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A Unifying Approach to Linear and Nonlinear Array Processing: A Tutorial

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The concept of linear filtering is the focus of a unifying approach to linear and nonlinear array processing methods. A number of well-known methods (conventional, optimal, maximum likelihood) are typically discussed within the context of linear filtering, while others (maximum entropy, linear predictor, generalized eigenvector) are not usually presented in this way. Through rederivation of these methods in terms of constrained filtering, we are able to relate performance of the methods to imposed (or neglected) constraints.

Each filter discussed is linearly constrained in the look direction and then optimized using quadratic forms based on noise and sidelobe structure. Each filter is more complicated than the ones preceding it, as we consider additional aims and impose additional constraints. Stability of the methods to mismatches is considered, and techniques for improving stability are presented.

Appendixes on Fourier transform approximation, array gain and detection, and complex Gaussian random variables are included.

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CONTENTS

1. INTRODUCTION 1

2. OPTIMAL NOISE SUPPRESSION 3

3. OPTIMAL SIDELOBE PLUS NOISE SUPPRESSION 4

4. INCORPORATING PRIOR INFORMATION 6

5. DATA ADAPTIVE FILTERING AND CAPON'S METHOD 8

6. INCREASING LEAKAGE SUPPRESSION 10

7. EXPLICIT GENERALIZED EIGENVECTOR METHODS 11

8. THE PROBLEM OF ROBUSTNESS 12

9. COMMENTS AND CONCLUSIONS 18

10. REFERENCES 18

APPENDIX A — The Fourier Transform and its Approximation 20

APPENDIX B — Complex Gaussian Random Variables 24

APPENDIX C — Array Gain, Detection, and Likelihood Ratios 26

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A UNIFYING APPROACH TO LINEAR AND NONLINEAR ARRAY PROCESSING: A TUTORIAL

1. INTRODUCTION

Array processing is the extraction of useful information about a spatiotemporal field from measurements taken using an array, or spatial configuration, of sensors. Practical restrictions on the number of sensors and their placement, as well as the typical low signal-to-noise ratio (S/N), have made the use of sophisticated algorithms to process the limited data an option of growing importance, particularly as computer power increases and costs decline. The purpose of this report is to present, in a unifying fashion, a number of recently developed array processing methods.

Methods for array processing fall into one of two broad categories, which, borrowing terminology from computing, are described as *on-line* and *batch*. On-line methods process in real time, as the data are received and any adaptation is incorporated through feedback loops. Batch methods accept as data the full set of spatiotemporal measurements, usually in the form of cross-sensor correlation matrices. Our concern here is exclusively with batch methods.

The information extracted from the data can vary; the typical view is that the data contain certain components, signals, which can be described well using a small number of parameters, and it is the values of these parameters that we seek. What the parameters are will depend on the physical model, although much of the development presented here is independent of the actual physical model. We use a planewave model in most cases, but this is not essential, either to the implementation or to the understanding of the methods discussed. We focus on bearing estimation and speak of steering or looking in a particular direction, but most of what we say applies to the estimation of other parameters as well.

Because many of the methods used for bearing estimation are related to mathematical procedures that serve other purposes (power spectrum estimation, statistical hypothesis testing) they are often discussed in the literature in a manner that obscures their relation to one another and their properties with regard to array processing. For example, Burg's maximum entropy method satisfies a number of information theoretic criteria, but what does it do to array data and why? We attempt, therefore, to rederive many of the well known linear and nonlinear methods, using the unifying framework of the linear filter. This is a natural framework, and one that has already been used in many cases. Put simply, we wish to linearly transform the data to enhance the information-bearing components relative to the others. The filters differ from one another with respect to the criteria of optimality that are imposed.

Several appendixes are included in this report to help make this tutorial self-contained.

We assume throughout that we have an arbitrarily configured array of M sensors, in vector locations s_m, $m = 1, \ldots, M$, that the received time series at each sensor is Fourier filtered to produce, at each successive block of time, a data vector $x = (x(1), \ldots, x(M))^T$ of single-frequency-bin complex values. By limiting the methods to a single frequency, we can represent the time delays associated with planewave arrival directions in terms of phase changes between sensors. If the sensor locations

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take the form $s_m = m\Delta\ v$, where $\Delta > 0$ and v is a fixed unit vector, the array will be called a *uniform line array.* If, in addition, the interelement spacing is Δ = one-half the wavelength, then the array is called *Nyquist.* We assume throughout that Δ is at most 1/2 wavelength, to avoid the problem of aliasing.

Most of the methods discussed here use the cross-sensor correlation matrix, $R = <x\ x^+>$, where $^+$ denotes conjugate transposition and $<\ >$ indicates averaging over the x associated with each time block. The linear filters we discuss operate on each x, but the optimality is measured in terms of average performance, hence the appearance of R.

A localized farfield source at frequency ω, with wavevector k and unit amplitude, produces at the array the "steering vector"

$$e(k) = (\exp(-i\omega u \cdot s_1/c), \ldots, \exp(-i\omega u \cdot s_M/c))^T, \tag{1.1}$$

where u is the unit vector normal to the planewave, c the speed of propagation, ω the (angular) frequency of interest, and $k = (\omega/c)u$. The usual model for x is

$$x = \Sigma a(k)e(k) + n, \tag{1.2}$$

where each $a(k)$ is random, see Appendix B, n is a noise vector, and the sum is taken over all the physically meaningful k. To allow us to use matrix notation, we restrict k to some large but finite family; this is not a significant alteration and does not affect the subsequent development. If the signals of interest do not correspond to planewaves, but to some other waveforms given by the physical model, these can be substituted in our formulas wherever $e(k)$ appears, and the k can be viewed as a vector parameter.

In most cases the $a(k)$are zero except for a very small number of the k and the goal is to determine how many are nonzero and which k these are. A straightforward method for solving this problem is based on the simple properties of the dot product of vectors. To illustrate, suppose $x = e(k_0)$ for some fixed k_0. We then form the function of k given by

$$P_1(k) = <|e(k)^+x/M|^2> = e(k)^+R\ e(k)/M^2. \tag{1.3}$$

This function attains its maximum value when $k = k_0$, from which the value of k_0 can be ascertained. This estimation P_1 is called the conventional estimator. If x consists of several of the $e(k)$, plus noise, as in Eq. (1.2), P_1 may not work as well. If components of x are not sufficiently dissimilar ($e(k_1)$ and $e(k_2)$, with k_1 close to k_2), then P_1 may not resolve these two signals; this happens particularly when M is small. Also the noise vectors can, in certain cases (such as when ω is small compared to the Nyquist frequency $c\pi/\Delta$), appear to come from localized sources.

The quantity inside the absolute value in Eq. (1.3),

$$y_1(k) = e(k)^+\ x/M \tag{1.4}$$

has the form of a linear filtering of x; that is, we can write Eq. (1.4) as $y_1(k) = y(k)$,

$$y(k) = b^+x = b(k)^+x \tag{1.5}$$

if we use $b = b(k) = e(k)/M$. The filter (or vector of weights) $b = (b(1), \ldots, b(M))^T$ depends on k, the value being tested at the moment. We filter x through each of the $b(k)$, hoping that, because of the way $b(k)$ has been designed, if x has any signal component corresponding to k this component will be passed, while others are suppressed. Because the vectors have finite length M, it is impossible to suppress completely all other components. In subsequent sections we consider ways of improving the filter. In the case of P_1 the filter is (essentially) $e(k)$ itself; in other methods $b(k)$ uses $e(k)$, but is not simply (a multiple of) $e(k)$, as it is here.

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Returning to Eq. (1.2), we can write it in matrix notation as

$$x = Ea + n, \tag{1.6}$$

where there are as many columns in E (and entries in a) as there are k in the large finite set we are using. Then Eq. (1.6) is an underdetermined system of noisy linear equations and there is a sizable literature on the subject of "solving" such systems [1]. In the array processing case, there are two aspects that make the problem distinctive:

- we usually have a collection of various x, the $a(k)$ are viewed as random and we desire only the average power $<|a(k)|^2>$, or the correlation, $<a(k)\ \overline{a(l)}>$ and
- we have the prior information that all but a few of the $a(k)$ are zero, and we seek ways to incorporate this information.

The following sections present a number of methods, based on linear filtering, to help us decide how many signal components are present and what the signal powers and cross correlations are. Those filters constrained by $b^+e(k) = 1$ will provide estimates of signal power, while others, not so constrained, provide only indicators of signal presence and do not simultaneously estimate power. At the end of the fourth section of this report we discuss these ideas again.

Because this is a tutorial, and not a survey paper, no attempt is made to provide extensive references to the literature as we proceed. There are a number of standard survey articles that should be consulted, which provide good bibliographies [2-4].

2. OPTIMAL NOISE SUPPRESSION

The conventional estimator, P_1 in Eq. (1.3), was derived earlier from a simple matching of x with each $e(k)$; if x consists of one signal component $e(k_0)$, this estimator will tell us what k_0 is. For certain noises, however, P_1 can produce misleading results. The purpose of this section is to consider filters designed to optimally suppress the effect of the noise vector n.

We now assume that the $a(k)$ each have mean zero, that n has mean zero, and that the $a(k)$ are independent of n (although not necessarily of each other). We also assume that the averaging time (number of time blocks) is sufficient to permit R to be written approximately as

$$R \approx R_0 = \sigma^2 E A_0 E^+ + \rho^2 N_0, \ \mathrm{trace}(A_0) = \mathrm{trace}(N_0) = M, \tag{2.1}$$

with $\sigma^2 A_0 = [<a(k)\ \overline{a(l)}>]$, $\rho^2 N_0 = [<n(m)\ \overline{n(j)}>]$. Our goal is to estimate A_0, which we do not assume to be diagonal, so as to include the possibility of correlated arrivals. In what follows we use A and N to denote prior estimates of $\sigma^2 A_0$ and $\rho^2 N_0$.

If all the $a(k)$ are zero (no signals) then the conventional estimator gives

$$P_1(k) = e(k)^+ R e(k)/M^2 \approx \rho^2 e(k)^+ N_0 e(k)/M^2, \tag{2.2}$$

so that if $N_0 = I$ (I the identity matrix) then $P_1(k)$ is constant for all k. If the noise is correlated between sensors ($N_0 \neq I$), however, $P_1(k)$ can exhibit local maxima that can be mistaken for low level signals.

Holding k fixed for the moment, consider two hypotheses: (H_0) $R = N_0$ (no signals); (H_1) $R = e(k)\ e(k)^+ + N_0$. Let b be any linear filter (complex M-vector) and consider its performance in each case. We have

$$(H_0):\ <|b^+x|^2> = b^+Rb = b^+N_0b; \text{ and} \tag{2.3}$$

$$(H_1):\ <|b^+x|^2> = |b^+e(k)|^2 + b^+N_0b. \tag{2.4}$$

We choose b to pass $e(k)$ undistorted ($b^+e(k) = 1$), while minimizing b^+Nb. It follows from vector calculus that

$$b = \lambda N^{-1}e(k), \text{ with } \lambda = 1/e(k)^+N^{-1}e(k). \tag{2.5}$$

Applying this filter to our data vectors, x produces output

$$y_2(k) = \lambda e(k)^+N^{-1}x; \tag{2.6}$$

the average magnitude squared output is our optimal noise suppression estimator,

$$P_2(k) = <|y_2(k)|^2> = \lambda^2 e(k)^+N^{-1}RN^{-1}e(k). \tag{2.7}$$

Note that if the noise is assumed to be uncorrelated ($N = I$), then $P_1(k) = P_2(k)$ for all k.

Correlated noise matrices are a common occurrence when a uniform line array is used in which the spacing Δ is smaller than the Nyquist rate (1/2 wavelength or $\pi c/\omega$). Cox [5] discusses various intersensor correlations that can be observed in commonly encountered ambient noise environments. In practice we usually do not know N_0 exactly, so we either model it using prior information or estimate it from neighboring frequency bins or signal-absent measurements, if available. One particular matrix that is used to model ambient (spherical) isotropic noise on a uniform line array is the *sinc* matrix, with entries

$$N_{m,j} = <n(m)\,\overline{n(j)}> = \sin[(m-j)\Delta\omega/c]/(m-j)\Delta\pi. \tag{2.8}$$

If the array is Nyquist, then this matrix is I; otherwise sensors are mutually correlated.

The estimator $P_2(k)$ requires the inversion of N. If N is the sinc matrix and Δ is considerably smaller than $\pi c/\omega$, then approximately $M(\Delta\omega/\pi c)$ of its eigenvalues are close to 1, the rest is close to zero and N is ill-conditioned [6]. If we use the sinc matrix in place of the true N_0 in computing $P_2(k)$, and if the true N_0 also contains a small amount of uncorrelated noise, then the estimator becomes unstable. It is always safer to add a little to the main diagonal of the sinc matrix for stability.

In deriving P_2 we held k fixed and considered two possibilities: no signal (H_0); and one signal in the k direction (H_1). Clearly there is another alternative; there can be signals in other directions. The filter Eq. (2.5) is not designed to take this into account. Consequently the presence of signals in nearby directions can cause a large output in the k direction, even when there is no signal in that direction. This can lead to loss of resolution and poor sidelobe structure.

The term "sidelobe structure" refers to the various values of an estimator $P(k)$ if k is held fixed, but $R = e(l)\,e(l)^+$ is varied as a function of l. That is, the sidelobe structure is the estimator output at k caused by a single signal $e(l)$, viewed as a function of l. Clearly it is good to have low sidelobes, so that signals in other directions do not produce high output in direction k. One problem with the P_1 estimator that persists, even when the noise is uncorrelated, is the dependence of the sidelobe structure on the array configuration. We consider this problem next.

3. OPTIMAL SIDELOBE PLUS NOISE SUPPRESSION

As before, we hold k fixed temporarily and design the filter $b = b(k)$. We define the sidelobe function of filter b to be

$$s(l) = b^+e(l), \tag{3.1}$$

the output of the filter when the input is $e(l)$. Subject to the constraint $b^+e(k) = 1$, we minimize

$$\sum_l |s(l)|^2 + b^+Nb. \tag{3.2}$$

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Note that because $s(k) = 1$ by constraint, it is not necessary to say "$l \neq k$" in the summation. We can write

$$\sum_l |s(l)|^2 = \sum_l s(l)\, s(l)^+ = b^+ \left[\sum_l e(l) e(l)^+\right] b = b^+ EE^+ b$$

so Eq. (3.2) becomes $b^+(EE^+ + N)b$. Our solution is then

$$b = \lambda (EE^+ + N)^{-1} e(k),\ \lambda = 1/e(k)^+(EE^+ + N)^{-1} e(k). \tag{3.3}$$

Applying this filter to the data gives

$$y_3(k) = \lambda e(k)^+ (EE^+ + N)^{-1} x, \tag{3.4}$$

and estimator

$$P_3(k) = <|y_3(k)|^2> = \lambda^2 e(k)^+ (EE^+ + N)^{-1} R (EE^+ + N)^{-1} e(k). \tag{3.5}$$

Implicit in our concern about sidelobe structure is our desire to control the average behavior of our estimator, as signals appear at the various l. Compare P_3 and P_1 in this average sense, with $N = 0 = 0I$.

For a given l and $R = e(l)\, e(l)^+$, the estimator $P_1(k)$ yields

$$P_1(k;l) = e(k)^+ e(l) e(l)^+ e(k)/M^2, \tag{3.6}$$

so that averaging over l we get (except for a constant)

$$\text{average } (P_1(k;l)) = e(k)^+ \left[\sum_l e(l) e(l)^+\right] e(k) = e(k)^+ EE^+ e(k). \tag{3.7}$$

Using P_3 instead we obtain

$$P_3(k;l) = \lambda^2 e(k)^+ (EE^+)^{-1} e(l) e(l)^+ (EE^+)^{-1} e(k), \tag{3.8}$$

so that averaging over l gives (except for a constant)

$$\begin{aligned} \text{average } P_3(k;l) &= \lambda^2 e(k)^+ (EE^+)^{-1}\, EE^+ (EE^+)^{-1} e(k) \\ &= 1/e(k)^+ (EE^+)^{-1} e(k). \end{aligned} \tag{3.9}$$

As we see in Section 5, Eq. (3.9) is the maximum likelihood or Capon's estimator of the field consisting of a uniform distribution of the $e(l)$. Examination of Eq. (3.9) for specific array configurations shows it to be essentially constant over k, whereas Eq. (3.7) can vary considerably as a function of k.

If we have some prior information about the relative power in signals and noise, we could modify the above procedure and instead

$$\text{minimize } \beta^2 b^+ EE^+ b + b^+ N b, \text{ subject to } b^+ e(k) = 1. \tag{3.10}$$

This leads to a generalized version of P_3:

$$P_3(k) = \alpha^2 e(k)^+ (\beta^2 EE^+ + N)^{-1} R (\beta^2 EE^+ + N)^{-1} e(k), \tag{3.11}$$

with $\alpha^2 = 1\, e(k)^+ (\beta^2 EE^+ + N)^{-1} e(k)$. Note that even when the noise is assumed to be uncorrelated ($N = I$), the P_1 estimator will not be an optimal sidelobe suppression estimator unless $EE^+ = I$ as well. The matrix EE^+ involves the geometry of the array and the chosen set of l. In fact EE^+ may be identical to a matrix one might use to model ambient noise; it might be a sinc matrix, for example. This is because one way to model ambient noise is by imagining a large number of independent far-field sources uniformly distributed in space or in a plane [5].

In this section we have improved upon the P_2 estimator by including sidelobe suppression as a constraint. A problem still remains, however. Because we do not yet know which signals are present, we do not know which sidelobes (which l) are troublesome. Sidelobes are a problem only when something is *out there* in that direction. In the next section of this report we look at how we might use prior information to suppress primarily those sidelobes that pose a problem.

4. INCORPORATING PRIOR INFORMATION

In the previous section, we derived the estimator P_3 (3.11) by including in that part of the data to be filtered out all signal components, corresponding to l different from k, that might be present. If we have prior knowledge of the locations and relative strengths of actual signals we can use this to modify our estimator. The resultant procedure is related to the optimal Bayesian solution for Eq. (1.6) [1].

Suppose that we have a prior estimate, A, of the signal correlation matrix A_0. For fixed k we seek filter b to minimize

$$b^+(EAE^+ + N)b, \text{ subject to } b^+e(k) = 1. \tag{4.1}$$

The desired b is

$$b = \lambda(EAE^+ + N)^{-1}e(k),\ \lambda = \lambda(k) = 1/e(k)^+\ (EAE^+ + N)^{-1}\ e(k). \tag{4.2}$$

The filter output is

$$y_4(k) = \lambda(k)\ e(k)^+\ (EAE^+ + N)^{-1}x, \tag{4.3}$$

and the averaged squared output is our estimator,

$$P_4\ (k) = \lambda(k)^2\ e(k)^+\ (EAE^+ + N)^{-1}\ R\ (EAE^+ + N)^{-1}\ e(k). \tag{4.4}$$

The output $y_4(k)$, viewed as an estimate of $a(k)$, is closely related to the Bayesian solution of Eq. (1.6), as we now show.

With the assumption that a and n are independent complex Gaussian random vectors (see Appendix B) with mean zero and covariance matrices A_0 and N_0, respectively, the probability density of x, given a, is

$$P(x|a) \sim \exp\ (-\ (x - Ea)^+\ (\rho^2 N_0)^{-1}(x - Ea)), \tag{4.5}$$

where $\sim$ means "is proportional to." The probability density of a itself is

$$p(a) \sim \exp\ (-\ a^+(\sigma^2 A_0)^{-1}a), \tag{4.6}$$

so that the probability density of a given x, is

$$P(a|x) \sim p(x|a)p(a), \tag{4.7}$$

or

$$p(a|x) \sim \exp\left[-\ (x - Ea)^+\ (\rho^2 N_0)^{-1}\ (x - Ea) - a^+(\sigma^2\ A_0)^{-1}a\right]. \tag{4.8}$$

The Bayesian approach tells us to select as our estimate of a the expected value of Eq. (4.8). But the expected value in this case is also the value of a for which Eq. (4.8) is largest. Consequently the Bayes estimate is $\hat{a}$, where $\hat{a}$ minimizes

$$G(a) = (x - Ea)^+\ (\rho^2\ N_0)^{-1}(x - Ea) + a^+\ (\sigma^2 A_0)^{-1}a. \tag{4.9}$$

From vector calculus it follows that

$$\hat{a} = \sigma^2 A_0 E^+\ (\sigma^2\ EA_0E^+ + \rho^2 N_0)^{-1}x. \tag{4.10}$$

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Consider for a moment the case in which A_0 is diagonal, $A_0 = \text{diag}\ \{A_0(l,l)\,|\text{all } l\}$. Then the kth entry of Eq. (4.10) becomes

$$\hat{a}(k) = \sigma^2 A_0\ (k,k)\ e(k)^+\ (\sigma^2\ EA_0\ E^+ + \rho^2 N_0)^{-1} x. \tag{4.11}$$

Here the statistics are known ($\sigma^2 A_0$ and $\rho^2 N_0$ assumed known) and are being used to estimate an individual a. In Eq. (4.3), A and N are estimates of $\sigma^2 A_0$ and $\rho^2 N_0$. Also $\lambda(k)$ is an estimate of $\sigma^2 A_0(k,k)$, which is the true signal power in the k direction. The expression for $\lambda(k)$ in Eq. (4.2) will be encountered again in Section 5 of this report; it is the Capon, or maximum likelihood (ML) estimator of k directional signal power, given $EAE^+ + N$ as the cross-sensor correlation matrix. So $\lambda(k)$ is a prior estimate of signal power, based on our prior estimate of R_0.

Returning to the general case, in which A_0 is not necessarily diagonal, it is interesting to note that the Bayes estimator Eq. (4.10) does not have the same form as (4.3). The kth entry of $\hat{a}$ can be written

$$\hat{a}(k) = \sigma^2 \left[\sum_l A_0\ (k,l)\ e(l)^+\right] (\sigma^2\ EA_0E^+ + \rho^2 N_0)^{-1} x, \tag{4.12}$$

which suggests that when signals are correlated, our constraint $b^+\ e(k) = 1$ is not the best thing to use. This makes sense; up to now our approach has been to hold k fixed, find $b(k)$, derive the estimates for $a(k)$ and $<|a(k)|^2>$, and then move to another k. We do not require the estimate $y(k)$ or $P(k)$ to have any particular global properties, as functions of k. But when the signals are correlated what happens at one k is related to what happens elsewhere; this has not been taken into account in the development so far.

Based on the Bayesian solution in the general case, we can modify estimators y_4 and P_4 as follows:

$$y'_4\ (k) = \left[\sum_l A(k,l) e\ (l)^+\right] (EAE^+ + N)^{-1} x,$$

$$P'_4(k) = <\ |y'_4(k)|^2>. \tag{4.13}$$

Unless we have sufficient prior information to warrant using a nondiagonal A as our estimate of A_0, these modified estimators reduce to those of Eqs. (4.3) and (4.4).

As previously mentioned, the $\lambda(k)$ used in Eq. (4.3) is a prior estimate of $\sigma^2 A_0(k,k)$. Suppose $A = \text{diag}\ \{A(l,l)\}$ and we consider the signals-only case. Then $\lambda(k) = 1/e(k)^+\ (EAE^+)^{-1} e(k)$. With $L = \text{diag}\ \{\lambda(l)\}$, is it true that $L = A$, that is, are our initial estimates of the signal powers the same as the $\lambda(k)$? No, they are different, generally. Writing

$$1/\lambda(k) = e(k)^+\ (EAE^+)^{-1}\ e(k), \tag{4.14}$$

we see that the diagonal entries of the matrix $E^+(EAE^+)^{-1}E$ are $1/\lambda(l)$, so that the diagonal entries of $AE^+(EAE^+)^{-1}E$ are $A(l,l)/\lambda(l)$. But the trace is the sum of the diagonal entires, so

$$\text{trace}(AE^+(EAE^+)^{-1}E) = \sum_l A(l,l)/\lambda(l). \tag{4.15}$$

But trace $(AE^+(EAE^+)^{-1}E) = \text{trace}\ (EAE^+)^{-1}\ EAE^+) = M$ so

$$M = \sum_l A(l,l)/\lambda(l). \tag{4.16}$$

If there are more than M nonzero $A(l,l)$ in the summation, then not all the terms can equal 1. We shall return to this point in our discussion of Capon's method in Section 5.

The linear methods presented in the first four sections of this report provide estimates of the power in the various directions for the components that are passed by the filters: the various $P(k)$ are

considered to be estimators of $<|a(k)|^2>$, so that when there is no signal in the k direction $P(k)$ should be nearly zero. The problem then is to constrain the filters $b(k)$ properly. Later in this report we consider nonlinear methods. It is not correct to view these estimators as giving power: they give only indications, through their peaks, of signal presence. Once we have used these methods to determine J, the number of actual sources, and their corresponding k we can include these quantities as prior information in a Bayesian approach to estimating A_0. We can, for example, take $A(k,k) = 1$ if k corresponds to an actual source, according to our best estimates, and 0 otherwise. Then rank (EAE^+) is J. Our estimate of $a(k)$ is then $y_4(k)$ in Eq. (4.3). To estimate $A_0(k,l)$ we can use $<y_4(k)\overline{y_4(l)}>$, where k and l are limited to those values already deemed to correspond to actual sources.

5. DATA ADAPTIVE FILTERING AND CAPON'S METHOD

We saw, in our discussion of the Bayesian method and its relation to the estimators in the last section, that the matrix $EAE + N$ that appears in Eqs. (4.3) and (4.4) is a prior estimate of the matrix $\sigma^2 EA_0E^+ + \rho^2 N_0$, which is our model for the true cross-sensor correlation matrix, R_0. Capon's idea is to dispense with prior estimates of R_0 and to use instead the actual measured R.

Holding k fixed, we find that the desired filter b is now

$$b = \lambda(k)R^{-1}e(k), \lambda(k) = 1/e(k)^+R^{-1}e(k), \tag{5.1}$$

the filter output is

$$y_5(k) = b^+x = \lambda(k)e(k)^+R^{-1}x \tag{5.2}$$

and the averaged squared output power is the estimator,

$$P_5(k) = \lambda(k)^2 e(k)^+R^{-1}RR^{-1}e(k) = 1/e(k)^+R^{-1}e(k) = \lambda(k). \tag{5.3}$$

This power estimate is Capon's maximum likelihood (ML) estimator. Note this estimator is not linear, that is although $y_5(k) = b^+x$ appears to be a linear function of x there is a hidden occurrence of x in the b^+ term, because b depends now on R, which depends on x. For this reason, ML is called a nonlinear estimator. It has also been referred to as a *data adaptive* procedure, because the filter adapts not to any prior estimate of what is *out there*, but to the actual data contained in R. It is helpful to note that the filtering scenario, whereby x is received and filtered by b to produce $y(k)$, cannot now occur in time, because of the dependence of b on the entire collection of data vectors x. The averaged magnitude squared of the filter output is no longer *power* because of the dual dependence on x.

How well does $P_5(k)$ estimate power in the kth signal component? To see, let us imagine a noise-free case, with uncorrelated signals, $R = \sigma^2 EA_0E^+$, $A_0 = \text{diag}\{A_0(l,l)\}$. Recalling our discussion concerning L and A at the end of the previous chapter, we know that we do not have $P_5(k) = \sigma^2 A_0(k,k)$ for each k, generally. Because the sum of $\sigma^2 A_0(k,k)/P_5(k)$ over all k must equal M Eq. (4.16), the $P_5(k)$ can equal the $\sigma^2 A_0(k,k)$ only if there are precisely M nonzero entries on the diagonal of A_0. If there are fewer than M actual signals, then $P_5(k)$ must underestimate some of the $\sigma^2 A_0(k,k)$, while if there are more than M signals, P_5 (k) will overestimate some of their powers. In most practical cases N_0 is present as well, giving the appearance of many smaller signals distributed in various directions, so typically $P_5(k)$ overestimates the $\sigma^2 A_0(k,k)$. Part of what it sees as signal is noise power in that direction.

Nonlinear methods have been studied carefully since the late 1960s when it was observed that Capon's ML and Burg's maximum entropy method (ME), to be discussed later, were able to resolve closely spaced sources that linear methods could not [7]. There are many different derivations of these nonlinear methods that shed some light on their superior resolving capability, but by far the most revealing approach to take is the analysis of the generalized eigenvector structure of R [8]. Not only

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does this route provide a clear understanding of why nonlinear methods tend to resolve better, it helps us see what can go wrong and suggests safer alternatives.

Assume, from now on, that there are only $J(<M)$ values of k for which $A_0(k,k) > 0$. Let these values of k be indexed k_j, $j = 1, \ldots, J$, and let $e(j) = e(k_j)$, $a(j) = a(k_j)$. Let B_0 be the J by J matrix obtained from A_0 by removing all rows and columns whose main diagonal entry is zero. Let F be the M by J matrix whose jth column is $e(j)$. Then R_0 can be written as

$$R_0 = \sigma^2 F B_0 F^+ + \rho^2 N_0. \tag{5.4}$$

To avoid technical difficulties we also assume that the rank of B_0 is J, that is although signals may be mutually correlated, there is no complete correlation that would reduce the rank of B_0.

The generalized eigenvectors of the pair (R_0, N_0) are the vectors $z_m, m = 1, \ldots, M$, that solve the following constrained maximization problem:

$$\text{maximize } z^+ R_0 z, \text{ subject to constraints } z^+ N_0 z = 1 \text{ and } z^+ z_n = 0, 1 \leqslant n < m. \tag{5.5}$$

The values

$$c_m = z_m^+ R_0 z_m \tag{5.6}$$

are called the generalized eigenvalues corresponding to the pair (R_0, N_0); note that $c_m \geqslant c_{m+1} \geqslant p^2$ for all m. The z_m can also be defined as the solution to the minimization problem:

$$z^+ R z \text{ subject to the constraints } z^+ N_0 z = 1,\ z^+ z_n = 0, m < n \leqslant M. \tag{5.7}$$

Because rank $(B_0) = J < M$, it follows that for $m \geqslant J + 1$ $c_m = \rho^2$ and $z_m^+ (F B_0 F^{+)} z_m = 0$. Writing for $m \geqslant J + 1$,

$$0 = z_m^+ (F B_0 F^+) z_m = (F^+ z_m)^+ B_0 (F^+ z_m); \tag{5.8}$$

it follows from rank $(B_0) = J$ *that* $F^+ z_m = 0, m \geqslant J + 1$. Therefore, for each $j = 1, \ldots, J$,

$$e(j)^+ z_m = 0. \tag{5.9}$$

If the model Eq. (5.5) is exactly equal to R, and we know N_0 exactly, we can calculate the c_m and z_m. We determine J and ρ^2 from the multiplicity and value of the lowest generalized eigenvalue, and the value k_j from the zeros of the functions $e(k)^+ z_m, m \geqslant J + 1$. How would we calculate these generalized eigenvectors?

The generalized eigenvectors for the pair (R_0, N_0) can be found by calculating ordinary eigenvectors of another matrix. Let $N_0 = V^+ V$ and set $T = (V^+)^{-1} R_0 V^{-1}$. Let u_m, λ_m, $m = 1, \ldots, M$ be orthonormal eigenvectors and eigenvalues of T with $\lambda_m \geqslant \lambda_{m+1}$ for all m. Then $z_m = V^{-1} u_m$ and $c_m = \lambda_m$, for all m.

In practice we do not have R_0 and N_0 but R and N. To avoid cumbersome notation let us use z_m, c_m for the pair (R, N), keeping in mind that N only approximates $\rho^2 N_0$, and that the computed c_m, $m \geqslant J + 1$, will not be precisely identical. Let us refer to the z_m, $m \geqslant J + 1$, as the *low* generalized eigenvectors (LGE) and the others as *high* (HGE). Similarly, the u_m, $m \geqslant J + 1$, are the *low* eigenvectors of T (LE), the others the *high* ones (HE). We now return to Capon's estimator P_5 and analyze it from the standpoint of the generalized eigenvectors of (R, N).

Set $C = \text{diag}\{c_m, m = 1, \ldots, M\}$ (recall $c_m = \lambda_m$), and let U be the M by M matrix whose mth column is u_m. Then $UCU^+ = T$ and $UU^+ = I$. Therefore

$$T^{-1} = UC^{-1} U^+ \tag{5.10}$$

and

$$R^{-1} = V^{-1}T^{-1}(V^{+})^{-1} = (V^{-1}U)C^{-1}(V^{-1}U)^{+} = ZC^{-1}Z^{+}, \tag{5.11}$$

where Z is the M by M matrix whose mth column is z_m. Note if $N \neq I$, then the decomposition $R^{-1} = ZC^{-1}Z^{+}$ is not the usual eigenvector decomposition of R^{-1}; in particular, the z_m are not orthogonal ($ZZ^{+} \neq I$).

From Eq. (5.11) we can write

$$R^{-1} = \sum_{m=1}^{M} c_m^{-1} z_m z_m^{+}, \tag{5.12}$$

so that $P_5(k)$ in Eq. (5.3) becomes

$$P_5(k) = 1\Big/\sum_{m=1}^{M} c_m^{-1}|e(k)^{\dagger} z_m|^2. \tag{5.13}$$

We know from Eq. (5.9) that $0 = e(j)^{+}z_m$ for the LGE so that if we did not have the HGE terms in Eq. (5.13) $P_5(k_j)$ would be $+\infty$. Because $c_m \geqslant c_{m+1}$ the reciprocal weighting of terms by c_m^{-1} emphasizes the desired LGE terms, and more so as the S/N increases.

The next step, then, in improving the performance of our estimators, is to reduce the weighting given to the HGE terms in Eq. (5.13). We can, of course, simply calculate the z_m (assuming we know N_0 and the model is accurate) and keep the terms we want. This is sometimes done and we discuss it in Section 7 of this report. First, however, we consider further improvement through constraint modification because this approach leads us to several other nonlinear methods that have appeared in the literature, including Burg's maximum entropy method (ME) [9], Johnson's linear predictive (LP) estimator [2], and the Byrne-Fitzgerald extensions of ME [10,11].

6. INCREASING LEAKAGE SUPPRESSION

Each of the estimators we have discussed so far performs well when there is a signal in the direction of look k, because of the constraint $b^{+}e(k) = 1$. However, when there is no signal in direction k, but signals nearby, there can be a high output in the k direction due to *leakage* from these nearby sources. Two signals close together can cause the estimator to give a higher reading at values of k between the two signals than it gives at either of the two themselves, causing loss of resolution. To improve resolution we must increase leakage suppression. This is related to decreasing the weighting of the HGE terms in Eq. (5.13), as we shall see shortly.

The LGE are orthogonal to each of the signal vectors $e(j)$, $j = 1, \ldots, J$, so the span of these $e(j)$ must coincide with the span of the HGE (*span* meaning the collection of all linear combinations of the indicated set of vectors). Therefore increased emphasis on signal suppression is equivalent to reducing the HGE weighting in Eq. (5.13). Writing $T = UCU^{+}$, $T^n = UC^nU^{+}$, $n = 1, 2, \ldots$, we see that by raising T to a power, we increase the relative size of c_m^{-1} for $m \geqslant J + 1$, compared to $m \leqslant J$. Suppose we replace $R = V^{+}TV$ with

$$R^{(n)} = V^{+}T^{n}V \tag{6.1}$$

in the P_5 estimator. From the equation

$$(R^{(n)})^{-1} = ZC^{-n}Z^{+}, \tag{6.2}$$

we see that we obtain

$$P_6^{(n)}(k) = 1\Big/\sum_{m=1}^{M} c_m^{-n}|e(k)^{+} z_m|^2, \tag{6.3}$$

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and that as n approaches $+\infty$ Eq. (6.3) approaches the explicit generalized eigenvector solution,

$$P_6^{\infty}(k) = 1\Big/\sum_{m=J+1}^{M} |e(k)^+ z_m|^2. \tag{6.4}$$

We shall return to Eq. (6.4) in the next section of this report. Now we want to consider some special cases of Eq. (6.3), particularly for $n = 2$.

Using $n = 2$ in Eq. (6.3), we obtain

$$P_6^{(2)}(k) = 1/e(k)^+R^{-1}NR^{-1}e(k). \tag{6.5}$$

If we replace N with a dyad, pp^+, we obtain an approximation of Eq. (6.5) that involves less computation;

$$P_6'(k) = 1/|e(k)^+R^{-1}p|^2. \tag{6.6}$$

If the noise N is highly correlated, then p could be, for example, the largest eigenvector of N. If we have a uniform line array and the noise is stationary, so that N is Toeplitz (constant on each diagonal), then N is determined by its first column, which could be taken to be p. In any case, from the point of view we have adopted, these estimators appear as approximations only. In some cases, however, Eq. (6.6) can be derived as an optimal estimator in its own right. If we take p to be the first column of the matrix I, then we get either Burg's ME (if R is Toeplitz) or Johnson's LP method (general R). If p is the first column of any positive-definite Toeplitz matrix (not necessarily I), then Eq. (6.6) is (essentially) the Byrne-Fitzgerald extension of ME, called the weighted reciprocal spectrum approximation (WRSA) [11].

It has been noted in the literature that ME has resolving capability that is superior to ML [7] and also that ME can give misleading results when the noise is correlated [12]. Both of these observations are supported by our analysis. Because ME is an approximation of Eq. (6.5) it has the added resolving power that comes from $n = 2$, instead of $n = 1$ for ML. If the noise is correlated, using the first row of I as p is a poor approximation of N by dyad pp^+; it is better to use the first row of N or its largest eigenvector. Hence the misleading results when $N \neq I$.

Section 7 considers the explicit generalized eigenvector estimators. Because, in practice there is model error, we must consider as well the stability of these estimators.

7. EXPLICIT GENERALIZED EIGENVECTOR METHODS

As previously noted, higher resolution is obtainable if we reduce the output power of our filters $b(k)$ for those k that do not correspond to actual signals. Let k be such a vector. In order that $b = b(k)$ reject entirely the signal component, it is necessary (and sufficient) that b be a linear combination of the LGE.

Let q be an arbitrary member of the span of the LGE; that is,

$$q = \sum_{m=J+1}^{M} d_m z_m. \tag{7.1}$$

Let $b = \lambda q$ for some λ. The usual constraint $b^+e(k) = 1$ tells us that

$$\lambda = \lambda(k) = 1\Big/\sum_{m=J+1}^{M} -d_m e(k)^+ z_m, \tag{7.2}$$

which is finite because k is not one of the k_j. The filter output is then

$$y_7(k) = \lambda(k) q^+ x, \tag{7.3}$$

and the averaged magnitude squared output is

$$P_7(k) = |\lambda(k)|^2 q^+ Rq = q^+ Nq \Big/ \left| \sum_{m=J+1}^{M} d_m e(k)^+ z_m \right|^2. \tag{7.4}$$

This estimator is finite unless $k = k_j$ for some j, in which case it is infinite.

It is important to reiterate the assumptions on which Eq. (7.4) is based. We assume that $R = R_0$, that $N = N_0$, that there are $J(<M)$ signals present and that B_0 has rank J. In practice these assumptions can be violated in numerous ways: the signal vectors $e(k)$ are not precisely planewaves; the averaging time for $\hat{R}$ is not long enough to eliminate cross terms between signals and noise, so that $R \neq R_0$; the noise matrix N_0 is not precisely known. Because the information we need is carried by the nulls of the functions $e(k)^+ z_m$ it is easily perturbed. The main problem with P_7 is robustness; that is, stability in the presence of model errors.

One particular source of perturbation is phase error, which can be systematic (in the electronics) or the result of wavefront curvature or array motion. These perturbations cause severe degradation of most nonlinear methods when the noise is correlated.

Phase errors can be modelled by taking

$$D = \text{diag}\,\{\exp\,(i\phi_m),\ m = 1, \ldots, M\}. \tag{7.5}$$

with $\{\phi_m\}$ M independent random variables, uniformly distributed on the interval $[-\epsilon, \epsilon]$, for some small $\epsilon > 0$. We then replace R with $\tilde{R} = DRD^+$ in all the estimators.

When $N_0 = I$, phase errors are not a serious problem for nonlinear estimators, so long as ϵ is moderate-sized. But as the noise becomes correlated even small values of ϵ (corresponding to, say, $\pm 3°$ at broadside) can cause severe degradation, even for high S/N. In Section 8 of this report, we consider why this is the case and what can be done about it.

8. THE PROBLEM OF ROBUSTNESS

Provided all our assumptions are exactly met, the generalized eigenvector methods Eq. (7.4) give perfect source localization and resolution, regardless of S/N level. In practice, instabilities are encountered so frequently that considerable concern has been voiced about the effectiveness of nonlinear methods in actual situations [13,14]. Much attention is being given to understanding the causes of instability and to developing ways to combat it. In this section we analyze one form of perturbation that can lead to severe estimator degradation, phase errors with correlated noise, and discuss some recently developed methods for stabilizing in the presence of such perturbations. Our analysis of the instability reveals that the information we seek is redundantly stored, that most often only partial loss occurs, but that most methods interrogate mainly the most unstable storage locations. The instability is caused by the sort of noise fields commonly encountered in sonar array processing and is not directly related to the particular perturbing effect, in this case phase errors, that triggers the degradation. The following discussion summarizes Refs. 15 and 16.

If we write $R = XLX^+$, where the columns of X are orthonormal eigenvectors of R and $L = \text{diag}\left\{\lambda_1, \ldots, \lambda_M\right\}$, λ_m the ordered eigenvalues of R, then Capon's ML estimator becomes

$$P_5(k) = 1 \Big/ \sum_{m=1}^{M} \lambda_m^{-1} |e(k)^+ x_m|^2, \tag{8.1}$$

with x_m the mth column of X. Clearly those terms for which λ_m is smallest are given the most weight. As the noise becomes correlated between sensors (N_0 not diagonal), the x_m for m near M begin to

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behave somewhat like the lowest eigenvectors of the noise-only matrix, N_0, and the low λ_m begin to separate into two groups, those in one much smaller than those in the other [6]. If the noise is isotropic (spherical) and the array is a uniform line array, then N_0 is as in Eq. (2.8). If the spacing is much less than 1/2 wavelength, the lowest eigenvectors of N_0 are nearly orthogonal to all vectors $e(k)$ corresponding to acoustically feasible planewave arrivals. Figure 8.1 illustrates what happens.

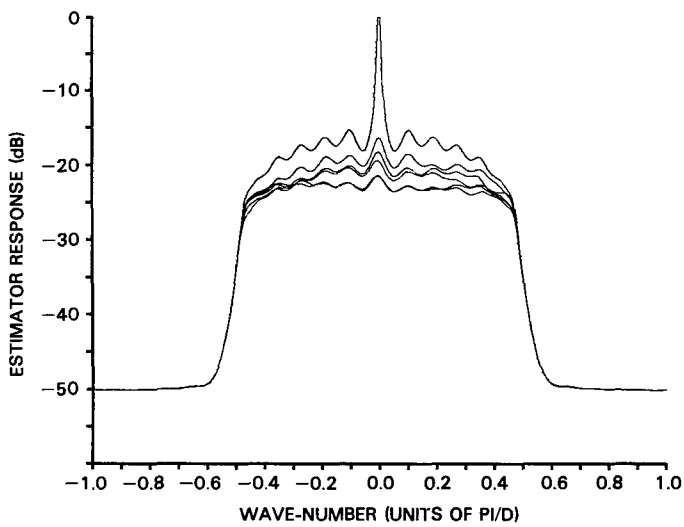

Fig. 8.1 — Estimator response using eigenvectors 13 to 25; 1 signal

In each of the first four figures we used a simulated uniform line array with $M = 25$ sensors and a spacing of 1/4 wavelength (so twice oversampled). The noise is spherically isotropic. In Fig. 8.1 we display the bearing response function

$$1\Big/\sum_{m=I}^{L} \lambda_m^{-1} |e(k)^+ x_m|^2 \tag{8.2}$$

for $I = 13$, $L = 25$, with a single planewave source at broadside. In each case the top curve corresponds to the case of no phase errors, while the five other curves involve independent simulations of random phase errors of at most $\pm 5^0$.

Figure 8.2 shows Eq. (8.2) in the same cases, but with $I = 2$, $L = 12$. The information about the source is still preserved in the (almost) nulls of the $e(k)^+ x_m$, $m = 2, \ldots, 25$, even though the noise is not white. However, for $m \geqslant 13$ these (almost) nulls are not much different from neighboring values and the relative sizes are easily disturbed when phase errors are introduced. Those for $m = 2, \ldots, 12$, on the other hand, are much smaller than their neighbors and the relative values are not disturbed, hence Fig. 8.2.

Figures 8.3 and 8.4 show the effect on resolution. Replacing the single source at broadside with two uncorrelated ones, at $\pm 0.0075\pi/\Delta$, the smallest eigenvector of R ($I = L = 25$ in (8.2)) fails to resolve even without phase errors (because of correlated noise) (Fig. 8.3), while the choice of $I = 8$, $L = 11$ (Fig. 8.4) provides stable resolution.

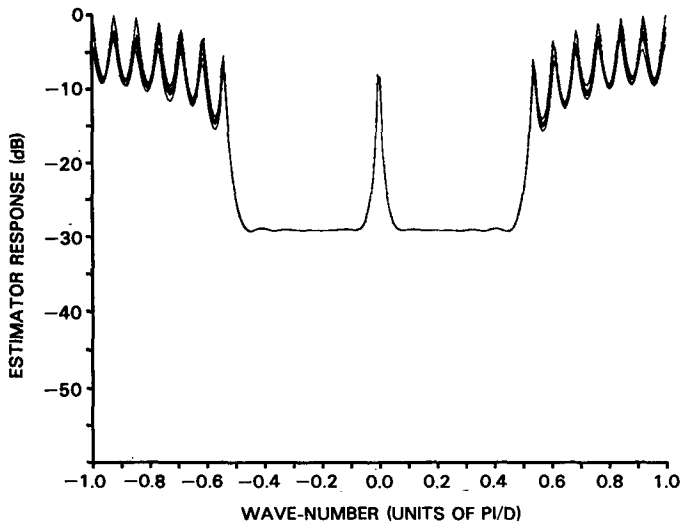

Fig. 8.2 — Estimator response using eigenvectors 2 to 12; 1 signal

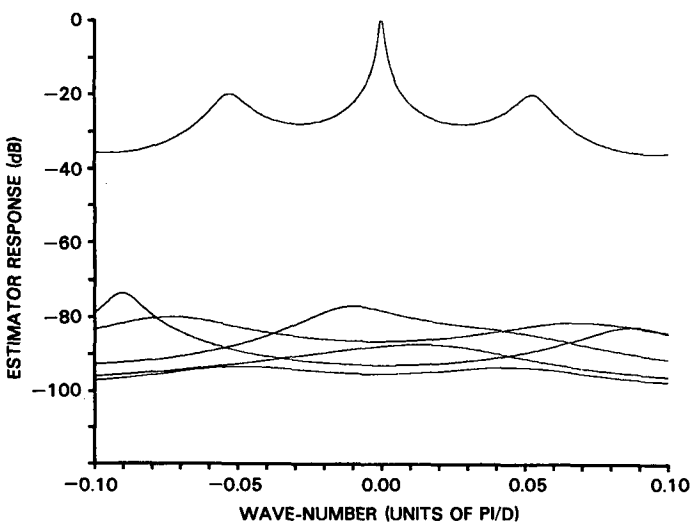

Fig. 8.3 — Estimator response using eigenvector 25; 2 signals

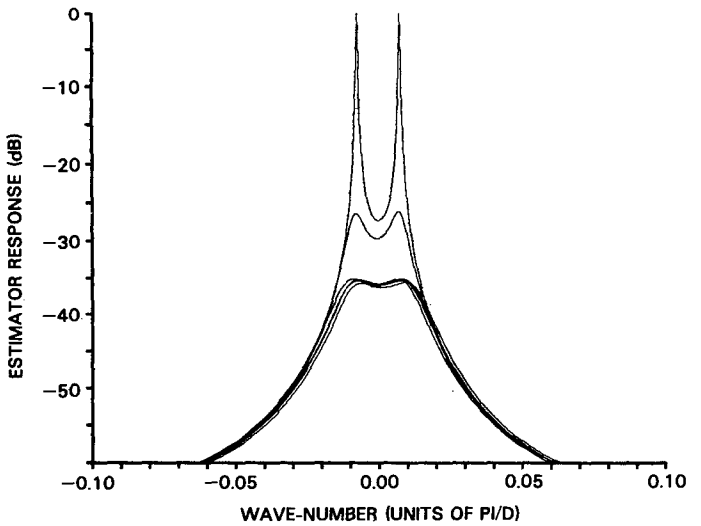

Fig. 8.4 — Estimator response using eigenvectors 8-11; 2 signals

If we wish to avoid the actual calculation of the individual eigenvectors of R and prefer an estimator with the computational cost of ML, we can still shift emphasis to the more stable eigenvectors, as we now show. The estimator

$$P_8(k) = 1/e(k)^+R^{-1/2}(R + \alpha^2R^{-1})^{-K}\ R^{-1/2}e(k) \tag{8.3}$$

can be written as

$$P_8(k) = 1\Big/\sum_{m=1}^{M}\left(\frac{\lambda_m^{K-1}}{(\lambda_m^2 + \alpha^2)^K}\right)|e(k)^+x_m|^2, \tag{8.4}$$

where $K = 0,1,2, \ldots$ and α are parameters chosen by the user. By normalizing R to have trace 1, we make 1 the average of the eigenvalues; taking α near 1 then emphasizes those in the middle. It is important that α be greater than the lowest eigenvalues corresponding to unstable eigenvectors, but not as large as λ_J, where J is the number of signals. As the noise becomes increasingly correlated, the number of stable eigenvectors decreases; by raising K we further emphasize just those eigenvalues near α.

Figures 8.5 through 8.8 simulate a uniform line array of $M = 25$ sensors, oversampled by a factor of 5 (Δ is one-tenth of a wavelength). Isotropic noise is 30 dB above the uncorrelated noise and there are two signals, at $(0.1 \pm 0.03)\pi/\Delta$, each with power −3 dB relative to the total noise power. Figure 8.5 shows (top curve) the conventional estimator, with and without phase errors (no difference), below it the ML without phase errors, and below that ML for the same five independent sets of phase errors used before.

Figure 8.6 shows estimator $P_8(k)$ for the same data; from top to bottom we have $K = 1$, 4, and 12. Here $\alpha^2 = 6.0$ and each of the three graphs is actually the five phase error cases superimposed on the errorless case, showing almost perfect stability. Remembering that $K = 0$ gives ML and comparing Fig. 8.5 with $K = 1$ in Fig. 8.6 we see how quickly stability can be achieved as we iterate beyond ML.

Figure 8.7 shows that the estimator $P_8(k)$ is not overly sensitive to the choice of α^2; here we have the same case as in Fig. 8.6, with $K = 6$ and $\alpha^2 = 6.0$, 1.0, and 0.5.

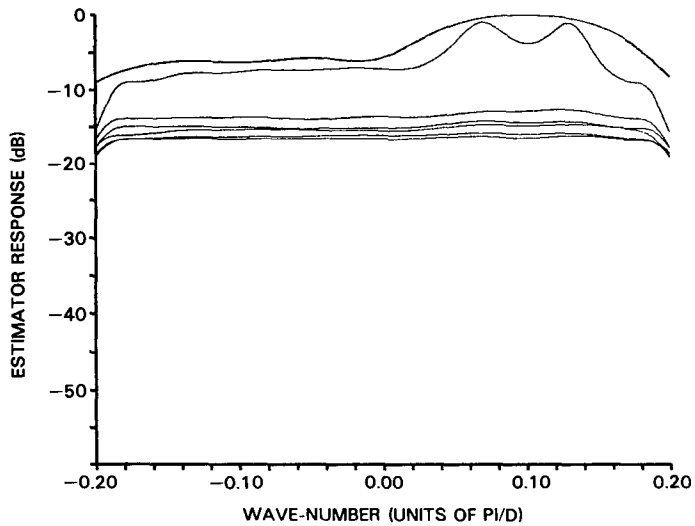

Fig. 8.5 — Conventional and MLM estimators; 2 uncorrelated signals

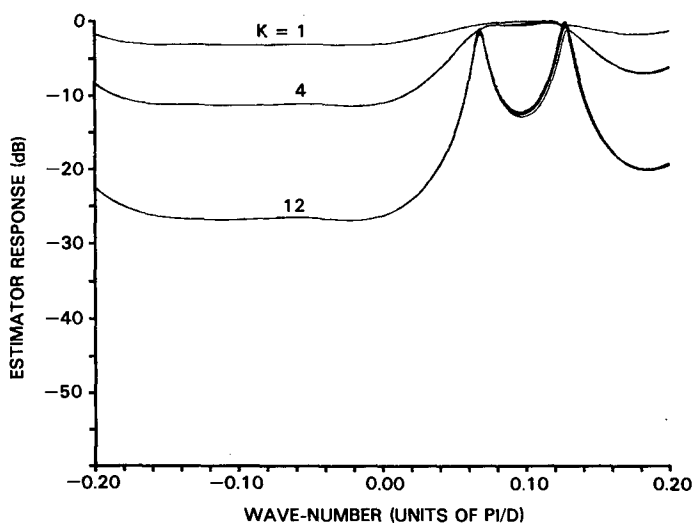

Fig. 8.6 — SNLM estimators; K = 1,4,12; 2 uncorrelated signals

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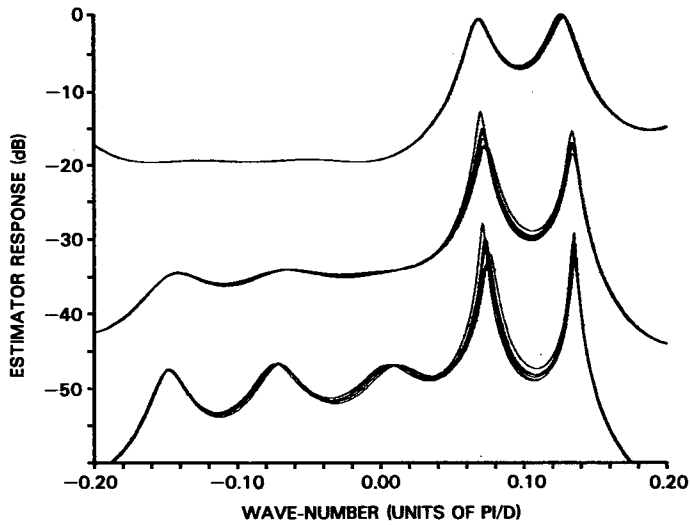

Fig. 8.7 — SNLM estimators; $\alpha^2 = 6,1,0.5$;
2 uncorrelated signals

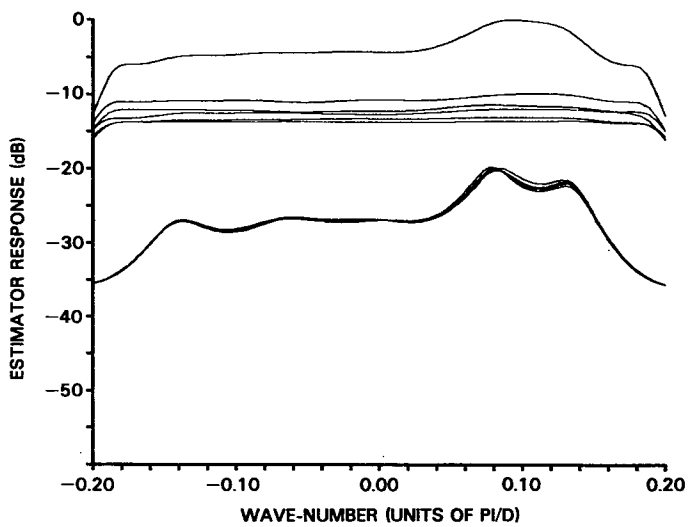

Fig. 8.8 — MLM and SNLM estimators;
2 correlated signals

When the signal arrivals are correlated most nonlinear methods degrade, either not resolving or giving biased estimates of arrival angles [17]. In Fig. 8.8 we have two signals with correlation coefficient of (0.9, 0°); shown are ML, with and without phase errors (top curves) and $P_8(k)$, $K = 12$, $\alpha^2 = 1.0$ for the same cases (superimposed lower curves). As the signals become more correlated, the second largest eigenvalue approaches those of the noise (in the uncorrelated noise case) and careful selection of α becomes more crucial.

The real point of these findings is not the particular methods themselves, but the fact that the desired information is still there in these highly perturbed cases. Once we know that the information is more robustly stored in the higher eigenvectors (the ones immediately below $m = J$), provided the noise is not too correlated, we can extract that information in several ways. The estimators $P_8(k)$ are just suggested ways of doing this.

9. COMMENTS AND CONCLUSIONS

We have derived a number of well-known array processing methods by using the notion of linear filter to provide a unifying approach. Certainly there are other ways to derive each of the methods; see Ref. 18, for example, where the optimal noise suppression method is derived in three different ways. There are other methods which we did not discuss (see Ref. 19, e.g.), but which can be understood in terms of filtering, eigenvector analysis and the other tools we have used in this report.

The last word has not yet been written on the subject. Each application brings with it its own peculiar requirements. Two directions of great interest at the moment, for sonar array processing, are statistical analyses of the performance of the various methods [20] and the use of more complicated physical models to guide the signal processing [21]. Discussion of either of these topics in any detail would have taken us too far afield, however.

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Appendix A

THE FOURIER TRANSFORM AND ITS APPROXIMATION

The Fourier transform (FT) of the function $f(t)$ is defined to be the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i\omega t)\, dt, \tag{A1}$$

for $-\infty < \omega < \infty$. Given $F(\omega)$ for all ω, we can recapture f by using the inversion formula,

$$f(t) = \int_{-\infty}^{\infty} F(\omega) \exp(-i\omega t)\, dw/2\pi. \tag{A2}$$

For the integrals in Eqs. (A1) and (A2) to exist, it is necessary that both f and F approach zero sufficiently rapidly as $|t|$ and $|\omega|$ approach ∞. However, it is useful to speak of the FT of periodic functions and for that the notion of the δ function is needed; we denote by $\delta(\omega)$ the FT of the function $f(t) = 1/2\pi$ for all t, so that $\delta(\omega - \alpha)$ is the FT of the function $f(t) = (1/2\pi) \exp(-i\alpha t)$. The δ function $\delta(\omega - \alpha)$ is usually depicted as a *spike* at α and thought of as a (generalized) function that is zero except at α but with integral $1/2\pi$. These generalized functions can be defined more rigorously by using limits of ordinary functions.

If we have only finitely many values of f, we cannot use Eq. (A1) directly; to obtain F we must either model f (or F) in some way or settle for an approximation of F. We consider two cases:

Case A (equispaced data): the data values are $f(m\Delta)$, $m = 1, \ldots, N$, for some $\Delta > 0$;

Case B (general data): the data values are $f(t_m)$, $m = 1, \ldots, N$, for some N distinct values $t = t_m$.

We begin with the modelling approach and then consider approximation.

The simplest sort of model for f is a linear combination of N known functions, g_n, with coefficients $a(n)$ to be determined from the data;

$$f(t) = \sum_{n=1}^{N} a(n)\, g_n(t), \tag{A3}$$

with

$$f(t_m) = \sum_{n=1}^{N} a(n)\, g_n(t_m), m = 1, \ldots, N, \tag{A4}$$

determining the $a(n)$ uniquely, so long as we have chosen the g_n so that the matrix,

$$G = [G(m,n) = g_n(t_m)], \tag{A5}$$

is invertible. Once the $a(n)$ are found the FT is

$$F(\omega) = \sum_{n=1}^{N} a(n)\, G_n(\omega), \tag{A6}$$

with G_n the FT of g_n. The problem with this approach is that unless we employ prior information to select the g_n in an appropriate manner, the whole procedure is completely arbitrary. If the model is reasonable, the FT in Eq. (A6) will be a good estimate; if the model is not accurate Eq. (A6) will be useless. Some methods that appear to be models are actually optimal approximations and as such are

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less arbitrary than they might appear. In other cases, when certain models are viewed as approximations they are suboptimal and should be replaced by optimal ones. One particular model that can also be viewed as an approximation (although not always an optimal one) is the discrete Fourier transform (DFT).

For case A the DFT model for f is

$$f_{\mathrm{DFT}}(t) = \sum_{n=1}^{N} a(n) \exp(-i\omega_n t), \tag{A7}$$

with $\omega_n = -\pi(1 - 2n/N)/\Delta$. The reason for this particular choice of ω_n is that the matrix $G = E$,

$$E = [\exp(i\omega_n\, m\, \Delta)], \tag{A8}$$

then has $E^{-1} = (1/N)E^{+}$, $+$ denoting conjugate transpose. We can then write the $a(n)$ in closed form: for $n = 1, \ldots, N$,

$$a(n) = (1/N) \sum_{m=1}^{N} f(m\Delta) \exp(i\omega_n\, m\, \Delta). \tag{A9}$$

The FT estimate is then

$$F_{\mathrm{DFT}}(\omega) = (2\pi/N) \sum_{n=1}^{N} \left[\sum_{m=1}^{N} f(m\Delta) \exp(i\omega_n\, m\Delta) \right] \delta(\omega - \omega_n). \tag{A10}$$

The DFT model (in case A) assumes that $F(\omega)$ is the sum of N delta functions, at $\omega = \omega_n$. We could, of course, select other values for the ω_n; all we would lose would be the computational convenience of the closed form solution Eq. (A9) and the fact that Eq. (A10) can be numerically computed rapidly, using a fast Fourier transform (FFT) algorithm.

The expression on the right-hand side of Eq. (A9) can be thought of as a function of general ω; let

$$H(\omega) = N^{-1} \sum_{m=1}^{N} f(m\Delta) \exp(i\omega m\Delta). \tag{A11}$$

Then $H(\omega_n) = a(n)$ and within the context of the modelling approach $H(\omega)$ has no significance for any ω except the ω_n. We see, however, that H does appear as an optimal approximation; this can cause considerable confusion if the distinction between modelling and approximation is not clearly made. A second source of confusion comes from the practice known as *zero-padding.*

Zero-padding refers to the practice of artificially enlarging the equispaced data set by appending a number of zero values; that is, for $m = N + 1, \ldots, N + K$ let $f(m\Delta) = 0$. When the larger set of values is used as data along with the DFT, the number N in Eqs. (A7) to (A10) is replaced by $N + K$ and the definition of ω_n changes accordingly. From a modelling standpoint we have simply changed the model. But the function $H(\omega)$ still is involved and gives the values of the new coefficients $a(n)$ at the new points ω_n, because the K new terms added to Eq. (A11) are all zero. Zero-padding really only makes sense within the context of approximation; creating new data values, equal to zero, and changing the model seem pointless and arbitrary from a modelling perspective.

Note that in Eq. (A10) the FT estimate at $\omega = \omega_n$ is not $H(\omega_n)$; it is $2\pi\, H(\omega_n)\, \delta(\omega - \omega_n)$. As we shall see, in approximating, that H itself is the estimate of F (except for a constant factor).

When we pass to the general case B, there is the temptation to modify $H(\omega)$ in Eq. (A11) to

$$J(\omega) = N^{-1} \sum_{m=1}^{N} f(t_m) \exp(i\omega t_m). \tag{A12}$$

The question now is: for which N values of ω is J to be used to provide the model coefficients? There is no model for which J provides the coefficients, in general. Also, as we shall see, J is not an optimal approximation either, in general. Although J is used in practice in the case B situations, it is an estimator of F that is devoid of mathematical foundation.

Turn now to approximation. Suppose we take an arbitrary weighting function $P(\omega) \geqslant 0$ and minimize the weighted mean squared error,

$$\int_{-\infty}^{\infty} |F(\omega) - P(\omega) \sum_{n=1}^{N} b(n) \exp(i\,\omega t_n)|^2 P^{-1}(\omega)\, d\omega, \tag{A13}$$

with the notational convention that if $P(\omega) = 0$ then $P^{-1}(\omega)$ is defined to be 0 [A1].

The $b(n)$ for which the error is least must satisfy the equations

$$f(t_m) = \sum_{n=1}^{N} b(n) \int_{-\infty}^{\infty} P(\omega) \exp(i\,\omega(t_n - t_m))\, d\,\omega/2\pi, \tag{A14}$$

$m = 1, \ldots, M$. The FT $F(\omega)$ is being approximated by a function of the form

$$\hat{F}(\omega) = P(\omega) \sum_{n=1}^{N} b(n) \exp(i\omega t_n); \tag{A15}$$

This is not a model. We are not saying that $F(\omega)$ looks like Eq. (A15), but we are using functions of that type to approximate $F(\omega)$, whatever $F(\omega)$ looks like. Note that by using the t_n in $\hat{F}(\omega)$ we are able to use our data values on the left-hand side of Eq. (A14); if we had chosen another form for $\hat{F}(\omega)$, the set of equations we would have obtained from minimizing the error would have involved values of f that we do not have. That is, if we want a minimum weighted error approximation, the form of the approximation is determined partially by the nature of the data (the t_m values). The $P(\omega)$ is chosen by us, and this allows us to tailor the approximation to fit whatever prior knowledge we may have about f and F. Some examples will help to illustrate the procedure.

Example 1. Suppose we are in case A and our prior knowledge is that f is a bandlimited function (F has bounded support) and that $F(\omega) = 0$ if $|\omega| \geqslant \pi/\Delta$. We select $P(\omega)$ to be the function that is 1 for $|\omega| < \pi/\Delta$, 0 otherwise. Then at least $\hat{F}(\omega)$ in Eq. (A15) will have the proper support. Then Eq. (A14) becomes

$$\begin{aligned} f(m\Delta) &= \sum_{n=1}^{N} b(n) \int_{-\pi/\Delta}^{\pi/\Delta} \exp(i\,\omega(n-m)\Delta)\, d\omega/2\pi, \\ &= (1/\Delta) b(m), m = 1, \ldots, M \end{aligned} \tag{A16}$$

so the estimator is

$$\hat{F}(\omega) = \Delta \sum_{m=1}^{N} f(m\Delta) \exp(i\omega m\,\Delta) = N\Delta\, H(\omega), \tag{A17}$$

for $|\omega| < \pi/\Delta$. Here Eq. (A17) makes sense for every ω, not just for a finite number, as before. Also, it is $N\Delta\, H(\omega)$ itself that now estimates $F(\omega)$. We can also see more clearly the role of zero-padding. In practice we cannot actually evaluate $\hat{F}(\omega)$ at all values $|\omega| < \pi/\Delta$, so we select some finite number. Suppose we want $N + K$ values, equispaced within $[-\pi/\Delta, \pi/\Delta]$. Then the calculations are

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exactly those we performed when we zero-padded in the modelling approach. The FFT algorithms are designed to compute DFT models but can be used to evaluate $\hat{F}(\omega)$ at $N + K$ equispaced points if we pretend that we had $N + K$ data values originally.

Example 2: Suppose we have further prior information that $F(\omega) = 0$ for $|\omega| > \Omega$, for some known Ω less than π/Δ. The estimate in Eq. (A17) approximated $F(\omega)$ on $[-\pi/\Delta, \pi/\Delta]$, but we know now that outside of the smaller interval $[-\Omega, \Omega]$ $F(\omega) = 0$. We then modify $P(\omega)$, making it 1 if $|\omega| < \Omega$,0 otherwise. Now Eq. (A14) becomes

$$f(m\Delta) = \sum_{n=1}^{N} b(n) \int_{-\Omega}^{\Omega} \exp\,(i\omega(n-m)\Delta)\, d\omega/2\pi$$

$$= \sum_{n=1}^{N} b(n) \frac{\sin\,(\Omega\,(m-n)\Delta)}{\pi(m-n)\Delta}. \tag{A18}$$

Solving Eq. (A18), our estimate of $F(\omega)$, valid for $|\omega| \leqslant \Omega$, becomes

$$\hat{F}(\omega) = \sum_{m=1}^{N} b(n) \exp\,(i\omega m\Delta), \tag{A19}$$

where the $b(n)$ are (generally) not equal to $(1/\Delta)f(m\Delta)$.

Example 3: Consider case B with the same prior knowledge as in example 1. Then Eq. (A14) becomes

$$f(t_m) = \sum_{n=1}^{N} b(n) \int_{-\pi/\Delta}^{\pi/\Delta} \exp\,(i\,\omega\,(t_n - t_m))\, d\omega/2\pi. \tag{A20}$$

The matrix that must be inverted to find the $b(n)$ is not just $(1/\Delta)I$, as it was in example 1 (unless, of course, $t_m = m\Delta$). Consequently our estimate of $F(\omega)$, valid for $|\omega| < \pi/\Delta$, will be

$$\hat{F}(\omega) = \sum_{m=1}^{N} b(m) \exp\,(i\omega t_m). \tag{A21}$$

The $b(m)$ will not be $(1/\Delta)f(t_m)$ generally, but some other value, dependent on all the $f(t_n)$. In particular Eq. (A21) will not be $J(\omega)$ unless we are in case A.

In all cases we have considered here the estimators of $F(\omega)$ are themselves consistent with the data; that is, if we apply the inversion formula Eq. (A2) and check the values at the $t = t_m$ or $t = m\Delta$, we get the data values back again. This can be troublesome if the data contain some additive noise and the matrix we invert is as in example 2, for Ω much less than π/Δ. To avoid instability in such cases, it is advisable to increase the values along the main diagonal by, say, 10%, before inverting. A more rigorous treatment of stability is beyond the scope of this appendix (see Refs. A2 and A3 and any other articles on "regularization)".

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Appendix B

COMPLEX GAUSSIAN RANDOM VARIABLES

If X and Y are real-valued Gaussian random variables, with means m_X, m_Y and variances σ_X^2, σ_Y^2, respectively, then the complex-valued random variable $Z = X + iY$ may not have any of the useful mathematical properties that we associate with the term Gaussian. For that reason it is necessary to make further assumptions about the real and imaginary parts of a complex random variable before calling it Gaussian. Consider here only the single variate case; the more general case is treated in detail in Goodman [B1].

With no further assumptions, it is the case that the mean of Z is $m_Z = m_X + im_Y$. The variance, σ_Z^2, is

$$\sigma_Z^2 = E|Z - m_Z|^2 = \sigma_X^2 + \sigma_Y^2, \tag{B1}$$

regardless of possible dependence between X and Y. To have the statistics of Z completely determined by its mean and variance (as in the real case), we must assume that X and Y are independent, which implies then that the real random vector (X, Y) is multivariate Gaussian with a diagonal correlation matrix.

The benefit that comes from using Gaussian statistics is due to the pleasant algebraic properties of the density function $\exp(-x^2)$. We would like Z to have a probability density function proportional to

$$p(z) = \exp(-|z - m_z|^2/\sigma_z^2), \tag{B2}$$

which is

$$p(z) = \exp(-[(x - m_X)^2 + (y - m_Y)^2]/(\sigma_X^2 + \sigma_Y^2)). \tag{B3}$$

If we assume that $\sigma_X^2 = \sigma_Y^2$, then Eq. (B3) becomes

$$p(z) = \exp(-\frac{1}{2}(x - m_X)^2/\sigma_X^2) \exp(-\frac{1}{2}(y - m_Y)^2/\sigma_Y^2), \tag{B4}$$

and the right-hand side of Eq. (B4) is (except for constants) the true probability density function for Z. Therefore, Z will have a density of the form Eq. (B3) provided we assume X and Y are independent, and $\sigma_X^2 = \sigma_Y^2$. In the multivariate case, similar assumptions are made about the covariance matrix of the random vector of real and imaginary parts [B1].

We say then that $Z = X + iY$ is a complex Gaussian random variable if X and Y are independent real Gaussian random variables with the same variance. It then follows that (using $\tilde{Z} = Z - m_Z$, and similarly for $\tilde{X}$, $\tilde{Y}$)

(a) $|\tilde{Z}|^2 = \tilde{X}^2 + \tilde{Y}^2$ is $\sigma_Z^2 \chi_2^2$, where χ_2^2 is a chi-squared random variable with two degrees of freedom;

(b) $|\tilde{Z}| = \sqrt{|\tilde{Z}|^2}$ is $\sigma_Z R$, where R is a Rayleigh random variable;

(c) If $\tilde{Z} = |\tilde{Z}| \exp(i\theta)$, then $\tan\theta = \tilde{Y}/\tilde{X}$ is a Cauchy random variable;

(d) $\theta = \arctan(\tilde{Y}/\tilde{X})$ is uniformly distributed over $(-\pi, \pi)$.

For further discussion of these results see Refs. B2 and B3.

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Appendix C

ARRAY GAIN, DETECTION, AND LIKELIHOOD RATIOS

The basic model used in the statistical theory of detection of a narrowband source is that the data vector x obtained from the array either has entries

$$x(m) = n(m),\ m = 1, \ldots, M,\ (H_0), \tag{C1}$$

or

$$x(m) = A\ e^{-i\tau_m} + n(m),\ (H_1), \tag{C2}$$

where the $n(m)$ are independent complex Gaussian random variables with mean 0 and variance ρ^2, A is complex Gaussian with mean 0 and variance σ^2, and τ_m is the time delay at the mth sensor corresponding to a single frequency planewave with arrival angle θ relative to broadside. The θ is fixed, the two hypotheses H_0 and H_1 are the only ones admitted, and the problem is to decide which one it is.

If we simply average the entries $x(m)$, we get

$$y_0 = M^{-1} \sum_{m=1}^{M} x(m); \tag{C3}$$

if H_0 is true, then y_0 is complex Gaussian with mean 0, variance ρ^2/M, and if H_1 is true, the variance becomes

$$\left| \sum_{m=1}^{M} e^{-i\tau_m} \right|^2 (\sigma^2/M^2) + (\rho^2/M). \tag{C4}$$

This is fine if all the delays are $\tau_m = 0$ (broadside arrival), because then the variance is $\sigma^2 + (\rho^2/M)$. However, if the arrival is not broadside, the variance in the signal component can be much less than σ^2. The significance of the variance lies in the distribution of $|y_0|^2$: if H_0, then $|y_0|^2$ is (ρ^2/M) times a χ_2^2 r.v. (chi-squared with two degrees of freedom having mean 1 and variance 2), so the mean of $|y_0|^2$ is (ρ^2/M). If H_1, then the mean of $|y_0|^2$ is given by Eq. (C4). The objective is to increase the difference in means as much as possible, so as to improve detection. For the off-broadside cases we must process the data differently if we wish to detect better.

One possibility is to replace y_0 with

$$y_1 = M^{-1} \sum_{m=1}^{M} x(m) e^{i\tau_m}. \tag{C5}$$

If H_0, then y_1 is still complex Gaussian with mean 0 and variance (ρ^2/M). If H_1, then

$$y_1 = A + M^{-1} \sum_{m=1}^{M} n(m) e^{i\tau_m}, \tag{C6}$$

and (assuming signal and noise are independent) y_1 has mean 0 and variance $\sigma^2 + (\rho^2/M)$. The r.v. $|y_1^2|$ is then a multiple of χ_2^2 in either case, with mean either (ρ^2/M) (if H_0) or $\sigma^2 + (\rho^2/M)$ (if H_1). This selective reduction of noise mean only is what is meant by array gain; comparing input S/N σ^2/ρ^2 to output SNR $\sigma^2/(\rho^2/M)$ we have the improvement in S/N of M, hence an array gain of M. What makes achievement of this gain possible is the ability to sum coherently the signal components in Eq. (C5), that is, to correct for the phase differences in the $Ae^{-i\tau_m}$ terms prior to summing.

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Careful analysis of what was just done shows that there are other options besides y_1 in Eq. (C5). If $b = (b(1), \ldots, b(M))^T$ is any complex vector such that

$$\sum_{m=1}^{M} b(m)e^{i\tau_m} = 1 \tag{C7}$$

then we can compute

$$y_2 = \sum_{m=1}^{M} \overline{b(m)}x(m) \tag{C8}$$

and get the same gain as before. There would be no point in this if the only cases that needed to be considered were H_0 and H_1 and we could assume that the $n(m)$ are always mutually independent. In most cases of interest, however, there can be signals with arrival angles other than θ present in x and the noises may be correlated between pairs of sensors. We can handle these more complicated cases by using the freedom to select various b, constrained by Eq. (C7), but chosen to optimize such things as sidelobe and noise suppression.

Note that our definition of array gain is based on the assumption that the $n(m)$ is independent (uncorrelated). Because array gain depends on input and output S/N, it is a function of the actual noise field and of the vector b. We must also note that if there are sources present with arrival angles other than θ, our procedure may or may not give us gain against those components. It depends on how b is designed.

Let us suppose that $N = [E(n(m)\ \overline{n(j)})]$ is the noise-only correlation matrix. If we use y_2 in Eq. (C8), then the variance of y_2^2 is either $b^+ Nb$ (H_0) or $\sigma^2 + b^+Nb$ (H_1), where + denotes conjugate transpose and we have assumed Eq. (C7). The input S/N is σ^2/ρ^2 and the output S/N is σ^2/b^+Nb, hence the gain obtainable from b will be

$$\text{gain}(b) = \rho^2/b^+Nb. \tag{C9}$$

To maximize our gain in this case, we select b so that b^+Nb is minimized, subject to Eq. (C7), which we write as $b^+e = 1$, $e = (\exp(i\tau_1), \ldots, \exp(i\tau_M))^+$. The solution is then

$$b = \lambda N^{-1}e,\ \lambda = 1/e^+N^{-1}e, \tag{C10}$$

and the optimal array gain available is

$$\text{optimal gain} = \rho^2(e^+N^{-1}e)\,. \tag{C11}$$

If $N = \rho^2 I$, then Eq. (C11) becomes M, as before. The maximum gain possible in this situation $(N \neq \rho^2 I)$ will depend on the spatial distribution of noise energy; $e^+N^{-1}e$ will be small if θ is in a sector where the noise is concentrated. In the extreme case in which the noise consists almost entirely of a single source at an angle near θ the gain will be almost zero.

All of the above analysis assumed tacitly that the b remains the same, whether H_0 or H_1 is true; the definition of array gain depended on the same b being used in each case. For nonlinear bearing estimation methods, such as maximum entropy and maximum likelihood, we must be careful when we describe them in terms of filters, as in Eq. (C8). The b used in each of those methods is a function of the measured cross sensor correlation matrix changes as we pass from H_0 to H_1. Trying to speak of array gain in such cases leads to a lot of confusion, and much of what has been said in the literature about these situations should be ignored. To analyze nonlinear methods properly, we must either drop the idea of array gain or substitute some other notion that quantifies the statistical advantage we hope to achieve. Array gain is about statistical behavior and it is only the latter that continues to concern us when using nonlinear methods.

In most array processing situations we have many independent realizations of vector x. The maximum detection probability for a fixed false alarm rate is obtained by placing a threshold on the likelihood ratio; the Gaussian assumptions tell us that

$$p(x|H_0) \sim \exp(-x^+N^{-1}x), \text{ and}$$
$$p(x|H_1) \sim \exp(-x^+(\sigma^2 ee^+ + N)^{-1}x),$$

so we maximize the (log of the) likelihood ratio

$$\Pi p(x|H_1)/\Pi p(x|H_0), \tag{C12}$$

where the product is over all realizations of x that we have obtained. Taking logs of Eq. (C12) and using the identity

$$(aa^+ + B)^{-1} = B^{-1} - (B^{-1}a)(B^{-1}a)^+/(1 + a^+B^{-1}a), \tag{C13}$$

we see that we must detect based on the size of the quantity $e^+N^{-1}RN^{-1}e$, where R is the average of xx^+ over all available x.

Using e^+Re, the so-called conventional processor, is optimal within the narrow confines of the two-hypothesis situation we have been using, with $N = \rho^2 I$. If we wish to improve sidelobe structure, we must include potential sources at other angles within the noise component N.