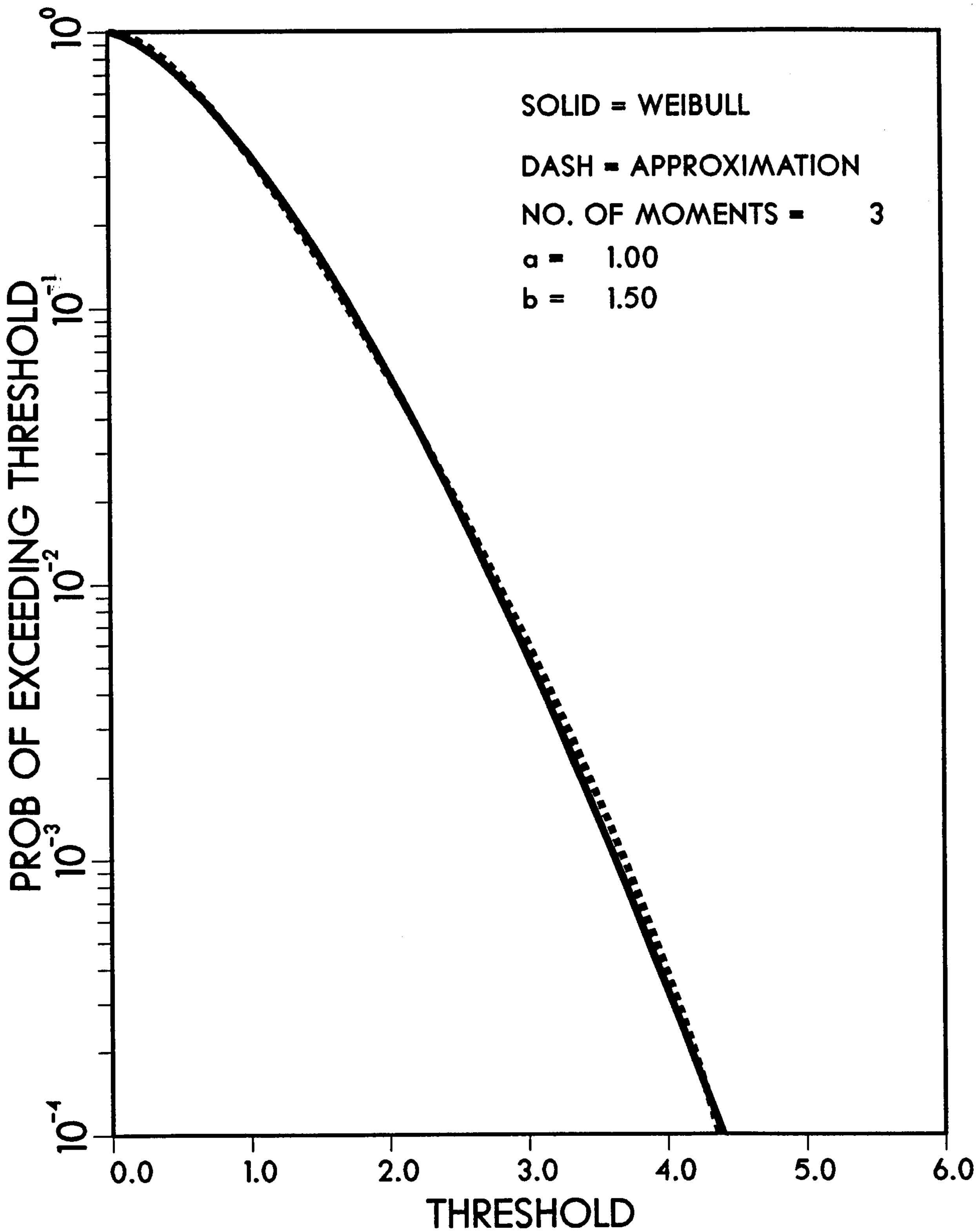

Fig. 2(b) — Approximation of Weibull ($a = 1$, $b = 1.5$) with 3 moments

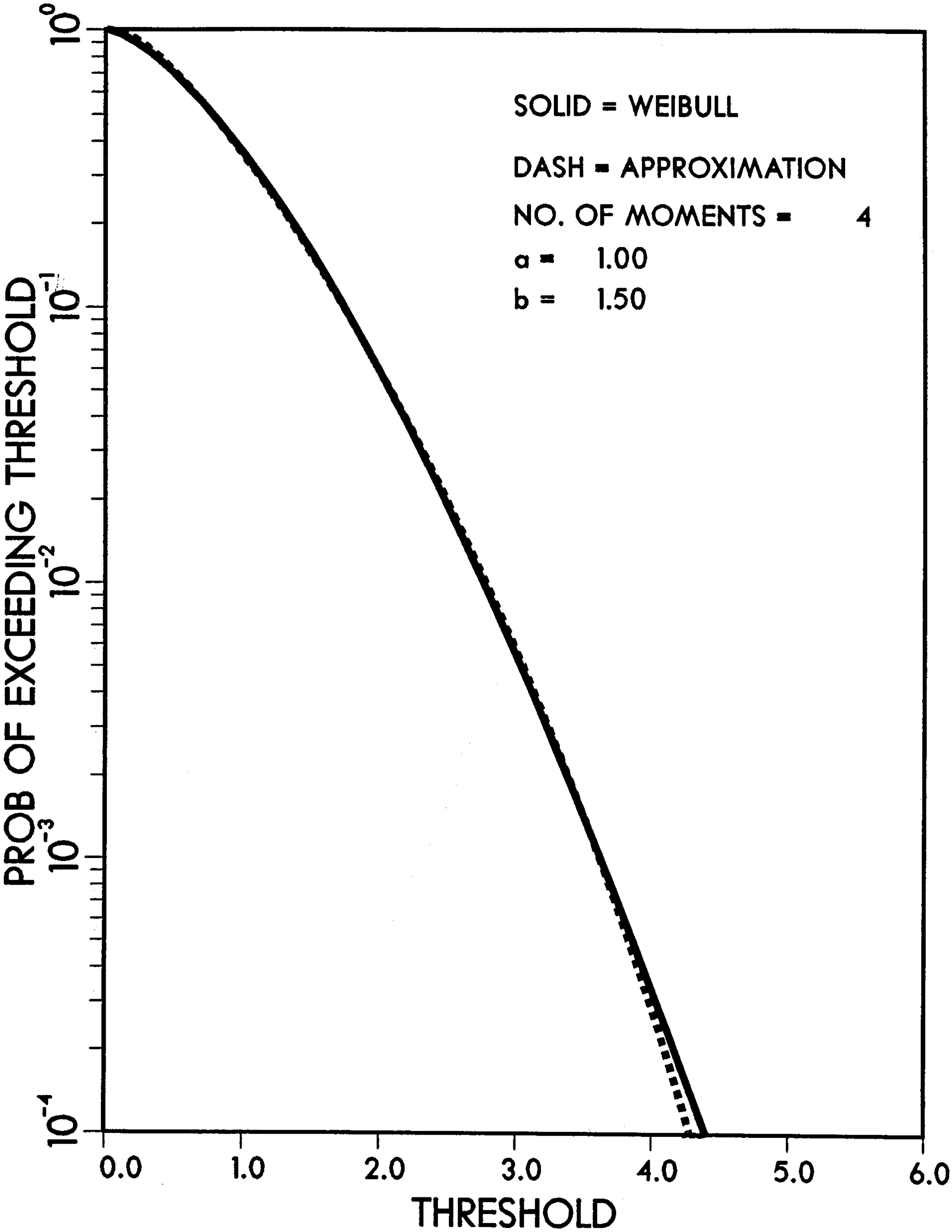

Fig. 2(c) — Approximation of Weibull ($a = 1$, $b = 1.5$) with 4 moments

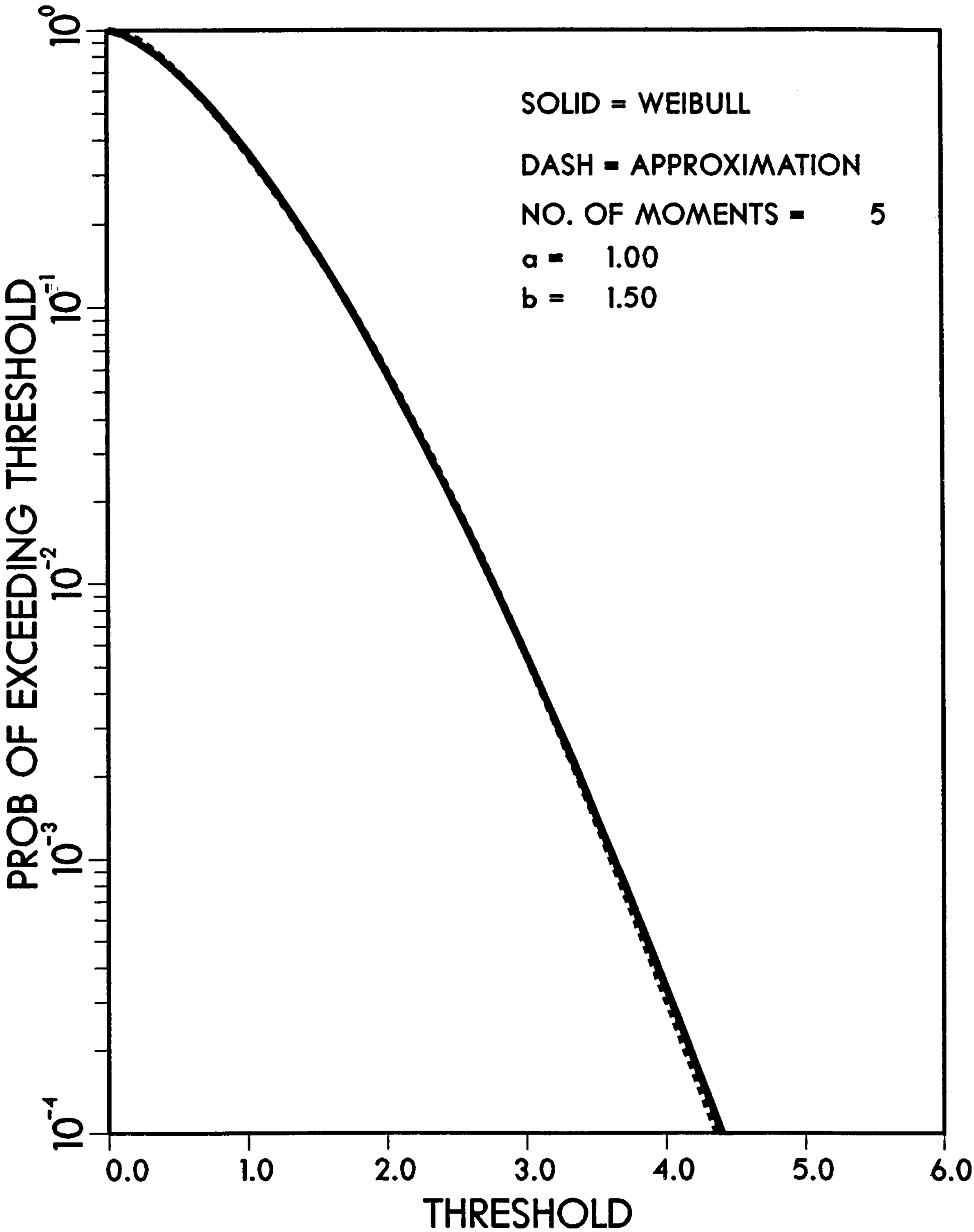

Fig. 2(d) — Approximation of Weibull ($a = 1$, $b = 1.5$) with 5 moments

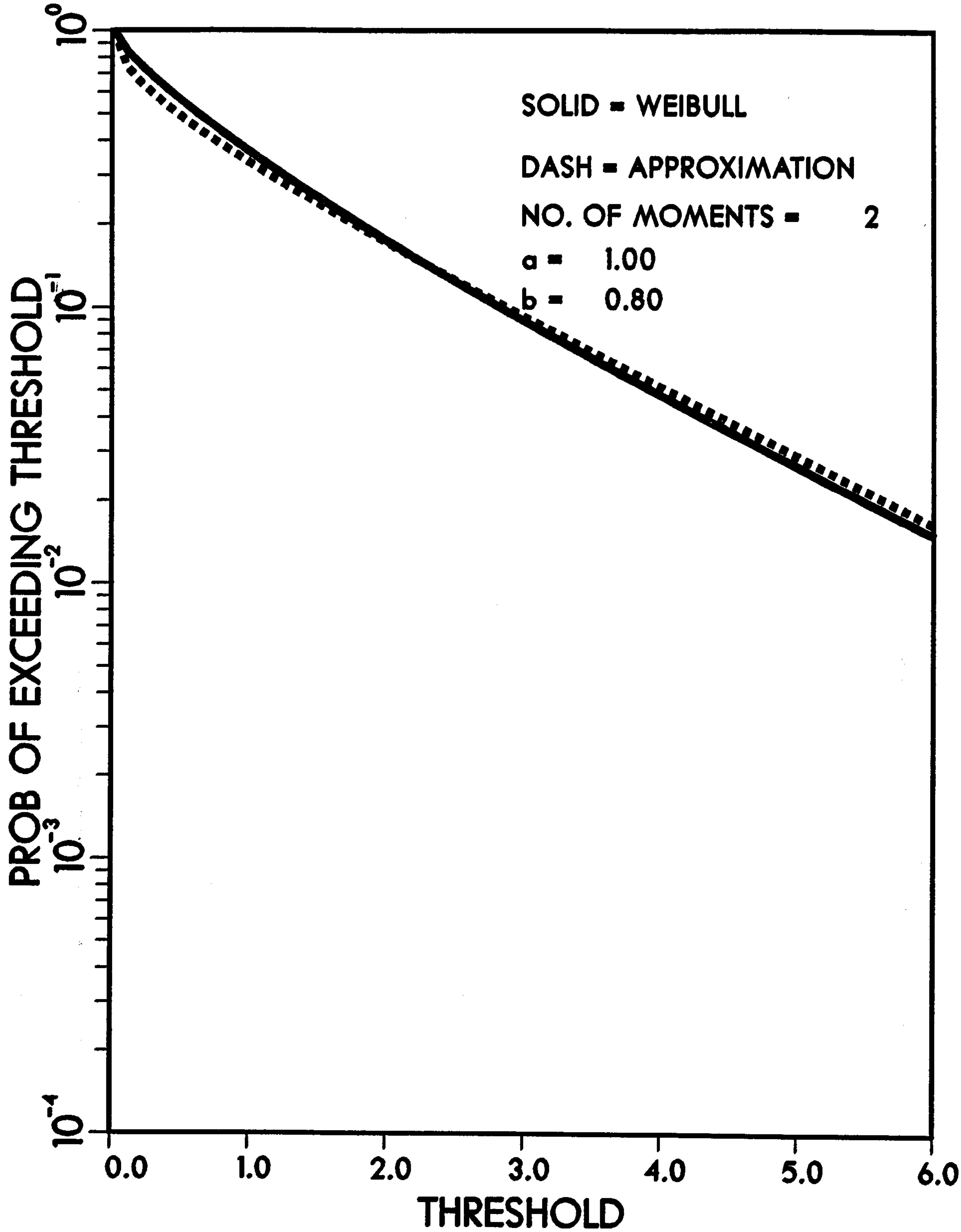

Fig. 3(a) — Approximation of Weibull ($a = 1$, $b = 0.8$) with 2 moments

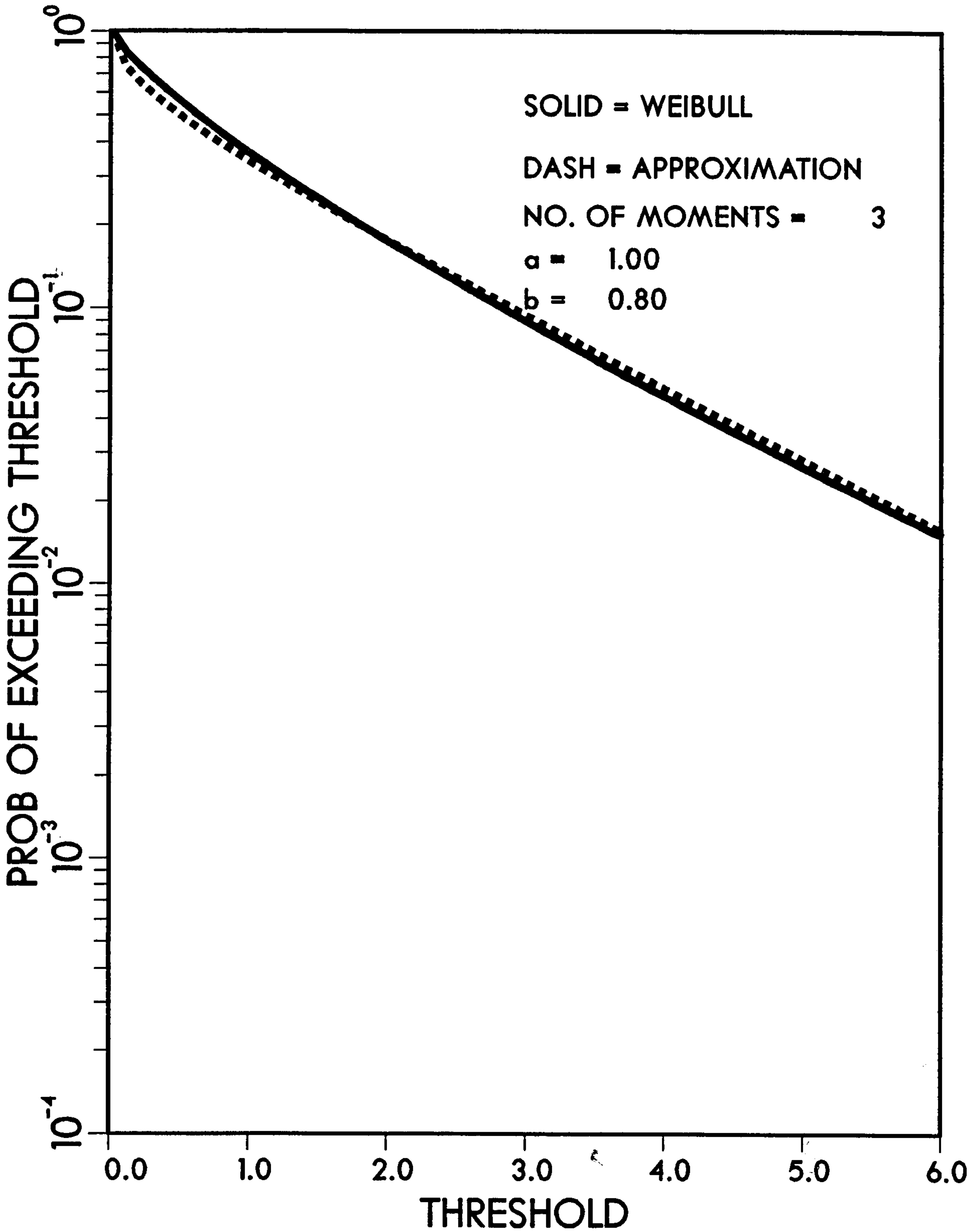

Fig. 3(b) — Approximation of Weibull (a = 1, b = 0.8) with 3 moments

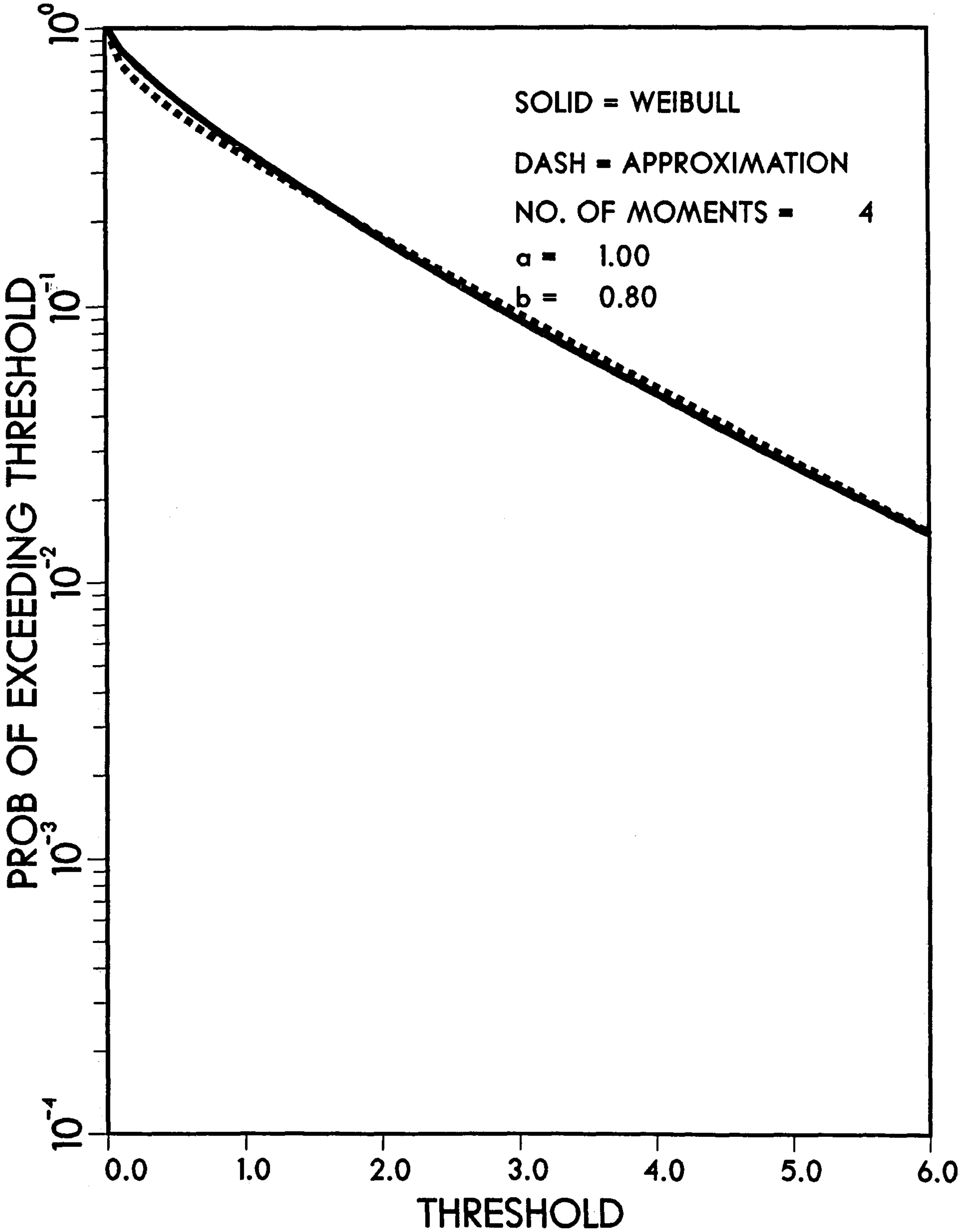

Fig. 3(c) — Approximation of Weibull ($a = 1$, $b = 0.8$) with 4 moments

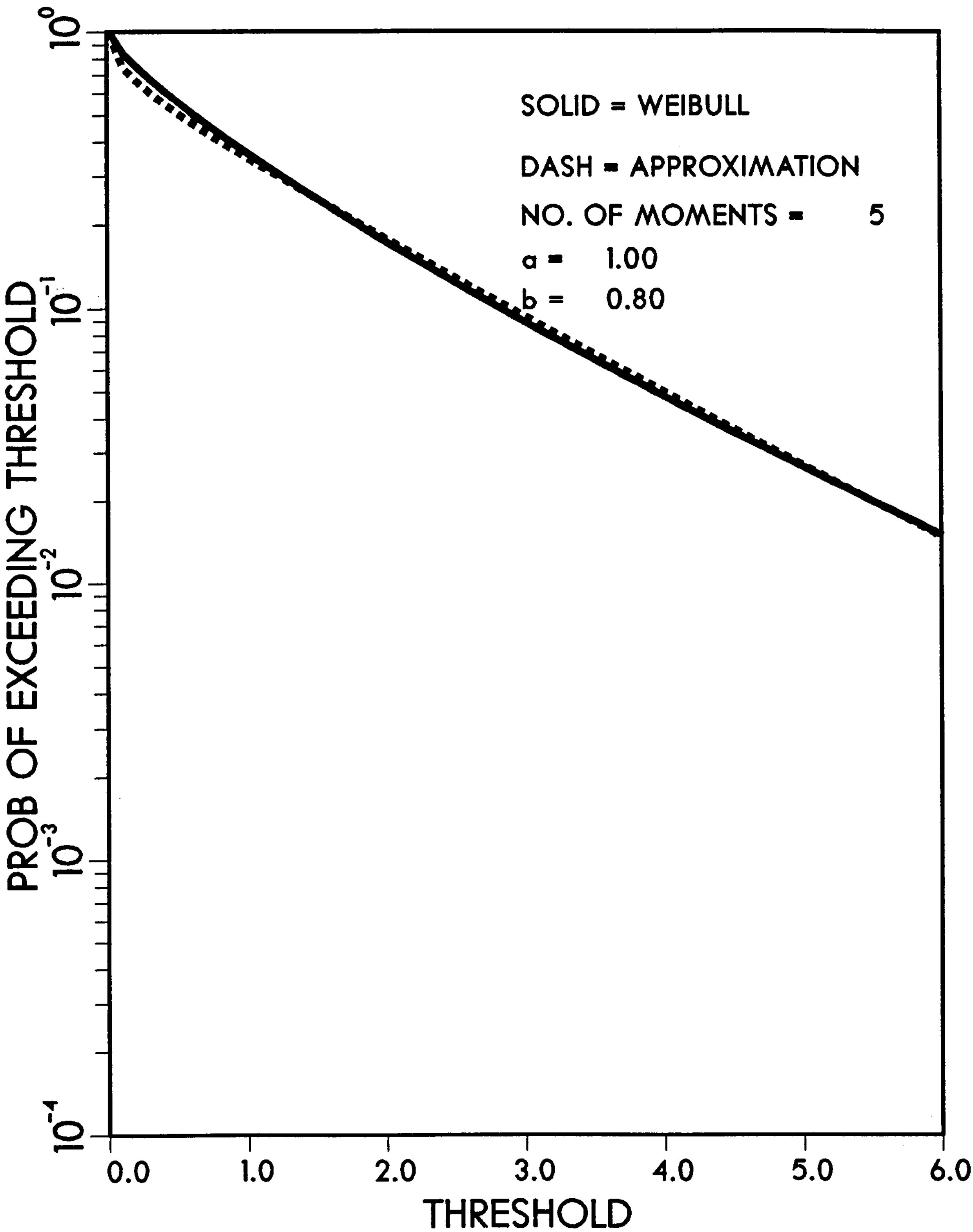

Fig. 3(d) — Approximation of Weibull ($a = 1$, $b = 0.8$) with 5 moments

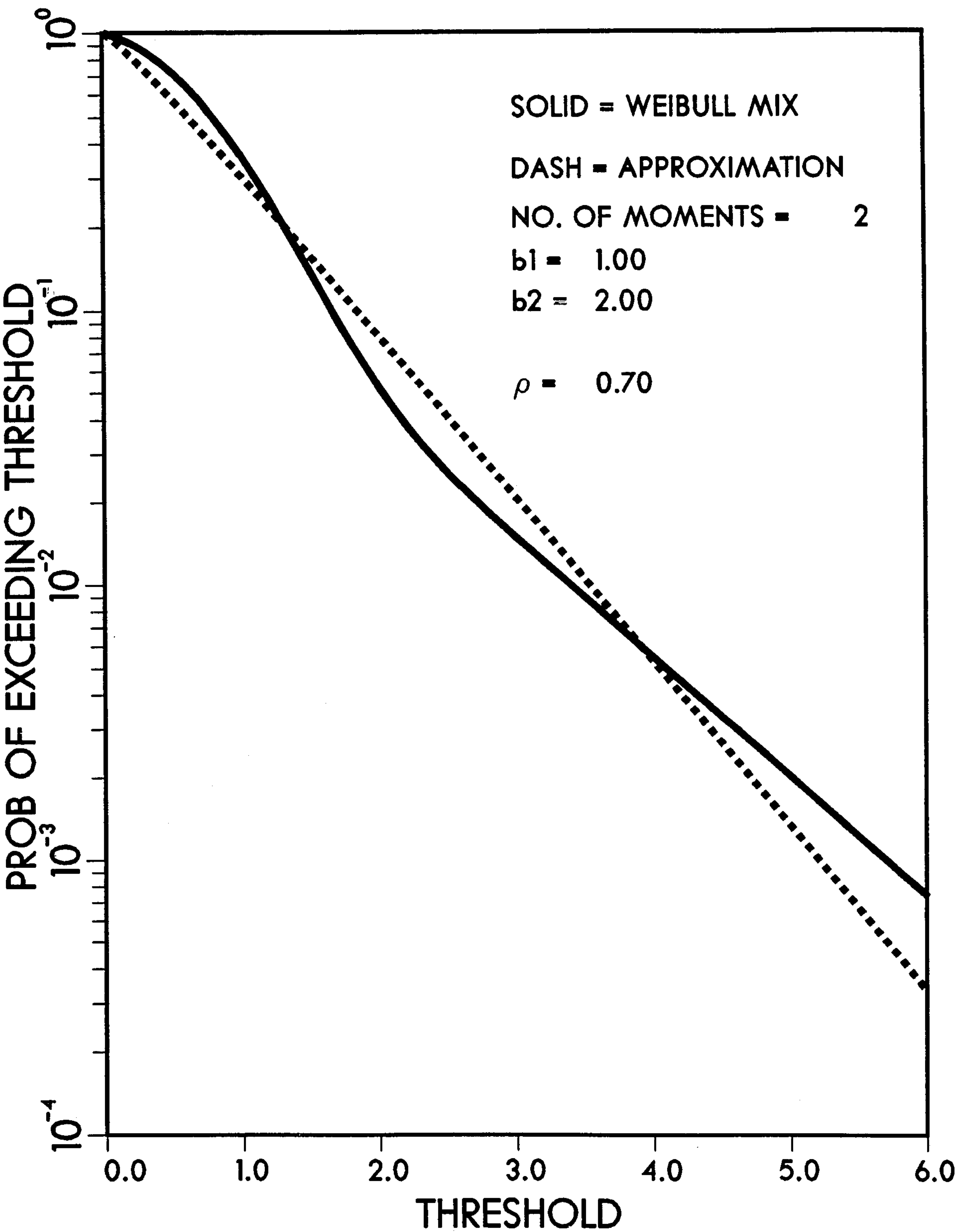

Fig. 4(a) — Approximation of Weibull mixture with 2 moments

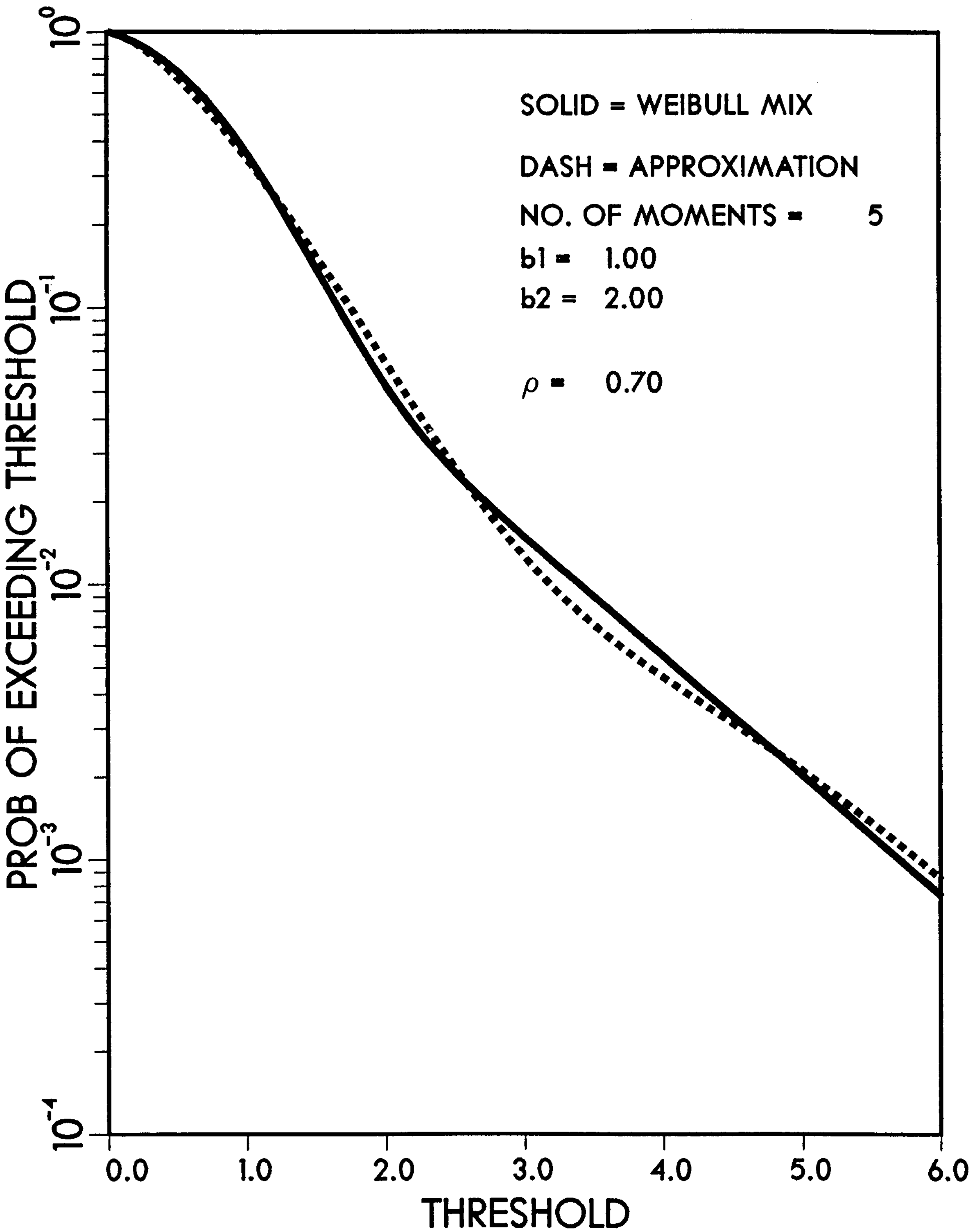

Fig. 4(b) — Approximation of Weibull mixture with 5 moments

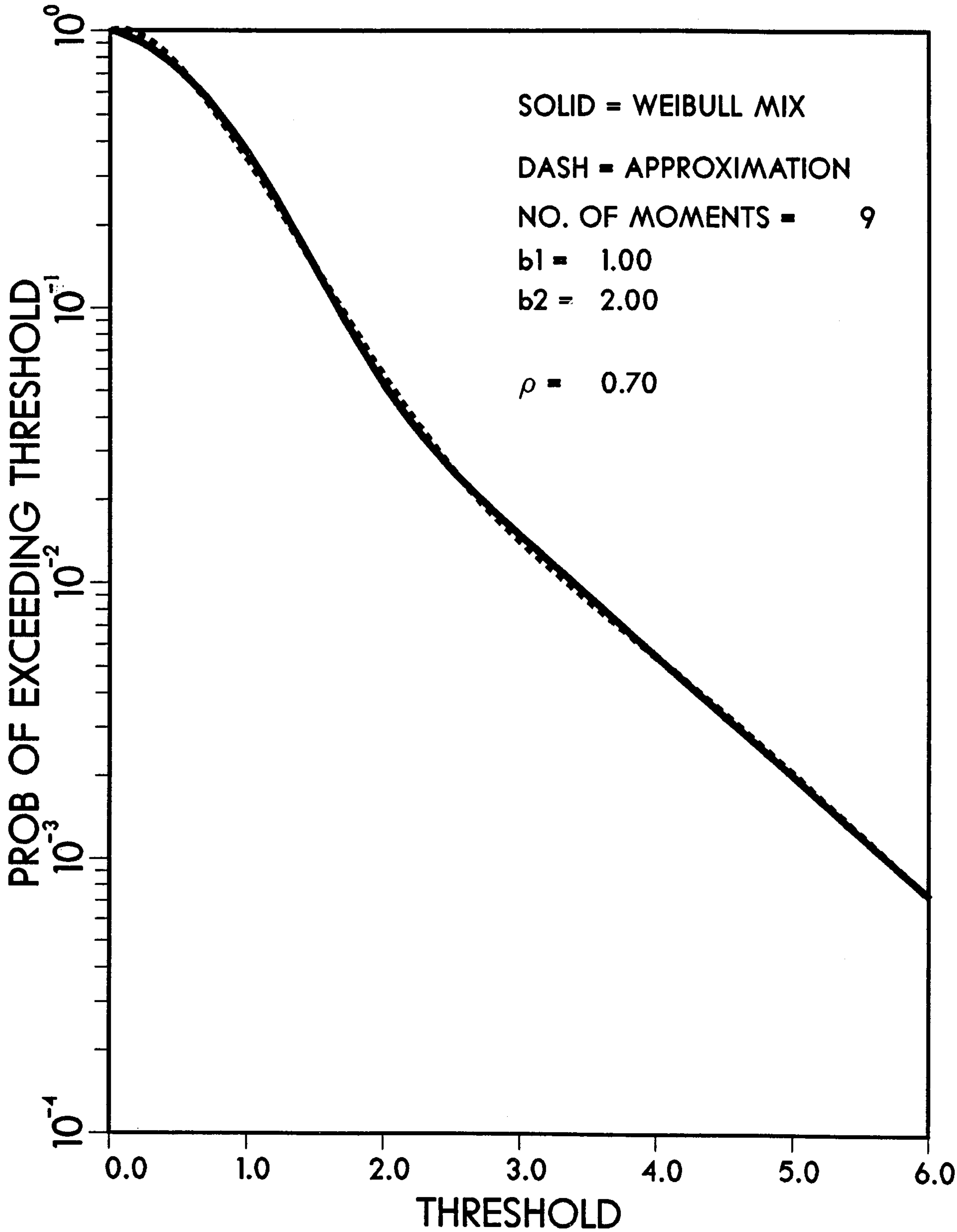

Fig. 4(c) — Approximation of Weibull mixture with 9 moments

5. MULTIDIMENSIONAL NON-GAUSSIAN CLUTTER MODELS

Conte and Longo (1987) present the idea of modelling the radar sea clutter process as a spherically invariant random process. This idea is attractive as it leads to a multidimensional model of the clutter, namely a multidimensional PDF, that includes correlation and need not be Gaussian. Moreover, they also show that most of the major models for non-Rayleigh amplitude statistics may be included in the framework of this model. As was discussed earlier in this report, the Weibull distribution, for which Conte and Longo only gave a partial result, is also consistent with this framework. This model, therefore, has merit as a multidimensional model of non-Gaussian clutter and deserves further investigation. In this section, application of the theorem presented in Section 2 is shown to lead to the same multidimensional PDF that is suggested by Conte and Longo. Since the theorem can be interpreted in a physical sense, this result gives a physical motivation for this approach. In addition, the shortcoming of the model in terms of the physical modelling of the problem emerges clearly from applying the theorem and suggests the next step to be taken to derive an even more realistic model.

To apply the phasor approach of Eq. (2) to the problem of modelling the scattered return from m pulses, consider the following phasor model:

$$E'_k(\bar{r},t) = \frac{e^{j\omega t}}{\sqrt{N_k}} \sum_{i=1}^{N_k} a_{ik}(\bar{r},\, t) e^{j\phi_{ik}(\bar{r},\, t)}, \quad k = 1, \ldots, m, \tag{101}$$

where E'_k is the scattered field for the kth of m pulses. The aspect of the problem that should be noted is that in general each pulse arises from a different instantaneous number of scatterers. To proceed further in the development of a statistical model for this situation, one would need a multidimensional model for the dynamics of the number fluctuations from pulse to pulse. Note, however, that if one assumes that the radar samples the environment much more rapidly than the timescale on which the number of scatterers changes, then each pulse return consists of the same number of scatterers, and the N_k's in Eq. (101) may be replaced by a single N. This situation may then be accommodated within the framework of the theorem as follows: Assemble the m elementary scatterer contributions from each of the N elementary scatterers into a random vector of length $2m$ (in which the polar representation of the phasor approach has been converted to an inphase and quadrature representation). The total return (i.e., from all m pulses) then will consist of the sum of these N random vectors, which is precisely the situation addressed in the theorem. If this vector of length $2m$ is represented as a complex random vector of length m, then the PDF of the random vector Y after application of the theorem may be written

$$f_Y(y) = \frac{1}{(\pi)^m |\Phi|} \int_0^\infty \frac{1}{\tau^m} e^{-\frac{y^H \Phi^{-1} y}{\tau}} \, dF_\tau(\tau), \tag{102}$$

where Φ represents the normalized correlation matrix that describes the pulse-to-pulse correlation between the pulses. (This matrix has been assumed to be invertible. For practical scenarios, this assumption is justified.) Note that in the application of the theorem, this quantity is assumed to be known. This assumption is consistent with the idea of measuring this quantity for application to practical problems.

Although the model described by Eq. (102) is the same model as described by Conte and Longo, the motivation for using this model here differs from the motivation of Conte and Longo, who sought to develop a model for the radar clutter *process*. They did seek to interpret the random process giving rise to such a multidimensional PDF physically in terms of a locally Gaussian process whose power level is modulated because of the radar's large-scale sampling of the environment. Their use of this model, nonetheless, was somewhat arbitrary. Herein, the problem was approached from the basic phenomenological picture of the scattering process with minimal assumptions about the statistics of the scatterers involved. The model then emerges as the end product of the limit theorem presented in Section 2. This approach complements the work of Conte and Longo by returning to basic physical principles to obtain the model; it is fully analogous to the classical approach of justifying the multivariate Gaussian model on the basis of the CLT. Unlike in the work of Conte and Longo, though, nothing is stated here about clutter as a random process. The result here is limited to the description of the statistics of a random vector of pulse returns. For this work to be extended to a description of the random process describing clutter returns, a dynamical model of the number fluctuations included in this modelling approach is needed. Because investigation of the role of number fluctuations in scattering problems would seem to be a new area of research, much more work in this area is needed before an accurate model of radar clutter as a random process can be formulated.

6. DETECTION OF TARGETS IN NON-GAUSSIAN NOISE

6.1 Structure of the Optimal Detector

In this section, the optimal detection structure for detecting a signal in additive noise modelled as above is studied. The resulting structure is a function of an optimal estimator of a random quantity. This structure reveals an intimate relationship between optimal detection and optimal estimation for this class of processes.

Consider the following hypothesis test:

$$H_0 : x = y$$

$$H_1 : x = y + s \tag{103}$$

where

x = observed m-dimensional complex data vector

y = m-dimensional complex noise vector

s = known m-dimensional complex signal vector

$= \beta\hat{s},\ \beta = \gamma e^{j\phi}$ = complex number.

If the noise is modelled as indicated above, then x has the following PDF under hypothesis $i\,(i = 0, 1)$:

$$f_i(x) = \frac{1}{(\pi)^m\,|\Phi|}\int_0^\infty \frac{1}{\tau^m} e^{-\frac{q_i}{\tau}}\, dF_\tau(\tau), \tag{104}$$

where

$$q_i = (x - s)^H \Phi^{-1}(x - s),\ i = 1$$

$$= x^H \Phi^{-1} x,\ i = 0$$

Φ = normalized correlation matrix (assumed to be invertible)

τ = variance of underlying Gaussian vector

$F_\tau(\tau)$ = CDF of τ

$|\ |$ = matrix determinant

H = complex conjugate transpose.

Equation (104) may be interpreted as the PDF of a conditional Gaussian random vector with variance τ averaged over the variation in τ. Then

$$\Phi = \frac{E[yy^H]}{E[\tau]} \tag{105}$$

where E denotes expected value.

The Neyman-Pearson likelihood ratio (LR) statistic is given by

$$\lambda(x) = \frac{f_1(x)}{f_0(x)}. \tag{106}$$

Define a function F_m as

$$F_m(\beta) = \int_0^\infty \frac{1}{\tau^m} e^{-\frac{\beta}{\tau}}\, dF_\tau(\tau). \tag{107}$$

Then the LR statistic is given by

$$\lambda(x) = \frac{F_m((x - s)^H \Phi^{-1}(x - s))}{F_m(x^H \Phi^{-1} x)}. \tag{108}$$

This structure forms the starting point for the analysis of detection performance given later in this report. First, however, an interpretation of the LR statistic as a so-called estimator-correlator (Schwartz 1975) is briefly reviewed. This interpretation is interesting as it reveals the LR statistic presented above to be equivalent to a matched filter compared to a data-dependent threshold.

6.1.1 Optimal Detector as an Estimator-Correlator

This section presents a brief review of a result previously given by Sangston and Gerlach (1989). It is known that the conditional Gaussian PDF is a multivariate member of the exponential family, i.e., the PDF is of the form

$$f_i(x \mid \tau) = c(\tau)\, h(x) \exp\left\{ \sum_{j=1}^{l} \eta_j(\tau) T_j(x) \right\}, \qquad (109))$$

where in this case

$$c(\tau) = \frac{1}{(\pi)^m \, |\Phi| \tau^m}$$

$$h(x) = 1$$

$$l = 1$$

$$\eta_1(\tau) = \frac{1}{\tau}$$

$$T_1(x) = \text{sufficient statistic for } \frac{1}{\tau} = -q_i .$$

Reparameterize the conditional Gaussian PDF by setting $\alpha = 1/\tau$ to obtain

$$f_i(x) = \frac{1}{(\pi)^m \, |\Phi|} \int_0^\infty \alpha^m e^{-\alpha q_i} \, dF_\alpha(\alpha). \tag{110}$$

Define a function $g(z)$ for $z \geq 0$ by

$$g(z) = \frac{1}{(\pi)^m |\Phi|} \int_0^\infty \alpha^m e^{-\alpha z} \, dF_\alpha(\alpha). \tag{111}$$

From Eq. (111) may be obtained

$$\frac{d \ln g(z)}{dz} = \frac{-1}{(\pi)^m |\Phi| g(z)} \int_0^\infty \alpha^{m+1} e^{-\alpha z} \, dF_\alpha(\alpha). \tag{112}$$

Since the minimum mean-square estimate of α under hypothesis i is given by the conditional mean estimate (CME) (Poor 1988),

$$E_i[\alpha | x] = \frac{1}{(\pi)^m |\Phi| f_i(x)} \int_0^\infty \alpha^{m+1} e^{-\alpha q_i} \, dF_\alpha(\alpha) \tag{113}$$

where q_i is as defined above, Eq. (112) combined with Eq. (113) yields

$$\begin{aligned} E_i[\alpha | x] &= -\frac{1}{g(q_i)} \frac{dg(q_i)}{dq_i} \\ &= -\frac{d \ln g(q_i)}{dq_i}. \end{aligned} \tag{114}$$

Since

$$\ln f_i(x) = \ln g(q_i) = \int_0^{q_i} \frac{d \ln g(z)}{dz} \, dz + \ln g(0), \tag{115}$$

then, with the obvious notation suggested by Eq. (115), the structure of the likelihood ratio is given by

$$\lambda(x) = \frac{f_1(x)}{f_0(x)} = \exp\left\{ \int_0^{q_0} E[\alpha | z] \, dz - \int_0^{q_1} E[\alpha | z] \, dz \right\}, \tag{116}$$

where $E[\alpha | z] = -d \ln g(z)/dz$. Equation (116) shows explicitly the relationship between optimal detection and optimal estimation for this type of noise.

Define a function $F(q)$ as

$$F(q) = \int_0^q E[\alpha | z]\, dz,\ q > 0. \tag{117}$$

Because $E[\alpha | z]$ is nonnegative for positive z and q is positive, $F(q)$ is a monotonically increasing function of q. In this new notation, the log-likelihood ratio may be rewritten as

$$F(q_0) - F(q_1) \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{118}$$

or equivalently,

$$F(q_0) - T \underset{H_0}{\overset{H_1}{\gtrless}} F(q_1). \tag{119}$$

Now, since $F(q)$ is monotonically increasing,

$$F^{-1}(F(q_0) - T) \underset{H_0}{\overset{H_1}{\gtrless}} q_1. \tag{120}$$

Finally, since

$$\begin{aligned} q_1 &= x^H \Phi^{-1} x - 2\mathrm{Re}(s^H \Phi^{-1} x) + s^H \Phi^{-1} s \\ &= q_0 - 2\mathrm{Re}(s^H \Phi^{-1} x) + s^H \Phi^{-1} s, \end{aligned} \tag{121}$$

substitution in Eq. (120) yields

$$2\mathrm{Re}(s^H \Phi^{-1} x) \underset{H_0}{\overset{H_1}{\gtrless}} q_0 - F^{-1}(F(q_0) - T) + s^H \Phi^{-1} s. \tag{122}$$

The optimal test is seen to be equivalent to a matched filter compared to a data-dependent threshold.

6.2 Generalized Likelihood Ratio Detector

Normally for the radar detection problem, the complex signal amplitude of the desired signal vector is unknown to the detector. If unknown parameters are present in the LR detector, a scheme known as the generalized likelihood ratio test (GLRT) is commonly used whereby the unknown parameters are replaced by maximum likelihood (ML) estimates under each hypothesis (Van Trees 1968). In this section, the GLRT for detection of a signal with unknown complex amplitude in additive noise modelled as described earlier is derived.

Set $s = as_0$, where a is the unknown complex amplitude of the desired signal s and s_0 is known and has magnitude one, i.e., $||s_0|| = s_0^H s_0 = 1$. The GLRT for this problem is thus given by

$$\lambda(x) = \frac{F_m((x - \hat{a}s_0)^H \Phi^{-1}(x - \hat{a}s_0))}{F_m(x^H \Phi^{-1} x)} \tag{123}$$

where

$$\hat{a} = arg \max_a F_m((x - as_0)^H \Phi^{-1}(x - as_0)) \tag{124}$$

is the maximum likelihood estimate of the complex amplitude a. As usual, the maximum of F_m with respect to a can be found by setting $dF_m/da = 0$ and solving for a. With $\beta = (x - as_0)^H \Phi^{-1}(x - as_0)$, this derivative may be written as

$$\frac{dF_m}{da} = \frac{dF_m}{d\beta}\frac{d\beta}{da}. \tag{125}$$

From Eq. (107), one may show

$$\frac{dF_m}{d\beta}(\beta) = -\int_0^\infty \frac{1}{\tau^{m+1}} e^{-\frac{\beta}{\tau}} \, dF_\tau(\tau). \tag{126}$$

Because $F_\tau(\tau)$ is nondecreasing it follows easily that $\frac{dF_m}{d\beta} < 0$. Thus, the solutions of interest here require $\frac{d\beta}{da} = 0$. Only one such solution exists and is given by

$$\hat{a}(x) = \frac{s_0^H \Phi^{-1} x}{s_0 \Phi^{-1} s_0}. \tag{127}$$

An interesting aspect of this particular result is that it does not depend on the nature of the CDF $F_\tau(\tau)$. Substitution of this result into the GLRT in place of a yields

$$\lambda(x) = \frac{F_m(x^H \Phi^{-1} x (1 - |\rho|^2))}{F_m(x^H \Phi^{-1} x)}, \tag{128}$$

where | | denotes magnitude, and

$$|\rho|^2 = \frac{|s_0^H \Phi^{-1} x|^2}{(x^H \Phi^{-1} x)(s_0^H \Phi^{-1} s_0)}. \tag{129}$$

It follows from the Schwartz Inequality that $0 \leq |\rho| \leq 1$.

If this estimate of the unknown complex signal amplitude is substituted into the matched filter interpretation of the LR test, the result is easily shown to be

$$\frac{|s_0^H \Phi^{-1} x|^2}{s_0^H \Phi^{-1} s_0} \underset{H_0}{\overset{H_1}{\gtrless}} q_0 - F^{-1}(F(q_0) - T). \tag{130}$$

This structure is again seen to be a relatively simple structure compared to a data-dependent threshold, where this same simple structure is compared to a fixed threshold if the noise is Gaussian. In this interpretation, the effect of deviating from Gaussian noise is to induce a variability in the setting of the threshold.

6.3 Equivalent GLRT

If Φ is nonsingular, then there exists a nonsingular matrix A such that the effect of processing the input vector x by A is to whiten the input noise vector, normalize the input noise power of each element in the input noise vector to one and

$$A s_0 = ((s_0 \Phi^{-1} s_0)^{1/2}, 0, \ldots, 0)^T = \tilde{s}_0 \tag{131}$$

(Horn and Johnson 1985). Note that all of the desired signal energy has been placed into the first element by the matrix transform. Thus if $u = Ax$, then under H_0

$$E[uu^H] = I_m \tag{132}$$

where I_m is the $m \ x \ m$ identity matrix. Transforming the input vector by the nonsingular matrix A does not change the statistical properties of the GLRT of the preceding section. Under this transformation, the GLRT becomes

$$\tilde{\lambda}(u) = \frac{F_m(u^H u (1 - |\tilde{\rho}|^2)}{F_m(u^H u)} \tag{133}$$

where

$$|\tilde{\rho}|^2 = \frac{|\tilde{s}_0^H u|^2}{(u^H u)(\tilde{s}_0^H \tilde{s}_0)} = \frac{|u_1|^2}{\sum_{n=1}^{m} |u_n|^2}. \tag{134}$$

The GLRT finally becomes

$$\tilde{\lambda}(u) = \frac{F_m(\sum_{n=2}^{m} |u_n|^2)}{F_m(\sum_{n=1}^{m} |u_n|^2)}. \tag{135}$$

Figure 5 shows a functional block diagram of the equivalent GLRT. In this figure, T is the detector threshold.

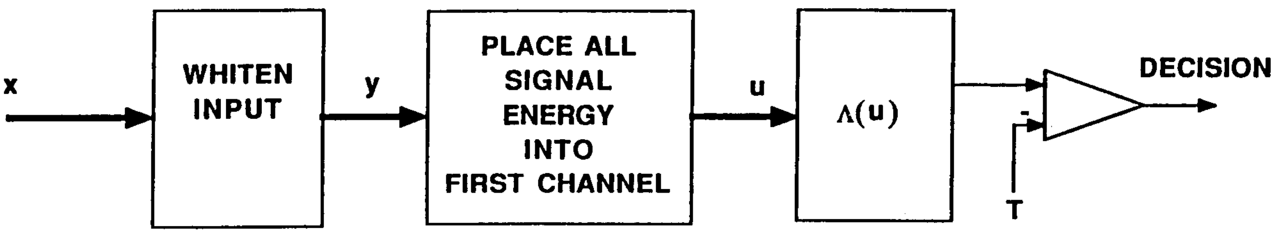

Fig. 5 — Equivalent Detector

6.4 Performance Assessment of Detector

This section presents closed-form solutions for the probabilities of detection P_D and false alarm P_F associated with the GLRT presented earlier. The desired signal amplitude is assumed to fluctuate statistically as a complex circular Gaussian process with input power equal to σ_s^2 (Helstrom 1968). As a result, the desired signal magnitude is Rayleigh distributed with variance σ_s^2. Under the H_1 hypothesis, the signal after the A matrix transformation is contained entirely in the first element or channel and is also a complex circular Gaussian process with power equal to $\tilde{\sigma}_s^2 = s_0^H \Phi^{-1} s_0 \, \sigma_s^2$. Note that at this point in the processing (after the A matrix transformation), the output of the first channel is the matched filter output. The signal-to-noise (S/N) ratio associated with the matched filter, denoted $(S/N)_{opt}$, can be expressed as

$$(S/N)_{opt} = \sigma_s^2 \, s_0^H \Phi^{-1} s_0. \tag{136}$$

We now analyze the statistically equivalent GLRT presented earlier. Let

$$w = |u_1|^2 \tag{137}$$

and

$$v = \sum_{n=2}^{m} |u_n|^2. \tag{138}$$

Note that the random variables $|u_1|^2$, $|u_2|^2$, . . . , $|u_m|^2$ when conditioned on τ are statistically independent. Thus, w and v when conditioned on τ are statistically independent. Furthermore, under H_0 or H_1, u_n ($n = 2, \ldots, m$) are i.i.d. circular Gaussian random variables with variance equal to τ. It is straightforward to show that v when conditioned on τ under H_0 and H_1 has a $2(N-1)$ chi-square PDF (Papoulis 1965) with $\sigma^2 = 0.5$ and is given by

$$f_{v|\tau}(v|\tau) = \frac{1}{\tau^{N-1}(N-2)!} v^{N-2} e^{-\frac{v}{\tau}} \tag{139}$$

where $f(\ |\tau)$ denotes that the PDF is conditioned on τ.

Under H_0, u_1 is a complex circular Gaussian random variable with variance equal to τ, and under H_1, it is the same except the variance equals $\tilde{\sigma}_s^{\,2} + \tau$. Thus, w has an exponential PDF with

$$f_{w|\tau}(w|\tau) = \begin{cases} \dfrac{1}{\tau} e^{-\frac{w}{\tau}} & H_0 \\[2ex] \dfrac{1}{\tilde{\sigma}_s^{\,2} + \tau} e^{-\frac{w}{\tilde{\sigma}_s^{\,2}+\tau}} & H_1. \end{cases} \tag{140}$$

The decision rule in terms of w and v is

$$\tilde{\lambda} = \frac{F_m(v)}{F_m(w+v)} \underset{H_0}{\overset{H_1}{\gtrless}} T. \tag{141}$$

It was shown earlier that F_m is strictly monotonically decreasing. As a result, $T \geq 1$. Since F_m is also a one-to-one function, an equivalent decision rule is

$$w \underset{H_0}{\overset{H_1}{\gtrless}} F_m^{-1}\left[\frac{F_m(v)}{T}\right] - v = T_0, \tag{142}$$

where F_m^{-1} is the inverse function of F_m. Define $P(D\,|v, \tau)$ to be the probability of detection when conditioned on v and τ. It is well-known (DiFranco and Rubin 1980) that the conditional probability of detection of a random variable with exponential PDF and variance $\sigma_s^2 + \tau$ is

$$P(D\,|v, \tau) = e^{-\frac{T_0}{\sigma_s^2 + \tau}}. \tag{143}$$

From Eqs. (136), (142), and (143), it follows that

$$P_D = \int_0^{\infty}\int_0^{\infty} f_{v|\tau}(v\,|\tau) f_\tau(\tau) \exp\left[-\frac{F_m^{-1}(F_m(v)\,/\,T) - v}{(S/N)_{opt} + \tau}\right] dv d\tau, \tag{144}$$

where $f_\tau(\tau)$ is specified and $f_{v|\tau}(v\,|\tau)$ is given by Eq. (139).

The P_F is found by setting $(S/N)_{opt} = 0$ in Eq. (144):

$$P_F = \int_0^{\infty}\int_0^{\infty} f_{v|\tau}(v\,|\tau) f_\tau(\tau) \exp\left[-\frac{F_m^{-1}(F_m(v)\,/\,T) - v}{\tau}\right] dv d\tau. \tag{145}$$

6.4.1 Results

In this section, we present some performance curves of P_F and P_D for a specific Rayleigh mixture model. The underlying mixing distribution is chosen to be the gamma distribution with PDF given by

$$f_\tau(\tau) = \frac{(\nu/\eta)^\nu}{\Gamma(\nu)} \tau^{\nu-1} e^{-\frac{\nu}{\eta}\tau}, \tag{146}$$

where Γ is the gamma function, η determines the mean of the distribution, and ν controls the deviation from Rayleigh statistics. Figure 6 shows curves that describe the behavior of the gamma PDF for different values of ν. As may be seen from the figure, the statistics of the Rayleigh mixture approach pure Rayleigh statistics as ν approaches ∞. Without loss of generality, η is set equal to 1, which merely normalizes the elemental noise power to a specific value. With this mixing PDF, the statistics of the univariate magnitude of x under the H_0 hypothesis, i.e., the Rayleigh mixture statistics, are described by the K distribution

$$f_{|x|}(|x|) = \frac{2\nu|x|}{\Gamma(\nu)}(|x|\sqrt{\nu/2})^{\nu-1} K_{\nu-1}(\sqrt{2\nu}\,|x|) \tag{147}$$

where K_ν is a modified Bessel function of the second kind.

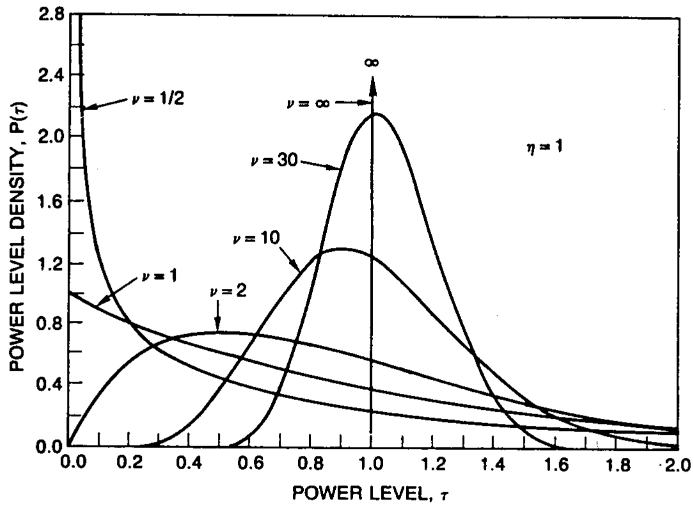

Fig. 6 — The gamma PDF

The P_F results are a function of m, ν, and T; the P_D results are a function of m, ν, $(S/N)_{opt}$, and P_F. In the latter case, m, ν, and P_F are specified, and a unique threshold T is determined from Eq. (145). This threshold is then used in Eq. (144) in conjunction with m, ν, and $(S/N)_{opt}$ to determine P_D.

For $\nu = \infty$, this corresponds to $f_\tau(\tau) = \delta(\tau - 1)$ where δ is the Dirac delta function. For this case, the complex input random variables are unconditionally Gaussian, and it is known (DiFranco and Rubin 1980) that

$$P_F = e^{-T} \tag{148}$$

and

$$P_D = e^{-\frac{T}{((S/N)_{opt} + 1)}}$$

$$= P_F^{-\frac{1}{((S/N)_{opt} + 1)}} . \tag{149}$$

Figures 7 and 8 present curves of P_F vs $\ln T$ for $\nu = 1, 2, 10, 30, \infty$, and $m = 2$ and 5, respectively. As expected, P_F is monotonically decreasing in T but the threshold must increase to maintain a constant P_F for an increase in the variance of τ (or equivalently, a decrease in ν.) Also comparison of Figs. 7 and 8 show that for constant P_F and ν, the threshold decreases for increasing m.

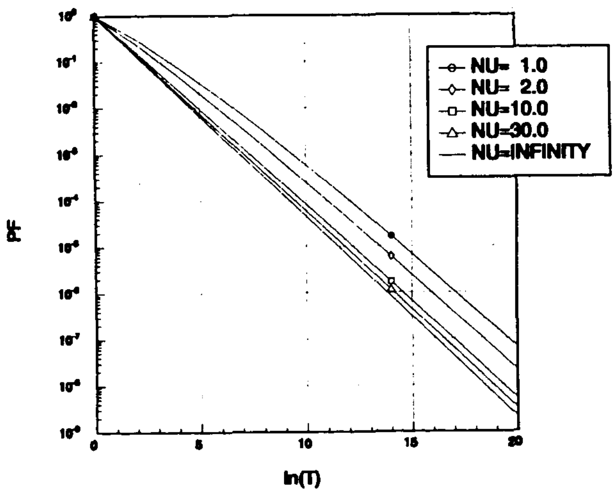

Fig. 7 — P_F for Rayleigh mixture, $m = 2$

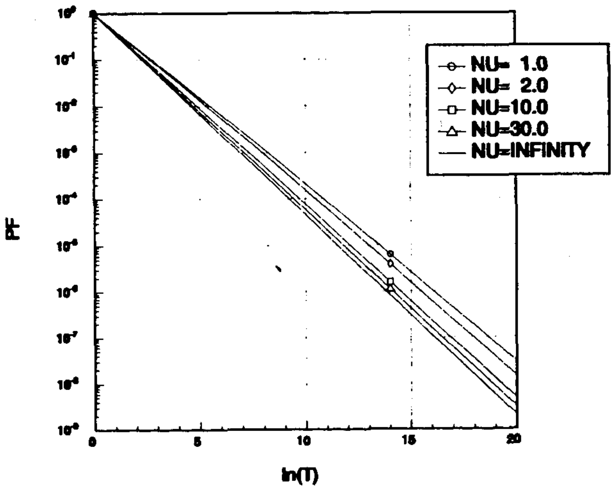

Fig. 8 — P_F for Rayleigh mixture, $m = 5$

Figures 9 and 10 present curves of P_D vs $(S/N)_{opt}$ for $\nu = 1, 2, 10, 30, \infty$, $P_F = 10^{-6}$, and $m = 2$ and 5, respectively. For constant $(S/N)_{opt}$, P_D decreases as the variance of τ increases. For $m = 2$ and for a constant level of P_D (in the range 0.1 to 0.9), there is approximately 7 dB difference in $(S/N)_{opt}$ from $\nu = 1$ to $\nu = \infty$. For $m = 5$ and for a constant level of P_D (in the range 0.1 to 0.9), there is approximately 4.5 dB difference in $(S/N)_{opt}$ from $\nu = 1$ to $\nu = \infty$.

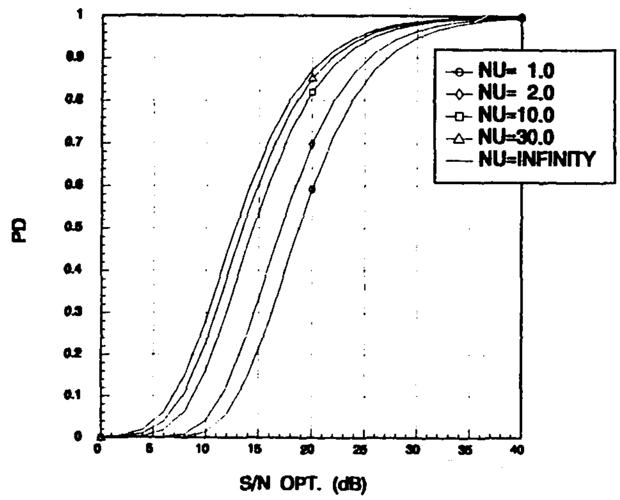

Fig. 9 — P_D for Rayleigh mixture, $m = 2$, $P_F = 10^{-6}$

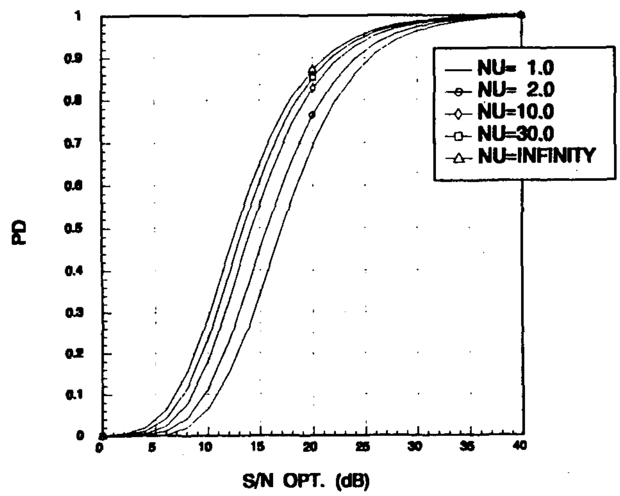

Fig. 10 — P_D for Rayleigh mixture, $m = 5$, $P_F = 10^{-6}$

7. SUMMARY AND CONCLUSIONS

To summarize, the problem of the detection of targets against a background of non-Gaussian, correlated clutter was studied. The investigation began by returning to the underlying phenomenological picture of the physical scattering process and deriving the required statistical model from that starting point. The statistics were obtained through the application of a new limit theorem from probability theory that was presented and proven herein. This theorem is an extension of the CLT to the situation in which the number N of random vectors being added together is itself a random variable. Since the case of nonrandom N may be viewed as a degenerate case of fluctuations, this new theorem includes the CLT. This theorem was then used to study the problem of modelling the one-dimensional statistics of clutter amplitude statistics and to formulate a multidimensional model of clutter statistics that incorporates correlation between pulse returns. This multidimensional model formed the starting point for an investigation into the problem of radar target detection against a background of non-Gaussian, correlated clutter.

The work presented herein raises a number of interesting questions. First, further justification of this type of analysis to scattering and propagation problems requires

a. rigorous demonstration that number fluctuations occur,
b. demonstration that $\bar{N}$ is large, and
c. a methodology for determining the PMF of the number fluctuations from the physics of the problem.

Limited results along these lines are available. For propagation of waves through continuous random media, Dashen (1987) uses the path integral formalism to draw some conclusions about a and b. With respect to c, Jakeman (1980) has shown that a birth-death-immigration process for number fluctuations leads to the negative binomial distribution. In a study of wave propagation through a turbulent atmosphere, Gochelashvili and Shishov (1978) assume the exponential mixture form for the distribution of intensity. They then obtain the form of $F_t(t)$ from considerations of the spectrum of turbulence. Through the mechanism of canonical PMF's presented above, a model for $F_t(t)$ may be used to obtain a model for the PMF of the number fluctuations. Thus the work of Gochelashavili and Shishov suggests that models for the number fluctuations may be obtainable from knowledge of the spectrum for problems of propagation through random media. Further work is needed for all three questions.

Further work on the limit theorem itself is also suggested. In particular, one would like to know if the assumption of identically distributed random vectors can be dropped (but still assume independence) or if the assumption of independence can be dropped (but still assume identically distributed random vectors.) Both of the assumptions have been relaxed in the case of the classical CLT, so the possibility exists that they may also be relaxed here. These results would greatly extend the applicability of these ideas to scattering problems.

Finally, an investigation into dynamical models for number fluctuations is needed if the model is to be extended to describe the radar clutter process, as opposed to the description only of the statistics of clutter as a random vector, as was done here.

For the detection problem, the interpretation of the optimal detector in terms of the estimator-correlator structure suggests various approaches to suboptimal implementation of the detector. These problems are as yet unexplored. Also, the structure of the optimal detector with the proper dynamics of number fluctuations taken into account remains an open question. Intuitively, one expects the structure to continue to be that of the estimator-correlator, although at this point this idea is only a conjecture. The detection structure for this problem appears to be a very interesting question.

ACKNOWLEDGMENTS

The authors thank Drs. Eric Mokole and Dennis Trizna of the Naval Research Laboratory for their help during the development of this work.

REFERENCES

Beckmann P. and A. Spizzichino. 1987. *The Scattering of Electromagnetic Waves from Rough Surfaces* (Artech House, Inc., Norwood MA), Ch. 7.

Bernstein S. 1929. "Sur les Fonctions Absolument Monotones," *Acta Mathematica* **52**(1), 1.

Boas R.P. 1939. "The Stieltjes Moment Problem for Functions of Bounded Variation," *Bull. Am. Math. Soc.* **45**(6), 399-404.

Bochner S. 1937a. "Completely Monotone Functions of the Laplace Operator for Torus and Sphere," *Duke Mathematical Journal* 3(5), 488-503.

—. 1937b. "Stable Laws of Probability and Completely Monotone Functions," *Duke Mathematical Journal* 3(9), 726-728.

Chow S.C. and H. Teicher. 1978. *Probability Theory: Independence, Interchangeability, Martingales* (Springer-Verlag, New York), Section 9.4.

Conte E. and M. Longo. 1987. "Characterization of Radar Clutter as a Spherically Invariant Random Process," *IEE Proc. Part F* **134**(2), 191-197.

Courant R. and D. Hilbert. 1953. *Methods of Mathematical Physics: Vol. 1* (John Wiley and Sons, New York), pp. 103-105.

Cramer H. 1970. *Random Variables and Probability Distributions,* 3rd ed. (Cambridge University Press, London), p. 103.

Dashen R. 1979. "Path Integrals for Waves in Random Media," *J. Math. Phys.* **20**(5), 894-920.

—. 1984. "Distribution of Intensity in a Multiply Scattering Medium", *Opt. Lett.* **10**(4), 110-112.

Difranco J.V. and W.L. Rubin. 1980. *Radar Detection* (Artech House Norwood, MA).

Flatte S.M., S.A. Reynolds, and R. Dashen. 1987. "Path Integral Treatment of Intensity Behavior for Rays in a Sound Channel," *J. Acoust. Soc. Am.* **82**(3), 967-972.

Gochelashvili K.S. and V.I. Shishov. 1978. "Strong Fluctuations of Laser Radiation Intensity in a Turbulent Atmosphere - the Distribution Function," *Sov. Phys. JETP* **47**(6), 1028-1030.

Grosjean C.C. 1989. "An Inverse Laplace Transform - Solution," *SIAM Review* **31**(4), 685-686.

Helstrom C.W. 1968. *Statistical Theory of Signal Detection,* 2nd. ed. (Pergamon, New York).

Horn R.A. and C.A. Johnson. 1985. *Matrix Analysis* (Cambridge Univ. Press, New York).

Hille E. and J. Tamarkin. 1933, 1934. "On Moment Functions," *Proc. Nat. Acad. Sci.* **19**, 902-912, **20**, 140-144.

Jakeman E. and P.N. Pusey. 1976. "A Model for Non-Rayleigh Sea Echo," *IEEE Trans. Ant. Prop.* **24**(6), 806-814.

—. 1978. "Significance of K Distributions in Scattering Experiments," *Phys. Rev. Lett.* **4**(9), 546-550.

Jakeman E. 1980. "On the Statistics of the K Distribution," *J. Phys. A.* **13**(1), 31-48.

Kolmogorov A.N. and S.V. Fomin. 1970. *Introductory Real Analysis* (Dover Books, New York).

Olin I.D. 1982. "Amplitude and Temporal Statistics of Sea Clutter," *IEE Conference Publication No. 216, Radar 82* (IEE Press, London).

Papoulis A. 1965. *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York).

Pollard H. 1946. "The Representation of $e^{-x^{\lambda}}$ as a Laplace Integral," *Bull. Amer. Math. Soc.* **52**, 908-910.

Poor H.V. 1988. *An Introduction to Signal Detection and Estimation* (Springer-Verlag, New York).

Prudnikov A.P., Y.A. Brychkov, and O.I. Marichev. 1986. *Integrals and Series: Vol. 1* (Gordon and Breach Science Publishers, New York).

Sangston K.J. and K. Gerlach. 1989. "Results on the Detection of Signals in Spherically Invariant Noise," NRL Report 9202.

Schoenberg I.J. 1938. "Metric Spaces and Completely Monotonic Functions," *Annals of Mathematics* **39**(4), 811-841.

Schwartz S.C. 1975. "Conditional Mean Estimates and Bayesian Hypothesis Testing," *IEEE Trans. Inf. Theory* **21**(6), 663-665.

Skolnik M.I. 1980. *Introduction to Radar Systems*, 2nd ed. (McGraw-Hill Book Company, New York), p. 496.

Tamarkin J.D. 1931. "On a Theorem of S. Bernstein-Widder," *Trans. Amer. Math. Soc.* **33**, 892-896.

Trizna D.B. 1988. "Measurement and Interpretation of North Atlantic Ocean Marine Radar Sea Scatter," NRL Report 9099.

—. 1989. "Pacific Marine Radar Sea Scatter Experimental Results," NRL Memorandum Report 6506.

Trunk G.V. 1972. "Radar Properties of Non-Rayleigh Sea Clutter," *IEEE Trans. Aerospace and Electronics Systems* **8**(2), 196-204.

—. 1976. "Non-Rayleigh Sea Clutter: Properties and Detection of Targets," NRL Report 7986.

Valenzuela G.R. and M. Laing. 1971. "On the Statistics of Sea Clutter," NRL Report 7349.

Van Trees H.C. 1968. *Detection, Estimation, and Modulation Theory, Part I* (John Wiley and Sons, New York).

Ward K.D. 1981. "Compound Representation of High Resolution Sea Clutter," *Electron. Lett.* **17**(6), 561-563.

—. 1982. "A Radar Sea Clutter Model and its Application to Performance Assessment," *IEE Conference Publication No. 216, Radar 82* (IEE Press, London).

Watson G.N. 1944. *A Treatise on the Theory of Bessel Functions,* 2nd ed. (Cambridge Univ. Press, New York).

Widder D.V. 1931. "Necessary and Sufficient Conditions for the Representation of a Function as a Laplace Integral," *Trans. Amer. Math. Soc.* **33**, 851-892.

—. 1934. "The Inversion of the Laplace Integral and the Related Moment Problem," *Trans. Amer. Math. Soc.* **36**, 107-200.

—. 1935. "An Application of Laguerre Polynomials," *Duke Mathematical Journal* **1**, 126-136.