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Results on the Detection of Signals in Spherically Invariant Random Noise

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A relationship between well-known estimator-correlator results from detection theory and the detection of signals in spherically invariant random noise is described. This relationship gives a general result concerning the structure of the likelihood ratio for this detection problem. Furthermore, an alternate formulation of both the likelihood ratio and the optimal estimator that arises in the estimator-correlator structure is given. This alternate formulation is important because it yields a closed-form solution for this optimal estimator without requiring explicit knowledge of a prior distribution for the unknown quantity being estimated. Since interest in modeling actual noise processes as spherically invariant random processes has recently appeared, the results here should help not only to give insight into the optimal detection structure in such noise but also to give guidance in formulating suboptimal detectors for problems of practical interest.

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CONTENTS

RESULTS ON THE DETECTION OF SIGNALS IN SPHERICALLY INVARIANT RANDOM NOISE

INTRODUCTION

Vershik [1] introduced the class of spherically invariant random processes (SIRP) as part of a study of properties exhibited by Gaussian random processes. Since that introduction, papers exploring various aspects of this class of processes have appeared. These studies have examined questions regarding representations of such processes [2–5] as well as estimation and detection in such processes [6–8]. Recently, this work has been applied to the problem of modeling certain types radar clutter processes [9,10].

In this study, we reexamine the structure of the optimal processor, i.e., the likelihood ratio, for detecting a known signal in additive noise modeled as a SIRP. The structure of this optimal detector is that of the estimator–correlator, a result that follows by direct application of a result attributable to Schwartz [11]. However, for the sake of clarity, a simple derivation of this result is presented. In addition, we discuss how the estimator–correlator structure leads to suboptimal structures that have been obtained by previous investigators. We also briefly point out that the locally optimum detector for this problem consists of the same optimal estimator used in a different manner than in the optimal detector. Finally, we derive an alternate formulation of the likelihood ratio and the optimal estimator that appears in the estimator–correlator structure. This alternate formulation is important because it allows the optimal estimate to be computed without knowledge of a prior density for the random parameter. Instead, knowledge of a certain marginal probability density function (pdf) is required. In practice, this pdf is more likely to be available to the investigator than is the prior density.

STRUCTURE OF THE OPTIMAL DETECTOR

In this section, the optimal detection structure for detecting an additive signal in a spherically invariant random process is derived and shown to be a general structure applicable to the whole class of processes. The resulting structure is a function of an optimal estimator of a random quantity. This structure reveals an intimate relationship between optimal detection and optimal estimation for this class of processes.

Consider the following hypothesis test:

$$H_o\colon\ x = y$$

$$H_1\colon\ x = y + s$$

where

y is an m–dimensional complex noise vector

s is an m–dimensional complex noise complex vector

$= \beta\hat{s},\ \beta = \gamma e^{j\phi} =$ complex number.

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From this point only complex processes are examined. However the corresponding results for real processes are easily obtainable from our approach.

If x is a sample from a SIRP, then x has the following pdf under hypothesis i ($i = 0,1$):

$$f_i(x) = \frac{1}{(2\pi)^m|\Phi|}\int_0^\infty \frac{1}{\tau^m}\exp\left\{-\frac{q_i}{2\tau}\right\} f_T(\tau)d\tau, \tag{1}$$

where

$$q_i = (x-s)^t\,\Phi^{-1}\,(x-s),\quad i=1$$
$$= x^t\Phi^{-1}x,\quad i=0$$

Φ is the normalized correlation matrix

τ is the variance of underlying Gaussian process

$f_T(\tau) = pdf$ of τ

$||$ is the matrix determinant

t is the complex conjugate transpose.

Clearly Eq. (1) represents the pdf of a conditional Gaussian process with random variance τ, averaged over the variation in τ. Close examination of the conditional Gaussian pdf shows that it is a multivariate member of the exponential family, i.e., the pdf is of the form

$$f_i(x|\tau) = c(\tau)h(x)\exp\left\{\sum_{j=1}^{l}\eta_j(\tau)T_j(x)\right\}, \tag{2}$$

where in our case

$$c(\tau) = \frac{1}{(2\pi)^m\ |\Phi|\,\tau^m}$$

$$h(x) = 1$$

$$l = 1$$

$$\eta_1(\tau) = \frac{1}{\tau}$$

$T_1(x)$ is a sufficient statistic for $\dfrac{1}{\tau} = -\dfrac{q_i}{2}$

Our first result follows by direct application of a general result given by Schwartz [11]. However, we now give a brief derivation.

If we reparameterize our conditional Gaussian pdf by setting $\alpha = 1/\tau$, we obtain

$$f_i(x) = \frac{1}{(2\pi)^m|\Phi|}\int_0^\infty \alpha^m \exp\{-\alpha z_i\} f_\alpha(\alpha) d\alpha, \tag{3}$$

where we have substituted $z_i = q_i/2$. If we now differentiate Eq. (3) by z_i, we obtain

$$\frac{df_i}{dz_i} = -\frac{1}{(2\pi)^m|\Phi|}\int_0^\infty \alpha^{m+1} \exp\{-\alpha z_i\} f_\alpha(\alpha) d\alpha. \tag{4}$$

Since the minimum mean–square estimate of α is given by the conditional mean estimate (CME) [12],

$$E_i[\alpha|x] = \frac{1}{(2\pi)^m|\Phi| f_i(x)}\int_0^\infty \alpha^{m+1} \exp\{-\alpha z_i\} f_\alpha(\alpha) d\alpha, \tag{5}$$

and we immediately obtain

$$E_i[\alpha|x] = -\frac{1}{f_i(x)}\frac{df_i}{dz_i}\Bigg]_{z_i=\frac{q_i}{2}} \tag{6a}$$

$$= -\frac{d\ln f_i}{dz_i}\Bigg]_{z_i=\frac{q_i}{2}}. \tag{6b}$$

At this point we note that we may use a dummy variable z in place of z_i in our formulation, i.e., we now use $z = q_i/2$ instead of $z_i = q_i/2$. From Eq. 6(b) we then obtain

$$\int_0^{\frac{q_i}{2}} E_i[\alpha|z]dz = -\ln f_i\Big]_{z=0}^{z=\frac{q_i}{2}}, \tag{7a}$$

where

$$E_i[\alpha|z] = -\frac{d\ln f_i}{dz_i}. \tag{7a}$$

Since under either hypothesis we have $f_i(x) = g(q_i)$, i.e., the subscript i indicates that we have a function of q_i , and since we are using $z = q_i/2$, it is easy to show that the form of $E_i[\alpha|z]$ is the same under both hypotheses. We denote this form $E[\alpha|z]$ and differentiate between the two hypotheses by evaluating this form at the different points, $z = q_i/2, i = 0, 1$. Therefore, since $-\ln f_i\Big]_{z_i=0}$ is the same under both hypotheses, we obtain finally

$$\ln \frac{f_1(x)}{f_o(x)} = \int_0^{\frac{q_0}{2}} E[\alpha|z]dz - \int_0^{\frac{q_1}{2}} E[\alpha|z]dz. \tag{8}$$

The structure of the likelihood ratio is therefore given by

$$\Lambda(x) = \frac{f_1(x)}{f_o(x)} = \exp\left\{\int_0^{\frac{q_0}{2}} E[\alpha|z]dz - \int_0^{\frac{q_1}{2}} E[\alpha|z]dz\right\}. \tag{9}$$

Equation (9) shows explicitly the relationship between optimal estimation and optimal detection for this class of processes.

A GEOMETRIC/FUNCTIONAL INTERPRETATION OF THE OPTIMAL DETECTION STRUCTURE

In this section, we interpret the optimal detection structure as the area under a curve in a certain coordinate system. This interpretation then leads to the conclusion that the optimal detection structure is equivalent to a comparison of the matched filter with a variable threshold, which is a function of q_0, the quadratic form in x, and the detection threshold T that yields the desired probability of false alarm.

From Eq. (8), we may write the optimal detection structure as

$$\int_{q_1/2}^{q_0/2} E[\alpha \mid z]\, dz \underset{H_0}{\overset{H_1}{\gtrless}} T. \tag{10}$$

Note that since Φ is in general a positive definite matrix, the limits of integration are always positive. Let us assume temporarily that $q_0 > q_1$ and $T > 0$. Also, since α is a nonnegative random variable, the integrand $E[\alpha \mid z]$, which is a function of z, is also nonnegative for positive z. Thus, the structure in Eq. (10) compares the area under a curve in the $E[\alpha \mid z] - z$ coordinate system to a threshold T. Figure 1 shows this comparison.

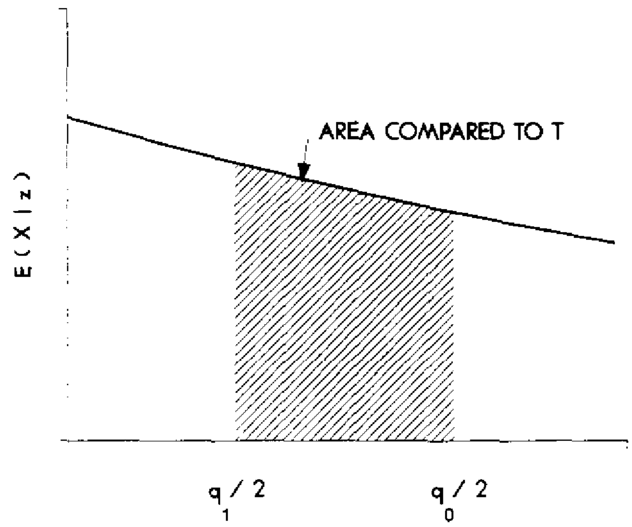

Fig. 1 — Geometric interpretation of optimal detector

Now, since $E[\alpha|z]$ is nonnegative for positive z and the limits $q_1/2$ and $q_0/2$ are always positive, this area is $< T$ if and only if $q_1/2 > q_0/2 - K$, i.e., $q_1/2$ is closer than K units to $q_0/2$. Note, however, that unless $E[\alpha|z]$ is a constant for positive z the number of units K is actually a function of the location of $q_0/2$ on the z axis. Thus, the comparison indicated in Eq. (10) is equivalent to the comparison

$$\frac{q_0}{2} - \frac{q_1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} f(q_0, T), \tag{11}$$

where $f(q_0, T)$ is a variable threshold that is a function of both q_0 and T. After substitution for the quadratic forms, Eq. (11) becomes

$$\mathrm{Re}(s^t\Phi^{-1}x) \underset{H_0}{\overset{H_1}{\gtrless}} f(q_0, T) + \frac{1}{2}\, s^t\Phi^{-1}s, \tag{12}$$

which is a matched filter compared to a variable threshold. Theoretically this function $f(q_0, T)$ may be obtained by finding the function $f(q_0)$ that satisfies

$$\int_{q_0/2-f(q_0)}^{q_1/2} E[\alpha|z]dz = T. \tag{13}$$

At this point, however, we abandon the geometric interpretation of this detection structure and simply examine the structure given by Eq. (10) from a functional point of view. Define a function $F(q)$ as

$$F(q) = \int_0^{\frac{q}{2}} E[\alpha|z]dz. \tag{14}$$

Since, as we have previously observed, $E[\alpha|x]$ is nonnegative for positive z and q is positive, the function $F(q)$ that we have defined is a monotonically increasing function of q. In this new notation, we rewrite the log-likelihood test given in Eq. (10) as

$$F(q_0) - F(q_1) \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{15a}$$

or equivalently,

$$F(q_0) - T \underset{H_0}{\overset{H_1}{\gtrless}} F(q_1). \tag{15b}$$

Nov , since $F(q)$ is monotonically increasing, we obtain

$$F^{-1}(F(q_0) - T) \underset{H_0}{\overset{H_1}{\gtrless}} q_1. \tag{16}$$

Finally, since

$$q_1 = x^t\Phi^{-1}x - 2\,\mathrm{Re}(s^t\Phi^{-1}x) + s^t\Phi^{-1}s \tag{17}$$

$$= q_0 - 2\,\mathrm{Re}(s^t\Phi^{-1}x) + s^t\Phi^{-1}s,$$

we obtain after substitution in Eq. (16)

$$\mathrm{Re}(s^t\Phi^{-1}x) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{q_0}{2} - \frac{1}{2}F^{-1}(F(q_0) - T) + \frac{1}{2}s^t\Phi^{-1}s. \tag{18}$$

Comparison of this result with the result in Eq. (12) reveals that

$$f(q_0, T) = \frac{q_0}{2} - \frac{1}{2}F^{-1}(F(q_0) - T). \tag{19}$$

This equation may or may not be solvable for a specific problem. However, the main advantage to this interpretation is the insight it gives into the operation of the optimal detector. Note that in this functional interpretation, we made no assumptions about either the ordering of q_0 and q_1 or the sign of T.

Observe that if, in Eq. (10), $T = 0$, which corresponds to a unity threshold for the likelihood ratio, then the log-likelihood ratio reduces to

$$q_0 \underset{H_0}{\overset{H_1}{\gtrless}} q_1, \tag{20}$$

which in turn is equivalent to the matched filter and is independent of the prior pdf $f_\alpha(\alpha)$. This result, which follows trivially from Eq. (10), was first given by Yao [3].

To further examine the consequences of the estimator-correlator structure, we now assume that $T > 0$, which is not overly restrictive in many situations. Immediately we may observe that H_0 is always chosen when $q_0 < q_1$. Let the functions $g(q)$ and $h(y)$ be defined as

$$g(q) = \frac{1}{(2\pi)^m|\Phi|} \int_0^\infty \alpha^m \exp\left\{-\frac{q}{2}\alpha\right\} f_\alpha(\alpha)d\alpha \tag{21}$$

and

$$h(y) = -\ln y. \tag{22}$$

With these definitions, we may write

$$F(q) = h(g(q)) + C, \tag{23}$$

where $F(q)$ is the function defined in Eq. (14) and C is the constant $h(g(0))$.

Straightforward application of the chain rule from calculus shows

$$\frac{d^2F(q)}{dq^2} = \frac{\left(\frac{dg}{dq}\right)^2 - g(q)\frac{d^2g}{dq^2}}{g^2(q)}. \tag{24}$$

Furthermore, since $g(q)$ is positive, we may show that this second derivative is ≤ 0 by showing that the numerator is ≤ 0. The numerator may be shown to be ≤ 0 by using the definition of the function $g(q)$ given by Eq. (21), performing the indicated differentiations, and applying the Schwartz inequality. Thus we have the condition

$$\frac{d^2F(q)}{dq^2} \leq 0. \tag{25}$$

At this point we assume $q_0 \leq q_1$ and we let $f(q) = dF(q)/dq$. From the result indicated in Eq. (25), it follows that $f(q)$ is a monotonically decreasing function of q and that

$$(q_0 - q_1)f(q_0) \leq \int_{q_1}^{q_0} f(q)dq. \tag{26}$$

Hence, if $(q_0 - q_1)f(q_0) > T$, the optimal test always chooses H_1. On the other hand, from Eq. (25) it also follows that

$$(q_0 - q_1)f(q_1) \geq \int_{q_1}^{q_0} f(q)dq. \tag{27}$$

Hence, if $(q_0 - q_1)f(q_1) < T$, the optimal test always chooses H_0. From these considerations we are led to the following observation. Whenever the detection threshold is such that $T > 0$, then the optimal detection procedure may be implemented in a sequential manner as follows:

1. If $q_0 \leq q_1$, choose H_0.
2. If $E_0[\alpha \mid x](q_0 - q_1) \geq T$, choose H_1.
3. If $E_1[\alpha \mid x](q_0 - q_1) < T$, choose H_0.
4. Otherwise, evaluate the log-likelihood ratio and decide accordingly.

From an implementation viewpoint, these considerations may reduct the computational load in making a decision, since the full log-likelihood ratio is not necessarily evaluated each time a decision is reached.

SUBOPTIMAL DETECTION STRUCTURES

A well-known consequence of the estimator-correlator structure is its implication for suboptimal detection. If the optimal estimator $E[\alpha|z]$ is unavailable or difficult to work with, a suboptimal detector can be formulated through the substitution of a suboptimal estimator in Eq. (9). For example, one could use a parametric representation of a prior density with a sufficient number of parameters to allow a large number of densities to be fit by this representation. The problem is then reduced to a problem in estimating the parameters of this prior density function, a task that could be implemented adaptively.

An alternate approach to suboptimal detection would be to use a maximum likelihood estimator (MLE) in place of the CME in the estimator-correlator structure. From the standard approach to finding MLEs [37], the MLE for α may be shown to be

$$\alpha_{\text{MLE}} = \left.\frac{m}{z}\right]_{z=\frac{q_i}{2}}, \quad i = 0.1. \tag{28}$$

Use of this estimator, instead of the CME, in Eq. (9) leads to

$$\Lambda_{\text{MLE}}(x) = \left[\frac{x'\Phi^{-1}x}{(x-s)'\Phi^{-1}(x-s)}\right]^m. \tag{29}$$

If we expand the quadratic form in the denominator and divide both the numerator and the denominator by the numerator, we obtain

$$\Lambda_{\mathrm{MLE}}(x) = \left[\frac{1}{1 - \dfrac{2\,\mathrm{Re}(s^t\Phi^{-1}x) - s^t\Phi^{-1}s}{x^t\Phi^{-1}x}}\right]^m . \tag{30}$$

For purposes of detection, use of this structure is equivalent to use of the following structure:

$$\frac{2\,\mathrm{Re}(s^t\Phi^{-1}x) - s^t\Phi^{-1}s}{x^t\Phi^{-1}x} . \tag{31}$$

Except that our result is for complex processes, our result is the same result obtained in Ref. 7 by an asymptotic argument. Therefore, the detector that is shown in Ref. 7 to be asymptotically optimal is shown here to be given by a suboptimal implementation of the estimator–correlator with an MLE substituted for the CME. This result is intuitively satisfying since in our problem we expect the MLE to be an estimator that is asymptotically equivalent to the CME.

A problem that is closely related to ours is the detection of signals of unknown amplitude and phase in Gaussian noise of unknown (but nonrandom) variance. Korado [14] examined this problem and presented the following structure as optimal for this problem:

$$\frac{|\hat{s}^H\Phi^{-1}x|}{\sqrt{x^H\Phi^{-1}x}} . \tag{32}$$

If we take Eq. (29) as a starting point, it is easy to show that the MLE of β (the unknown signal amplitude and phase) is given by

$$\beta_{\mathrm{MLE}} = \frac{\hat{s}^t\Phi^{-1}x}{\hat{s}^t\Phi^{-1}\hat{s}} . \tag{33}$$

Substitution of this result into Eq. (29) leads to

$$\left[\frac{1}{1 - \dfrac{|\hat{s}^t\Phi^{-1}x|^2}{\hat{s}^t\Phi^{-1}\hat{s}\; x^t\Phi^{-1}x}}\right]^m . \tag{34}$$

For purposes of detection, this structure is equivalent to Korado's result.

Finally a rather simplistic estimator is to estimate α by a constant α_0. Use of this estimator in the estimator–correlator leads to the following test:

$$\Lambda_c(x) = \exp\left\{\frac{\alpha_0}{2}[x^t\Phi^{-1}x - (x-s)^t\Phi^{-1}(x-s)]\right\}, \tag{35}$$

which easily is seen to lead to the classical matched filter. Clearly if the pdf of α is $\delta(\alpha - \alpha_0)$ (i.e., the variance is a known constant), then the estimator is correct and we obtain the expected result.

At this point we briefly examine the structure of another suboptimal detector, namely the locally optimal detector [15]. This detector is given in general by

$$\Lambda_{lo}(x) = \frac{d}{d\gamma} \; \Lambda(x) \Big]_{\gamma=0}, \tag{36}$$

where γ is the signal amplitude. By straightforward calculation, one may show that for our problem this structure is given by

$$\Lambda_{lo}(x) = E_0[\alpha \,|\, x] \text{ Re } (e^{j\phi} \hat{s}^t \Phi^{-1} x), \tag{37}$$

i.e., the locally optimum test is equal to the normalized matched filter multiplied by the CME of α.

Examination of the suboptimal tests given above shows them all to be essentially matched filters that are modified in some way to account for the unknown or statistical behavior of α . This observation coupled with our result that the optimal detector is the normalized matched filter compared to an adaptive threshold (i.e., Eq. (18)) leads us to conjecture that good suboptimal tests may be obtained by implementing the form

$$\text{Re}(s^t \Phi^{-1} x) \underset{H_0}{\overset{H_1}{\gtrless}} f_{so}(q_0), \tag{38}$$

where $f_{so}(q_0)$ is a good suboptimal function of q_0. However, we do not explore this issue any further here.

The results obtained in this study also emphasize a point about the detection structures for these types of processes (SIRPs). Even though sample vectors from these processes may be thought of as samples from a Gaussian process with an unknown and varying variance [9], the detection processing should involve estimation of the quantity α, not of the quantity τ (i.e., the estimate should not be of the variance directly). Since the Gaussian distribution is generally parameterized by its variance, the intuitive notion is to estimate the unknown variance; the results here show that this intuitive notion does not lead to an optimal detection structure. In effect, the parameter α is a more natural parameter than the variance (a conclusion that follows immediately if we consider the Gaussian distribution in the context of the exponential family with the so-called natural parameterization).

ALTERNATE FORMULATION OF LIKELIHOOD RATIO AND OPTIMAL ESTIMATOR

At this point we also present an alternate formulation of the likelihood ratio. From this formulation, we may see immediately a formulation of $E[\alpha|z]$ that does not require explicit knowledge of the prior pdf $f_\alpha(\alpha)$. Often, a prior pdf, being a pdf of a parameter rather than a pdf of directly observable data, is difficult to obtain. Fortunately, in the class of SIRPs, an alternate pdf, namely the marginal amplitude pdf of the complex process, may be used to solve the problem without knowledge of the prior pdf. In effect, the marginal amplitude pdf provides information that is equivalent to that of the prior pdf for this class of processes. This equivalence is fortunate, since in practice the marginal amplitude pdf, being a pdf of directly observable data, is usually easier to obtain than the prior pdf.

Since we are examining a complex SIRP, we have also under hypothesis H_0 the following relationship:

$$h(|x_j|) = \int_0^\infty \alpha |x_j| \exp\left\{-\alpha \frac{|x_j|^2}{2}\right\} f_\alpha(\alpha) d\alpha, \; j = 1 \ldots m \tag{39}$$

where

$|x_j|$ is the amplitude of the jth component of x
$h(|x_j|)$ = pdf of $|x_j|$, $j = 1 \ldots m$.

The interpretation of Eq. (39) is that since x (under H_0) represents a vector from a complex zero-mean Gaussian random process with random variance, the marginal pdf of the amplitude of each of the components of this vector is identical and Rayleigh with the same randomization of the Rayleigh parameter τ $(= 1/\alpha)$.

From Eq. (39) we may let $p = \dfrac{|x_j|^2}{2}$ and rearrange terms to obtain

$$g(p) = \frac{h(\sqrt{2p})}{\sqrt{2p}} = \int_0^\infty \exp\{-p\alpha\} \alpha f_\alpha(\alpha) d\alpha. \tag{40}$$

Differentiating with respect to p, we obtain in a straightforward fashion

$$(-1)^{m-1} \frac{d^{m-1}}{dp^{m-1}} g(p) = \int_0^\infty \alpha^m f_\alpha(\alpha) \exp\{-p\alpha\} d\alpha. \tag{41}$$

Comparing this result with Eq. (3), we obtain finally

$$f_i(x) = \frac{(-1)^{m-1}}{(2\pi)^m |\Phi|} \frac{d^{m-1}}{dp^{m-1}} \left[\frac{h(\sqrt{2p})}{\sqrt{2p}}\right]_{p=z_i=\frac{q_i}{2}} \tag{42}$$

Thus we see that knowledge of $h(|x_j|)$ is sufficient to represent the likelihood ratio. Often, knowledge of $h(|x_j|)$ and Φ is what an investigator may reasonably assume; for instance see Ref. 9.

To show how this result relates to the estimator-correlator structure, we examine the CME for α:

$$E_i[\alpha|x] = \frac{1}{(2\pi)^m |\Phi| f_i(x)} \int_0^\infty \alpha^{m+1} \exp\left\{-\alpha \frac{q_i}{2}\right\} p(\alpha) d\alpha. \tag{43}$$

Note that Eq. (43) is essentially Eq. 3 in which we have substituted $m + 1$ for m. Therefore, following the same derivation that leads to Eq. (42) yields

$$E_i[\alpha|x] = \frac{(-1)^m}{f_i(x)(2\pi)^m |\Phi|} \frac{d^m}{dp^m} \left[\frac{h(\sqrt{2p})}{\sqrt{2p}}\right]_{p=\frac{q_i}{2}}. \tag{44}$$

Substitution of the result in Eq. (42) into Eq. (44) yields

$$E_i[\alpha \mid x] = \left\{ \frac{-\dfrac{d^m}{dp^m}\left[\dfrac{h(\sqrt{2p})}{\sqrt{2p}}\right]}{\dfrac{d^{m-1}}{dp^{m-1}}\left[\dfrac{h(\sqrt{2p})}{\sqrt{2p}}\right]} \right\}_{p = \frac{q_i}{2}} \tag{45a}$$

for the optimal estimate of α or

$$E[\alpha \mid p] = \frac{-\dfrac{d^m}{dp^m}\left[\dfrac{h(\sqrt{2p})}{\sqrt{2p}}\right]}{\dfrac{d^{m-1}}{dp^{m-1}}\left[\dfrac{h(\sqrt{2p})}{\sqrt{2p}}\right]} \tag{45b}$$

for the form of the optimal estimator that we use in the estimator–correlator. Note that the estimate $E_i[\alpha|x]$ is obtained from the estimator $E[\alpha|p]$ by evaluating the estimator at the data point $p = q_i/2$.

A simple check shows that the substitution of this representation of $E[\alpha|p]$ (i.e., $E[\alpha|z]$) into Eq. (9) leads to a consistent result. As we stated above, this result is important because in practice knowledge of $h(|x_j|)$ is generally available whereas knowledge of $f_\alpha(\alpha)$ is generally not available. However, one question that remains open is what are necessary and sufficient conditions on $h(|x_j|)$ such that it has the representation given in Eq. (39). Such knowledge is required because not all pdfs $h(|x_j|)$ have such a representation. Thus to use these results presently, the investigator must verify in some way that the $h(|x_j|)$ of interest is compatible with such a representation.

CONCLUSIONS

In this report we reexamine the problem of signal detection in the class of spherically invariant random processes and show that this problem falls within the framework of the estimator–correlator structure. This structure reveals the optimal detector to be a function of the optimal estimator of a random quantity associated with the spherically invariant random process. We show how this structure leads to the interpretation that the optimal detector is an adaptive matched filter. We then demonstrate how some previously obtained results are easily derived as suboptimal implementations of the estimator–correlator. Finally, we also show how the required optimal estimator may be obtained without explicit knowledge of a prior pdf provided a different pdf, namely the marginal amplitude pdf of the complex process, which is generally easier to obtain than the prior pdf, is known.

The suboptimal detectors examined here may be classified in two ways. In one approach, the optimal detector structure is retained from the estimator–correlator, but a suboptimal estimator is used in place of the optimal estimator. In the other approach, the optimal estimator is retained from the estimator–correlator, but the function of this estimator that forms the detector is different from that of the optimal structure. Interestingly, each of these suboptimal approaches results in a detector structure that is in essence a variation on the classical matched filter. Since this class of random processes is a generalization of the Gaussian random process, this result is intuitively reasonable.

Finally, since this class of processes is of interest as a model for some types of practical noise processes, the results presented here should lead to optimal and suboptimal detection schemes for practical problems.

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