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# Convergence Performance of Adaptive Detectors, Part 2

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Two schemes for adaptive detection are compared: Kelly's generalized likelihood ratio test (GLRT) and the mean level adaptive detector (MLAD). Detection performance $P_D$ is predicted for the two schemes under the assumptions that the input noises are zero-mean Gaussian random variables that are temporally independent but spatially correlated; and the desired signal is Rayleigh distributed. $P_D$ is computed as a function of the false alarm probability, the number of input channels, the number of independent samples-per-channel, and the matched filtered output signal-to-noise (S/N) power ratio. In this analysis, the GLRT is shown to have better detection performance than the MLAD. The difference in detection performance increases as one uses fewer input samples. However, the required number of samples necessary to have only a 3 dB detection loss for both detection schemes is approximately the same. This is significant since, for the present, the MLAD is considerably less complex to implement than the GLRT.

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# CONVERGENCE PERFORMANCE OF ADAPTIVE DETECTORS, PART 2

## 1.0 INTRODUCTION

The general problem of detecting signals in a background of Gaussian noise for an adaptive array was addressed by using the techniques of statistical hypothesis testing [1]. In Ref. 1, the problem is formulated as a binary hypothesis test where one hypothesis is noise only and the other is signal plus noise. A given input data vector (called the primary data vector) is tested for signal presence. Another set of signal-free data vectors (called the secondary data vectors) is available that shares the unknown covariance matrix $M$ of the noise in the primary data vector. A likelihood ratio decision rule is derived, and its performance is evaluated for the two hypothesis.

Kelly's detector [1] uses the maximum likelihood (ML) estimates for the unknown parameters of the likelihood ratio test (LRT). The unknown parameters are the spatial covariance matrix and the unknown signal's complex amplitude (assumed in Kelly's analysis to be a nonrandom constant). This detection scheme is commonly referred to as the generalized likelihood ratio test (GLRT) and is referred to here as such.

Convergence results for $P_D$ and $P_F$ are presented in Ref. 1. Expressions for $P_D$ and $P_F$ are derived that are a function of the number of statistically independent secondary data vectors; the number of input channels $N$; the detector threshold; and the input signal-to-noise (S/N) power ratio. Note that $P_F$ did not depend on $M$ (a statistical measure of the external noise environment). Hence, this detector exhibited the desirable constant-false-alarm-rate (CFAR) property of having the $P_F$ be independent of the covariance matrix. Additional research in the area is contained in Refs. 2 through 5.

A mean level adaptive detector (MLAD) is a more easily implementable adaptive detection scheme. The MLAD is essentially an adaptive matched filter (AMF) followed by a mean level detector (MLD) [6,7]. Input samples used in determining the MLD threshold are derived from a block of data passing through the AMF. The squared magnitude of each of these same samples as processed through the AMF is used as a test statistic and is compared against an MLD threshold (an average of the instantaneous powers) that does not contain the given test statistic sample. We further clarify the implementation terminology by calling this an MLAD with concurrent data samples. In Part 1 of this report [8], an analysis is performed for an MLAD with nonconcurrent data; i.e., the MLD test statistic is derived from a set of data that are statistically independent of the test statistic of the primary data vector.

Here we compare the detection performance of the two detection schemes: GLRT vs MLAD. For this analysis, we assume that the complex, desired signal amplitude is a complex, zero-mean

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Gaussian random variable (RV) of unknown variance with independent and identically distributed real and imaginary parts (the magnitude of this amplitude is Rayleigh distributed). Under the GLRT, we would have had to reformulate Kelly's detector with the variance of the unknown signal amplitude as an unknown parameter and find the ML estimate of this quantity. This is mathematically tedious. In lieu of implementing this new GLRT, we chose to evaluate Kelly's GLRT as it is defined in his paper. As noted by Kelly, no optimality properties are claimed for this test. The form of the test is, however, reasonable.

## 2.0 GENERALIZED LIKELIHOOD RATIO TEST (GLRT)

### 2.1 Detector Form

Kelly [1] gives a mathematical formulation of the adaptive detection problem that leads to the GLRT. We now summarize that formulation. Two sets of input data—primary and secondary—are used. We assume that the secondary inputs do not contain the desired signal. Set

$X = N \times K$ matrix of secondary input data. The $n$th row represents the $K$ samples of data on the $n$th channel, where $n = 1, 2, \ldots, N$. The samples in the $k$th column are assumed time-coincident, where

$\mathbf{x}$ is the primary data vector of length $N$ and

$\mathbf{s}$ is the desired steering vector of length $N$.

Consider the two hypothesis

$$H_0 : \mathbf{x} = \mathbf{n}$$

$$H_1 : \mathbf{x} = \mathbf{n} + a\,\mathbf{s},$$

where $H_0$ is the noise only hypothesis, $\mathbf{n}$ is a noise vector of length $N$, and $H_1$ is the signal-plus-noise hypothesis, where $a$ is the unknown, complex signal amplitude. We assume the following:

(1) Input noises are complex, zero-mean stationary Gaussian RVs. The real and imaginary parts of a given input noise sample are independent and identically distributed (IID) with respect to each other. An RV with these characteristics is called a circular Gaussian process.

(2) Input noise samples are temporally statistically independent.

(3) The secondary data is statistically independent of the primary data.

(4) The desired signal is present in the primary data vector. It is not in the secondary data.

The GLRT is formulated as follows. Find the probability density function (PDF) under each hypothesis over all measured data. For this problem, this is straightforward since the sample vectors are assumed to be independent and each vector has an associated $N$-dimensional Gaussian PDF. If

there are any unknown parameters, maximize the PDF of the inputs over all unknown parameters for each of the two hypothesis. The maximizing parameter values are, by definition, the ML estimators of the parameters. Hence, obtain the maximized PDFs by replacing the unknown parameters by their ML estimates. Find the ratio of the resultant maximum of PDFs (the ratio of the PDF under $H_1$ to the PDF under $H_0$). Check this ratio to see if it exceeds a preassigned threshold $t$.

Kelly shows that the GLRT for the adaptive detection problem is given by

$$\frac{|\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}|^2}{(\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{s})\,[1 + \mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}]} \underset{H_0}{\overset{H_1}{\gtrless}} t, \tag{1}$$

where

$$\hat{R}_x = XX^{\mathrm{H}}, \tag{2}$$

and H denotes the conjugate transpose matrix operation. We recognize $\hat{R}_x$ as proportional to the ML estimate of the input covariance matrix. We note also that the desired signal's unknown complex amplitude $a$ has been estimated and is accounted for in Eq. (1).

### 2.2 Statistically Equivalent GLRT

Here we derive a statistically equivalent GLRT that allows us to formulate simply the detection and false alarm probabilities of the adaptive detector. As in Refs. 1 and 9, we can matrix transform the input vectors by an $N \times N$ matrix $A$, which has the properties that the input noise vectors are spatially whitened, each input element has noise power normalized to one, and

$$A\mathbf{s} = (0,0, \cdots 0, (\mathbf{s}^{\mathrm{H}} M^{-1} \mathbf{s})^{1/2}) \equiv \mathbf{s}_0, \tag{3}$$

where all of the desired signal has been placed into $N$th channel (note in Ref. 1, the signal was placed into the first channel; for analysis purposes, we place the signal into the $N$th channel).

In addition, set

$$\mathbf{z} = A\mathbf{x}, \text{ and} \tag{4}$$

$$Z = AX. \tag{5}$$

The elements of vector $\mathbf{z}$ (under $H_0$) and the elements of the vectors representing the columns of $Z$ (each column represents the transformed secondary data across the array at a given instant of time), are now spatially independent with each element having power equal to one. As shown by Ref. 1, the transformed GLRT is given by

$$\frac{|\mathbf{s}_0^{\mathrm{H}}\,\hat{R}_z^{-1}\,\mathbf{z}|^2}{(\mathbf{s}_0^{\mathrm{H}}\,\hat{R}_z^{-1}\,\mathbf{s}_0)\,(1+\mathbf{z}^{\mathrm{H}}\,\hat{R}_z^{-1}\,\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} t, \tag{6}$$

where

$$\hat{R}_z = ZZ^{\mathrm{H}}. \tag{7}$$

We note that if $Q$ is any $K \times K$ unitary matrix (i.e., $QQ^{\mathrm{H}} = I_K$ where $I_K$ is the $K \times K$ identity matrix), then

$$\hat{R}_z = ZZ^{\mathrm{H}} = ZQQ^{\mathrm{H}}\,Z^{\mathrm{H}}. \tag{8}$$

In the forthcoming development, we will transform $Z$ by a series of unitary matrix transforms that zero many of the elements of $Z$. This development is similar to that of Rader and Steinhardt [10].

Set $Z = (z_{mn}^{(0)}) \equiv Z_0$. We can show by construction that there exists a $K \times K$ unitary matrix $\Psi_0$ such that

$$Z_0\,\Psi_0 = Z_1 = \begin{bmatrix} y_{11} & 0 & \dots & 0 \\ z_{21}^{(1)} & z_{22}^{(1)} & \dots & z_{2K}^{(1)} \\ z_{21}^{(1)} & z_{32}^{(1)} & \dots & z_{3K}^{(1)} \\ \cdot & & \cdot & \\ \cdot & & \dots & \\ \cdot & & \cdot & \\ z_{N1}^{(1)} & z_{N2}^{(1)} & \dots & z_{NK}^{(1)} \end{bmatrix}, \tag{9}$$

where $\Psi_0$ is a function of only $z_{1k}^{(0)}$, $k = 1, 2, \dots, K$, and

$$y_{11} = \left[\sum_{k=1}^{K} |z_{1k}^{(0)}|^2\right]^{1/2}. \tag{10}$$

Note that after this transformation, $\hat{R}_z$ is dependent only on the RVs, $y_{11}$, $z_{nk}^{(1)}$, $n = 2, \dots, N$, $k = 1, 2, \dots, K$. Hence, we have reduced the total number of RVs on which $R_z$ is dependent by $K - 1$. In addition, $y_{11}$ is statistically independent of the $z_{nk}^{(1)}$, and the elements of the second through $N$th rows are statistically independent with each element having a power equal to one. (This follows because a vector of IID-complex, circular Gaussian RVs is the same statistically after transformation by a unitary matrix). We note that $y_{11}$ has a $\chi$ PDF of the order $2(K - 1)$ with $\sigma^2 = 0.5$.

Furthermore, there exists a $K \times K$ unitary matrix $\Psi_1$ such that

$$Z_2 = Z_1 \Psi_1, \tag{11}$$

$$\Psi_1 = \begin{bmatrix} 1 & 0 \\ 0 & \Psi_1' \end{bmatrix}, \tag{12}$$

and

$$Z_2 = \begin{bmatrix} y_{11} & 0 & 0 & \dots & 0 \\ z_{21}^{(1)} & y_{22} & 0 & \dots & 0 \\ z_{31}^{(1)} & z_{32}^{(2)} & z_{33}^{(2)} & \dots & z_{3K}^{(2)} \\ . & & & & \\ . & & & & \\ . & & & & \\ z_{N1}^{(1)} & z_{N2}^{(2)} & z_{N3}^{(2)} & \dots & z_{NK}^{(2)} \end{bmatrix}, \tag{13}$$

where $\Psi_1'$ is a $(K-1) \times (K-1)$ unitary matrix whose elements depend only on $z_{2K}^{(1)}$, $k = 2, 3, \dots K$, and

$$y_{22} = \left( \sum_{k=2}^{K} | z_{2k}^{(1)} |^2 \right)^{1/2}. \tag{14}$$

Again, note that we have reduced the number of RVs on which $\hat{R}_z$ depends. Also, all of the elements of $Z_2$ are independent. The $z_{nk}^{(1)}$, i = 1, 2 are IID circular Gaussian RVs with power equal to one and $y_{22}$ is a $\chi$ PDF of the order $2(K - 2)$ with $\sigma^2 = 0.5$.

This procedure can be reiterated so that the final, transformed matrix is given by

$$Z_N = [\tilde{Z}_N \,|\, \mathbf{0}], \tag{15}$$

where $\tilde{Z}_N$ is an $N \times N$ matrix defined as

$$\tilde{Z}_N = \begin{bmatrix} y_{11} & 0 & 0 & \dots & 0 \\ z_{21}^{(1)} & y_{22} & 0 & \dots & 0 \\ z_{31}^{(1)} & z_{32}^{(2)} & y_{33} & \dots & 0 \\ . & . & . & . & . \\ . & . & . & \dots & . \\ . & . & . & . & \\ z_{N1}^{(1)} & z_{N2}^{(2)} & z_{N3}^{(3)} & \dots & y_{NN} \end{bmatrix} \tag{16}$$

We note that the $N + 1$ through $K$th columns of $Z_N$ are zero-filled and are denoted by $\mathbf{0}$. Hence, $\hat{R}_z$ only depends on the $N(N + 1)/2$ RVs. All of the elements of $Z_N$ are statistically independent. The $z_{nk}^{(i)}$, $i = 1, 2, \ldots, N - 1$ are IID circular Gaussian RVs with power equal to one and $y_{nn}, n = 1, 2, \ldots, N$ is a $\chi$ PDF of the order $2(K - n)$ with $\sigma^2 = 0.5$. Thus,

$$\hat{R}_z = Z_N Z_N^{\mathrm{H}} = \tilde{Z}_N \tilde{Z}_N^{\mathrm{H}}. \tag{17}$$

We perform one last matrix transform on the input data. We multiply $\mathbf{z}$ by an $N \times N$ unitary matrix transform $B$, such that

$$B = \begin{bmatrix} B' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{18}$$

$$B\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ v_1 \\ v_2 \end{bmatrix} \equiv \mathbf{z}_B, \tag{19}$$

$$v_1 = \left\{ \sum_{n=1}^{N-1} |z_n|^2 \right\}^{1/2}, \tag{20}$$

and

$$v_2 = z_N, \tag{21}$$

where $B'$ is an $(N - 1) \times (N - 1)$ unitary matrix. We note that $B\mathbf{s}_0 = \mathbf{s}_0$, and if we set $Z_B = BZ$, then

$$\mathbf{s}_0^{\mathrm{H}} \hat{R}_z^{-1} \mathbf{s}_0 = \mathbf{s}_0^{\mathrm{H}} \hat{R}_{z_B}^{-1} \mathbf{s}_0,$$

$$\mathbf{s}_0^{\mathrm{H}} \hat{R}_z^{-1} \mathbf{z} = \mathbf{s}_0^{\mathrm{H}} \hat{R}_{z_B}^{-1} \mathbf{z}_B, \tag{22}$$

and

$$\mathbf{z}^{\mathrm{H}} \hat{R}_z^{-1} \mathbf{z} = \mathbf{z}_B^{\mathrm{H}} \hat{R}_{z_B}^{-1} \mathbf{z}_B,$$

where $\hat{R}_{Z_B} = Z_B Z_B^{\mathrm{H}}$. The statistical properties of $Z_B$ are identical to $Z$ and, hence, $Z_B$ can be reduced as given by Eqs. (15) and (16). We point out that $v_1$ and $v_2$ are independent and that $v_2$ is a circular Gaussian RV with its power depending on the given hypothesis (under $H_0$, the power equals one), and $v_1$ is a $\chi$ PDF of order $2(N-1)$ with $\sigma^2 = 0.5$.

As a result of the transformations of the input data, we see that Eq. (6) can be rewritten as

$$\frac{|\mathbf{s}_0^{\mathrm{H}} (\tilde{Z}_N \tilde{Z}_N^{\mathrm{H}})^{-1} \mathbf{z}_B|^2}{[\mathbf{s}_0^{\mathrm{H}} (\tilde{Z}_N \tilde{Z}_N^{\mathrm{H}})^{-1} \mathbf{s}_0] [1 + \mathbf{z}_B^{\mathrm{H}} (\tilde{Z}_N \tilde{Z}_N^{\mathrm{H}})^{-1} \mathbf{z}_B]} \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{23}$$

Define $u_{11} = y_{N-1,N-1}$, $u_{22} = y_{NN}, u_{21} = z_{N,N-1}^{(N-1)}$,

$$U = \begin{bmatrix} u_{11} & 0 \\ u_{21} & u_{22} \end{bmatrix},$$

$\mathbf{v} = (v_1, v_2)^{\mathrm{T}}$, and $\mathbf{1}_0 = (0,1)^{\mathrm{T}}$. It is straightforward to show that Eq. (23) is equivalent to

$$\frac{|\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{v}|^2}{(\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{1}_0) [1 + \mathbf{v}^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v}]} \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{24}$$

In going from Eq. (23) to Eq. (24), we have taken advantage of the fact that if $C$ is a nonsingular $N \times N$ lower triangular matrix that is partitioned as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where $C_{22}$ is a $n \times n$ matrix, and

$$C^{-1} = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix},$$

where $C^{22}$ is a $n \times n$ matrix, then $C^{22} = C_{22}^{-1}$.

Now

$$\mathbf{1}_0^T (UU^{\mathrm{H}})^{-1} \mathbf{1}_0 = \frac{1}{u_{22}^2}, \tag{25}$$

$$|\mathbf{1}_0^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v}|^2 = \frac{1}{u_{11}^2 u_{22}^4} \left[ |u_{11} v_2 - u_{21} v_1|^2 \right], \tag{26}$$

and

$$\mathbf{v}^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v} = \frac{1}{u_{11}^2 \, u_{22}^2} \left[ |u_{11} v_2 - v_{21} \, v_1|^2 + u_{22}^2 v_1^2 \right]. \tag{27}$$

By inserting Eqs. (23) through (27) into Eq. (24), we show that the GLRT is equivalent to

$$|u_{11} \, v_2 - v_2 \, u_{21}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} (u_{11}^2 + v_1^2) \, u_{22}^2 T, \tag{28}$$

where

$$T = \frac{t}{1-t}. \tag{29}$$

### 2.3 Probability of Detection

Under the $H_1$ hypothesis, we assume that the desired signal's amplitude (or magnitude) is Rayleigh distributed and that the signal's phase is uniformly distributed between $(0, 2\pi)$. This implies that the desired signal itself is a complex circular Gaussian RV; therefore, let the desired signal's input power-per-channel before any matrix transformation be equal to $\tilde{\sigma}_s^2$. After the $A$ matrix transformation (whitening and placing the signal into the $N$th channel), the signal power in the $N$th channel is $\sigma_s^2 = \tilde{\sigma}_s^2 \, \mathbf{s}^{\mathrm{H}} \, M^{-1}\mathbf{s}$. Thus under $H_1$, $v_2$ is a complex circular Gaussian RV with power equal to $\sigma_s^2 + 1$, where the 1 represents the noise power-level-per-channel after the $A$ matrix transformation. Also, $u_{21}$ is a complex circular Gaussian RV with power equal to one. We can rewrite Eq. (28) as

$$|\alpha|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T', \tag{30}$$

where

$$\alpha = \frac{u_{11} \, v_2 - v_1 \, u_{21}}{(u_{11}^2 \, (\sigma_s^2 + 1) + v_1^2)^{1/2}} \text{ , and} \tag{31}$$

$$T' = \frac{u_{11}^2 + v_1^2}{u_{11}^2 \, (\sigma_s^2 + 1) + v_1^2} \, u_{22}^2 \, T. \tag{32}$$

It is straightforward to show that $\alpha$, when conditioned on $u_{11}$ and $v_1$, is a complex circular Gaussian RV with power equal to one. It is well known [11] that the conditional probability of detection is given by

$$P(D \mid u_{11}, u_{22}, v_1) = e^{-T'} = \exp\left\{-\frac{u_{11}^2 + v_1^2}{u_{11}^2(\sigma_s^2 + 1) + v_1^2} u_{22}^2 T\right\}, \tag{33}$$

where $P(D \mid \cdot, \cdot)$ denotes the conditional probability of detection.

We set

$$\eta = u_{22}^2, \tag{34}$$

$$\mu = u_{11}^2, \tag{35}$$

$$\nu = v_1^2, \text{ and} \tag{36}$$

$$r = \frac{\nu}{\mu}. \tag{37}$$

Then Eq. (33) becomes

$$P(D \mid r, \eta) = \exp\left\{-\frac{1 + r}{\sigma_s^2 + 1 + r} \eta T\right\}. \tag{38}$$

The PDFs of $\eta, \mu$, and $N$ are $\chi^2$ of order $2(K - N + 1)$, $2(K - N + 2)$, and $2(N - 1)$, respectively with $\sigma^2 = 0.5$, and are given by

$$p_\eta(\eta) = \frac{1}{(K - N)!} \eta^{K-N} e^{-\eta} \qquad , \eta \geq 0, \tag{39}$$

$$p_\mu(\mu) = \frac{1}{(K - N + 1)!} \mu^{K-N+1} e^{-\mu} \quad , \mu \geq 0, \tag{40}$$

and

$$p_\nu(\nu) = \frac{1}{(N - 2)!} \nu^{N-2} e^{-\nu} \quad , \nu \geq 0. \tag{41}$$

By using the elementary probability theory, it is straightforward to show that

$$p_r(r) = \int_0^\infty \frac{\beta}{r^2} p_\nu(\beta) p_\mu \left(\frac{\beta}{r}\right) d\beta. \tag{42}$$

By inserting expressions for $p_\mu$ and $p_\nu$ as given by Eqs. (40) and (41), respectively, and by simplifying results in

$$p_r(r) = \frac{K!}{(N-2)!\,(K-N+1)!}\,\frac{r^{N-2}}{(1+r)^{K+1}}\,,\ r \geq 0. \tag{43}$$

If we set $q = 1/(1+r)$, it is straightforward to show that

$$p_q(q) = \frac{K!}{(N-2)!\,(K-N+1)!}\,(1-q)^{N-2}\,q^{K-N+1},\ 0 \leq q \leq 1, \tag{44}$$

which is the PDF derived by Reed et al., [9] for the normalized instantaneous S/N power ratio that results if the sampled matrix inversion (SMI) algorithm is used. By substituting $q$ for $r$ in Eq. (38), we obtain

$$p(D \mid q,\eta) = \exp - \left[\frac{1}{q\,\sigma_s^2 + 1}\right]\eta T. \tag{45}$$

Now

$$P(D \mid q) = \int_0^\infty \frac{1}{(K-N)!}\,\eta^{K-N}\exp - \left[1 + \frac{T}{q\,\sigma_s^2 + 1}\right] d\eta$$

$$= \left[1 + \frac{T}{q\,\sigma_s^2 + 1}\right]^{-(K-+1)}, \tag{46}$$

or

$$P_D = \int_0^1 \frac{K!}{(N-2)!\,(K-N+1)!}\left[1 + \frac{T}{q\left(\frac{S}{N}\right)_{opt} + 1}\right]^{-(K-N+1)} (1-q)^{N-2}\,q^{K-N+1}dq, \tag{47}$$

where $\sigma_s^2 = (S/N)_{opt}$. We note in Eq. (47) that we have replaced $\sigma_s^2$ with $(S/N)_{opt}$, where $(S/N)_{opt}$ is the optimal (S/N) output power ratio of the matched filter ($K = \infty$). We can write this in this way because the output noise power of the $N$th channel has been normalized to one and the output of the $N$th channel is the optimal matched filter output.

### 2.4 Probability of False Alarm

The probability of false alarm $P_F$ is easily found from Eq. (47) by setting $(S/N)_{opt} = 0$. It is found that

$$P_F = \frac{1}{(T+1)^{K-N+1}}. \tag{48}$$

An equivalent expression for $P_F$ as given by Eq. (48) was also derived by Refs. 1 and 4.

## 3.0 MEAN LEVEL ADAPTIVE DETECTOR (MLAD)

### 3.1 Detector Form

An intuitive form of adaptive detection is found by implementing MLAD. Figure 1 shows that the MLAD is essentially an AMF followed by an MLD. The MLAD is designed to perform detections over a block of data by using just this block of data in determining the AMF weights and the MLD threshold. The MLAD works as follows. Let there be $N$ channels and $K + 1$ samples per channel. Define

$\mathbf{x}$ as the primary $N$-length data vector,

$\mathbf{x}_k$ as the secondary $N$-length data vectors, $k = 1, 2, \ldots, K$,

$X_{\text{aug}}$ as the $(\mathbf{x} \mid X)$ = augmented $N \times (K + 1)$ matrix of input data, and

$\hat{R}_0$ as the $X_{\text{aug}}\, X_{\text{aug}}^{\text{H}}$.

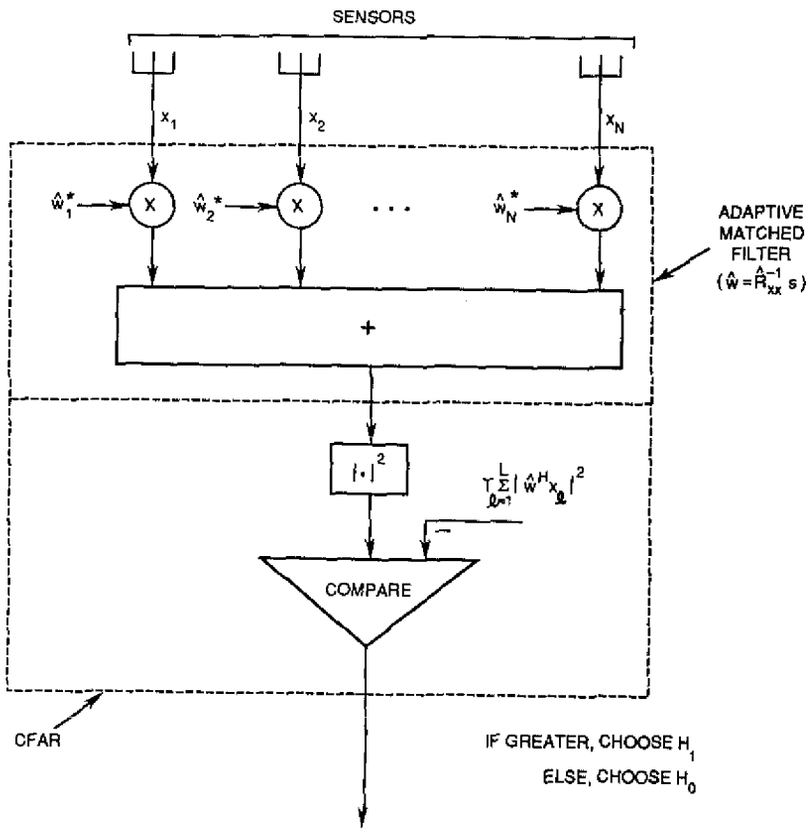


Fig. 1 — Mean level adaptive detector

The $N$-length weighting vector $\hat{\mathbf{w}}$ for the AMF is found by the SMI algorithm and is given by

$$\hat{\mathbf{w}} = \hat{R}_0^{-1}\ \mathbf{s}. \tag{49}$$

This weight is used in the detection rule given by

$$|\hat{\mathbf{w}}^{\mathrm{H}}\ \mathbf{x}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_0 \sum_{k=1}^{K} |\hat{\mathbf{w}}^{\mathrm{H}}\ \mathbf{x}_k|^2, \tag{50}$$

where $T_0$ is chosen to control the false alarm probability. We see that Eq. (49) is the algorithmic representation of the AMF and Eq. (50), the MLD.

Note that we have included the primary data vector in the $\hat{R}_0$ estimate and, hence, in the $\hat{\mathbf{w}}$ estimate. In a practical situation, this might be done since it is more numerically efficient to compute one weighting vector over the entire data block than it is to compute a distinct weighting vector for each point in the block. However, the presence of the desired signal (under $H_1$) will affect detection. In Eq. (50), the primary data vector is varied across the $K + 1$ data snapshots, where the $\mathbf{x}_k$ used on the right side of Eq. (50) does not include a selected primary data vector.

It is straightforward to show that

$$T_0 \sum_{k=1}^{K} |\hat{\mathbf{w}}^{H}\ \mathbf{x}_k|^2 + T_0\ |\hat{\mathbf{w}}^{\mathrm{H}}\ \mathbf{x}|^2 = T_0\ \mathbf{s}^{\mathrm{H}}\ \hat{R}_0^{-1}\ \mathbf{s}. \tag{51}$$

Thus Eq. (50) is equivalent to

$$\frac{|\mathbf{s}^{\mathrm{H}}\ \hat{R}_0^{-1}\ \mathbf{x}|^2}{\mathbf{s}^{\mathrm{H}}\ \hat{R}_0^{-1}\mathbf{s}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{T_0}{T_0 + 1} \equiv T_1, \tag{52}$$

where we note that $0 \leq T_1 \leq 1$.

Now we can write

$$\hat{R}_0 = \hat{R}_x + \mathbf{x}\mathbf{x}^{\mathrm{H}}, \tag{53}$$

where $\hat{R}_x = XX^{\mathrm{H}}$. By using the matrix inversion lemma [12] and after some simplification, we can show that

$$\mathbf{s}^{\mathrm{H}} \hat{R}_0^{-1} \mathbf{s} = \mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{s} - \frac{|\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}|^2}{1 + \mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}}, \tag{54}$$

$$\mathbf{s}^{\mathrm{H}} \hat{R}_0^{-1} \mathbf{x} = \frac{s^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}}{1 + \mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}}, \tag{55}$$

and

$$\mathbf{x}^{\mathrm{H}} \hat{R}_0^{-1} \mathbf{x} = \frac{\mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}}{1 + \mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}}. \tag{56}$$

By substituting Eqs. (54) through (56) into Eq. (52), the equivalent detection rule becomes

$$|\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_1 \left[ (\mathbf{s}^{\mathrm{H}} R_x^{-1} \mathbf{s})(1 + \mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x})^2 + |\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}|^2 (1 + \mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}) \right]. \tag{57}$$

### 3.2 Statistically Equivalent MLAD

We make the same assumptions on the input distributions and statistics as given for the GLRT discussed in section 2. Hence, we can use the matrix transformations here that were used to transform the GLRT analysis into a simpler problem. Specifically, we can show that the test given by Eq. (57) is statistically equivalent to

$$|\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{v}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_1 \Big[ \mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{1}_0 (1 + \mathbf{v}^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v}) + |\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}}) \mathbf{v}|^2 (1 + \mathbf{v}^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v})^2 \Big], \tag{58}$$

where in Eq. (57) we have replaced $\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}$ with $\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{v}$, $\mathbf{s}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{s}$ with $\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{1}_0$, and $\mathbf{x}^{\mathrm{H}} \hat{R}_x^{-1} \mathbf{x}$ with $\mathbf{x}^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v}$. The quantities $U$ and $\mathbf{v}$ are defined in section 2. Equations (25) through (27) give expressions for $\mathbf{1}_0^{\mathrm{T}} (UU^{\mathrm{H}})^{-1} \mathbf{1}_0$, $|\mathbf{1}_0^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v}|^2$, and $\mathbf{v}^{\mathrm{H}} (UU^{\mathrm{H}})^{-1} \mathbf{v}$, respectively. If these are substituted into Eq. (58), after somewhat extensive simplification, the following detection rule results

$$|u_{11} v_2 - v_1 u_{21}|^2 \left[ 1 - T_1 \left( 1 + \frac{v_1^2}{u_{11}^2} \right) \right] \underset{H_0}{\overset{H_1}{\gtrless}} \left( 1 + \frac{v_1^2}{u_{11}^2} \right)^2 u_{11}^2 u_{22}^2 T_1. \tag{59}$$

We note for $T_1 > 0$, that if

$$1 - T_1 \left[ 1 + \frac{v_1^2}{u_{11}^2} \right] \leq 0, \tag{60}$$

then $H_0$ is declared.

### 3.3 Probability of Detection

Again under the $H_1$ hypothesis, we assume that the desired signal is a complex, circular Gaussian RV (the amplitude is Rayleigh distributed). As in our analysis of the GLRT, $\sigma_s^2 = \tilde{\sigma}_s^2 \, s^H M^{-1} s$. Assume Eq. (60) is not true. We can write the decision rule given by Eq. (59) as

$$|\alpha|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_1' \tag{61}$$

where

$$\alpha = \frac{u_{11} v_2 - v_1 u_{21}}{(u_{11}^2 (\sigma_s^2 + 1) + v_1^2)^{1/2}}, \tag{62}$$

and

$$T_1' = \frac{1}{(q - T_1)(q \sigma_s^2 + 1)} \eta T_1, \tag{63}$$

$\eta$ is defined by Eq. (34), and $q = (1 + r)^{-1}$. As before under $H_1$, $v_1$ is a complex, circular Gaussian RV with power equal to $\sigma_s^2 + 1$, and $u_{21}$ is the same with power equal to one. Furthermore, $\alpha$ is the same with power equal to one. Thus

$$P(D \mid q, \eta) = \begin{cases} 0 & \text{if} \quad q \leq T_1 \\ e^{-T_1'} & \text{otherwise.} \end{cases} \tag{64}$$

The PDFs of $\eta$ and $q$ are given by Eqs. (39) and (44), respectively. After some simplification, we find that

$$P_D = \int_{T_1}^{1} \int_0^{\infty} P(D \mid q, \eta) p_\eta(\eta) p_q(q) \, d\eta dq$$

$$= \int_{T_1}^{1} \frac{K!}{(N-2)!(K-N+1)!} \left[ 1 + \frac{T_1}{(q - T_1)\left[ q \left( \frac{S}{N} \right)_{opt} + 1 \right]} \right]^{-(K-N+1)} (1-q)^{N-2} q^{K-N+1} dq, \tag{65}$$

where $\left( \frac{S}{N} \right)_{opt} = \sigma_s^2$.

### 3.4 Probability of False Alarm

The $P_F$ is found by setting $(S/N)_{opt} = 0$ in Eq. (65). The following equation results:

$$P_F = \int_{T_1}^{1} \frac{K!}{(N-2)!(K-N+1)!} \left[1 + \frac{T_1}{q - T_1}\right]^{-(K-N+1)} (1-q)^{N-2} q^{K-N+1} dq. \quad (66)$$

If we change the integration dummy variable to $q_0 = (q - T_1)/(1 - T_1)$, Eq. (66) reduces to

$$P_F = (1 - T_1)^K. \quad (67)$$

## 4.0 RESULTS

Here, we present results on the detection probability $P_D$ for the GLRT and MLAD vs the independent parameters: the probability of false alarm $P_F$; the steady state ($K = \infty$) S/N output power ratio of the matched filter $(S/N)_{opt}$; the number of independent samples per channel $K$ of the secondary data; and the number of input channels $N$. We set $K = MN$, where $M$ is a positive integer, and $M$ is an independent parameter called the degrees of freedom (DOF) factor. Figures 2 through 17 show plots of $P_D$ vs $(S/N)_{opt}$ for $P_F = 10^{-6}$, $10^{-10}$, $N = 2, 5, 10, 30$, and the two detection schemes.

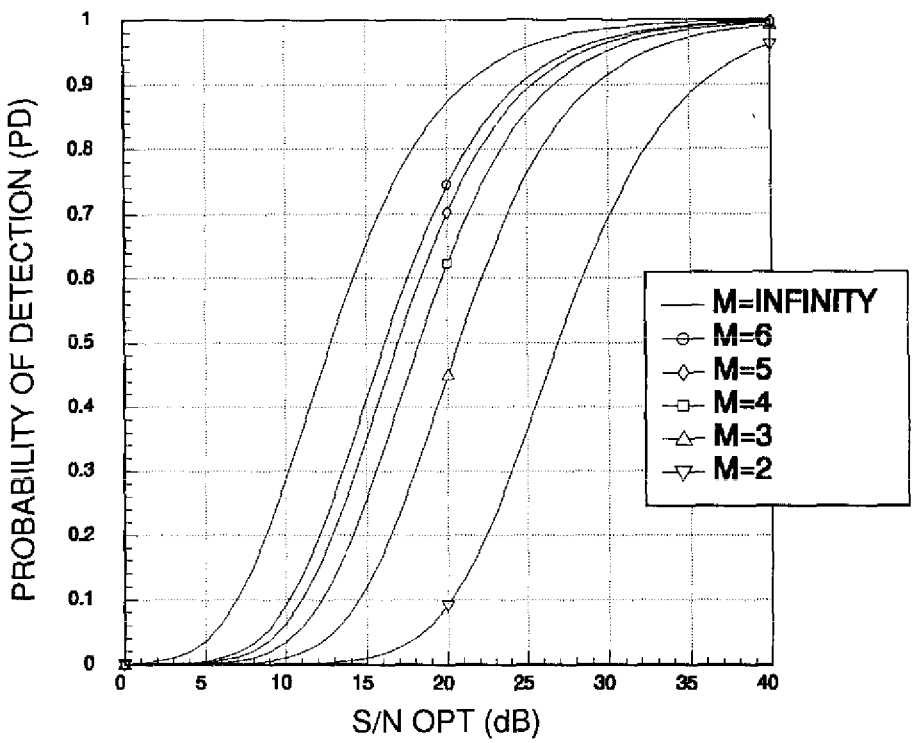


Fig. 2 — GLRT: $P_D$ for $N = 2$, $P_F = 1.D\text{-}6$

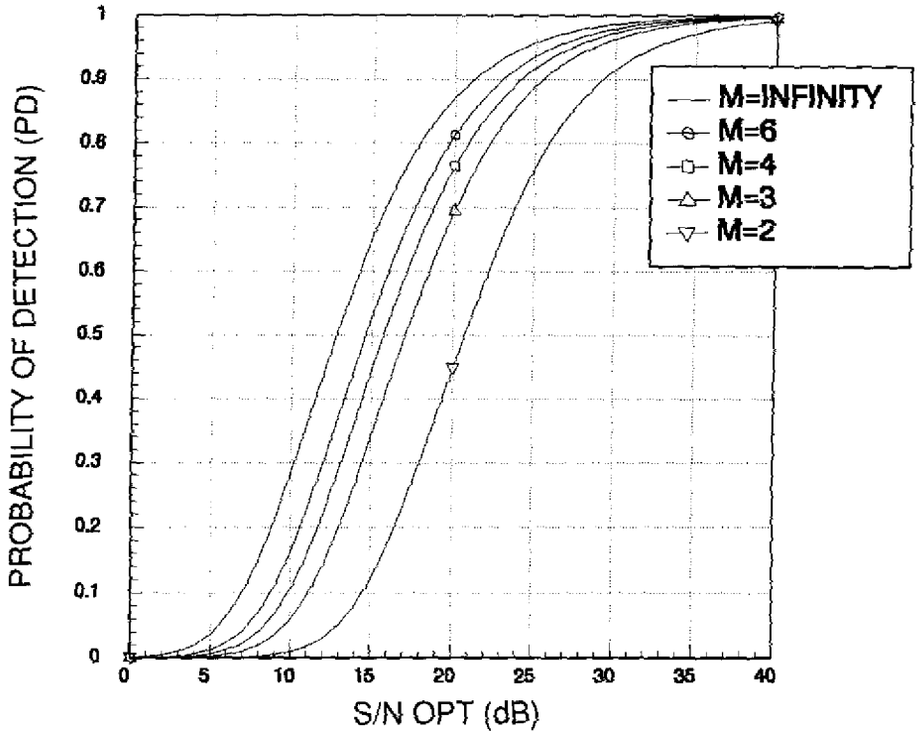


Fig. 3 — GLRT: $P_D$ for $N = 5$, $P_F = 1.D\text{-}6$

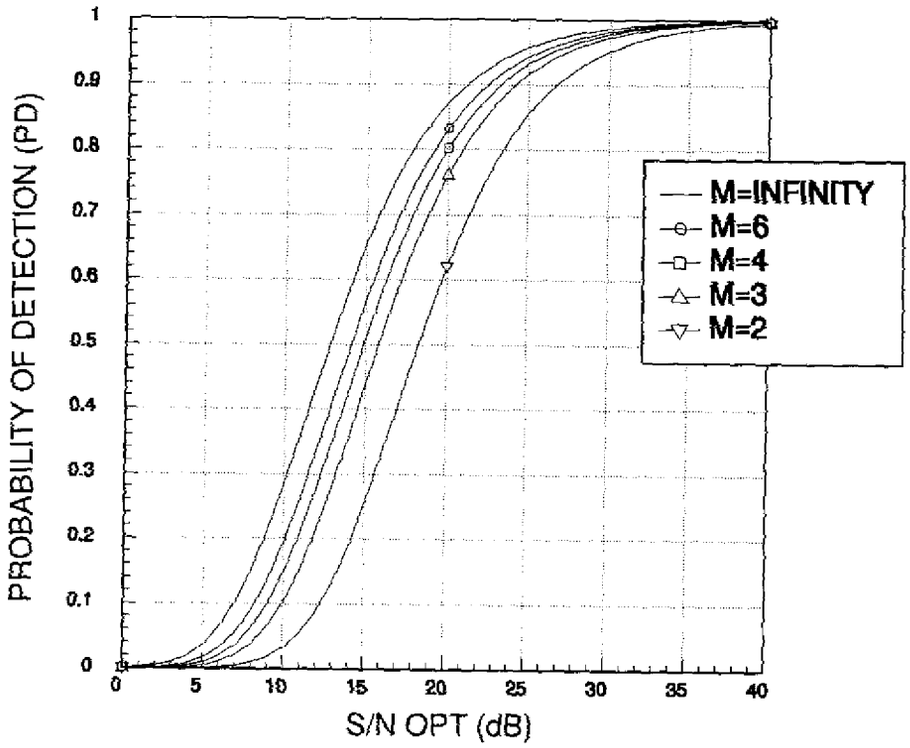


Fig. 4 — GLRT: $P_D$ for $N = 10$, $P_F = 1.D\text{-}6$

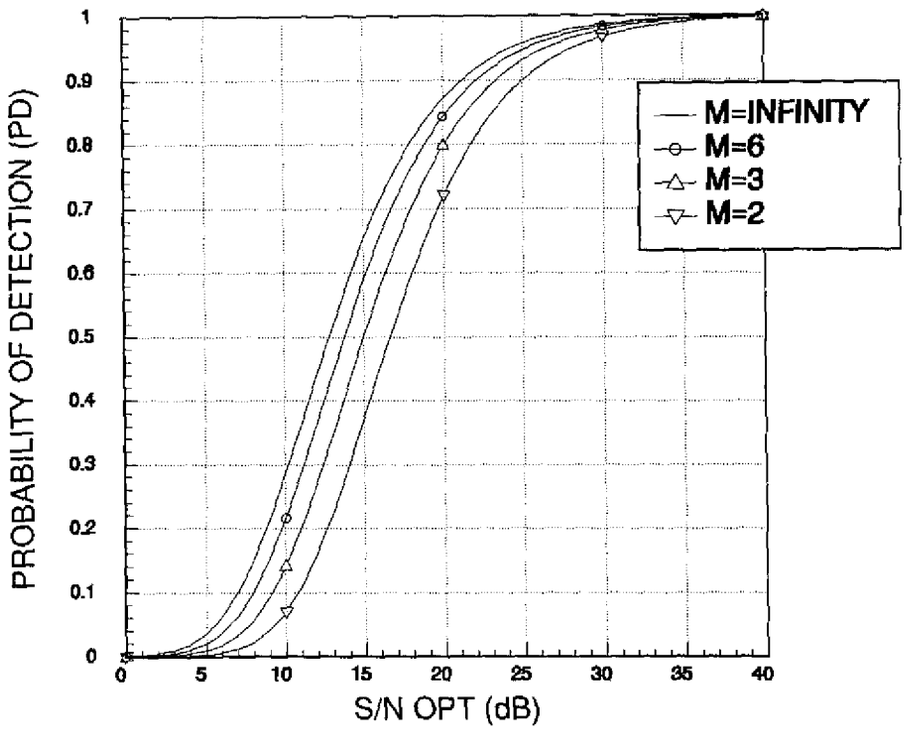


Fig. 5 — GLRT: $P_D$ for $N = 30$, $P_F = 1.D\text{-}6$

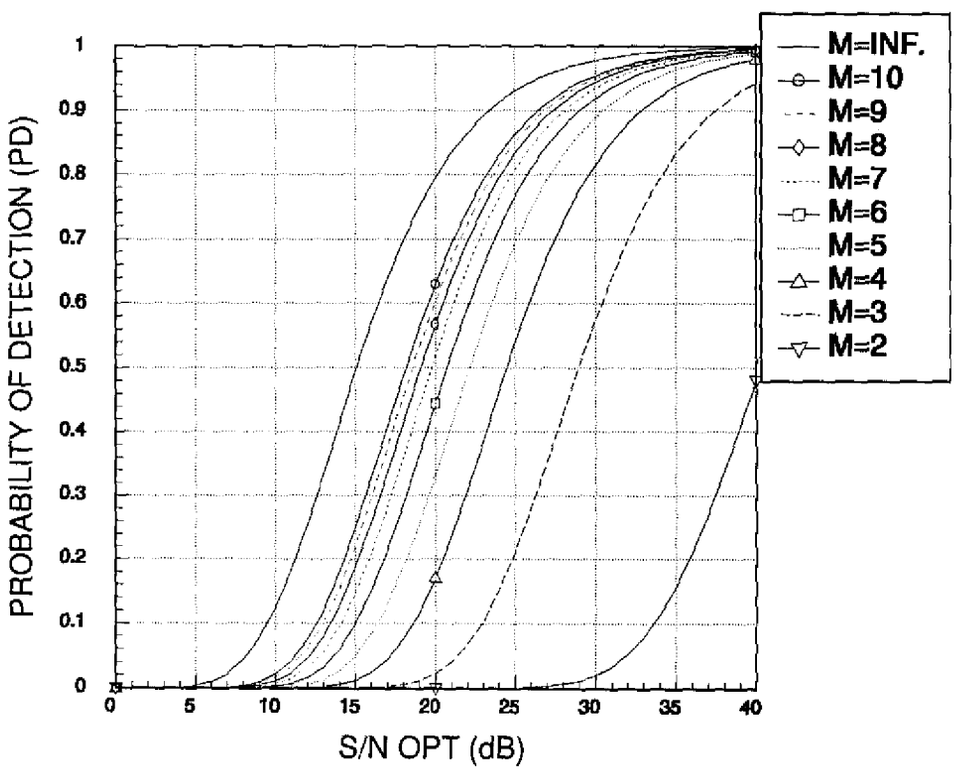


Fig. 6 — GLRT: $P_D$ for $N = 2$, $P_F = 1.D\text{-}10$

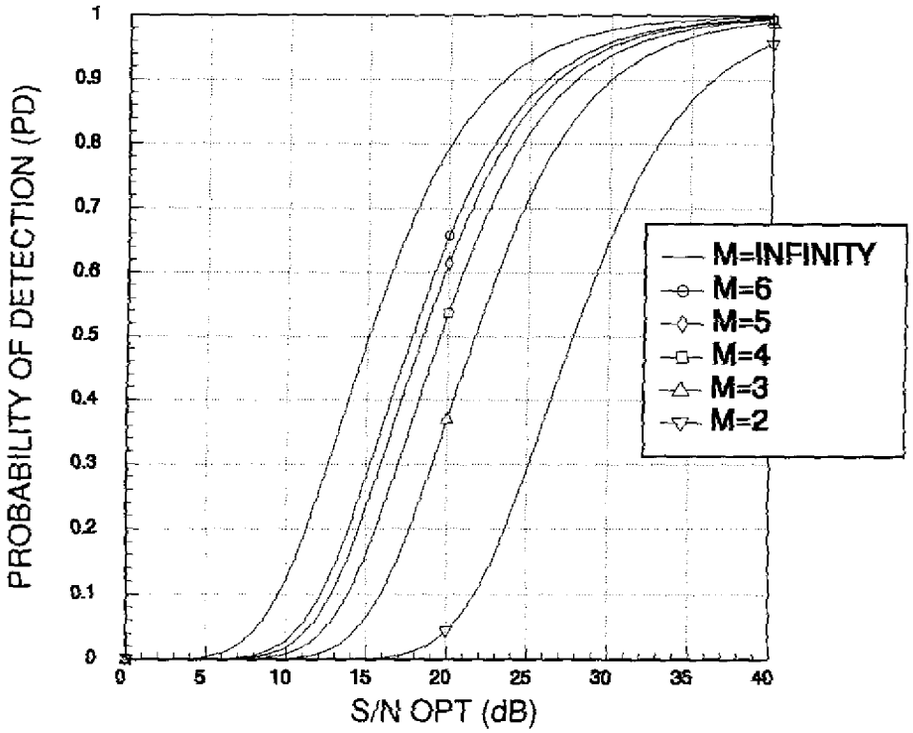


Fig. 7 — GLRT: $P_D$ for $N = 5$, $P_F = 1.D\text{-}10$

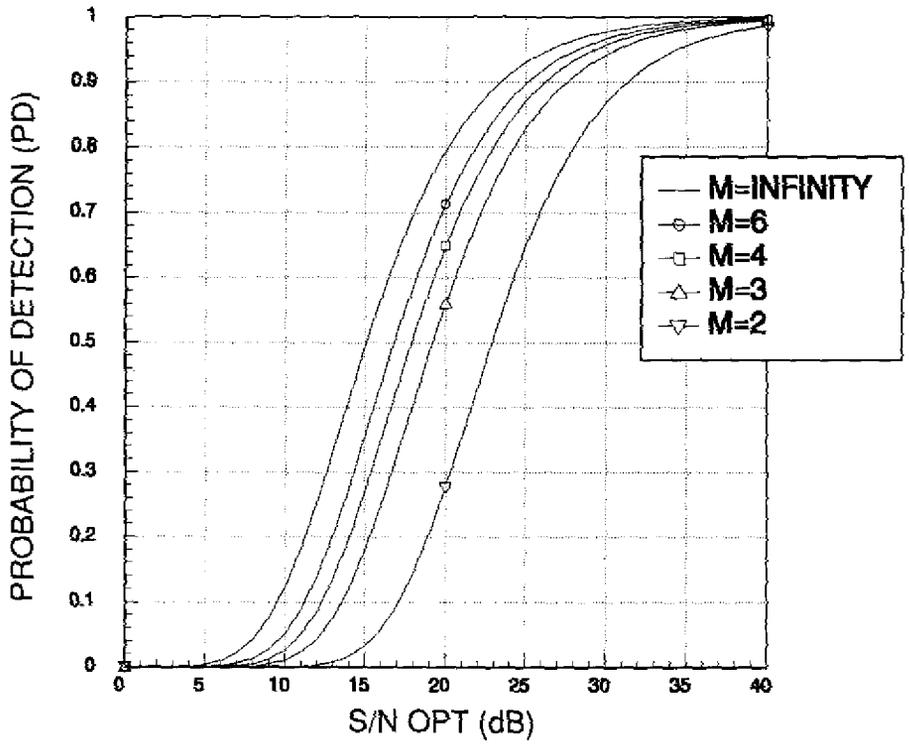


Fig. 8 — GLRT: $P_D$ for $N = 10$, $P_F = 1.D\text{-}10$

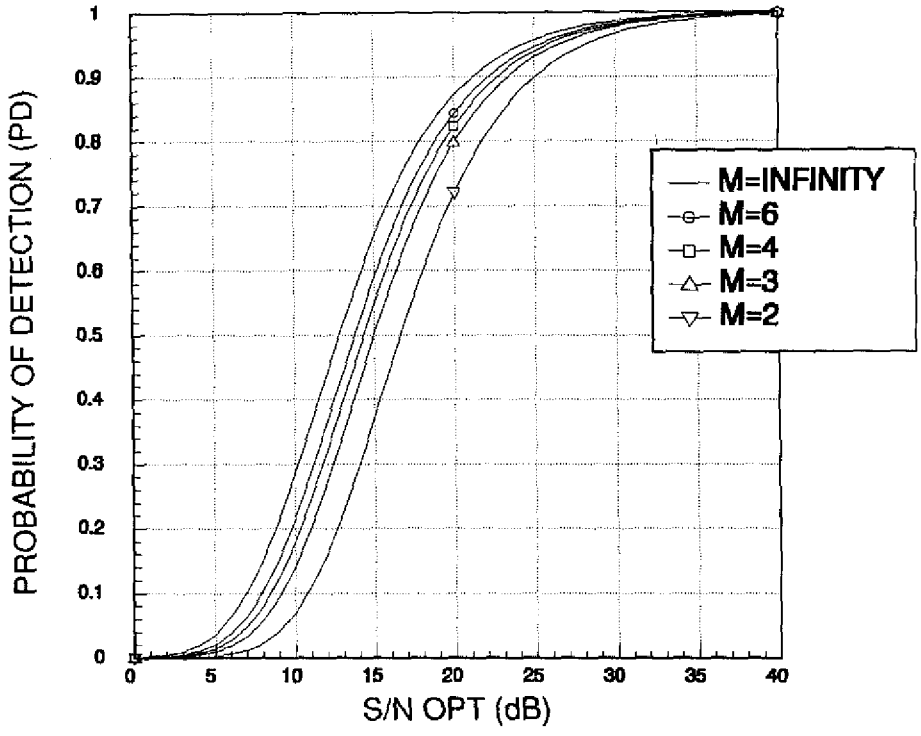


Fig. 9 — GLRT: $P_D$ for $N = 30$, $P_F = 1.D\text{-}10$

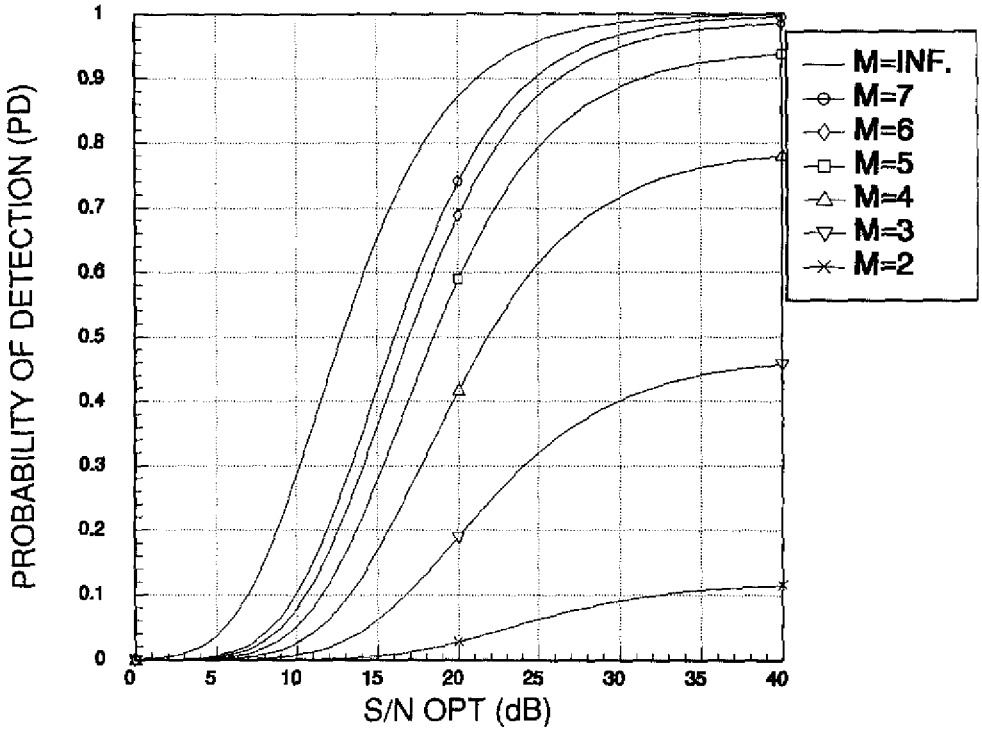


Fig. 10 — MLAD: $P_D$ for $N = 2$, $P_F = 1.D\text{-}6$

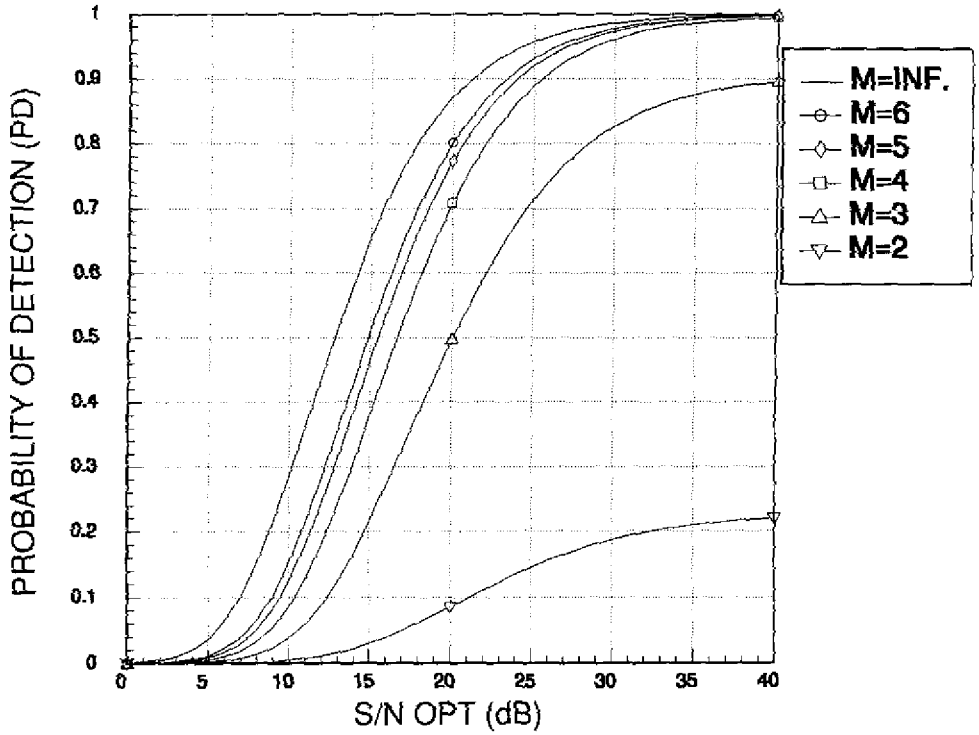


Fig. 11 — MLAD: $P_D$ for $N = 5$, $P_F = 1.D\text{-}6$

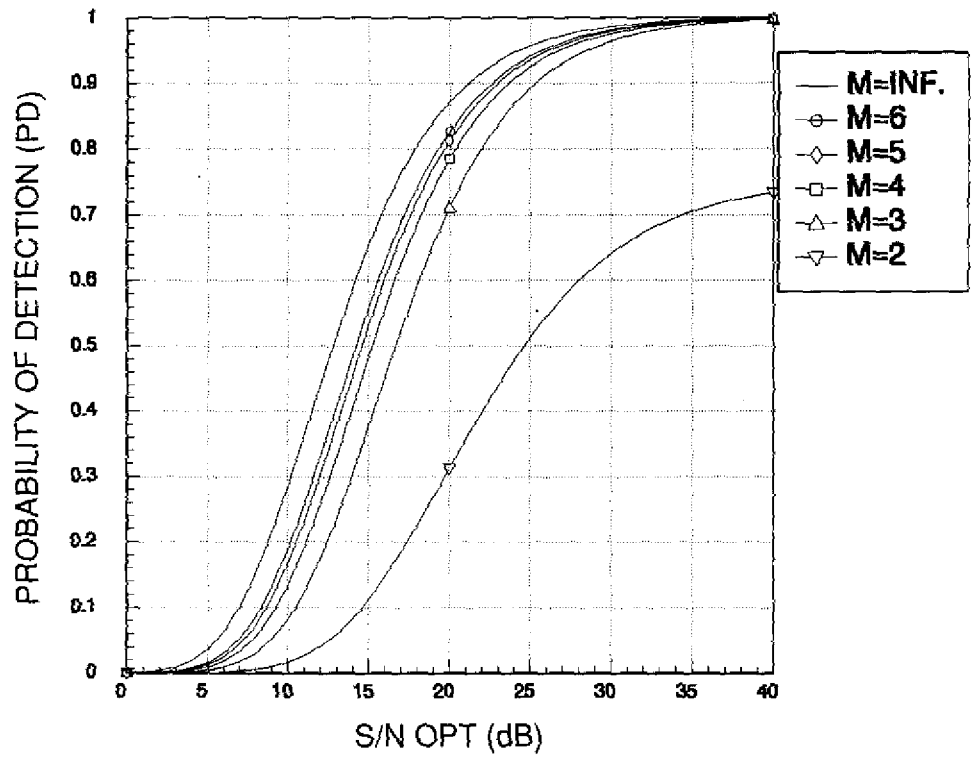


Fig. 12 — MLAD: $P_D$ for $N = 10$, $P_F = 1.D\text{-}6$

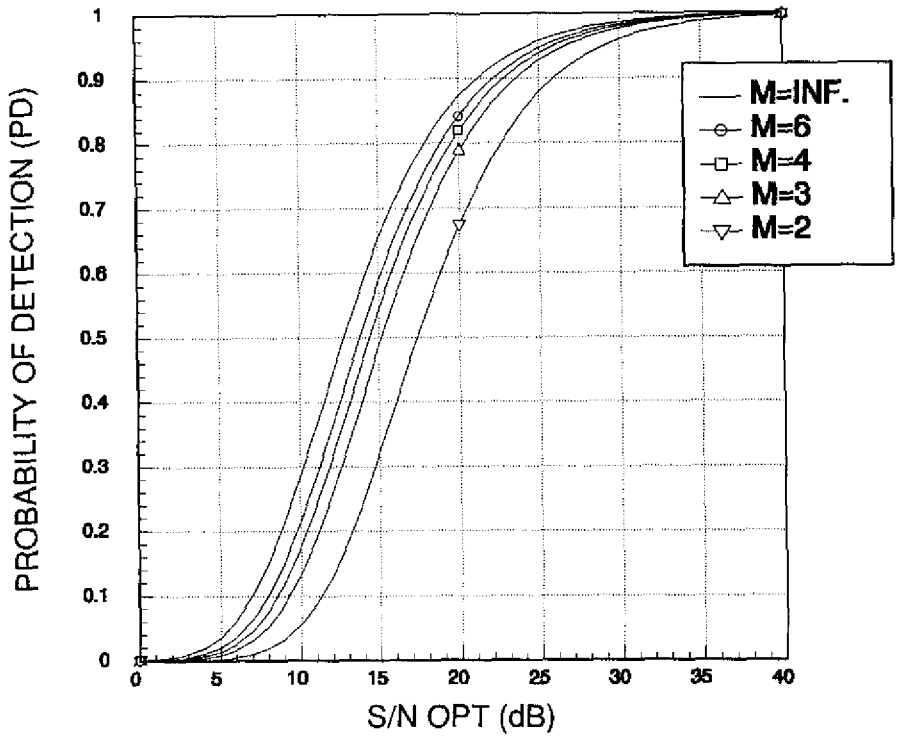


Fig. 13 — MLAD: $P_D$ for $N = 30$, $P_F = 1.D\text{-}6$

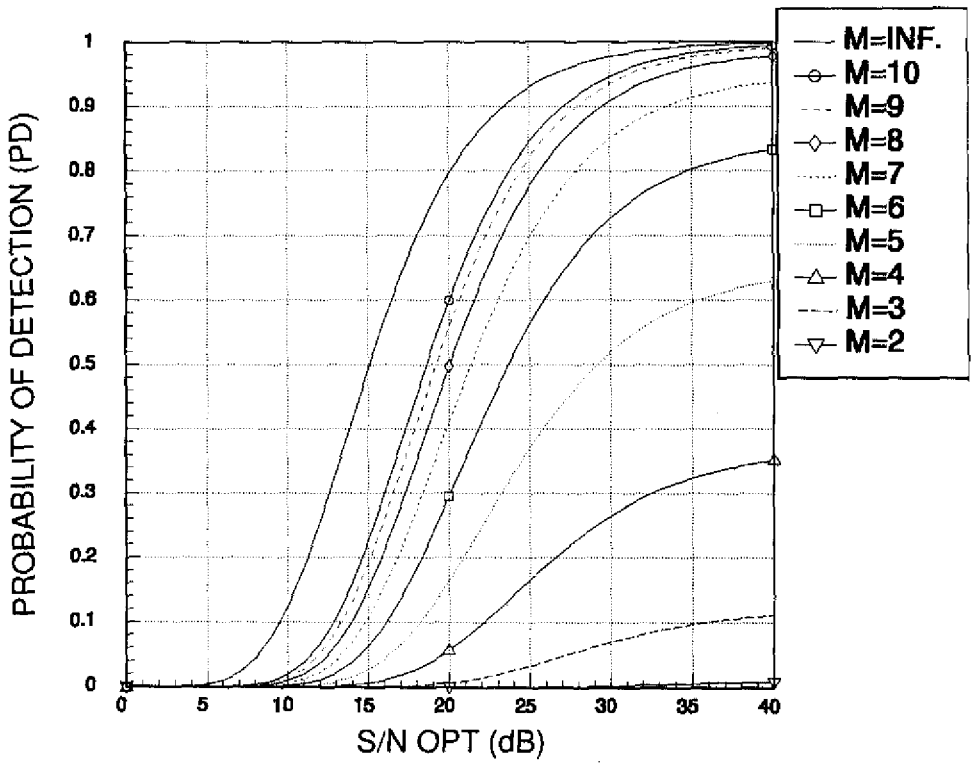


Fig. 14 — MLAD: $P_D$ for $N = 2$, $P_F = 1.D\text{-}10$

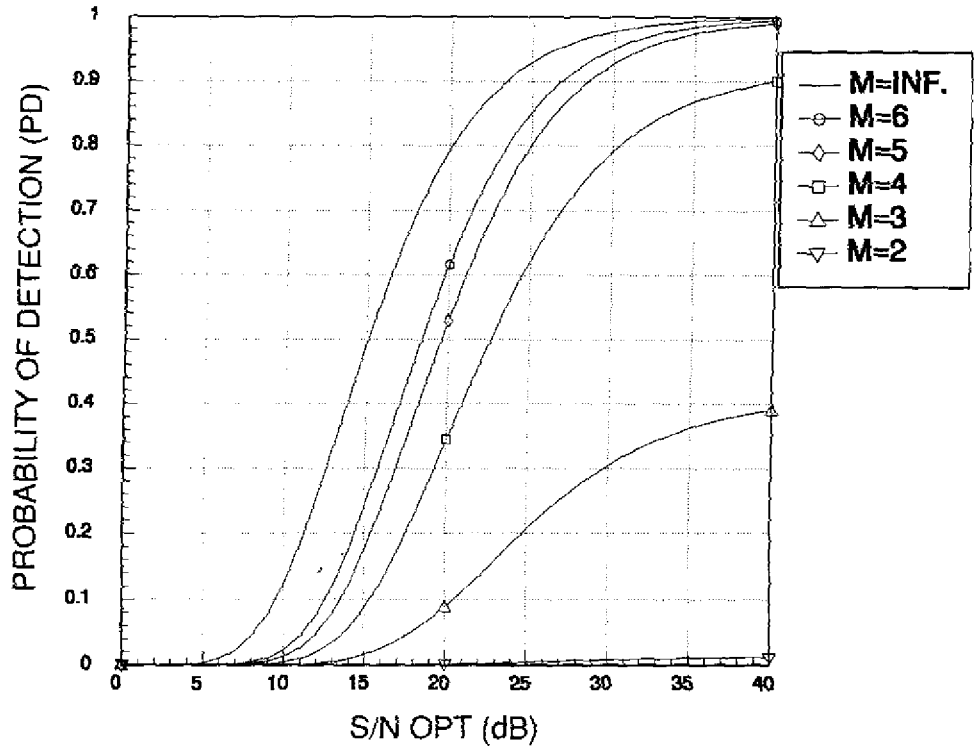


Fig. 15 — MLAD: $P_D$ for $N = 5$, $P_F = 1.D\text{-}10$

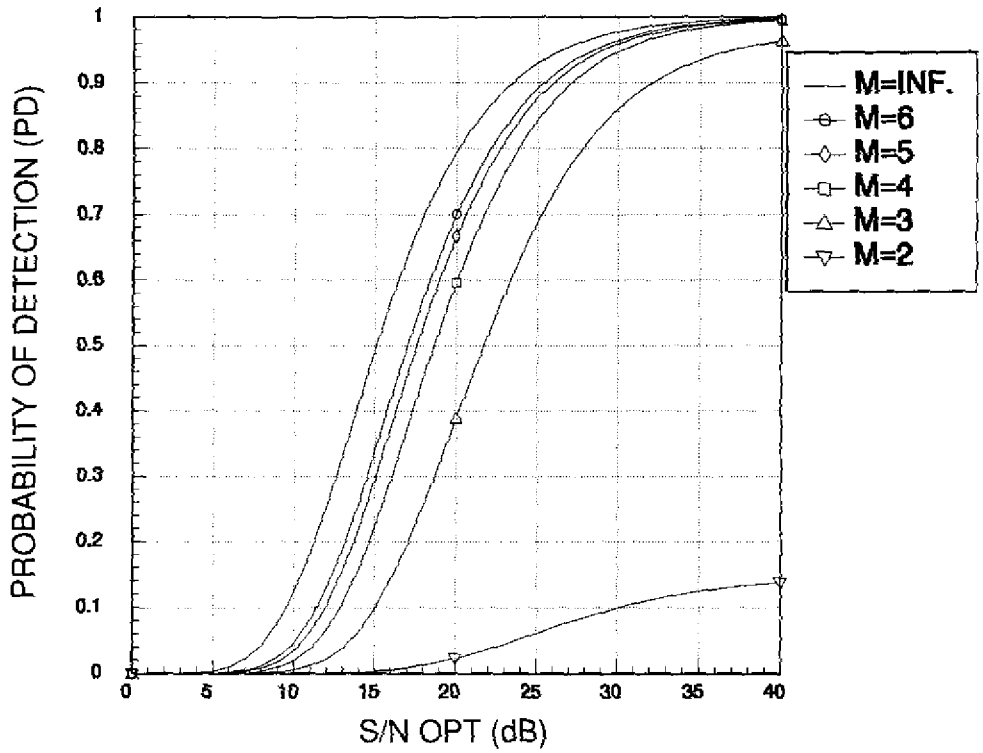


Fig. 16 — MLAD: $P_D$ for $N = 10$, $P_F = 1.D\text{-}10$

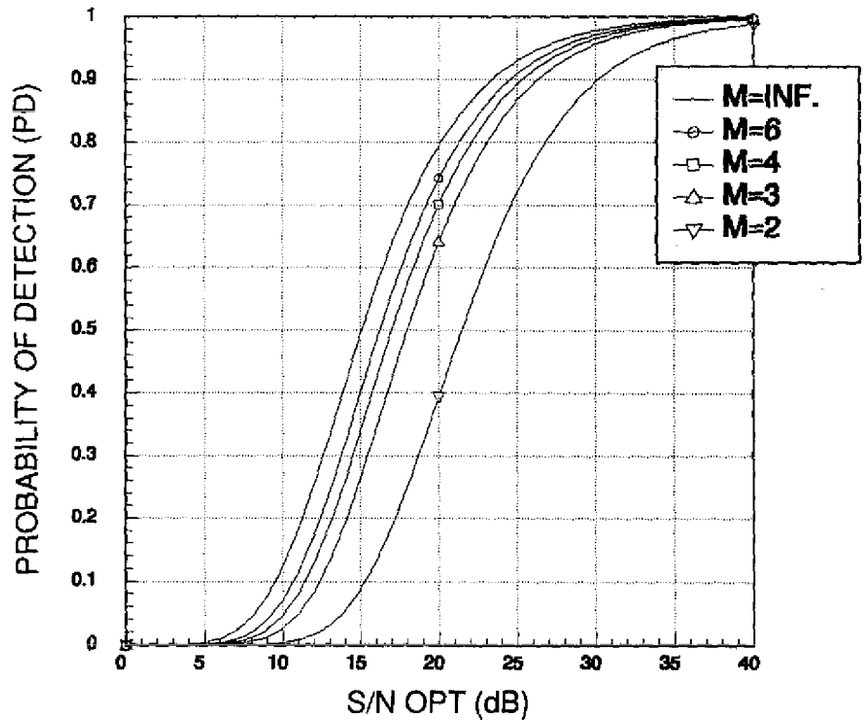


Fig. 17 -- MLAD: $P_D$ for $N = 30$, $P_F = 1.D\text{-}10$

Kelly [1] defines the (S/N) loss of an adaptive detector as the difference required to obtain a given $P_D$ between a steady-state ($M = \infty$) detector and the transient state with all other independent parameters being equal. We note that as $K \rightarrow \infty$ (or $M \rightarrow \infty$), the GLRT and MLAD are identical and, hence, their $P_D$ performance is identical. We define $M_{3\,\mathrm{dB}}$ to be the DOF factor such that the (S/N) loss is nearest to 3 dB. For both the GLRT and MLAD, we make the following observations from Figs. 2 through 17.

(1) Both detectors are slower to converge to their optimal value ($M = \infty$) for the lower ordered matched filters. For example, for the GLRT for most $P_D$s (0.1 – 0.9), $P_F = 10^{-6}$, if $N = 2$, then $M_{3\,\mathrm{dB}} = 6$, if $N = 30$, $M_{3\,\mathrm{dB}} = 2$.

(2) There is a diminishing return in convergence by using a larger DOF factor, i.e., there is more improvement in performance in going from $M$ to $M + 1$ than $M + 1$ to $M + 2$.

(3) Convergence slows for decreasing $P_F$. For example, for the MLAD for most $P_D$ (0.1 – 0.9) and $N = 5$, if $P_F = 10^{-6}$, then $M_{3\,\mathrm{dB}} = 5$; if $P_F = 10^{-10}$, then $M_{3\,\mathrm{dB}} = 6$.

We note that these trends were also observed by Kelly [1] for the GLRT, where the signal is modeled as having a constant amplitude.

In comparing the GLRT with the MLAD, we make the following observations from Figs. 2 through 17:

(4) The $P_D$ performance for the GLRT is always superior to the $P_D$ performance of the MLAD with all independent parameters being equal between the two detectors.

(5) For low DOF factor ($M = 2, 3$), the GLRT has much better $P_D$ performance than the MLAD with all other independent parameters being equal between the two detectors. For example, if $N = 2$, $P_F = 10^{-6}$, $(S/N)_{opt} = 30\ dB$, and $M = 2$ for the two detectors, then for the GLRT, $P_D = 0.7$ and for the MLAD, $P_D = 0.09$.

(6) The $M_{3\,dB}$ for the GLRT and MLAD are approximately the same with the greatest difference occurring for low DOF factors ($N = 2$). For example, for most $P_D$s ($0.1 \sim 0.9$), $P_F = 10^{-6}$, if $N = 2$, then $M_{3\,dB} = 6$ for the GLRT and $M_{3\,dB} = 7$ for the MLAD. For the same case but $N = 30$, the $M_{3\,dB}$ is identical for both the GLRT and MLAD.

The GLRT outperforms the MLAD because the MLAD's MLD samples $\hat{\mathbf{w}}^H \mathbf{x}_k$, $k = 1, 2, \ldots, K$ are being contaminated by the desired signal in $\mathbf{x}$ since $\hat{\mathbf{w}}$ is computed by using a sampled covariance matrix $\hat{R}_0$, which contains the desired signal in some form. This results in an increased average power in the MLD's samples, which results in an increase in the MLD's threshold, which results in fewer detections. The increase in the residue power in the range cells (for the radar problem) around the primary range cell because of signal contamination was also reported in Lewis and Kretschmer [13] and Gerlach [14].

Of the three comparisons between the GLRT and the MLAD, observation (6) above is possibly the most significant. As we will show in section 5, the GLRT requires more numerical computions to implement than the MLAD. However, as we note from the observation (6), the required number of samples will be approximately the same for both detector schemes in order to have only a 3 dB S/N detection loss.

## 5.0 COMPUTATIONAL CONSIDERATIONS

Here we briefly compare the computational requirements for the MLAD and GLRT detection schemes. First we consider the GLRT. We modify the form given by Eq. (1). Now

$$\hat{R}_x^{-1} = (\hat{R}_0 - \mathbf{x}\,\mathbf{x}^H)^{-1} = \hat{R}_0^{-1} + \frac{\hat{R}_0^{-1}\,\mathbf{x}\,\mathbf{x}^H\,\hat{R}_0^{-1}}{1 - \mathbf{x}^H\,\hat{R}_0^{-1}\,\mathbf{x}}. \tag{68}$$

If the expression on the right-hand side of Eq. (68) is substituted into Eq. (1) for $\hat{R}_x^{-1}$, after some simplification, the GLRT detection rule becomes

$$\frac{|\,\mathbf{s}^H\,\hat{R}_0\,\mathbf{x}\,|^2}{\mathbf{s}^H\,\hat{R}_0^{-1}\,\mathbf{s}} \cdot \frac{1}{(1 - \mathbf{x}^H\,\hat{R}_0^{-1}\,\mathbf{x})} \underset{H_0}{\overset{H_1}{\gtrless}} t', \tag{69}$$

where $t' = t/(1-t)$. If we compare this equivalent form of the GLRT to the MLAD form as given by Eq. (52), we see that the GLRT test statistic is merely the MLAD test statistic multiplied by $(1 - \mathbf{x}^H \hat{R}_0^{-1} \mathbf{x})^{-1}$. It is straightforward to show that $0 \leq 1 - \mathbf{x}^H \hat{R}_0 \mathbf{x} \leq 1$, and thus the GLRT's test statistic is always greater than the MLAD's test statistic under $H_0$ or $H_1$.

We see that it is the additional computation of $1 - \mathbf{x}^H \hat{R}_0^{-1} \mathbf{x}$ for the GLRT that is the major computational difference between the two detection schemes. There are many schemes for implementing the adaptive matched filter in open-loop fashion: SMI, Gram-Schmidt, Givens rotation, Householder, etc., combined with a block processing or sliding-window technique (of which there are many variations). No efficient method has been proposed in implementing the GLRT, although a variant that uses the normalized, fast orthogonalization network described in Refs. 15 and 16 could be used to compute $\hat{R}_0^{-1} \mathbf{x}$ and, thus, $\mathbf{x}^H \hat{R}_0^{-1} \mathbf{x}$. However the implementation complexity of this scheme is significantly greater than that of the AMF. The efficient implementation of the GLRT is left as a topic of future research.

## 6.0 SUMMARY

Two schemes for adaptive detection have been analyzed and compared: Kelly's GLRT and the MLAD. Detection performance $P_D$ was predicted for the two schemes under the assumptions that the input noises were Gaussian random variables that were temporally independent but spatially correlated, and the desired signal was Rayleigh distributed. $P_D$ was computed as a function of the false alarm probability, the number of input channels, the number of independent samples-per-channel, and the matched filtered output S/N power ratio.

In this analysis, the GLRT was shown to have better detection performance than the MLAD. The difference in detection performance increases as one uses fewer input samples. However, the required number of samples necessary to have only a 3 dB detection loss for both detection schemes is approximately the same. This is significant since for the present, the MLAD is considerably less complex to implement than the GLRT.

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