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Sidelobe Level of an Adaptive Array Using the SMI Algorithm

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The transient sidelobe level of an adaptive array is a function of the external noise environment, the number of adaptive antenna elements, the adaptive algorithm employed, auxiliary antenna gain margins, and the number of samples used to calculate the adaptive weights. In this report, an analytical result for the adaptive sidelobe level is formulated for when the adaptive algorithm is the open-loop Sampled Matrix Inversion (SMI) algorithm. This result is independent of whether the SMI algorithm is implemented using concurrent or nonconcurrent data processing. It is shown that the transient sidelobe level is eigenvalue dependent and increases proportionally to the gain margin of the auxiliary antenna elements with respect to the quiescent main antenna sidelobe level. Techniques that reduce this transient sidelobe level are discussed, and it is theoretically shown (as other researchers have discovered) that injecting independent noise into the auxiliary channels significantly reduces the transient sidelobe level. Also it is demonstrated that using this same technique reduces the SMI noise power residue settling time.

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SIDELOBE LEVEL OF AN ADAPTIVE ARRAY USING THE SMI ALGORITHM

I. INTRODUCTION

An adaptive antenna array adjusts its antenna element settings so as to null out interfering sources while maintaining a beam in a desired signal direction [1]. Two such adaptive array configurations are the "full up" adaptive array and the sidelobe canceller (SLC) illustrated in Figs. 1(a) and 1(b), respectively. The full up adaptive array adjusts all the weights on identical antenna array elements while only the auxiliary antenna weights are adjusted on the SLC configuration. The full up adaptive array may have some mainbeam constraints associated with it so that a desired mainbeam can be maintained [1].

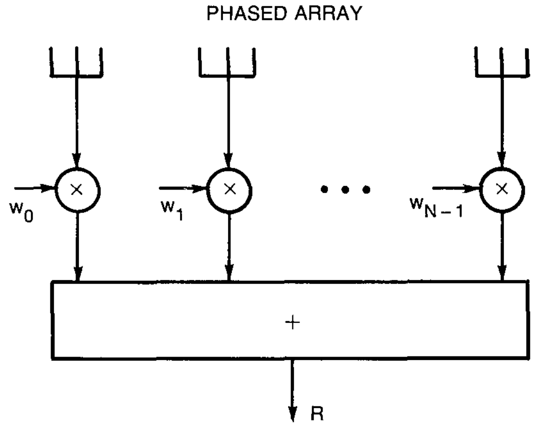

Fig. 1a — Full up adaptive array

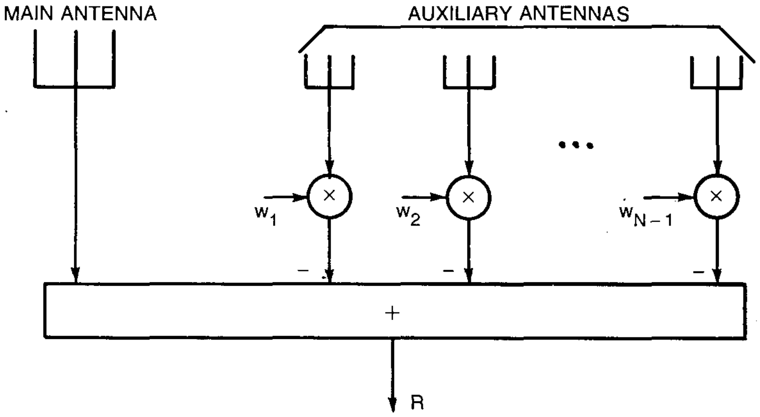

Fig. 1b — SLC configuration

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The weights on the full up adaptive array are adjusted so as to maximize the output signal-to-noise power ratio, and the weights on the SLC are adjusted so as to minimize the output noise power residue. If the noise environment is not known a priori, these weight settings cannot be set a priori but must be estimated from a finite set of incoming data on the input channels. Thus the weight settings will have perturbations about the quiescent optimum weight settings. These perturbations result in a rise in the adaptive array sidelobe level above the quiescent adaptive array sidelobe level. We call the nonquiescent sidelobe level that is due to finite sampling and averaging, the transient sidelobe level.

This report presents an analytical result for predicting this rise in the adaptive array sidelobe level by use of the Sampled Matrix Inversion (SMI). The SMI algorithm [2] is an open-loop, rapidly converging, adaptive array implementation whose convergence rate is independent of the external noise environment. For many years, it has been considered a baseline for fast converging adaptive array algorithms. Brennan [3] has presented some theoretical and computer simulation results for the effects of the SMI algorithm on the sidelobe level. His theoretical results pertain to the single auxiliary SLC. We generalize these results by examining an array with any number of inputs. Moreover, many of his observations made as a result of computer simulations are given a theoretical basis.

The SMI algorithm is briefly reviewed in Section II, and pertinent SMI theorems are presented in Section III. An exact expression for the transient sidelobe level is developed in Sections IV and V. A discussion of this result appears in Section VI, techniques for lowering the sidelobe level are presented in Sections VII and VIII, and other techniques that can be used to reduce the transient sidelobe level are discussed in Section IX.

II. THE SMI ALGORITHM

In this section, we briefly review the SMI algorithm and establish the notation and assumptions relevant to the succeeding development. Applebaum [4] has shown that the optimal (maximization of the average output signal-to-noise ratio) adaptive weighting of an N element array is given by

$$\mathbf{w} = \mu M^{-1}\mathbf{s}^* \tag{1}$$

where $\mathbf{w}$ is the optimal weighting vector of length N, $\mathbf{s}$ is a normalized steering vector related to the direction of arrival of the desired signal, M is the steady state covariance matrix of the inputs, μ is an arbitrary constant, and * denotes the complex conjugate. More formally we write

$$\mathbf{w} = (w_0, w_1, \ldots, w_{N-1})^T, \tag{2}$$

$$\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})^T, \tag{3}$$

and

$$M = E\{\mathbf{X}^*\mathbf{X}^T\}, \tag{4}$$

where T denotes the vector (or matrix) transpose operation, $E\{\cdot\}$ denotes the expected value, and X_0, $X_1, \cdots, X_{N-1}$ are the inputs of the adaptive array. Because M is generally not known a priori, it and thus the adaptive weights must be estimated from sampled data. Hence, time samples of the nth channel, $X_n(k)$, are taken where k indexes the sampled data. Note that the data on each channel for a given k are time coincident.

For this development, we make the following assumptions unless otherwise noted (the same assumptions were made in Ref. 2).

1. The $X_1, X_2, \ldots, X_N$ are zero mean stationary Gaussian random variables (r.v.).

2. $X_{n_1}(k_1)$ is independent of $X_{n_2}(k_2)$ for $n_1 \neq n_2$ or $k_1 \neq k_2$.

3. The estimated adaptive weights are computed from an input data set that is independent of the data that are adaptively weighted by the adaptive array. (We call this nonconcurrent processing.)

4. The desired signals are not present during the adaptive weight computation.

With respect to Assumption 3, if the weights are computed and applied to the same set of data, we call this concurrent processing and the output residue is called the concurrent output.

We write the sampled input vector as

$$\mathbf{X}(k) = (X_0(k), X_1(k), \ldots, X_{N-1}(k))^T. \tag{5}$$

For the SMI algorithm, the covariance matrix M is estimated as

$$\hat{M} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{X}^*(k)\mathbf{X}^T(k), \tag{6}$$

where the caret over the M denotes an estimate, K is the total number of independent samples per input channel, and $K \geq N$ so that M is not necessarily a singular matrix. Note that $\hat{M}$ is calculated in "batch" style from a block of K by N input data. Thus after $\hat{M}$ is calculated, we find the estimate of $\mathbf{w}$ as

$$\hat{\mathbf{w}} = \mu\hat{M}^{-1}\mathbf{s}^*. \tag{7}$$

As $K \to \infty$, $\hat{M} \to M$, so that for an infinite number of samples, the optimal array weighting vector is obtained.

In many instances in the following discussion we refer to the SLC configuration of an adaptive array. For this configuration

$$\mathbf{s} = (1, 0, \ldots, 0)^T \qquad \text{and} \quad w_0 = 1; \tag{8}$$

i.e., the desired signal is present only in the 0th channel, which we call the main channel. The $N-1$ other channels referred to as the auxiliary channels are used to cancel unwanted signals (noise) from the main channel. Note that the condition that $w_0 = 1$ is a constraint on the optimal weighting vector.

We define

$$\mathbf{X}_a = (X_1, X_2, \ldots, X_{N-1})^T, \tag{9}$$

$$\mathbf{w}_a = (w_1, w_2, \ldots, w_{N-1})^T, \tag{10}$$

$$\mathbf{X}_a(k) = (X_1(k), X_2(k), \ldots, X_{N-1}(k))^T, \tag{11}$$

$$M_a = E\{\mathbf{X}_a^* \mathbf{X}_a^T\}, \tag{12}$$

and

$$\mathbf{r}_{am} = E\{\mathbf{X}_a^* X_0\}, \tag{13}$$

where the subscript a refers to the auxiliary channels. It can be shown that the optimal weighting vector of the auxiliary channels for the SLC configuration is given by

$$\mathbf{w}_a = + M_a^{-1}\mathbf{r}_{am}. \tag{14}$$

Again with no a priori information, it is necessary to estimate M_a and $\mathbf{r}_{am}$. These estimates are given by the expressions

$$\hat{M}_a = \frac{1}{K}\sum_{k=1}^{K} \mathbf{X}_a^*(k)\mathbf{X}_a^T(k) \tag{15}$$

and

$$\hat{\mathbf{r}}_{am} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{X}_a^*(k)X_0(k). \tag{16}$$

For the steady state weights or optimal weights, we can write

$$X_0 = \sum_{n=1}^{N-1} w_n X_n + e, \tag{17}$$

where

$$E\{e\mathbf{X}^*{}_a\} = 0. \tag{18}$$

Equation (18) is merely an exemplification of the orthogonality principle.

III. SMI THEOREMS

In this section, we present and prove (or cite references for) a variety of theorems related to the SMI algorithm. We use these theorems in later sections to derive closed form solutions for the transient sidelobe level of an adaptive array.

Theorem 1: If the input vector $\mathbf{X}$ *is transformed by a nonsingular matrix* A*, then the estimated transformed weights* $\hat{\mathbf{w}}'$ *are related to the estimated untransformed weights* $\hat{\mathbf{w}}$ *by the relationship*

$$\hat{\mathbf{w}} = A^T\hat{\mathbf{w}}'. \tag{19}$$

Proof: We know that

$$\hat{\mathbf{w}}' = \hat{M}'^{-1}\mathbf{s}'^*, \tag{20}$$

where

$$\hat{M}' = \frac{1}{K}\sum_{k=1}^{K} [A\mathbf{X}(k)]^*\, [A\mathbf{X}(k)]^T \tag{21}$$

and

$$\mathbf{s}' = A\mathbf{s}. \tag{22}$$

Using Eq. (22), we can show that

$$\hat{M}' = A^*\hat{M}A^T. \tag{23}$$

Using Eqs. (21) and (22) in Eq. (20), we find that

$$\begin{aligned} \hat{\mathbf{w}}' &= [A^*\hat{M}A^T]^{-1}(A\mathbf{s})^* \\ &= (A^T)^{-1}\,\hat{M}^{-1}\mathbf{s}^* \\ &= (A^T)^{-1}\hat{\mathbf{w}}. \end{aligned} \tag{24}$$

The theorem follows from Eq. (24).

Similarly, we can show

Theorem 2: If the adaptive array is in the SLC configuration and the auxiliary input vector $\mathbf{X}_a$ *is multiplied by a nonsingular matrix transform* A*, then the estimated auxiliary weights* $\hat{\mathbf{w}}'$ *are related to the estimated untransformed auxiliary weights* $\mathbf{w}_a$ *by the relationship*

$$\hat{\mathbf{w}}_a = A^T\hat{\mathbf{w}}_a'. \tag{25}$$

Theorem 3 is cited in Ref. 2.

Theorem 3: There exists a unitary matrix U *that transforms the steering vector* $\mathbf{s}$ *into the vector* $(1, 0, \ldots, 0)^T$*, where* $\mathbf{s}^t\mathbf{s} = 1$.

Theorem 3 is important to our development because it allows us to transform any unconstrained adaptive array into the SLC configuration.

Theorem 4: If the input data vector $\mathbf{X}$ or $\mathbf{X}_a$ *is multiplied by a unitary matrix* U*, then*

$$E\{\hat{\mathbf{w}}^t\hat{\mathbf{w}}\} = E\{\hat{\mathbf{w}}'^t\hat{\mathbf{w}}'\} \tag{26}$$

or for the SLC configuration

$$E\{\hat{\mathbf{w}}_a^t\hat{\mathbf{w}}_a\} = E\{\hat{\mathbf{w}}_a'^t\hat{\mathbf{w}}_a'\}. \tag{27}$$

Equations (26) and (27) follow directly from the fact that $UU^t = I$.

Theorem 5: If $X_0 = \sum_{n=1}^{N-1} c_n X_n$ *and the adaptive array is in the SLC configuration, then the concurrent output of the SMI canceller is exactly zero independent of* K*, the number of independent samples per channel taken..*

Proof: We write

$$X_0(k) = \sum_{n=1}^{N-1} c_n X_n(k) = \mathbf{X}_a^T(k)\mathbf{c}, \tag{28}$$

where we define

$$\mathbf{c} = (c_1, c_2, \ldots, c_{N-1})^T. \tag{29}$$

Thus

$$\hat{\mathbf{r}}_{ma} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{X}_a^*(k)\mathbf{X}_a^T(k)\mathbf{c} \tag{30}$$

$$= \hat{M}_a \mathbf{c}.$$

Hence if we solve $\hat{M}_a \hat{\mathbf{w}}_a = \hat{\mathbf{r}}_{ma} = \hat{M}_a \mathbf{c}$, we see that $\hat{\mathbf{w}}_a = \mathbf{c}$, and the theorem follows.

Theorem 6 results from Theorem 5.

Theorem 6: If $X_0 = \sum_{n=1}^{N-1} c_n X_n + e$, then the concurrent output noise residue is independent of $\sum_{n=1}^{N-1} c_n X_n$.

Theorem 6 is exemplified by Fig. 2. Here, we implement two parallel cancellers; one cancels $\sum_{n=1}^{N-1} c_n X_n$, and the other cancels e. Note that separate estimated weighting vectors are computed for each canceller. As a result of Theorem 5, the left-hand canceller always has zero residue. Hence the final residue R is dependent on only the right-hand canceller's output residue.

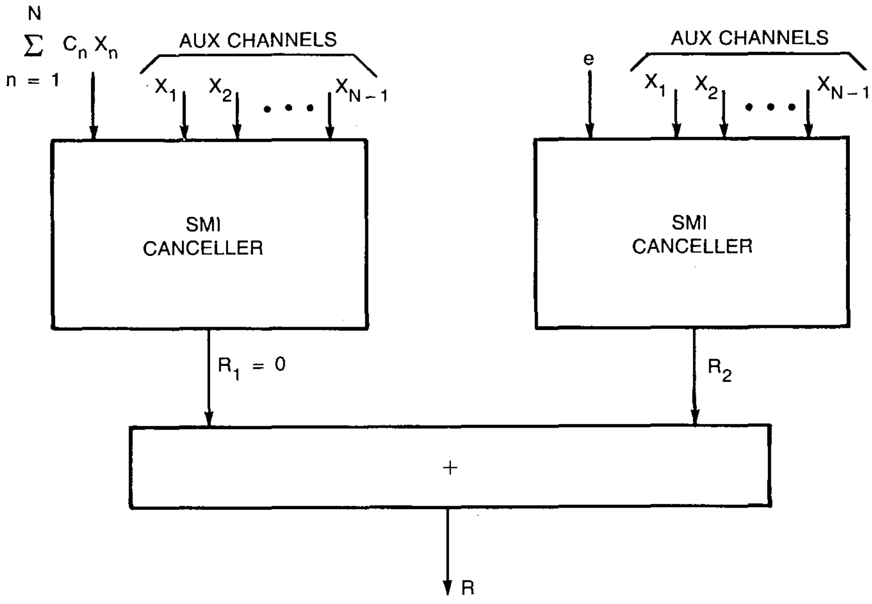

Fig. 2 — Equivalent SMI canceller

Theorem 7 was first proved by Reed et al. [2].

Theorem 7: If assumptions 1 through 4 are satisfied and

$$z = \frac{(\hat{S}/N)}{(S/N)_{opt}}, \tag{31}$$

where $(\hat{S}/N)$ is the sampled signal to noise ratio given by the expression

$$(\hat{S}/N) = \hat{\mathbf{w}}^t M \hat{\mathbf{w}} \tag{32}$$

and S/N) is the optimal signal-to-noise ratio given by

$$(S/N) = \mathbf{w}^t M \mathbf{w}, \tag{33}$$

then the probability density function (p.d.f.) of z is

$$p(z) = \frac{K!}{(N-2)!\,(K+1-N)!}\,(1-z)^{N-2} z^{K+1-N}\,, \quad 0 \le z \le 1. \tag{34}$$

From this theorem, Brennan and Reed [5] showed that

Theorem 8: If assumptions 1 through 4 are satisfied and the adaptive array is in the SLC configuration, then

$$\frac{\sigma^2_{\mathrm{SMI}}(K,N)}{\sigma^2_{\min}} = \frac{K}{K-N+1}\,; \quad K \ge N, \tag{35}$$

where $\sigma^2_{\mathrm{SMI}}(K, N)$ is the average output noise power residue of the N input SMI canceller using K samples, and $\sigma^2_{\min}$ is the minimum output noise power residue ($K = \infty$).

We can show that

$$\sigma^2_{\mathrm{SMI}}(K, N) = E\{|X_0|^2\} - E\{\hat{\mathbf{w}}_a^t M_a \hat{\mathbf{w}}_a\} \tag{36}$$

and

$$\sigma^2_{\min} = E\{|X_0|^2\} - \mathbf{w}_a^t M_a \mathbf{w}_a\,. \tag{37}$$

IV. SIDELOBE LEVEL MEASURE

In this section, we derive an expression for the sidelobe level contribution of the adaptive array that comes as a result of estimating the optimal adaptive weights. We consider the two adaptive array configurations, full up and SLC.

We simplify the full up adaptive array analysis by transforming the full up array into an SLC configuration by invoking Theorem 3. (The appendix gives a simple procedure for finding U when $\sqrt{N}\,\mathbf{s} = (1, e^{j\phi}, e^{2j\phi}, \cdots, e^{(N-1)j\phi})^T$). Let U be the unitary transform such that

$$U\mathbf{s} = (1, 0, 0, \ldots, 0)^T. \tag{38}$$

We place the SLC constraint ($w_0 = 1$) on the adaptive weights that are computed after this transformation. We shall see that this unitary transformation does not change the transient sidelobe level.

Let

$g_q(\phi)$ be the quiescent antenna pattern with no adaptation ($0 \le \phi < 2\pi$),

$g_{qa}(\phi)$ be the quiescent adaptive antenna pattern ($K = \infty$),

$g_a(\phi)$ be the adaptive antenna pattern (finite K),

$\mathbf{w}_q$ be the quiescent adaptive weighting vector ($K = \infty$, SLC configuration),

$\mathbf{w}$ be the estimated adaptive weighting vector (finite K), a random variable,

$\Delta\mathbf{w} = \mathbf{w} - \mathbf{w}_q$, a random variable,

$\mathbf{v}(\phi) = \frac{1}{\sqrt{N}} (1, e^{j\phi}, e^{2j\phi}, \ldots, e^{(N-1)j\phi})^T$, and

$\mathbf{u}_n$ be the nth row of U.

The auxiliary inputs have the following antenna patterns:

$$g_n(\phi) = \mathbf{u}_n^T \mathbf{v}(\phi), \qquad n = 1, 2, \ldots, N-1. \tag{39}$$

We define a vector of length $N - 1$ of these auxiliary antenna patterns:

$$\mathbf{g}(\phi) = (g_1(\phi), g_2(\phi), \cdots, g_{N-1}(\phi))^T. \tag{40}$$

By using the above defined quantities, it is simple to show that

$$\begin{aligned} g_a(\phi) &= g_q(\phi) - \mathbf{w}^T \mathbf{g}(\phi) \\ &= g_q(\phi) - (\mathbf{w}_q + \Delta\mathbf{w})^T U \mathbf{v}(\phi). \end{aligned} \tag{41}$$

Taking the expected value over the $\Delta\mathbf{w}$ of the magnitude squared of Eq. (41) results in

$$E_{\Delta w}\{|g_a(\phi)|^2\} = |g_q(\phi) - \mathbf{w}_q^T U \mathbf{v}(\phi)|^2 + E_{\Delta w}\{\Delta\mathbf{w}^t U^* \mathbf{v}^*(\phi) \mathbf{v}^T(\phi) U^T \Delta\mathbf{w}\}. \tag{42}$$

The fact that $E_{\Delta\mathbf{w}}\{\Delta\mathbf{w}\} = 0$ was used in the above derivation. Note that

$$g_{qa}(\phi) = g_q(\phi) - \mathbf{w}_q^T U \mathbf{v}(\phi), \tag{43}$$

so that

$$E_{\Delta w}\{|g_a(\phi)|^2\} = |g_{qa}(\phi)|^2 + E_{\Delta w}\{\Delta\mathbf{w}^t U^* \mathbf{v}^*(\phi) \mathbf{v}^T(\phi) U^T \Delta\mathbf{w}\}. \tag{44}$$

Thus we see that the adaptive transient sidelobe level contribution, ΔSL_a, is given by

$$\Delta SL_a = E_{\Delta w}\{\Delta \mathbf{w}^t U^* \mathbf{v}^*(\phi)\mathbf{v}^T(\phi)U^T \Delta \mathbf{w}\}. \tag{45}$$

The average contribution is found by averaging ΔSL_a over $0 \leq \phi < 2\pi$. Because

$$E_\phi\{\mathbf{v}^*(\phi)\mathbf{v}^T(\phi)\} = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{v}^*(\phi)\mathbf{v}^T(\phi)d\phi \tag{46}$$

$$= I,$$

where I is the $(N - 1) \times (N - 1)$ identity matrix, it follows that

$$\Delta\overline{SL}_a = E\{\Delta \mathbf{w}^t \Delta \mathbf{w}\}, \tag{47}$$

where $\Delta\overline{SL}_a$ is the average transient sidelobe level for a full up adaptive array, and we have dropped the $\Delta\mathbf{w}$ subscript from the expected value notation.

Consider the SLC configuration as shown in Fig. 1(b). Here

$$g_n(\phi) = Ge^{jm_n\phi}, \qquad n = 1, 2, \ldots, N - 1, \tag{48}$$

where $m_n \epsilon \{0, 1, \ldots, N - 1)$, the m_n are distinct for each n, and G is an arbitrary gain associated with the auxiliary antenna elements. Similar to the preceding development for the full up array, we can show that

$$\Delta\overline{SL}_a = G^2 E\{\Delta \mathbf{w}^t \Delta \mathbf{w}\}. \tag{49}$$

Hence the SLC configuration's average sidelobe level expression is the same as the full up adaptive array except for the factor G^2. We use Eq. (49) for both configurations by defining $G = 1$ for the full up array.

Note that as a result of Theorem 4, the unitary matrix transformation of the input channels does not change $\Delta\overline{SL}_a$. Also due to the form of $\Delta\overline{SL}_a$ given by Eq. (49), Assumption 3 as given in Section II, need not hold.

V. SIDELOBE LEVEL DERIVATION

In this section, we derive an exact expression for the average transient sidelobe level, $\Delta\overline{SL}_a$. In the derivation we consider only the SLC configuration, but the result is applicable for the full up array.

Consider the concurrent output residue of the SMI canceller as shown in Fig. 3. Here

$$R = X_0 - \hat{\mathbf{w}}^T \mathbf{X}_a \tag{50}$$

$$= X_0 - (\mathbf{w}_q + \Delta\mathbf{w})^T \mathbf{X}_a.$$

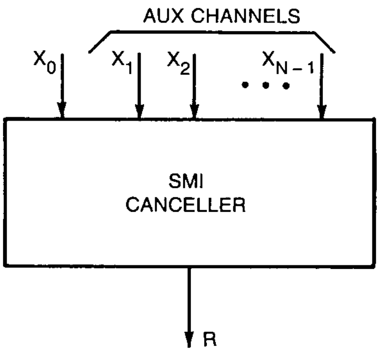

Fig. 3 — Representation of SMI canceller

Using Eq. (17), we can write

$$R = e - \Delta \mathbf{w}^T \mathbf{X}_a , \tag{51}$$

where e is statistically orthogonal to $\mathbf{X}_a$. In fact, invoking Theorem 6, we see that

$$\Delta \mathbf{w} = \hat{M}_a^{-1} \hat{\mathbf{r}}_{ae} , \tag{52}$$

where

$$\hat{\mathbf{r}}_{ae} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{X}_a^*(k) e(k) \tag{53}$$

and $e(k)$, $k = 1, 2, \ldots, K$ are the time samples of e.

Since the auxiliary input covariance matrix is hermitian, we can write

$$M_a = \Phi^t \Lambda \Phi , \tag{54}$$

where Λ is a diagonal matrix of the eigenvalues of M and Φ is a matrix of eigenvectors of M. Note that we assume that all powers are referenced to the internal noise level, which we set equal to one. The eigenvalues can then be divided into two classes: significant eigenvalues $>> 1$, and noise eigenvalues that are approximately equal to 1 or the internal noise power level.

We transform the auxiliary input by the matrix transform $\Lambda^{-1/2} \Phi^*$ as shown in Fig. 4, where $\Lambda^{-1/2}$ designates a diagonal matrix whose diagonal elements are equal to the square root of the corresponding diagonal elements of the inverse of Λ. Using Theorem 1, we see that

$$\begin{aligned} \Delta \mathbf{w} &= (\Lambda^{-1/2} \Phi^*)^T \, \Delta \mathbf{w}' \\ &= \Phi^t \Lambda^{-1/2} \Delta \mathbf{w}' . \end{aligned} \tag{55}$$

Thus

$$E\{\Delta \mathbf{w}^t \Delta \mathbf{w}\} = E\{\Delta \mathbf{w}'^t \Lambda^{-1/2} \Phi \Phi^t \Lambda^{-1/2} \Delta \mathbf{w}'\} . \tag{56}$$

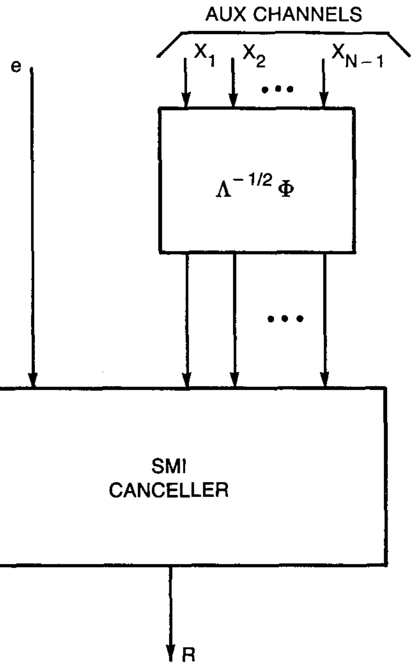

Fig. 4 — Orthogonalization of the auxiliary channels

Now let

$$\Delta \mathbf{w}' = (\Delta w_1', \Delta w_2', \ldots, \Delta w_{N-1}')^T, \tag{57}$$

$$\hat{\mathbf{w}}_0 = (1, \Delta w_1', \Delta w_2', \ldots, \Delta w_{N-1}')^T, \tag{58}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_{N-1}$ be the eigenvalues of M_a (and the diagonal elements of Λ). Because $\Phi\Phi^t = I$, we can rewrite Eq. (56) as

$$E\{\Delta \mathbf{w}^t \Delta \mathbf{w}\} = \sum_{n=1}^{N-1} \frac{1}{\lambda_n} E\{\,|\Delta w_n'|^2\}. \tag{59}$$

Because of symmetry, we know that

$$E\{\,|\Delta w_1'|^2\} = E\{\,|\Delta w_2'|^2\} = \ldots = E\{\,|\Delta w_{N-1}'|^2\}. \tag{60}$$

Also, the nonconcurrent output noise power residue is given by

$$\sigma_{\mathrm{SMI}}^2(K, N) = E\{\hat{\mathbf{w}}_0^t M' \hat{\mathbf{w}}_0\}, \tag{61}$$

where M' is the covariance matrix of the inputs (main and auxiliaries) of the SMI canceller shown in Fig. 3. Because all the inputs are independent random variables, M' is an $N \times N$ diagonal matrix with the first diagonal element equal to $\sigma_{\min}^2$, and all other diagonal elements equal to 1 so that

$$\sigma_{\mathrm{SMI}}^2(K, N) = \sigma_{\min}^2 + \sum_{n=1}^{N-1} E\{\,|\Delta w_n'|^2\}. \tag{62}$$

From Theorem 8 and Eq. (62) it follows that

$$\sum_{n=1}^{N-1} E\{|\Delta w_n'|^2\} = \frac{N-1}{K-N+1}\,\sigma_{\min}^2, \quad K \geq N. \tag{63}$$

From Eqs. (60) and (63), it follows that

$$E\{|\Delta w_n'|^2\} = \frac{1}{K-N+1}\,\sigma_{\min}^2, \quad n = 1, 2, \ldots, N-1. \tag{64}$$

Hence, substituting Eq. (64) into Eq. (59), we have shown that

$$E\{\Delta \mathbf{w}^t \Delta \mathbf{w}\} = \frac{\sigma_{\min}^2}{K-N+1} \sum_{n=1}^{N-1} \frac{1}{\lambda_n} \tag{65}$$

or the average transient sidelobe level contribution is given by

$$\Delta \overline{SL}_a = \frac{G^2 \sigma_{\min}^2}{K-N+1} \sum_{n=1}^{N-1} \frac{1}{\lambda_n}, \quad K \geq N. \tag{66}$$

VI. DISCUSSION

We see from the expression derived for the average transient sidelobe level, Eq. (66), that $\Delta \overline{SL}_a$ is eigenvalue dependent: the more significant eigenvalues ($\lambda >> 1$) there are, the smaller $\Delta \overline{SL}_a$ is. In addition, we observe that $\Delta \overline{SL}_a$ increases with the gain on the auxiliary elements G. As Brennan [3], pointed out, there is a tradeoff in selecting G in that although $\Delta \overline{SL}_a$ increases with G, the output noise residue after cancellation decreases because the adaptive weights are smaller, resulting in less amplification of the internal noise in the auxiliary channels. The effect of G^2 on the eigenvalues is to increase the significant eigenvalues proportionally (those $>> 1$) but leave the noise eigenvalues unchanged and approximately equal to one. Finally, we see from Eq. (56) that as expected, $\Delta \overline{SL}_a \rightarrow 0$ as $K \rightarrow \infty$, and that the transient sidelobe level increases as the steady state residue $\sigma_{\min}^2$ increases.

We note that for the full up adaptive array the eigenvalues we are referring to are those of the $N-1$ auxiliary inputs. Hence these depend on the unitary matrix transform U, which created those auxiliary inputs. However, since the number of significant eigenvalues is generally equal to the number of degrees of freedom (DOFs), N_{DOF}, needed to effectively suppress a given external environment noise, we see that the number of significant eigenvalues is a constant. Hence, if $\lambda_1 > \lambda_2 > \ldots \lambda_{N_{\mathrm{DOF}}} >> 1$ and $\lambda_n \approx 1$ for $N > N_{\mathrm{DOF}}$, then we can approximate $\Delta \overline{SL}_a$ as

$$\Delta \overline{SL}_a = G^2 \sigma_{\min}^2 \frac{N - N_{\mathrm{DOF}} - 1}{K-N+1}. \tag{67}$$

Equation (67) can be used to properly specify the number of independent samples per channel K necessary for the adaptive array sidelobes to settle within some arbitrary value of the quiescent adaptive sidelobe level. If $\overline{SL}_a$ is the average sidelobe level of $g_a(\phi)$ and $\overline{SL}_{qa}$ is the average quiescent sidelobe level of, $g_{qa}(\phi)$, then we can write

$$\overline{SL}_a = \overline{SL}_{qa} + G^2 \sigma^2_{\min} \frac{N - N_{\mathrm{DOF}} - 1}{K - N + 1}. \tag{68}$$

We find the K to be such that $\overline{SL}_a$ is within 3 dB of the quiescent adaptive sidelobe level, SL_{qa}. This is found simply to be

$$K_{3\mathrm{dB}} = (N - 1) \left[1 + \frac{G^2 \sigma^2_{\min}}{\overline{SL}_{qa}} \right] - N_{\mathrm{DOF}} \frac{G^2 \sigma^2_{\min}}{\overline{SL}_{qa}}. \tag{69}$$

If $\overline{SL}_q$ is the average quiescent sidelobe level when the sidelobe cancellor is disabled (all weights set equal to zero), then normally the adaptive array is designed so that the quiescent adaptive sidelobe level is equal to $\overline{SL}_q$ and $\sigma^2_{\min} = 1$. If we define $G_M = G^2/\overline{SL}_q$ to be the gain margin of the auxiliary antenna elements with respect to the main antenna's average sidelobe, then we can show by using Eq. (69) that

$$K_{3\mathrm{dB}} = (N - 1)(1 + G_M) - N_{\mathrm{DOF}} G_M. \tag{70}$$

When there is no external interference, then $\overline{SL}_a$ is at a maximum. For this worst case, $N_{\mathrm{DOF}} = 0$, so that

$$\overline{SL}_a \leq G^2 \frac{N - 1}{K - N + 1}. \tag{71}$$

In addition, for this worst case, $K_{3\mathrm{dB}}$ is at a maximum, $\overline{SL}_q$ is exactly equal to $\overline{SL}_{qa}$, and $\sigma^2_{\min} = 1$. We use this worst case scenario to specify $K_{3\mathrm{dB}}$:

$$K_{3\mathrm{dB}} = (N - 1)(1 + G_M). \tag{72}$$

We set $N_{aux} = N - 1$, the number of auxiliary input channels, and plot $K_{3\mathrm{dB}}/N_{aux}$ vs G_M in Fig. 5. Note that for $G_M >> 1$, $K_{3\mathrm{dB}}$ is directly proportional to the gain margin. Hence large gain margins result in long settling times for the adaptive sidelobes.

VII. SIDELOBE LEVEL REDUCTION BY NOISE INJECTION

In this section, we give the theoretical basis to a technique discussed in Refs. 3, 6, and 7, which significantly reduces the settling times of the adaptive sidelobes. For this technique, arbitrary independent noise is injected into each auxiliary channel (SLC configuration). This noise is independent from sample-to-sample, channel-to-channel, and has noise power equal to σ^2. It can be shown that in the steady state the adaptive auxiliary weights are given by

$$\mathbf{w}_a = (M_a + \sigma^2 I)^{-1} \mathbf{r}_{ma}. \tag{73}$$

After the auxiliary weights have been calculated by using the SMI algorithm, these weights are then applied to auxiliary data (concurrent or nonconcurrent) that do *not* contain the injected noise. The noise power of the injected noise, σ^2, is chosen so that the steady state noise power residue is not significantly increased (normally within 1 dB). Brennan [3], using computer simulation, showed that σ^2 can be chosen to be much greater than one (the referenced internal noise power level) without seriously degrading cancellation performance.

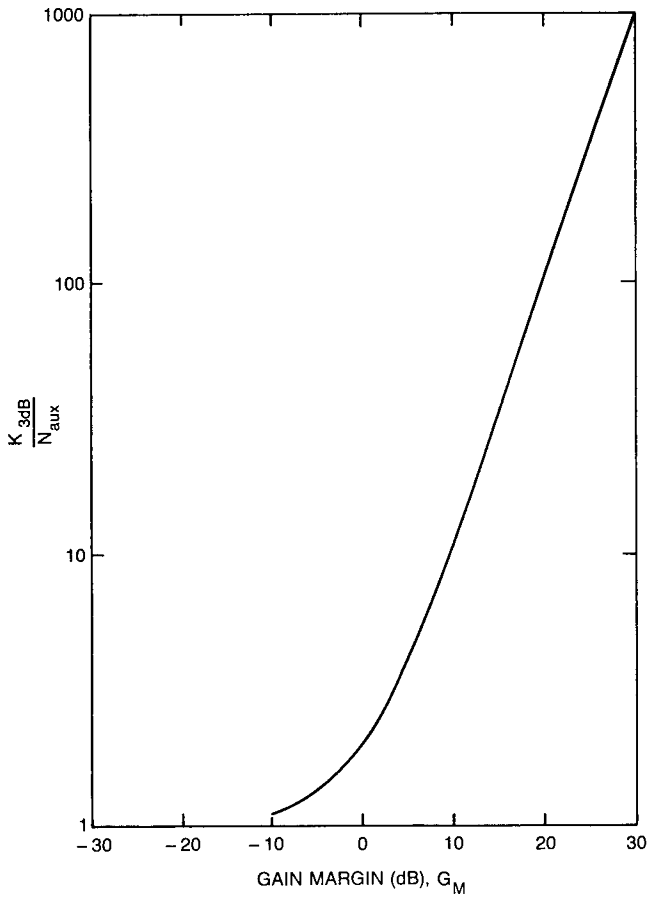

Fig. 5 — K_{3dB}/N_{aux} vs the gain margin

Figure 6 shows the SLC configuration with noise injection. The effect of injecting noise is to increase each of the auxiliary eigenvalues by σ^2 so that

$$\Delta\overline{SL}_a = \frac{G^2\sigma_{\min}^2}{K - N + 1} \sum_{n=1}^{N-1} \frac{1}{\lambda_n + \sigma^2}. \tag{74}$$

Thus for $\sigma^2 >> 1$, and $\sigma^2 << \lambda_1, \lambda_2, \ldots, \lambda_{N_{DOF}}$,

$$\Delta\overline{SL}_a = G^2\sigma_{\min}^2 \frac{N - N_{DOF} - 1}{K - N + 1} \frac{1}{\sigma^2} \tag{75}$$

and

$$K_{3dB} = (N - 1)\left[1 + G_M \frac{\sigma_{\min}^2}{\sigma^2}\right] - N_{DOF} G_M \frac{\sigma_{\min}^2}{\sigma^2}. \tag{76}$$

Note that $\Delta\overline{SL}_a$ is significantly decreased (by $\approx 1/\sigma^2$) by using noise injection. Also the sidelobe settling time is similarly decreased.

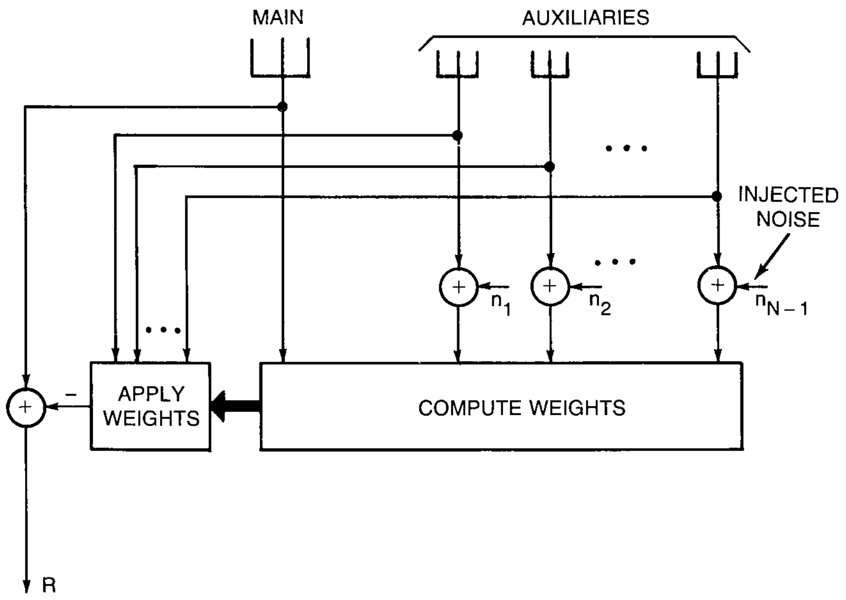

Fig. 6 — SMI canceller using noise injection

Again if we consider the worst case (no external noise), then

$$\frac{K_{3\mathrm{dB}}}{N_{aux}} = 1 + \frac{G_M}{\sigma^2}. \tag{77}$$

VIII. SETTLING TIME REDUCTION BY NOISE INJECTION

If we use the noise injection technique described in the preceding section, we can also decrease the settling time of the nonconcurrent output noise power residue. To see this we define the following quantities:

r lw(5.5i). $E\{|R|^2\}$ T{ is the steady state ($K = \infty$) output noise power residue by use of the noise injection technique, T}

$E\{|\hat{R}|^2\}$ T{ is the output noise power residue for finite sampling by use of the noise injection technique, T}

$E\{|R_{in}|^2\}$ T{ is the steady state output noise power residue if weights are applied to auxiliaries with injected noise, T}

$E\{|\hat{R}_{in}|^2\}$ T{ is the output noise power residue for finite sampling is the weights are applied to auxiliaries with injected noise, T}

$\mathbf{w}_{a,in}$ T{ is the steady state auxiliary weights by use of the noise injection technique, and T}

$\hat{\mathbf{w}}_{a,in}$ T{ is the estimated auxiliary weights by use of the noise injection technique. T}

$$\Delta\mathbf{w}_{a,in} = \hat{\mathbf{w}}_{a,in} - \mathbf{w}_{a,in} \tag{78}$$

$$M_{a,in} = M_a + \sigma^2 I. \tag{79}$$

Now we know that

$$E\{|R|^2\} = E\{|X_0 - \mathbf{w}_{a,in}^T \mathbf{X}_a|^2\}, \tag{80}$$

and we can show that

$$\begin{aligned} E\{|\hat{R}|^2\} &= E\{|X_0 - \hat{\mathbf{w}}_{a,in}^T \mathbf{X}_a|^2\} \\ &= E\{|X_0 - (\mathbf{w}_{a,in} + \Delta\mathbf{w}_{a,in})^T \mathbf{X}_a|^2\} \\ &= E\{|R|^2\} + E\{\Delta\mathbf{w}_{a,in}^t M_a \Delta\mathbf{w}_{a,in}\} \\ &= E\{|R|^2\} + E\{\Delta\mathbf{w}_{a,in}^t M_{a,in} \Delta\mathbf{w}_{a,in}\} + \sigma^2 E\{\Delta\mathbf{w}_{a,in}^t \Delta\mathbf{w}_{a,in}\}. \end{aligned} \tag{81}$$

Using Theorem 8, we can show that

$$E\{|R_{in}|^2\} + E\{\Delta\mathbf{w}_{a,in}^t M_{in} \Delta\mathbf{w}_{a,in}\} = E\{|R_{in}|^2\} \left(\frac{K}{K - N + 1} \right) \tag{82}$$

or

$$E\{\Delta\mathbf{w}_{a,in}^t M_{in} \Delta\mathbf{w}_{a,in}\} = E\{|R_{in}|^2\} \frac{N - 1}{K - N + 1}. \tag{83}$$

In Section V, we show that

$$E\{\Delta\mathbf{w}_{a,in}^t \Delta\mathbf{w}_{a,in}\} = \frac{E\{|R_{in}|^2\}}{K - N + 1} \sum_{n=1}^{N-1} \frac{1}{\lambda_n + \sigma^2}. \tag{84}$$

Hence, substituting Eqs. (83) and (84) into Eq. (81) and normalizing,

$$\frac{E\{|\hat{R}|^2\}}{E\{|R|^2\}} = 1 + \frac{E\{|R_{in}|^2\}}{E\{|R|^2\}} \left[\frac{N - 1}{K - N + 1} - \frac{\sigma^2}{K - N + 1} \sum_{n=1}^{N-1} \frac{1}{\lambda_n + \sigma^2} \right]. \tag{85}$$

Now normally σ^2 is chosen so that $E\{|R_{in}|^2\} \approx E\{|R|^2\}$. We then rewrite Eq. (85) as

$$\frac{E\{|\hat{R}|^2\}}{E\{|R|^2\}} = \frac{K}{K - N + 1} - \frac{\sigma^2}{K - N + 1} \sum_{n=1}^{N-1} \frac{1}{\lambda_n + \sigma^2}. \tag{86}$$

We see from Eq. (86) that the noise eigenvalues ($\lambda \approx 1$) in the summation will dominate, and if $\sigma^2 >> 1$, then

$$\frac{E\{|\hat{R}|^2\}}{E\{|R|^2\}} < \frac{K}{K - N + 1} - \frac{N - 1 - N_{\mathrm{DOF}}}{K - N + 1}, \tag{87}$$

or

$$\frac{E\{|\hat{R}|^2\}}{E\{|R|^2\}} < \frac{K - N + N_{\mathrm{DOF}} + 1}{K - N + 1}. \tag{88}$$

If we calculate how many samples $K'_{3\text{dB}}$ are needed so that the output noise power residue is within 3 dB of the quiescent value, we find

$$K'_{3\text{dB}} \approx N - 1 + N_{\text{DOF}}. \tag{89}$$

Note that, $N \leq K'_{3\text{dB}} \leq 2N$, depending on the number of DOFs used. Also, Eq. (89) implies that the less DOFs used, the faster the settling time. With no noise injection, $K'_{3\text{dB}} = 2N - 2$ [3]. Hence noise injection decreases the adaptive algorithm settling time.

Another method to reduce the settling time of the SMI algorithm is to estimate the number of DOFs needed and set the number of auxiliaries equal to N_{DOF}. One method for estimating N_{DOF} is to count the number of significant eigenvalues of the estimated auxiliary covariance matrix. A less computationally intensive technique is to use the open-loop Gram-Schmidt (GS) canceller [8,9] to implement the SMI. Here, the noise power at each level of the main channel in the GS structure is monitored by using finite averaging. Note that for a finite K that if $N > N_{\text{DOF}}$, the output noise power in the main channel decreases and then increases at the successive levels through the GS structure. Thus the cancellation process in the main channel is terminated at the level where the noise power is measured to be a minimum. Hence the number of levels prior to this termination point is estimated to be equal to N_{DOF}. If we compute $K'_{3\text{dB}}$ for the DOF's monitoring technique, then

$$K'_{3\text{dB}} = 2N_{\text{DOF}} - 2. \tag{90}$$

As a result for the three techniques

$$\text{normal}: \quad \frac{K'_{3\text{dB}}}{N_{aux}} = 2 \tag{91}$$

$$\text{noise injection}: \quad \frac{K'_{3\text{dB}}}{N_{aux}} = 1 + \frac{N_{\text{DOF}}}{N_{aux}} \tag{92}$$

$$\text{DOF monitoring}: \quad \frac{K'_{3\text{dB}}}{N_{aux}} = 2\,\frac{N_{\text{DOF}}}{N_{aux}} \tag{93}$$

where $N_{aux} = N - 1$.

We plot these quantities vs N_{DOF}/N_{aux} in Fig. 7. Note that the DOF monitoring technique is faster converging than is the noise injection technique. Hence, DOF monitoring with the GS canceller implementation offers a computationally efficient and fast converging adaptive canceller algorithm. Also, because $K'_{3\text{dB}}$ is a measure of how perturbed the adaptive weights are about their optimal values, we see that the transient sidelobe settling times should be faster by the use of DOF monitoring.

IX. OTHER SIDELOBE LEVEL REDUCTION TECHNIQUES

In this section, we briefly discuss other techniques that can be used to reduce the average transient sidelobe level. The basis of these techniques is to reduce the number of auxiliary input channels $N - 1$ of the SLC to the necessary number of DOFs, N_{DOF}.

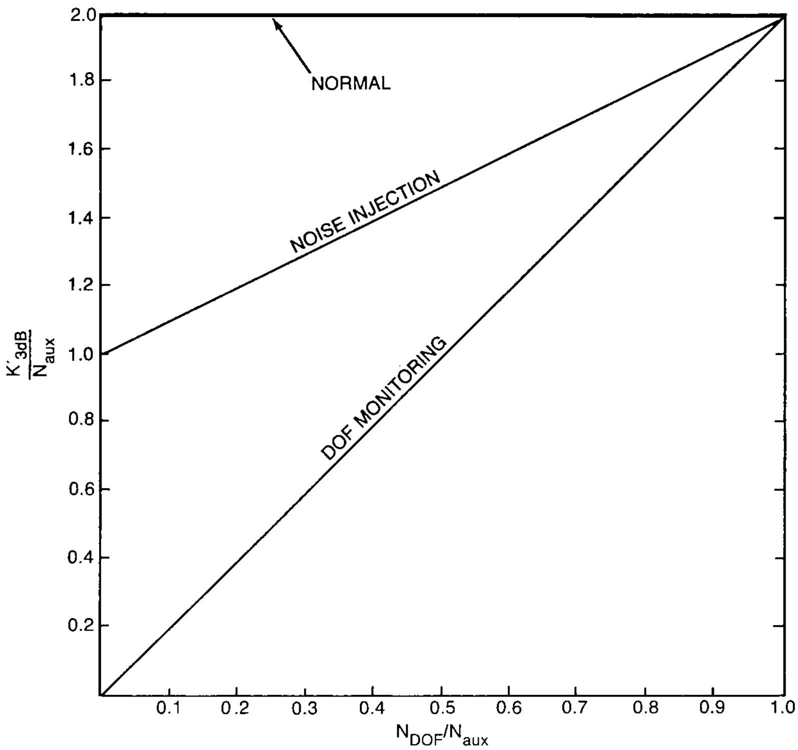

Fig. 7 — $K'_{3\mathrm{dB}}/N_{aux}$ vs N_{DOF}/N_{aux} for three SLC techniques

The first technique is to form $N - 1$ orthogonal auxiliary beams that cover the 0 to 2π angular space. This can be done by setting the unitary matrix transform U equal to the Butler matrix B where

$$B = \left[\frac{1}{\sqrt{N}} \Gamma_{N-1}^{(n-1)(m-1)} \right], \quad n, m = 1, 2, \cdots, N - 1 \tag{94}$$

and

$$\Gamma_{N-1} = e^{-j\frac{2\pi}{N-1}}, \quad j = \sqrt{-1}. \tag{95}$$

Thereafter, only those beams that contain high levels of external interference are inputted into the SLC SMI canceller. Hence the number of nonsignificant eigenvalues is reduced so that $\Delta\overline{SL}_a$ will be reduced. Also, the effects on the main channel's sidelobe level are localized around where (in angle) the various external interferences appear. Problems involved with this technique are the significant amount of hardware necessary to generate $N - 1$ beams and the assurance that enough beams are used to suppress the external noise environment (there may be multiple interferences in a single beam).

A second technique related to the first technique is to generate auxiliary beams that point at the various external interferences. Even though the auxiliary beams may not be orthogonal, the effects on the main channel's sidelobe level are localized around where in angle the various interferences appear. A prerequisite for this technique is that the interference's approximate (within an auxiliary antenna beamwidth) angular position be known. This can be done by use of a variety of techniques (for example, superresolution [10]).

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Appendix

UNITARY MATRIX TRANSFORM OF A STEERING VECTOR

In this appendix, we give a simple procedure for generating a unitary matrix transform U, which transforms the following steering vector

$$\mathbf{s} = \frac{1}{\sqrt{N}}(1, e^{j\phi}, e^{2j\phi}, \ldots, e^{(N-1)j\phi})^T \tag{A1}$$

into the $(1, 0, 0, \ldots, 0)^T$ vector. First, we define an $N \times N$ diagonal matrix S, such that the diagonal elements are equal to the corresponding elements of the steering vector $\mathbf{s}$. Next, we select any unitary transformation U_0, such that all of the first row elements equal $1/\sqrt{N}$. Examples of this kind of matrix are the Butler (see Eq. (90)) and the Hadamard matrices. Finally we set

$$U = U_0 S^* . \tag{A2}$$

It is elementary to verify that $UU^t = I$ and that $U_0 S^* \mathbf{s} = (1, 0, 0, \ldots, 0)^T$.