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An Incomplete Lipschitz-Hankel Integral of K_0 Part II

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Various representations for an incomplete Lipschitz-Hankel integral of K_0 and related integrals have been given in terms of elementary, cylindrical, and Kampé de Fériet functions. In addition some properties of the Kampé de Fériet functions associated with these integrals are derived.

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AN INCOMPLETE LIPSCHITZ-HANKEL INTEGRAL OF K_0 PART II

INTRODUCTION

An incomplete Lipschitz-Hankel integral of cylindrical functions of order zero, C_0, may be defined by

$$C_{e_0}(a, z) \equiv \int_0^z e^{at} C_0(t)\, dt$$

Of interest in applications are the functions $J_{e_0}(a, z)$, $I_{e_0}(a, z)$, $N_{e_0}(a, z)$, and $K_{e_0}(a, z)$ where J denotes the Bessel function of the first kind, I denotes the modified Bessel function, N denotes the Bessel function of the second kind or Neumann function, and K denotes the MacDonald function or Bessel function of imaginary argument. $J_{e_0}(a, z)$ and $N_{e_0}(a, z)$ occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function $I_{e_0}(a, z)$ plays an important role in the study of oscillating wings in supersonic flow and arises in the study of resonant absorption in media with finite dimensions [1, p. 195]. $K_{e_0}(a, z)$ occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of interest in the study of the scattered coherent reflected field from the sea surface [2].

In this report we are interested in

$$K_{e_0}(a, z) \equiv \int_0^z e^{at} K_0(t)\, dt$$

It is shown in Ref. 3 that $K_{e_0}(a, z)$ can be represented in closed form in terms of elementary, MacDonald, and Kampé de Fériet double hypergeometric functions when $|a| \leqslant 1$, $a \neq \pm 1$:

$$\begin{aligned} K_{e_0}(a, z) = z\, K_1(z) &\left\{ z\, L[\tfrac{1}{2}, 1; \tfrac{3}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] - a\, M[1, 1; \tfrac{3}{2}, 1; \tfrac{a^2z^2}{4}, a^2] \right\} \\ + z\, K_0(z) &\left\{ L[\tfrac{1}{2}, 1; \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] - \tfrac{a^3 z}{3} M[2, 1; \tfrac{5}{2}\ 2; \tfrac{a^2z^2}{4}, a^2] \right\} \\ + (1 - a^2)^{-1/2} &\sin^{-1} a \end{aligned} \tag{1}$$

Here, following the notation of Srivastava and Panda [4, p. 63] L and M are Kampé de Fériet functions defined by

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$$L[\alpha, \beta; \gamma, \delta; x, y] \equiv F\,{}^{0:1;1}_{2:0;0}\left[\begin{matrix} - : \alpha\,;\ \beta\,; \\ \gamma, \delta : -\,;\ -\,; \end{matrix}\ x, y\right] \qquad |x| < \infty,\ |y| < \infty$$

$$M[\alpha, \beta; \gamma, \delta; x, y] \equiv F\,{}^{1:0;1}_{1:1;0}\left[\begin{matrix} \alpha : -\,;\ \beta\,; \\ \gamma :\ \delta\,;\ -\,; \end{matrix}\ x, y\right] \qquad |x| < \infty,\ |y| < 1$$

We remark that the exact region of convergence for Kampé de Fériet functions may be determined by using Horn's theorem for double series.

Equation (1) can be used together with properties of L and M (see Ref. 3) to give the well-known results

$$K_{e_0}(1, z) = z \exp(z)[K_0(z) + K_1(z)] - 1 \tag{2}$$

$$K_{e_0}(-1, z) = z \exp(-z)[K_0(z) - K_1(z)] + 1 \tag{3}$$

Here the former result is known as King's Integral (1914).

DEFINITIONS AND PRELIMINARY RESULTS

It is the purpose of this report to give representations for $K_{e_0}(a, z)$ that are valid in the finite complex a-plane, i.e., to extend Eq. (1) outside the unit disk. To this end we define

$$Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] \equiv F\,{}^{0:2;1}_{2:1;0}\left[\begin{matrix} - : \alpha, \beta\,;\ \gamma\,; \\ \mu, \nu :\ \lambda\,;\ -\,; \end{matrix}\ x, y\right] \qquad |x| < \infty,\ |y| < \infty$$

and make the following *observation*:

$$M[\alpha, 1; \gamma, \delta; tx, t] = 1 + {}_0F_1[-; \delta; x]\,\{{}_2F_1[\alpha, 1; \gamma; t] - 1\} - \frac{\alpha t\, x^2}{2\gamma\delta(\delta + 1)}\, Q[\alpha + 1, 1, 1; \delta + 2, 3, \gamma + 1; tx, x] \tag{4}$$

Proof:

$$M[\alpha, 1; \gamma, \delta; tx, t] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}} \frac{(1)_n}{(\delta)_m} \frac{(tx)^m}{m!} \frac{t^n}{n!}$$

$$= 1 + \sum_{p=1}^{\infty} \frac{(\alpha)_p (1)_p}{(\gamma)_p} \frac{t^p}{p!} \sum_{k=0}^{p} \frac{1}{(\delta)_k} \frac{x^k}{k!}$$

$$= 1 + \sum_{p=1}^{\infty} \frac{(\alpha)_p (1)_p}{(\gamma)_p} \frac{t^p}{p!} \left\{ {}_0F_1[-; \delta; x] - \sum_{k=p+1}^{\infty} \frac{1}{(\delta)_k} \frac{x^k}{k!} \right\}$$

$$= 1 + {}_0F_1[-; \delta; x]\,\{{}_2F_1[\alpha, 1; \gamma; t] - 1\} - \sum_{p=1}^{\infty} \sum_{k=p+1}^{\infty} \frac{(\alpha)_p (1)_p}{(\gamma)_p (\delta)_k} \frac{t^p}{p!} \frac{x^k}{k!}$$

Calling the latter double sum S we have

$$S = \sum_{p=0}^{\infty} \sum_{k=p+2}^{\infty} \frac{(\alpha)_{p+1}(1)_{p+1}}{(\gamma)_{p+1}\,(\delta)_k} \frac{t^{p+1}}{(p+1)!} \frac{x^k}{k!}$$

$$= \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_{p+1}(1)_{p+1}}{(\gamma)_{p+1}\,(\delta)_{p+2+k}} \frac{t^p\, t}{(p+1)!} \frac{x^{p+2+k}}{(p+2+k)!}$$

$$= tx^2 \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_{p+1}(1)_p(1)_k}{(\gamma)_{p+1}\,(\delta)_{p+2+k}\,(1)_{p+2+k}} \frac{(tx)^p}{p!} \frac{x^k}{k!}$$

$$= \frac{\alpha t x^2}{2\gamma\delta(\delta+1)} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha+1)_p\,(1)_p\,(1)_k}{(\delta+2)_{p+k}\,(3)_{p+k}\,(\gamma+1)_p} \frac{(tx)^p}{p!} \frac{x^k}{k!}$$

and the result is proved.

We observe from Eq. 4 that the behavior of $M[\alpha, 1; \gamma, \delta; tx, t]$ on $|t| = 1$ is easily deduced from that of ${}_2F_1[\alpha,1;\gamma;t]$. In particular we obtain from Eq. 4

$$M[1, 1; \tfrac{3}{2}, 1; \tfrac{a^2z^2}{4}, a^2] = 1 - \frac{a^2z^4}{96} Q[2, 1, 1; 3, 3, \tfrac{5}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + I_0(z) \left\{ \frac{\sin^{-1} a}{a\sqrt{1-a^2}} - 1 \right\} \tag{5}$$

$$M[2, 1; \tfrac{5}{2}, 2; \tfrac{a^2z^2}{4}, a^2] = 1 - \frac{a^2z^4}{240} Q[3, 1, 1; 4, 3, \tfrac{7}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{2}{z} I_1(z) \left\{ \frac{3 \sin^{-1} a}{2a^3\sqrt{1-a^2}} - \frac{3}{2a^2} - 1 \right\} \tag{6}$$

Observing that

$$\lim_{\delta\to 0} \delta\, M[\alpha, \beta; \gamma, \delta; x, y] = (\alpha/\gamma)x\, M[\alpha+1, \beta; \gamma+1, 2; x, y]$$

$$\delta\, {}_0F_1[-; \delta; x] = \delta + x\, {}_1F_2[1; 2, \delta+1; x]$$

$${}_2F_1[\alpha, 1; \gamma; t] - 1 = (\alpha t/\gamma)\, {}_2F_1[\alpha+1, 1; \gamma+1; t]$$

$$\Gamma(\nu+1)\, I_\nu(z) = (z/2)^\nu\, {}_0F_1[-; \nu+1; z^2/4]$$

we may use Eq. 4 to obtain

$$z^3 Q[\alpha+1, 1, 1; 2, 3, \gamma+1; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] = 8\{2I_1(z) - z\} + \frac{a^2z^5}{24} \frac{\alpha+1}{\gamma+1} Q[\alpha+2, 1, 1; 4, 3, \gamma+2; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] \tag{7}$$

This latter equation may also be obtained directly from the definition of Q.

We shall also need the following generating relations:

$$L[\alpha, \beta; \gamma, \delta; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m (\delta)_m} \frac{x^m}{m!} \, {}_1F_2[\beta; m+\gamma, m+\delta; y] \tag{8}$$

$$Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\mu)_m (\nu)_m (\lambda)_m} \frac{x^m}{m!} \, {}_1F_2[\gamma; m+\mu, m+\nu; y] \tag{9}$$

Observe that L is a special case of Q, i.e.,

$$Q[\alpha, \lambda, \beta; \gamma, \delta, \lambda; x, y] = L[\alpha, \beta; \gamma, \delta; x, y]$$

which also follows from the definition of Q.

REPRESENTATIONS FOR $K_{e_0}(a, z)$ AND RELATED INTEGRALS

Using the result

$$K_0(z)I_1(z) + K_1(z)I_0(z) = 1/z$$

after substitution of Eqs. 5 and 6 into Eq. 1 we obtain

$$K_{e_0}(a, z) = z\,K_0(z)A(a, z) + z\,K_1(z)B(a, z) + a \tag{10}$$

where (on using Eq. 7)

$$A(a, z) \equiv L[\tfrac{1}{2}, 1; \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{a^5z^5}{720} Q[3, 1, 1; 4, 3, \tfrac{7}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{a^3}{3}\{2I_1(z) - z\}$$

$$= L[\tfrac{1}{2}, 1; \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{a^3z^3}{24} Q[2, 1, 1; 2, 3, \tfrac{5}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}]$$

$$B(a, z) \equiv z\,L[\tfrac{1}{2}, 1; \tfrac{3}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{a^3z^4}{96} Q[2, 1, 1; 3, 3, \tfrac{5}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] - a$$

We observe that not only have the singularities at $a = \pm 1$ in Eq. 1 been removed, but Eq. 10 is valid everywhere in the complex a-plane.

We may obtain a somewhat simpler representation for $K_{e_0}(a, z)$ by using [5, p. 89]

$$\int_0^z t^m K_0(t)\,dt = \frac{z^{m+1}}{m+1} K_0(z) \, {}_1F_2[1; \tfrac{m+1}{2}, \tfrac{m+3}{2}; \tfrac{z^2}{4}] + \frac{z^{m+2}}{(m+1)^2} K_1(z) \, {}_1F_2[1; \tfrac{m+3}{2}, \tfrac{m+3}{2}; \tfrac{z^2}{4}] \tag{11}$$

which is valid for all nonnegative integers m. Since

$$K_{e_0}(a, z) \equiv \int_0^z e^{at} K_0(t)\,dt = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \int_0^z t^{2n} K_0(t)\,dt + \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(2n+1)!} \int_0^z t^{2n+1} K_0(t)\,dt$$

we obtain after a straightforward computation using Eqs. 8, 9, and 11

$$K_{e_0}(a, z) = z\,K_0(z)\,\hat{A}(a, z) + z^2 K_1(z)\,\hat{B}(a, z) \qquad (12)$$

where

$$\hat{A}(a, z) \equiv L[\tfrac{1}{2}, 1; \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{az}{2}\,Q[1, 1, 1; 1, 2, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}]$$

$$\hat{B}(a, z) \equiv L[\tfrac{1}{2}, 1; \tfrac{3}{2}, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}] + \frac{az}{4}\,Q[1, 1, 1; 2, 2, \tfrac{3}{2}; \tfrac{a^2z^2}{4}, \tfrac{z^2}{4}]$$

In addition we easily obtain from Eqs. 10 and 12

$$\int_0^z \cos at\,K_0(t)\,dt = z\,K_0(z)\,L[\tfrac{1}{2}, 1; \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}] + z^2K_1(z)\,L[\tfrac{1}{2}, 1; \tfrac{3}{2}, \tfrac{3}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]$$

$$\begin{aligned}
\int_0^z \sin at\,K_0(t)\,dt &= a - az\,K_1(z)\left\{1 + \frac{a^2z^4}{96}\,Q[2, 1, 1; 3, 3, \tfrac{5}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]\right\}\\
&\quad + \frac{a^3z}{3}K_0(z)\left\{z - 2\,I_1(z) + \frac{a^2z^5}{240}\,Q[3, 1, 1; 4, 3, \tfrac{7}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]\right\}\\
&= a - az\,K_1(z)\left\{1 + \frac{a^2z^4}{96}\,Q[2, 1, 1; 3, 3, \tfrac{5}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]\right\}\\
&\quad - \frac{a^3z^4}{24}\,K_0(z)\,Q[2, 1, 1; 2, 3, \tfrac{5}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]\\
&= \frac{az^2}{2}\left\{K_0(z)\,Q[1, 1, 1; 1, 2, \tfrac{3}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]\right.\\
&\quad \left. + \frac{z}{2}\,K_1(z)\,Q[1, 1, 1; 2, 2, \tfrac{3}{2}; \tfrac{-a^2z^2}{4}, \tfrac{z^2}{4}]\right\}
\end{aligned}$$

REDUCTION FORMULA FOR Q[α, 1, 1; γ, δ, β; x, x]

We may write

$$\begin{aligned}
Q[\alpha, 1, 1; \gamma, \delta, \beta; x, x] &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m (1)_m (1)_n}{(\gamma)_{m+n}(\delta)_{m+n}(\beta)_m}\,\frac{x^{m+n}}{m!n!}\\
&= \sum_{p=0}^{\infty}\left[\sum_{m=0}^{p}\frac{(\alpha)_m}{(\beta)_m}\right]\frac{x^p}{(\gamma)_p(\delta)_p}
\end{aligned}$$

Then using [6, Eq. 7.1.1, p. 151] the result

$$\sum_{m=0}^{p} \frac{(\alpha)_m}{(\beta)_m} = \frac{1}{1-\beta+\alpha}\left[1-\beta+\alpha\,\frac{(\alpha+1)_p}{(\beta)_p}\right]$$

we easily obtain the reduction formula

$$Q[\alpha, 1, 1; \gamma, \delta, \beta; x, x] = \frac{1-\beta}{\alpha-\beta+1}\,{}_1F_2[1; \gamma, \delta; x] + \frac{\alpha}{\alpha-\beta+1}\,{}_2F_3[1, \alpha+1; \gamma, \delta, \beta; x]$$

In particular we have

$$Q[a-1, 1, 1; a, 3, a-\tfrac{1}{2}; x, x] = (3-2a)\,{}_1F_2\,[1; 3, a; x] - 2(1-a)\,{}_1F_2[1; 3, a-\tfrac{1}{2}; x]$$

It is easy to verify that

$$x^2\,{}_1F_2[1; 3, a; x] = 2(a-1)(a-2)\left\{{}_0F_1[-; a-2; x] - \frac{x}{a-2} - 1\right\}$$

so that

$$Q[a-1, 1, 1; a, 3, a-\tfrac{1}{2}; x, x] =$$
$$\frac{(a-1)(2a-3)}{x^2}\left\{1 + (2a-5)\,{}_0F_1[-; a-\tfrac{5}{2}; x] - 2(a-2)\,{}_0F_1[-; a-2; x]\right\}$$

from which we obtain

$$Q[2, 1, 1; 3, 3, \tfrac{5}{2}; \tfrac{\alpha^2}{4}, \tfrac{\alpha^2}{4}] = \frac{96}{\alpha^4}\,\{1 + \cosh\alpha - 2I_0(\alpha)\} \tag{13}$$

$$Q[3, 1, 1; 4, 3, \tfrac{7}{2}; \tfrac{\alpha^2}{4}, \tfrac{\alpha^2}{4}] = \frac{240}{\alpha^4}\left\{1 + 3\,\frac{\sinh\alpha}{\alpha} - \frac{8I_1(\alpha)}{\alpha}\right\} \tag{14}$$

We may obtain also

$$Q[1, 1, 1; 1, 2, \tfrac{3}{2}; \tfrac{\alpha^2}{4}, \tfrac{\alpha^2}{4}] = \frac{2}{\alpha}\,\{\sinh\alpha - I_1(\alpha)\} \tag{15}$$

$$Q[1, 1, 1; 2, 2, \tfrac{3}{2}; \tfrac{\alpha^2}{4}, \tfrac{\alpha^2}{4}] = \frac{4}{\alpha^2}\,\{\cosh\alpha - I_0(\alpha)\} \tag{16}$$

$$Q[2, 1, 1; 2, 3, \tfrac{5}{2}; \tfrac{\alpha^2}{4}, \tfrac{\alpha^2}{4}] = \frac{24}{\alpha^3}\,\{\sinh\alpha - 2I_1(\alpha)\} \tag{17}$$

Equations 13-17 may be used with Eqs. 10 or 12 to obtain Eqs. 2 and 3.

ASYMPTOTIC FORMULAS FOR L, M, Q

A computation similar to the one employed in obtaining Eq. 4 gives

$$Q[\alpha, 1, 1; \mu, \nu, \lambda; tx, x] = {}_2F_1[\alpha, 1; \lambda; t]\,{}_1F_2[1; \mu, \nu; x]$$
$$-\frac{\alpha}{\lambda}\,t\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha+1)_{m+n}}{(\lambda+1)_{m+n}}\,\frac{(1)_m(1)_n}{(\mu)_m(\nu)_m}\,\frac{(tx)^m}{m!}\,\frac{t^n}{n!}$$

From this we easily obtain

$$Q[\alpha, 1, 1; \mu, \nu, \lambda; tx, x] = {}_2F_1[\alpha, 1; \lambda; t]\, {}_1F_2[1; \mu, \nu; x] - \frac{\alpha}{\lambda}\, t \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(\lambda+1)_n}\, t^n\, {}_2F_3[1, \alpha+1+n; \lambda+1+n, \mu, \nu; tx] \tag{18}$$

Now using asymptotic expansions for ${}_pF_q$ for $|z| \to \infty$ and $0 \leqslant p \leqslant q$ developed by Meijer in 1946 [5, p. 7-12] we obtain for $\alpha + 1$, $\lambda + 1$, μ, ν not a negative integer or zero

$${}_2F_3[1, \alpha+1+n; \lambda+1+n, \mu, \nu; t^2x^2/4] \sim \frac{1}{2}\, \frac{\lambda\Gamma(\lambda)\Gamma(\mu)\Gamma(\nu)}{\alpha\Gamma(\alpha)\Gamma(1/2)}\, \frac{(\lambda+1)_n}{(\alpha+1)_n}\, e^{tx}\, (tx/2)^{\rho} \left\{1 + \frac{c_1}{tx} + \frac{c_2}{(tx)^2} + \ldots\right\}$$

where

$$\rho = 3/2 + \alpha - \lambda - \mu - \nu, \quad |tx| \to \infty, \quad |\arg tx| < \pi/2$$

Using this result with Eq. 18 while ignoring the subdominant terms

$$c_m/(tx)^m = c_m(\alpha, \lambda, \mu, \nu, n)/(tx)^m, \; m = 1, 2, 3, \ldots$$

gives for $0 < |t| < 1$, $|x| \to \infty$, $|\arg tx| < \pi/2$

$$Q\left[\alpha, 1, 1; \mu, \nu, \lambda; \frac{t^2x^2}{4}, \frac{x^2}{4}\right] \sim {}_2F_1[\alpha, 1; \lambda; t^2]\, {}_1F_2\left[1; \mu, \nu; \frac{x^2}{4}\right] - \frac{1}{2}\, \frac{\Gamma(\mu)\Gamma(\nu)\Gamma(\lambda)}{\Gamma(1/2)\Gamma(\alpha)}\, \frac{t^2}{1-t^2}\, e^{tx}\, (tx/2)^{3/2+\alpha-\lambda-\mu-\nu}$$

and as a corollary we have for $\lambda = 1$

$$L\left[\alpha, 1; \mu, \nu; \frac{t^2x^2}{4}, \frac{x^2}{4}\right] \sim (1-t^2)^{-\alpha}\, {}_1F_2\left[1; \mu, \nu; \frac{x^2}{4}\right] - \frac{1}{2}\, \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(1/2)\Gamma(\alpha)}\, \frac{t^2}{1-t^2}\, e^{tx}\, (tx/2)^{1/2+\alpha-\mu-\nu}$$

We may also show that for $0 < t < 1$, $x \to \infty$, $\alpha, \mu, \nu, \lambda > 0$, $\rho = 3/2 + \alpha - \lambda - \mu - \nu$

$$Q\left[\alpha, 1, 1; \mu, \nu, \lambda; -\frac{t^2x^2}{4}, \frac{x^2}{4}\right] \sim {}_2F_1[\alpha, 1; \lambda; -t^2]\, {}_1F_2\left[1; \mu, \nu; \frac{x^2}{4}\right] + \frac{\Gamma(\mu)\Gamma(\nu)\Gamma(\lambda)}{\Gamma(1/2)\Gamma(\alpha)}\, \frac{t^2}{1+t^2}\, (tx/2)^{\rho} \cos\left(tx + \frac{\pi}{2}\rho\right)$$

and for $\lambda = 1$, $\rho = 1/2 + \alpha - \mu - \nu$

$$L\left[\alpha, 1; \mu, \nu; -\frac{t^2x^2}{4}, \frac{x^2}{4}\right] \sim (1+t^2)^{-\alpha}\, {}_1F_2\left[1; \mu, \nu; \frac{x^2}{4}\right] + \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(1/2)\Gamma(\alpha)}\, \frac{t^2}{1+t^2}\, (tx/2)^{\rho} \cos\left(tx + \frac{\pi}{2}\rho\right)$$

In addition we have the following:

$$M[\alpha, \beta; \gamma, \delta; \frac{a^2z^2}{4}, a^2] \sim \frac{1}{2}\,\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(1/2)\Gamma(\alpha)}\,\frac{e^{az}(az/2)^{1/2+\alpha-\gamma-\delta}}{(1-a^2)^{\beta}}$$

$$0 < |a| < 1,\quad |z| \to \infty,\quad |\arg az| < \pi/2$$

$$M[\alpha, \beta; \gamma, \delta; \frac{-a^2z^2}{4}, -a^2] \sim \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(1/2)\Gamma(\alpha)}\,\frac{(az/2)^{1/2+\alpha-\gamma-\delta}\cos\left[az + \frac{\pi}{2}\left(\frac{1}{2} + \alpha - \gamma - \delta\right)\right]}{(1+a^2)^{\beta}}$$

$$0 < a < 1,\quad z \to \infty,\quad \alpha, \beta, \gamma, \delta > 0$$

Although these relations *per se* may be of some interest, they are not particularly useful in obtaining asymptotic expansions for, say, $K_{e_0}(a, z)$. However, we may easily obtain, for example, the complete Lipschitz-Hankel integral $K_{e_0}(-a, \infty) = \cos^{-1}a/\sqrt{1-a^2}$, Re $a > -1$.

SUMMARY

Various representations for the incomplete Lipschitz-Hankel integral $K_{e_0}(a, z)$ and related integrals have been given in terms of elementary, cylindrical, and Kampé de Fériet functions. In addition some properties of the Kampé de Fériet functions associated with these integrals are derived.

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