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Angular Resolution of Coherent and Noncoherent Sources

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The problem of resolving radiating sources in angle is formulated as a binary hypothesis test. A generalized likelihood ratio test is developed and positive results are obtained. Using simulation techniques, the probability of resolution was calculated for both coherent and noncoherent sources. Two 20-dB noncoherent Rayleigh fluctuating sources separated by 0.1 beamwidth can be resolved at a resolution probability of 0.9 and at a false-alarm rate of 0.01 by using samples from an 8-element array and 32 range cells. A one-term approximation to the generalized likelihood ratio, which is equivalent to the square error residue from fitting one source to the data, is only slightly less accurate than the likelihood approach.

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Angular Resolution of Coherent and Noncoherent Sources

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CONTENTS

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ANGULAR RESOLUTION OF COHERENT AND NONCOHERENT SOURCES

INTRODUCTION

When attempting to resolve targets, one can either resolve them in angle or range or a combination of both coordinates. However, the range coordinate is a much more powerful discriminant for separating targets. For example, two targets separated by 1000 ft are separated by two 1-μs pulsewidths but by only 1/20 of a 2° beamwidth at a range of 100 nmi. On the other hand, when attempting to resolve radiating sources (e.g. jammers), one must resolve them in angle.

The probability of resolving sources is not only a function of their separation but also a function of their strengths and phase differences. There have been many articles written on resolution and reprints of some of the most important ones can be found in Ref. 1. In an early article Ksienski and McGhee [2], using a decision theoretic approach, indicate that targets within a quarter of a beamwidth can be resolved. Lately, the problem of angular resolution has been investigated by using superresolution techniques [3]. All of these investigations solve the problem by using estimation approaches. We approach the resolution problem as a binary-hypothesis problem where the two hypothesis are:

- H_1: One Source Present
- H_2: Two Sources Present

The development is similar to the one used for the range resolution of targets [4].

GENERALIZED LIKELIHOOD RATIO TEST—NONCOHERENT SOURCES

We now formulate the resolution problem as a binary-hypothesis test. The received samples, from the two sources impinging on the array elements shown in Fig. 1, are

$$\begin{aligned} X_{ij} &= n_{X_{ij}} + A_{j1} \cos\left(\phi_{j_1} + \pi(i-1)\sin\theta_1\right) \\ &\qquad + A_{j2}\cos\left(\phi_{j2} + \pi(i-1)\sin\theta_2\right), \end{aligned}$$

and (1)

$$\begin{aligned} Y_{ij} &= n_{Y_{ij}} + A_{j1} \sin\left(\phi_{j1} + \pi(i-1)\sin\theta_1\right) \\ &\qquad + A_{j2}\sin\left(\phi_{j2} + \pi(i-1)\sin\theta_2\right), \end{aligned}$$

where

- X_{ij} and Y_{ij} are the inphase and quadrature samples from the ith element and jth range cell,
- $n_{X_{ij}}$ and $n_{Y_{ij}}$ are independent inphase and quadrature Gaussian noise samples with mean zero and variance σ^2,
- A_{j1} and A_{j2} are source amplitudes in the jth range cell,
- ϕ_{j1} and ϕ_{j2} are the phase angles in the jth range cell,
- θ_1 and θ_2 are the angles of arrival, and
- the receiver elements are 1/2 wavelength apart.

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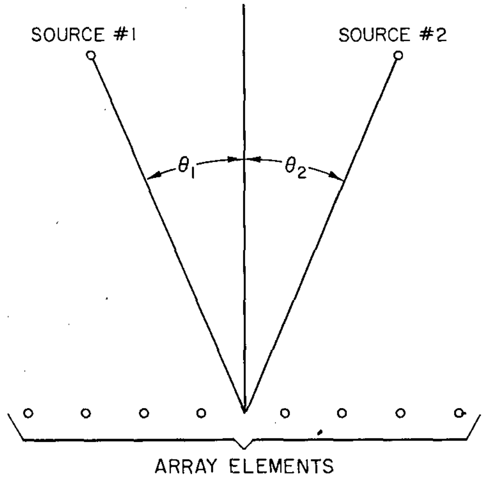

Fig. 1 — Signals from two sources received by an 8-element linear array

The angular resolution problem is equivalent to deciding which hypothesis is true:

$$H_1: \; A_{j1} > 0 \text{ and } A_{j2} = 0$$

or

$$H_2: \; A_{j1} > 0 \text{ and } A_{j2} > 0.$$

If $\{A_{j1}, A_{j2}, \phi_{j1}, \phi_{j2}, \theta_1, \theta_2\}$ are all known, the optimal detector (in the Neyman Pearson sense) is the likelihood ratio. Since in the problem of interest the parameters $\{A_{j1}, A_{j2}, \phi_{j1}, \phi_{j2}, \theta_1, \theta_2\}$ are unknown, in the likelihood ratio we replace the parameters with their maximum likelihood estimates.

The maximum-likelihood estimates are the values of the parameters which minimize the square error

$$L = \sum_{j=1}^{m} \sum_{i=1}^{n} (X_{ij} - A_{j1} \cos (\phi_{j1} + \pi(i-1) \sin \theta_1) - A_{j2} \cos (\phi_{j2} + \pi(i-1) \sin \theta_2))^2$$
$$+ \sum_{j=1}^{m} \sum_{i=1}^{n} (Y_{ij} - A_{j1} \sin (\phi_{j1} + \pi(i-1) \sin \theta_1) - A_{j2} \sin (\phi_{j2} + \pi(i-1) \sin \theta_2))^2, \tag{2}$$

where n is the number of array elements and m is the number of range cells. Making the following substitutions;

$$B_j = A_{j1} \cos \phi_{j1},$$
$$C_j = A_{j2} \sin \phi_{j1},$$
$$W_i = \cos (\pi(i-1) \sin \theta_1),$$
$$U_i = \sin (\pi(i-1) \sin \theta_1),$$
$$D_j = A_{j2} \cos \theta_{j2},$$
$$E_j = A_{j2} \sin \theta_{j2},$$
$$S_i = \cos (\pi(i-1) \sin \theta_2),$$

and

$$T_i = \sin (\pi(i-1) \sin \theta_2),$$

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Eq. (2) can be written as

$$L = \sum_{j=1}^{m} \sum_{i=1}^{n} (X_{ij} - B_j W_i + C_j U_i - D_j S_i + E_j T_i)^2$$
$$+ \sum_{j=1}^{m} \sum_{i=1}^{n} (Y_{ij} - C_j W_i - B_j U_i - E_j S_i - D_j T_i)^2. \tag{3}$$

For any given value of ϕ_1 and ϕ_2, W_i, U_i, S_i, and T_i are known, and the maximum-likelihood estimates of B_j, C_j, D_j, and E_j can be found by solving the equations

$$\frac{\partial L}{\partial B_j} = \frac{\partial L}{\partial C_j} = \frac{\partial L}{\partial D_j} = \frac{\partial L}{\partial E_j} = 0 \quad j = 1, \ldots, m.$$

The Appendix of this report shows that the solutions for the one source case (that is, assuming H_1 is true) are

$$D_j = E_j = 0,$$
$$B_j = \sum_{i=1}^{n} (W_i X_{ij} + U_i Y_{ij})/n, \tag{4}$$

and

$$C_j = \sum_{i=1}^{n} (W_i Y_{ij} - U_i X_{ij})/n.$$

In the Appendix it is also shown that the solutions for the two-source case (that is, assuming H_2 is true) are

$$B_j = (nH_j - FJ_j - GK_j)/(n^2 - F^2 - G^2),$$
$$C_j = (nI_j + GJ_j - FK_j)/(n^2 - F^2 - G^2),$$
$$D_j = (nJ_j - FH_j + GI_j)/(n^2 - F^2 - G^2), \tag{5}$$

and

$$E_j = (nK_j - GH_j - FI_j)/(n^2 - F^2 - G^2),$$

where

$$F = \sum_{i=1}^{n} (W_i S_i + U_i T_i),$$
$$G = \sum_{i=1}^{n} (U_i S_i - W_i T_i),$$
$$H_j = \sum_{i=1}^{n} (W_i X_{ij} + U_i Y_{ij}),$$
$$I_j = \sum_{i=1}^{n} (W_i Y_{ij} - U_i X_{ij}), \tag{6}$$
$$J_j = \sum_{i=1}^{n} (S_i X_{ij} + T_i Y_{ij}),$$

and

$$K_j = \sum_{i=1}^{n} (S_i Y_{ij} - T_i X_{ij}).$$

A direct search technique [5] is used to estimate the source directions θ_1 and θ_2.

Finally, the log likelihood ratio can be written as

$$\Lambda = \underset{\substack{\theta_1, B_j, C_j \\ D_j = E_j = 0}}{\text{Min}} \{L\} - \underset{\theta_1, \theta_2, B_j, C_j, D_j, E_j}{\text{Min}} \{L\}. \tag{7}$$

Calculation of the log likelihood involves two minimizations. For the first, one searches on θ_1 and uses Eq. (4) for the other parameters. For the second, one searches on θ_1 and θ_2 and uses Eq. (5) for the other parameters. Previous work [4] on range resolution indicate that results similar to those obtained with Eq. (7) can be obtained by using only the first term. Consequently, we also evaluate

$$\Lambda' = \underset{\substack{\theta_1, B_j, C_j \\ D_j = E_j = 0}}{\text{MIN}} \{L\}.$$

This procedure is equivalent to locating the radiating source which minimizes the square error and then comparing the minimum square error to a threshold.

To proceed with a hypothesis test, it is first necessary to calculate a threshold to yield a desired probability of false alarm equal to α. The thresholds T and T' are set so that

$$P_r \{\Lambda \geqslant T | H_1\} = \alpha$$

and

$$P_r \{\Lambda' \geqslant T' | H_1\} = \alpha.$$

In this report we have selected $\alpha = 0.01$. Because of this high false-alarm rate, Monte Carlo simulations can be used to estimate T and T'. Two thousand repetitions were run for the case where there were $n = 8$ array elements and $m = 32$ range cells. The signal-to-noise ratio (S/N) was 20 dB at the output of the antenna and the signal exhibited independent Rayleigh fluctuations range cell to range cell. For each repetition the jammer location was located randomly (uniformly distributed) within the main beam of the array. The estimated values for the threshold were 523 for T' and 103 for T.

PROBABILITY OF RESOLUTION—NONCOHERENT SOURCES

The probability of resolving two Rayleigh fluctuating sources at a false-alarm rate of $\alpha = 0.01$ was found by simulation and is shown in Figs. 2 and 3 for the likelihood ratio and the one-term approximation of the likelihood ratio respectively. Each figure shows the probability of resolution as a function of S/N for source separations of 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidths. After comparing Figs. 2 and 3, one concludes that the approximation to the likelihood ratio is only 1 or 2 dB worse than the likelihood ratio results. Figure 4 compares the likelihood results with those obtained [3] for an adaptive array, linear prediction least mean square superresolution algorithm. The likelihood results correspond to a 0.9-probability of resolution. The linear prediction results correspond to the appearance of a double peak. Furthermore, the number of range cells is variable for the linear prediction method—enough cells are used so that the asymptotic behavior is obtained. (The performance does not increase with the number of range cells since the method has an inherent noise bias. A nonbiased eigen-analysis algorithm [6] would have improved performance.) For sources separated by 0.1° beamwidth, the likelihood approach requires approximately 15 dB less S/N. For larger separations, the difference is less.

PROBABILITY OF RESOLUTION—COHERENT SOURCES

If the two sources are coherent (for instance, one source is just a multipath bounce), the signals are not independent, but are related by

$$A_{j2} = \rho A_{j1}$$

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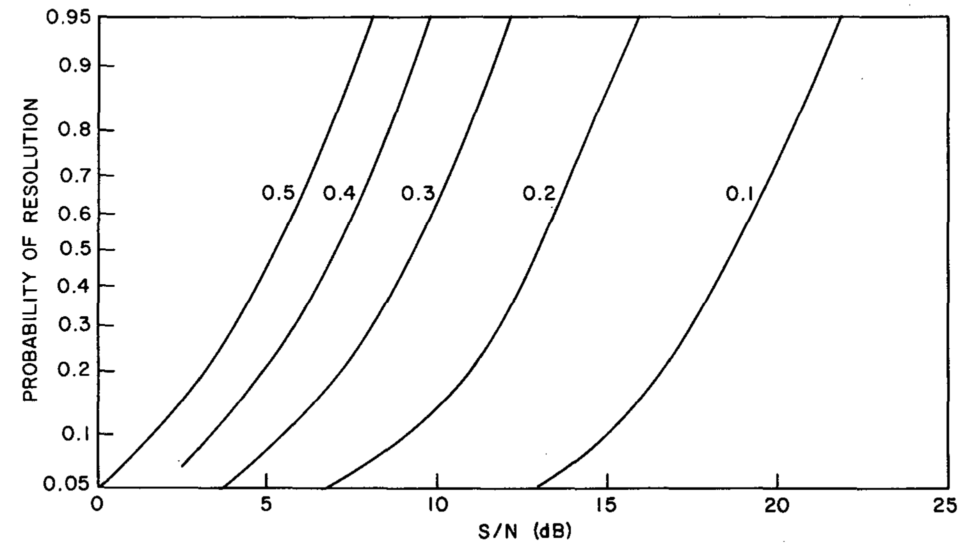

Fig. 2 — Probability of resolving two noncoherent Rayleigh fluctuating sources as a function of the S/N at the output of the array. Generalized likelihood ratio procedure; source separations are 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidth.

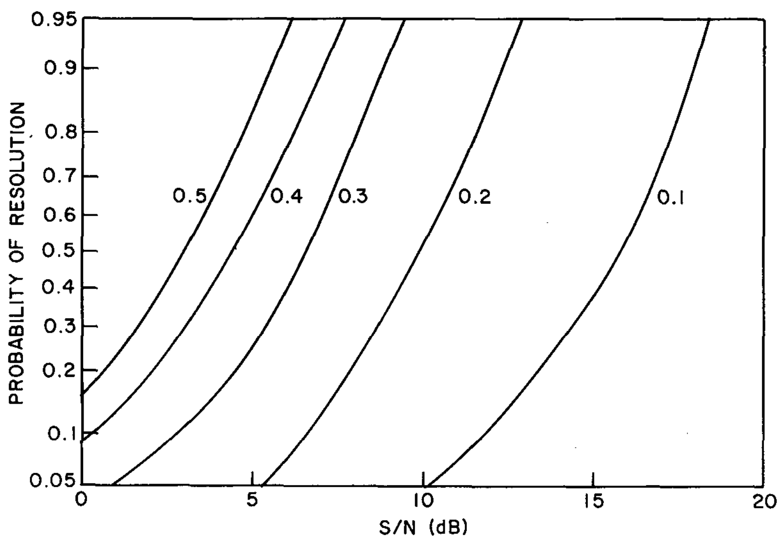

Fig. 3 — Probability of resolving two noncoherent Rayleigh fluctuating sources as a function of the S/N at the output of the array. Approximation to generalized likelihood ratio procedure; source separations are 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidth.

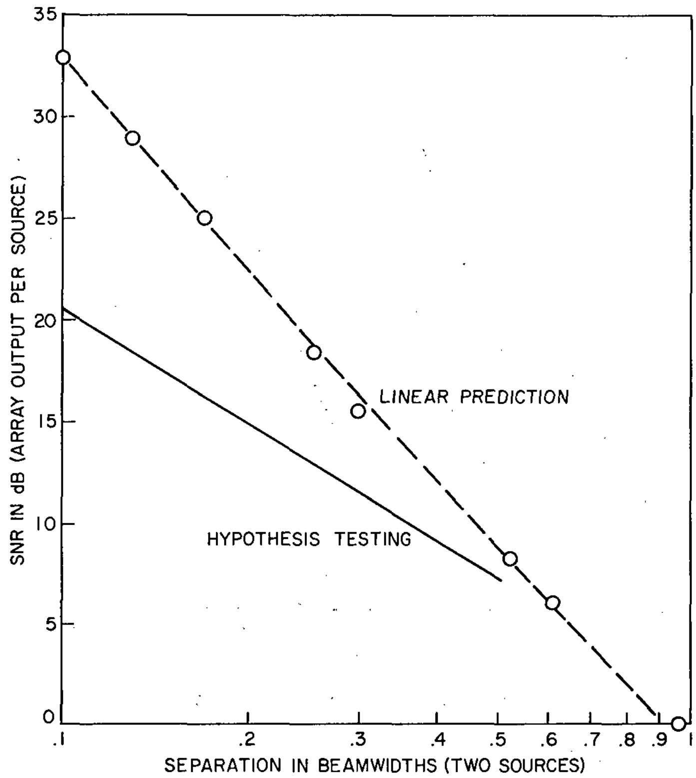

Fig. 4 — Comparison of the resolution capability of the generalized likelihood ratio test with a superresolution estimating algorithm

and

$$\phi_{j2} = \phi_{j1} + \beta + (n - 1)\pi(\sin\theta_1 - \sin\theta_2)/2. \tag{8}$$

This complicated phase equation is required so that the phase difference is β at the center of the aperture. For the noncoherent case, we minimized L [Eq. (2)] with respect to $4m + 2$ parameters: $(A_{j1}, A_{j2}, \phi_{j1}, \phi_{j2}; j = 1, \ldots, m)$, θ_1 and θ_2. However, we were able to solve for the 4m parameters in terms of θ_1 and θ_2. Consequently, only a two-dimensional search was required. For the coherent case, we must minimize L, Eq. (2), with respect to $2m + 4$ parameters: $(A_{j1}, \phi_{j1}; j = 1, \ldots, m)$, ρ, β, θ_1, and θ_2. In this case, one would be able to solve for the 2m parameters in terms of ρ, β, θ_1, and θ_2. Consequently, a four-dimensional search would be required and this would require a rather long computation time. To avoid this long computation, we will use the likelihood for noncoherent sources for the coherent source case. Obviously, this is a suboptimal procedure.

The probability of resolving two coherent, Rayleigh fluctuating sources at a false alarm rate of 0.01, using the likelihood ratio for noncoherent sources, was found by simulation and is shown in Figs. 5 and 6 for phase differences of $\beta = 0°$ and $\beta = 90°$. Figures 7 to 10 show the results for the one-term approximation to the like lihood ratio for noncoherent sources for phase differences of $\beta = 0°$, $\beta = 45°$, $\beta = 90°$, and $\beta = 180°$. Comparing the results for the two tests, one concludes that the likelihood ratio is between 2 and 4 dB better than the one-term approximation of the likelihood ratio.

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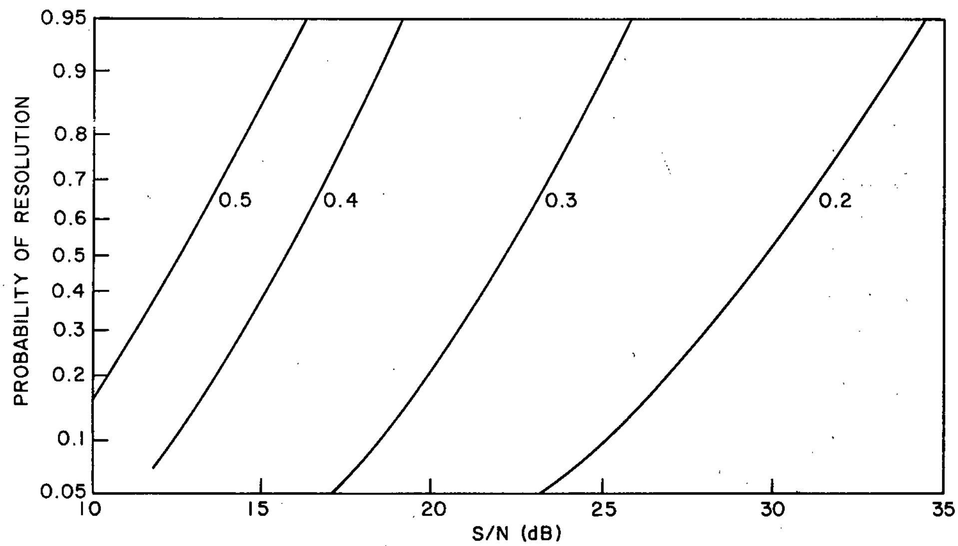

Fig. 5 — Probability of resolving two coherent, Rayleigh fluctuating sources as a function of the S/N at the output of the array. Generalized likelihood ratio procedure; source separations are 0.2, 0.3, 0.4, and 0.5 beamwidth. Phase difference is 0°.

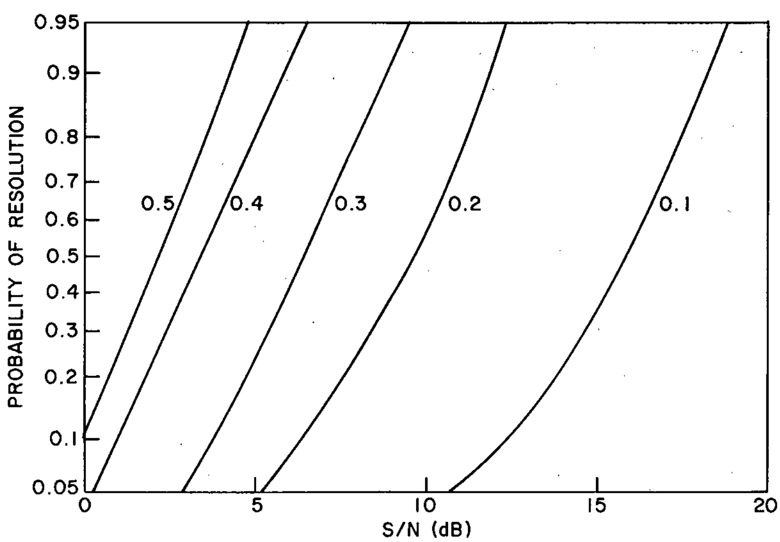

Fig. 6 — Probability of resolving two coherent, Rayleigh fluctuating sources as a function of the S/N at the output of the array. Generalized likelihood ratio procedure; source separations are 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidth. Phase difference is 90°.

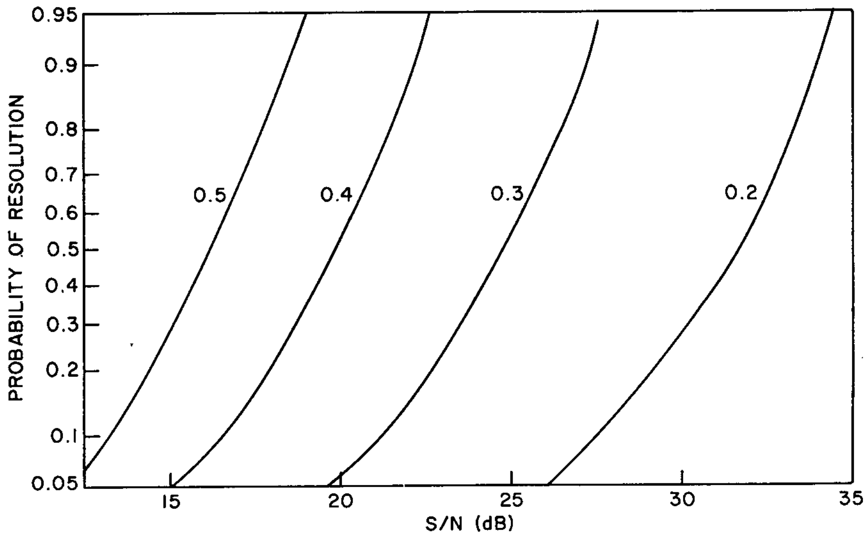

Fig. 7 — Probability of resolving two coherent, Rayleigh fluctuating sources as a function of the S/N at the output of the array. Approximation to generalized likelihood ratio procedure; source separations are 0.2, 0.3, 0.4, and 0.5 beamwidth. Phase difference is 0°.

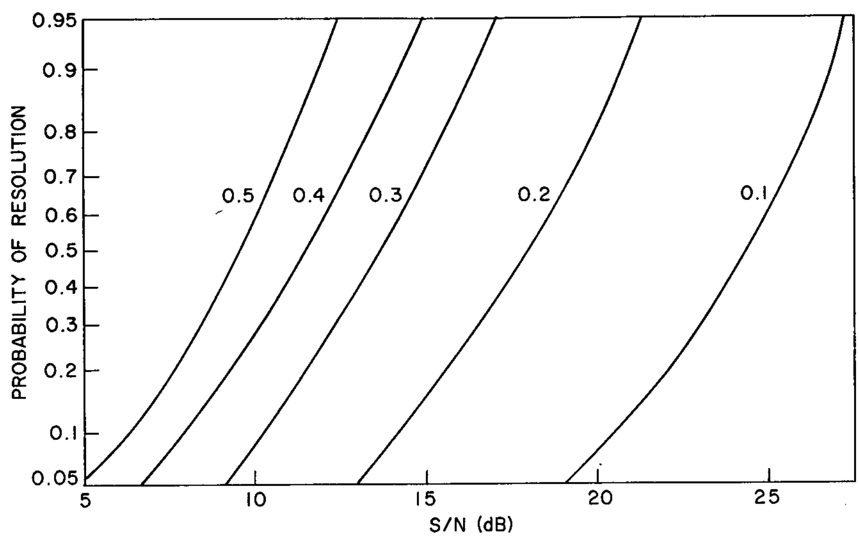

Fig. 8 — Probability of resolving two coherent, Rayleigh fluctuating sources as a function of the S/N at the output of the array. Approximation to generalized likelihood ratio procedure; source separations are 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidth. Phase difference is 45°.

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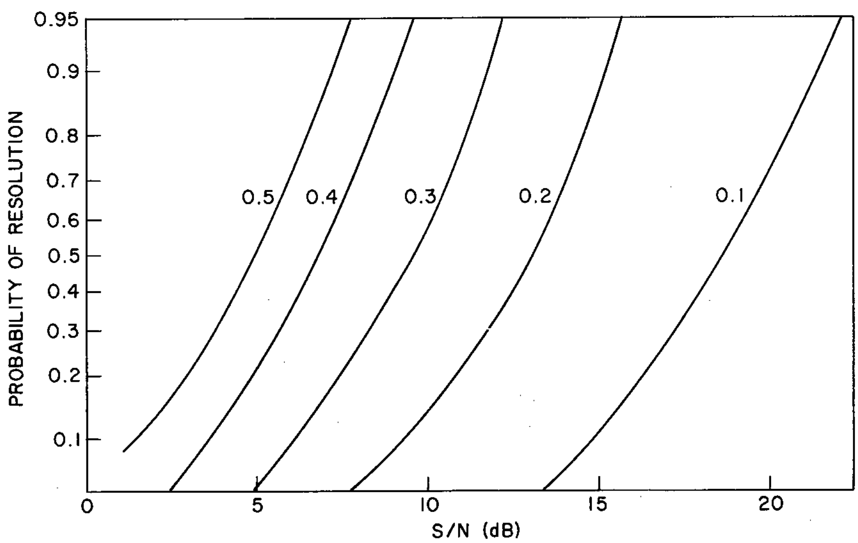

Fig. 9 — Probability of resolving two coherent, Rayleigh fluctuating sources as a function of the S/N at the output of the array. Approximation to generalized likelihood ratio procedure; source separations are 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidth. Phase difference is 90°.

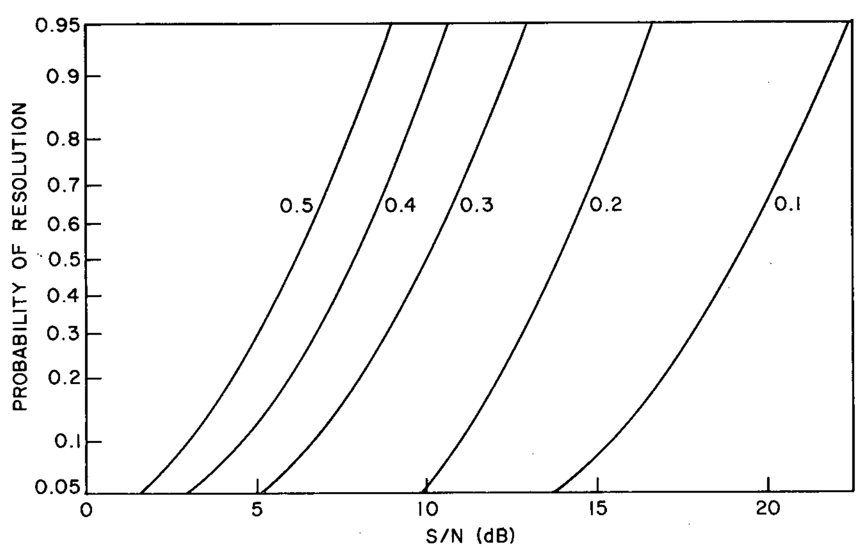

Fig. 10 — Probability of resolving two coherent, Rayleigh fluctuating sources as a function of the S/N at the output of the array. Approximation to generalized likelihood ratio procedure; source separations are 0.1, 0.2, 0.3, 0.4, and 0.5 beamwidth. Phase difference is 180°.

SUMMARY

The problem of resolving radiating sources in angle is formulated as a binary hypothesis test. A generalized likelihood ratio test was developed for noncoherent sources and used to resolve both noncoherent and coherent sources. Two 20-dB noncoherent Rayleigh fluctuating sources separated by 0.1 beamwidth can be resolved at a resolution probability of 0.9 and at a false alarm rate of 0.01 by using samples from an 8-element array and 32 range cells. The likelihood method was compared and found to be superior to a linear predictive least mean square superresolution estimation algorithm. A one-term approximation to the generalized likelihood ratio, which is equivalent to the square error residue from fitting one source to the data, is only slightly less accurate than the likelihood approach.

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Appendix

MAXIMUM LIKELIHOOD ESTIMATES

The likelihood function is given by

$$L = \sum_{j=1}^{m} \sum_{i=1}^{n} (X_{ij} - B_j W_i + C_j U_i - D_j S_i + E_j T_i)^2$$
$$+ \sum_{j=1}^{m} \sum_{i=1}^{n} (Y_{ij} - C_j W_i - B_j U_i - E_j S_i - D_j T_i)^2. \tag{A1}$$

If H_1 is true, $A_{j2} = 0$ and hence $D_j = E_j = 0$. In this case, Eq. (A1) reduces to

$$L = \sum_{j=1}^{m} \sum_{i=1}^{n} (X_{ij} - B_j W_i + C_j \; U_i)^2$$
$$+ \sum_{j=1}^{m} \sum_{n=1}^{n} (Y_{ij} - C_j W_i - B_j U_i)^2.$$

Then the maximum likelihood estimates are found by solving the equations

$$\frac{\partial L}{\partial B_j} = \frac{\partial L}{\partial C_j} = 0.$$

Taking the derivatives yields the equations

$$\frac{\partial L}{\partial B_j} = \sum_{i=1}^{n} - 2 W_i (X_{ij} - B_j W_i + C_j U_i) + \sum_{i=1}^{n} - 2 U_i (Y_{ij} - C_j W_i - B_j U_i) = 0$$

and

$$\frac{\partial L}{\partial C_j} = \sum_{i=1}^{n} 2 U_i (X_{ij} - B_j W_i + C_j U_i) + \sum_{i=1}^{n} - 2 W_i (Y_{ij} - C_j W_i - B_j U_i) = 0.$$

Rearranging terms and noting $W_i^2 + U_i^2 = 1$ yields the solution

$$B_j = \sum_{i=1}^{n} (W_i X_{ij} + U_i Y_{ij})/n,$$

$$C_j = \sum_{i=1}^{n} (W_i Y_{ij} - U_i X_{ij})/n.$$

If H_2 is true, the maximum likelihood estimates are found by solving the equations

$$\frac{\partial L}{\partial B_j} = \frac{\partial L}{\partial C_j} = \frac{\partial L}{\partial D_j} = \frac{\partial L}{\partial E_j} = 0.$$

Taking the derivatives yields the equations

$$\frac{\partial L}{\partial B_j} = \sum_{i=1}^{n} - 2 W_i (X_{ij} - B_j W_i + C_j U_i - D_j S_i + E_j T_i)$$
$$+ \sum_{i=1}^{n} - 2 U_i (Y_{ij} - C_j W_j - B_j U_i - E_j S_i - D_j T_i) = 0,$$

$$\frac{\partial L}{\partial C_j} = \sum_{i=1}^{n} 2U_i\,(X_{ij} - B_j W_j + C_j U_i - D_j S_i + E_j T_i)$$
$$+ \sum_{i=1}^{n} - 2W_i(Y_{ij} - C_j W_i - B_j U_i - E_j S_i - D_j T_i) = 0,$$
$$\frac{\partial L}{\partial D_j} = \sum_{i=1}^{n} - 2S_i(X_{ij} - B_j W_i + C_j U_i - D_j S_i + E_j T_i)$$
$$+ \sum_{i=1}^{n} - 2T_i(Y_{ij} - C_j W_i - B_j U_i - E_j S_i - D_j T_i) = 0,$$
$$\frac{\partial L}{\partial E_j} = \sum_{i=1}^{n} 2T_i(X_{ij} - B_j W_i + C_j U_i - D_j S_i + E_j T_i)$$
$$+ \sum_{i=1}^{n} - 2S_i(Y_{ij} - C_j W_i - B_j U_i - E_j S_i - D_j T_i) = 0.$$

Making the following substitutions

$$F = \sum_{i=1}^{n} (W_i S_i + U_i T_i),$$
$$G = \sum_{i=1}^{n} (U_i S_i - W_i T_i),$$
$$H_j = \sum_{i=1}^{n} (W_i X_{ij} + U_i Y_{ij}),$$
$$I_j = \sum_{i=1}^{n} (W_i Y_{ij} - U_i X_j),$$
$$J_j = \sum_{i=1}^{n} (S_i X_{ij} + T_i Y_{ij}),$$
$$K_j = \sum_{i=1}^{n} (S_i Y_{ij} - T_i X_{ij}),$$

and rearranging terms yields

$$\begin{aligned} nB_j + D_j F + E_j G &= H_j, \\ nC_j - D_j G + E_j F &= I_j, \\ B_j F - C_j G + nD_j &= J_j, \end{aligned} \qquad \text{(A2)}$$

and

$$B_j G + C_j F + nE_j = K_j.$$

Defining the following vectors and matrices

$$N = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \qquad M = \begin{pmatrix} F & G \\ -G & F \end{pmatrix}$$
$$P = \begin{pmatrix} B_j \\ C_j \end{pmatrix} \qquad Q = \begin{pmatrix} D_j \\ E_j \end{pmatrix}$$
$$R = \begin{pmatrix} H_j \\ I_j \end{pmatrix} \qquad Z = \begin{pmatrix} J_j \\ K_j \end{pmatrix},$$

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allows Eq. (A2) to be rewritten as

$$\begin{aligned} NP + MQ &= R \\ M^T P + NQ &= Z, \end{aligned} \tag{A3}$$

where M^T is the transpose of matrix M. Solving Eq. (A3) yields

$$P = (NR - MZ)/(NN - MM^T)$$

and

$$Q = (NZ - M^T R)/(NN - M^T M).$$

Substituting for the matrices and vectors yields

$$B_j = (nH_j - FJ_j - GK_j)/(n^2 - F^2 - G^2),$$
$$C_j = (nI_j + GJ_j - FK_j)/(n^2 - F^2 - G^2),$$
$$D_j = (nJ_j - FH_j + GI_j)/(n^2 - F^2 - G^2),$$

and

$$E_j = (nK_j - GH_j - FI_j)/(n^2 - F^2 - G^2).$$