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The Ambiguity Function of Random Binary-Phase-Coded Waveforms

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The cross-correlation technique for signal detection and parameter estimation is widely used in communication and radar systems. For popular binary waveforms, the properties of their ambiguity functions have been studied for years. However, the results are generally restricted to specific codes which are either deterministic or pseudorandom. From the uncooperative reception viewpoint, without *a priori* knowledge of the emitted signal in many cases, the cross-correlating signals may be considered truly random. The report investigates the ambiguity function of truly random binary-phase-coded waveforms as an approximation to those waveforms commonly employed in binary-modulated pseudorandom systems/encoded radar systems.

Here, in a statistical sense, the ambiguity function of a truly random binary-phase-coded waveform is analytically derived in which the normally used deterministic cross-correlation process is replaced by its ensemble average. Various doppler responses are presented and discussed. The results are compared with those obtained

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19. ABSTRACT (Continued)
by transmitting an aperiodic maximum-length pseudorandom sequence. It is shown that the ambiguity function of the latter case is closely represented by the ensemble-average response of the truly random binary signal of equal length. With the transmitted signal unknown to a cross-correlator, the characteristics of the ambiguity function derived here provides predictive characteristics useful for practical cross-correlator system design.

CONTENTS

THE AMBIGUITY FUNCTION OF RANDOM BINARY-PHASE-CODED WAVEFORMS

INTRODUCTION

Since the early development of simple continuous wave (CW) and pulse radar systems, the radar waveform concept associated with matched filter, pulse-compression, or coded-waveform techniques has been one of the most critical elements in radar design [1]. It is well known [1,2] that the use of the matched filter in radar systems can be considered as optimum predetection processing for white Gaussian noise interference, and the output of the filter is indeed the noise-corrupted input signal cross-correlated with a replica of the transmitted signal. The matched filter receiver is mathematically equivalent to a cross-correlation receiver, or simply a cross-correlator. Generally, the detected signal out of the matched filter or the cross-correlator is characterized by the so-called "ambiguity function" which is determined by the transmitted waveform.

The cross-correlation technique for signal detection and parameter estimation is widely used. A typical example is the detection of the emitted signal presented at two sensors remotely located in a transmission medium [3]. Interesting problems may include the geolocation of the transmitter by cross-correlating the signals received over two separate paths. In this case, the extracted parameters of interest would be time difference of arrival (TDOA) and differential doppler (DD). In general, precise extraction of parameters requires the received signals to be cross-correlated over a two-dimensional ambiguity surface on TDOA and DD domains. Obviously, predictive measures of the observer's resolution, accuracy, processing gain, and interference rejection will be provided if the ambiguity function of the received signal can be determined.

For popular discrete-coded waveforms, the properties of their ambiguity functions have been studied for years. The results are generally restricted to specific codes either deterministic or pseudorandom such as Barker codes, maximum-length shift-register sequences, and others [1,4]. However, in many cases without *a priori* knowledge of the transmitted signal, one would like to have similar results for cross-correlating waveforms incurred in the binary-modulated pseudonoise systems or encoded radar systems.

Indeed, in order to simultaneously produce a high data link and reduce the bit energy, a longer transmission is practically applied. From the reception viewpoint, the discretely coded waveform of finite code length can thus be considered aperiodic. Here we considered random binary-phase-coded waveforms as an approximation to these pseudorandom waveforms and replaced the cross-correlation processing by its ensemble average.

In this report, the ambiguity function of a random binary-phase-coded signal, which is considered *aperiodic* and *finite*, is statistically formulated and discussed. In addition, the responses of various doppler filters are obtained and compared to those generated through an aperiodic maximum-length pseudorandom sequence [5].

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FORMULATION OF THE AMBIGUITY FUNCTION OF RANDOM BINARY-PHASE-CODED WAVEFORMS

In radar or digital communication systems, the transmitted signal in complex representation is generally given by $s(t,f_0) = u(t) \exp(j2\pi f_0 t)$ where f_0 is the carrier frequency and $u(t)$ is the complex transmitted modulation with $u(t) = a(t) \exp(jm(t))$. Here $a(t)$ and $m(t)$ are the amplitude and phase/frequency modulation. For discrete binary-phase-coded waveforms of interest, $u(t)$ can then be expressed by

$$u(t) = \sum_{n=1}^{N} P_n(t) \exp(j\,\theta_n), 0 \leqslant t \leqslant N\delta \tag{1}$$

where N is a finite integer, $\theta_n = 0$ or π, δ is the pulsewidth, and $P_n(t) = 1$ with $(n\text{-}1)\,\delta \leqslant t < n\,\delta$ and $n = 1, 2, \ldots, N$. For convenience, the expression $\exp(j\,\theta_n)$ is denoted by c_n throughout the report. Clearly, $\{c_n\}$ indicates a sequence of 1's and -1's. $N\delta$ and $1/\delta$ are the code length and the pulse repetition frequency.

When the doppler effect can be treated as a simple shift of the carrier, it is easy to obtain the ambiguity function through the cross-correlation of $s(t,f_0)$ with $s(t+\tau, f_0+f)$, where τ and f are the time delay and the doppler shift. Generally, the function is characterized by

$$X(\tau,f) = \int_{-\infty}^{\infty} u^*(t)\,u(t+\tau) \exp(-j2\pi\,\mathrm{ft})\,\mathrm{dt} \tag{2}$$

where $u^*(t)$ is the complex conjugate of $u(t)$. For the binary-phase-coded waveform, the ambiguity function can be obtained by directly substituting Eq. (1) into the above equation. The result is [1]:

$$\begin{aligned} X(\tau,f) &= X_p(\hat{\tau},f) \sum_{n=1}^{N-k} c_n c_{n+k} \exp(-j2\pi f(n-1)\delta) \\ &+ X_p(\delta - \hat{\tau},f) = \sum_{n=1}^{N-(k+1)} c_n c_{n+k+1} \exp(-j2\pi fn\,\delta) \end{aligned} \tag{3}$$

where $-\infty < f < \infty$, $\tau = k\,\delta + \hat{\tau}$ with $0 \leqslant \hat{\tau} \leqslant \delta$ and $k = 0, 1, 2, \ldots N-1$, and

$$X_p(\hat{\tau},f) = \int_0^{\delta-\tau} \exp(-j2\pi\,\mathrm{ft})\,\mathrm{dt}.$$

The generalized form of Eq. (3) is of particular interest since it includes both continuous- and discrete-time cases and can be applied to an aperiodic waveform. As cited in the first section of this report, we consider $\{c_n\}$ as an approximation to pseudorandom sequences, truly random, and redefine the ambiguity function from a statistical viewpoint. In other words, we assume

$$E\left[\sum_i c_i\right] = E\left[\sum_{i,j} c_i\,c_j\right] = \ldots = 0,\ (i \neq j) \text{ and } E\left[\sum_i c_i^2\right] = E\left[\sum_i c_i^4\right] = 1, \text{and replace}$$

$X(\tau,f)$ of Eq. (2) by its ensemble average, i.e., $\hat{X}^2(\tau,f) = E[X(\tau,f)\,X^*(\tau,f)]$. By directly substituting the result of Eq. (3) into $\hat{X}^2(\tau,f)$ and replacing $j2\pi f\delta$ by y, one obtains

$$\begin{aligned} \hat{X}^2(\tau,f) &= E\left[|X_p(\hat{\tau},f)|^2 \sum_{n=1}^{N-k} \sum_{m-2}^{N-k} c_n c_m^* c_{n+k} c_{m+k}^* e^{-y(n-m)}\right] \\ &+ E\left[|X_p(\delta-\hat{\tau},f)|^2 \sum_{n=1}^{N-(k+1)} \sum_{m=1}^{N-(k+1)} c_n c_{n+k+1} c_m^* c_{m+k+1}^* e^{-y(n-m)}\right] \\ &+ E\left[X_p(\hat{\tau},f)\,X_p^*(\delta-\hat{\tau},f) \sum_{n=1}^{N-k} \sum_{m-1}^{N-(k+1)} c_n c_m^* c_{n+k} c_{m+k+1}^* e^{-y(n-m+1)}\right] \\ &+ E\left[X_p(\delta-\hat{\tau},f)\,X_p^*(\hat{\tau},f) \sum_{n-1}^{N-(k+1)} \sum_{m=1}^{N-k} c_n c_m^* c_{n+k+1} c_{m+k}^* e^{-y(n-m+1)}\right]. \end{aligned} \tag{4}$$

With the random characteristics of an aperiodic code sequence previously discussed, it is easy to verify that the third and fourth terms in the right-hand side of Eq. (4) are zeros since the coefficients are mutually "exclusive" at all times; i.e., the conditions such that the terms $c_n c_m^* c_{n+k} c_{m+k+1}^*$ and $c_n c_m^* c_{n+k+1} c_{m+k}^*$ contain no odd powers of the parameters for $1 \leqslant i \leqslant N - k$ do not exist. For the first and second terms, there are only two cases resulting in nonzero mean, that is, (a) $k = 0$ and (b) $k \neq 0$ and $n = m$. Therefore Eq. (4) can be reduced to

$$\hat{X}^2(\tau,f) = \begin{cases} |X_p(\hat{\tau},f)|^2\left[\sum\limits_{n=1}^{N} e^{-yn}\left(\sum\limits_{m=1}^{N} e^{ym}\right)\right] + |X_p(\delta-\hat{\tau},f|^2\left[\sum\limits_{n=1}^{N-(k+1)} e^{-yn}(e^{yn})\right]; k = 0 \\ |X_p(\hat{\tau},f)|^2\left[\sum\limits_{n=1}^{N-k} e^{-yn}(e^{yn})\right] + |X_p(\delta-\hat{\tau},f)|^2\left[\sum\limits_{n=1}^{N-(k+1)} e^{-yn}(e^{yn})\right]; k \neq 0, \end{cases} \tag{5}$$

or after simplification

$$\hat{X}^2(\tau,f) = \begin{cases} |X_p(\hat{\tau},f)|^2\dfrac{(1-e^{NY})(1-e^{-NY})}{(1-e^{y})(1-e^{-y})} + |X_p(\delta-\hat{\tau},f)|^2\,(N-1); k = 0 \text{ and } f \neq \text{PRF} \\ |X_p(\hat{\tau},f)|^2 N^2 + |X_p(\delta-\hat{\tau},f)|^2\,(N-1); k = 0 \text{ and } f = \text{PRF} \\ |X_p(\hat{\tau},f)|^2 (N-k) + |X_p(\delta-\hat{\tau},f)|^2\,(N-k-1);\ k \neq 0. \end{cases} \tag{6}$$

In the above, $X_p(\hat{\tau},f)$ and $X_p(\delta - \hat{\tau},f)$ can be obtained by direct integration as defined in Eq. (3), yielding $|Sa(\pi f(\delta - \hat{\tau})|\ (\delta - \hat{\tau})$ and $|Sa(\pi f\hat{\tau})|\hat{\tau}$. Here $Sa(x) = \sin x/x$ as commonly defined. As a result, the ambiguity function of a binary-coded signal is expressed as follows:

$$\hat{X}^2(\tau,f) = \begin{cases} \dfrac{Sa^2(\pi f(\delta-\hat{\tau}))Sa^2(\pi fN\delta)N^2(\delta-\hat{\tau})^2}{Sa^2(\pi f\delta)} + Sa^2(\pi f\hat{\tau})(N-1)\hat{\tau}^2; k = 0 \\ Sa^2(\pi f(\delta-\hat{\tau}))(N-k)(\delta-\hat{\tau})^2 + Sa^2(\pi f\hat{\tau})(N-k-1)^2; k = 1,2,\ldots,N-1 \end{cases} \tag{7}$$

where $\tau = k\delta + \hat{\tau}$, $0 \leqslant \hat{\tau} \leqslant \delta$, and N and δ are the number of code elements and the pulse width. Note that the above expressions are valid under the blind speed condition; i.e., the doppler frequency is the same as the PRF or an integer multiple thereof.

With Eq. (7) normalized to $\bar{\tau} = \hat{\tau}/\delta$, $\bar{f} = fN\delta$, and $\bar{X}^2(\bar{\tau},\bar{f}) = \hat{X}^2(\bar{\tau},\bar{f})/N^2\delta^2$, the responses of $F_0, F_{1/2}, F_1, F_{3/2}$, and F_2, doppler filters for $N = \text{PRF} \cdot \text{T} = 127$ are shown in Figs. 1 through 5. The code element of $N = 127$ is specifically chosen here so that the responses can be compared with those obtained in Ref. [5] for the case of pseudorandom waveforms (see Discussion and Conclusion). To be consistent with the name "ambiguity function" used in radar systems, the term "doppler filter" as observed in matched-filter-bank implementation (doppler-steered pulse compressor) of radar detection is adopted here. For F_i filters, the doppler shift is such that the total phase shift across the whole code is $2\pi i$. In the special case of zero doppler shift, the mainlobe response can be obtained from Eq. (7) with both f and k set to zero. That is,

$$\hat{X}^2(\bar{\tau},0)/N^2\delta^2 = (1-\bar{\tau})^2 + \bar{\tau}^2(N-1)/N^2;\ 0 \leqslant \bar{\tau} \leqslant 1. \tag{8}$$

The response reduces to half of its peak value at $\bar{\tau} = 0.29$ (or $\tau = 0.29\delta$) for the typical case of $N = 4800$ (Fig. 6). For $k \neq 0$, the sidelobe response is:

$$\bar{X}^2(\bar{\tau},0)/N^2\delta^2 = (1-\bar{\tau})^2(N-k)/N^2 + \bar{\tau}^2(N-k-1)/N^2; \begin{matrix} 0 \leqslant \bar{\tau} \leqslant 1 \text{ and} \\ k = 1,2,\ldots,N-1 \end{matrix} \tag{9}$$

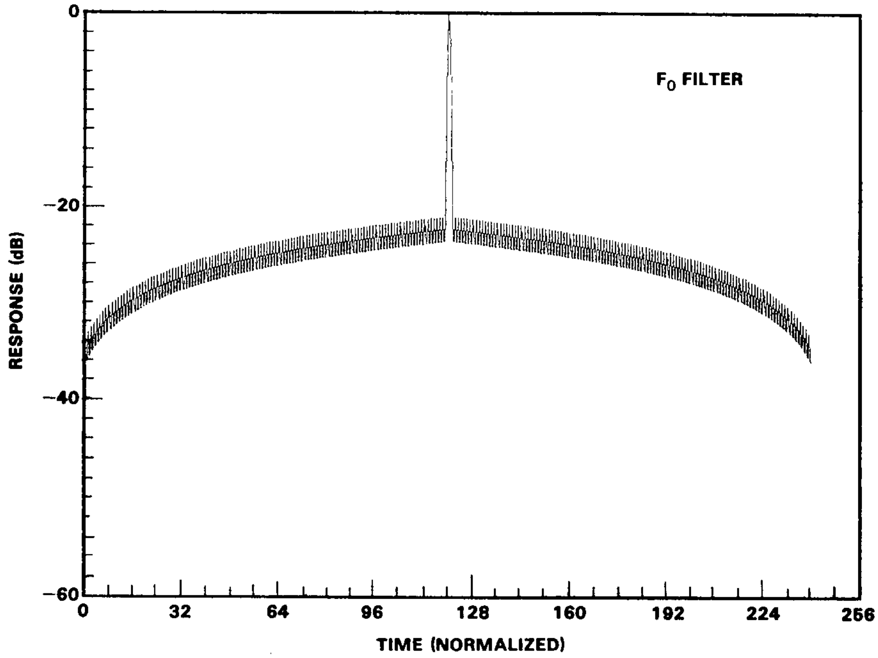

Fig. 1 — F_0 doppler-filter response for a random binary-phase code with the code element $N = 127$

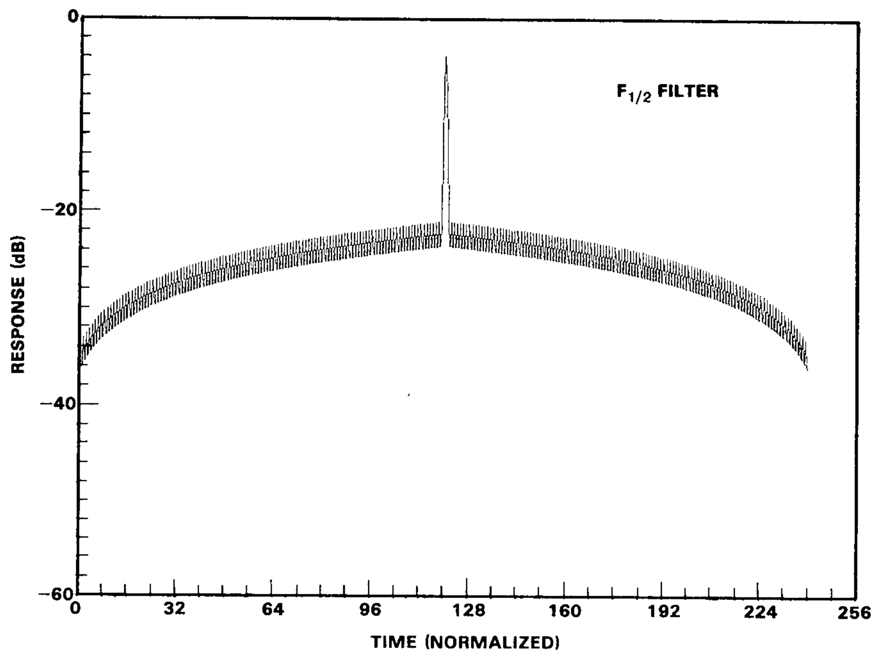

Fig. 2 — $F_{1/2}$ doppler-filter response for a random binary-phase code with the code element $N = 127$

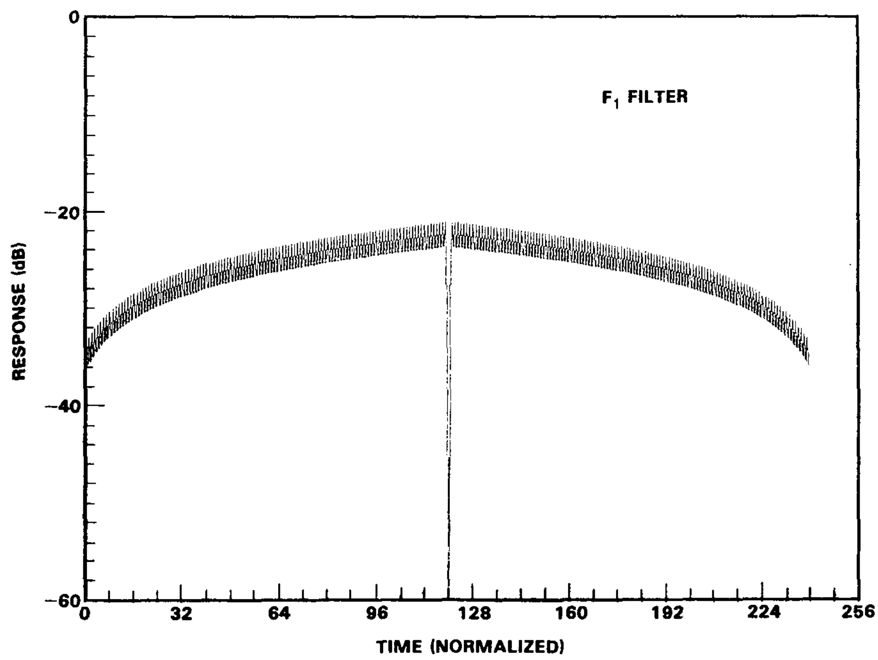

Fig. 3 — F_1 doppler-filter response for a random binary-phase code with the code element $N = 127$

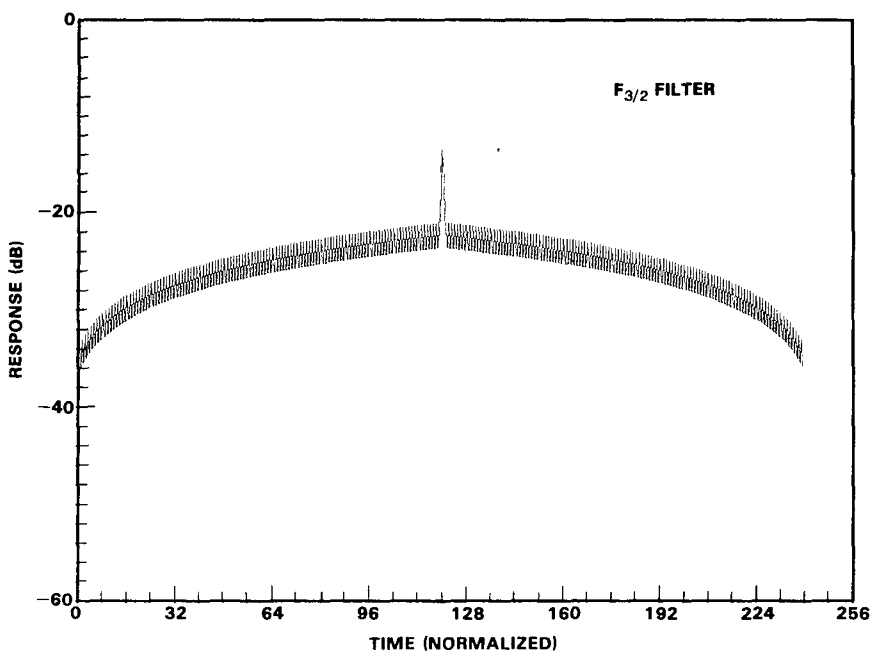

Fig. 4 — $F_{3/2}$ doppler-filter response for a random binary-phase code with the code element $N = 127$

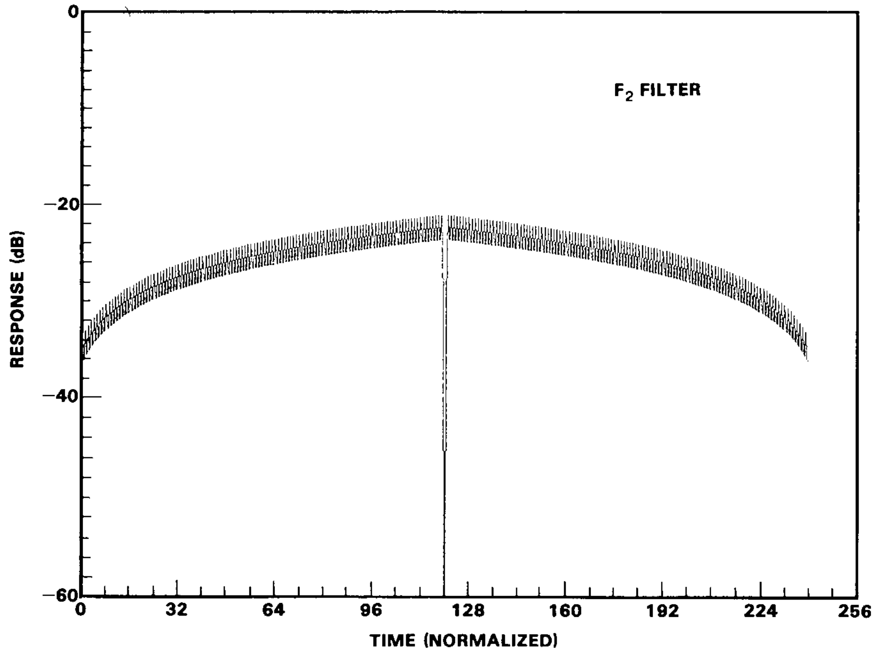

Fig. 5 — F_2 doppler-filter response for a random binary-phase code with the code element $N = 127$

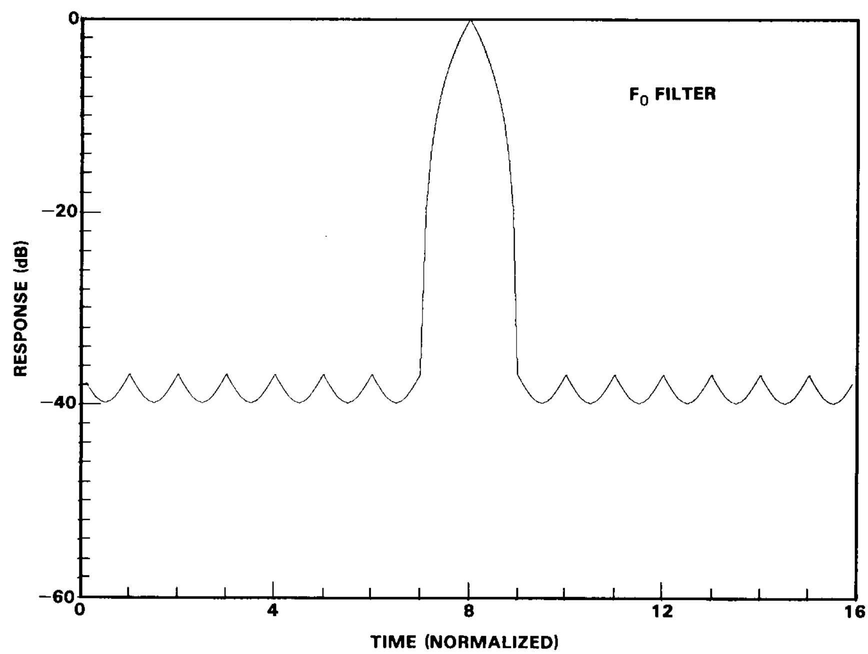

Fig. 6 — F_0 doppler-filter response for a random binary-phase code with the code element $N = 4800$

which varies approximately from 36.8 to 39.2 dB below the main peak for the first eight cycles. Clearly, the function governed by Eqs. (8) and (9) converges to a triangular one near the origin as N becomes large. This is the trivial case for an infinitely long random sequence. In the special case of zero time delay by setting $k = 0$ and $\hat{\tau} = 0$ in Eq. (7), the frequency response of the ambiguity is simply a $\sin^2(\pi f N \delta)/(\pi f N \delta)^2$ function (Fig. 7) which is identical to the spectrum as seen in pseudorandom waveforms [6].

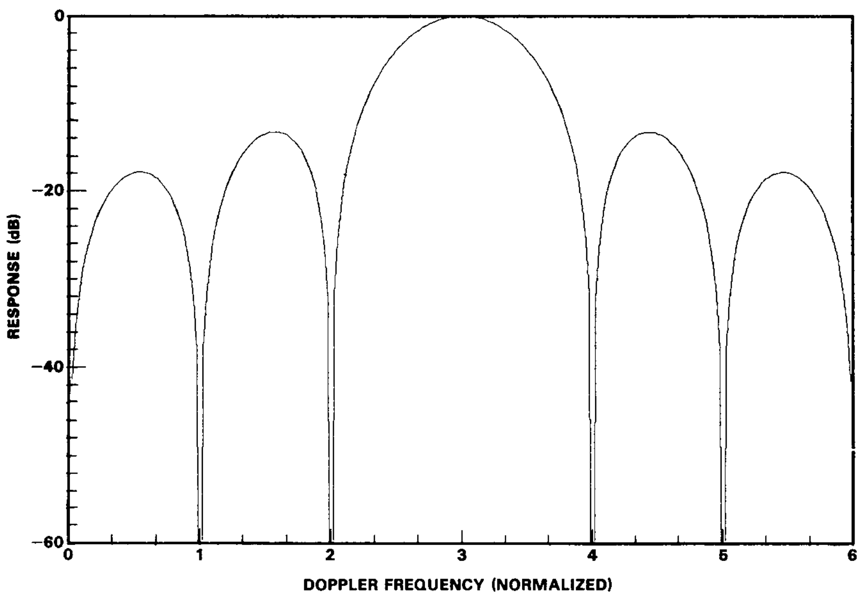

Fig. 7 — Ambiguity function frequency response for a random binary-phase code with the code element $N = 4800$

DISCUSSION AND CONCLUSION

In communications and radar, the emitted signals are either pseudorandom with a very long cycle time or coded. From the reception viewpoint, the discretely coded waveform to be cross-correlated has a finite code length and can be considered aperiodic. Without *a priori* knowledge of the emitted signal in applications, one may consider the waveform truly random and replace the cross-correlation processing by its ensemble average. Herein, from a statistical sense, the ambiguity function of a random binary-phase-coded signal is analytically derived. Although the signal detection relies on the realization of cross-correlation processing, the limitation of the transmitter power, the threshold of the received signal-to-noise ratio, and other practical considerations, the characteristics of the ambiguity function derived here provide predictive measures and specification requirements for practical system design.

The derived ambiguity function in an analytical form can also be used as a baseline in relation to other deterministic/pseudorandom correlation functions of interest. For example, responses of F_0, F_1, and F_2 doppler filters for a 127-element maximum-length shift-register code were investigated in Ref. [5] (Fig. 8). Indeed, their fundamental properties such as resolution, processing gain, and sidelobe level can be easily predicted through the corresponding doppler filter responses of a random binary-phase code. The responses of various doppler filters directly obtained from Eq. (7) with the code element of $N = 127$ are presented in Figs. 1 through 5. It is observed that the responses of F_1- and F_2-filters are quite comparable to those obtained by transmitting an aperiodic maximum-length pseudorandom sequence. Note that, in Fig. 8, only discrete sample points were approximately calculated and the specific code sequence was generated from a seven-stage shift register with an initial condition of 217 in octal.

For further applications, the derivation of the main result presented here may include the case that coded sequences have more complex occurrence statistics. The consideration of target fluctuation which causes a spread in time and frequency can also be included. In the latter case, the output of the matched filter is then the emitter scattering function convolved with the ambiguity function derived here.

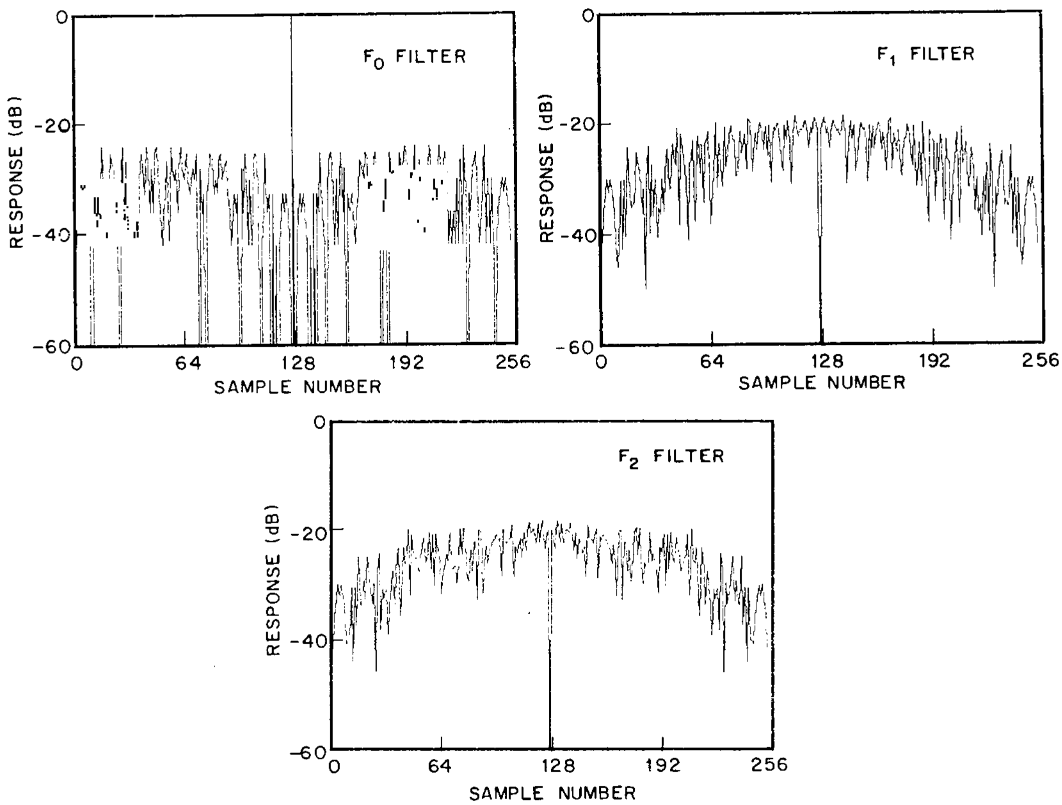

Fig. 8 — Responses of the doppler filters with a zero doppler target for a 127-element maximum-length pseudorandom binary code

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