

A New Derivation for the Rough Surface Reflection Coefficient and for the Distribution of Sea Wave Elevations

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this report the rough surface reflection coefficient for the coherent reflected field is derived. The result obtained assumes a plane wavefront incident on a Gaussian collection of sinusoidal surface waves with uniform phase distribution. The theoretical result obtained is compared with experimental data. In addition, the probability density function of sea wave elevations is obtained.		

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A NEW DERIVATION FOR THE ROUGH SURFACE REFLECTION COEFFICIENT AND FOR THE DISTRIBUTION OF SEA WAVE ELEVATIONS

In 1961 C. I. Beard [1] found that experimental values of the coherent reflected field, $R = |\bar{E}/E_\delta\Gamma|$ were larger than the values given by the generally accepted theoretical formula

$$\left| \frac{\bar{E}}{E_\delta\Gamma} \right|_{S_A} = \exp(-2(2\pi g)^2) \quad (1)$$

for values of $g = (\sigma_h \sin \chi)/\lambda$ greater than 0.1 radian. Here \bar{E} is the average electric field due to the "sea surface" S_A , E_δ is the field due to the direct wave, Γ is the smooth sea reflection coefficient, σ_h is the standard deviation of the sea-surface elevation, χ is the grazing angle, and λ is the electromagnetic wavelength (Fig. 1).

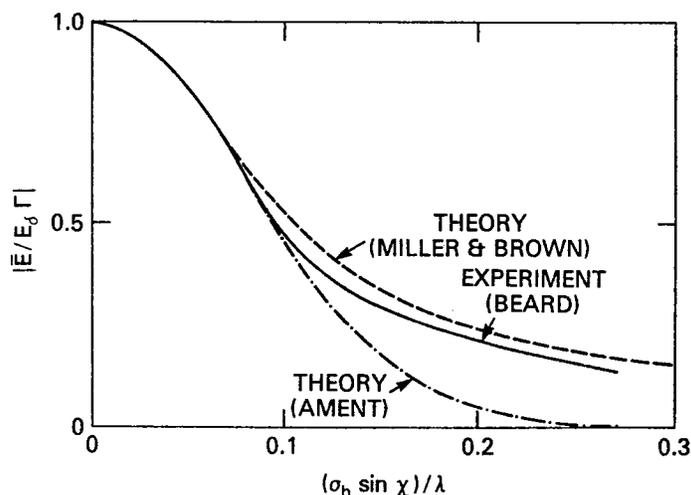


Fig. 1 — Comparison of theoretical and experimental results for coherent forward scattering

This expression for $|\bar{E}/E_\delta\Gamma|_{S_A}$ was first published by W.S. Ament [2] in 1953 and was obtained from the equation

$$R = \int_{-\infty}^{\infty} \exp\left[\frac{4\pi i}{\lambda} y \sin \chi\right] D(y) dy, \quad (2)$$

essentially also derived in Ref. 2. Here $D(y)$ is the normalized probability density function for sea wave elevations, y . Equation (1) was obtained from Eq. (2), by assuming that sea waves were flat surfaces with sea height H , distributed normally, i.e., $y = H$ for the "sea surface" S_A , and

$$D(y) = \frac{1}{\sqrt{2\pi} \sigma_H} \exp\left[-\frac{1}{2} \frac{H^2}{\sigma_H^2}\right]. \quad (3)$$

One might note that Eq. (2) is the Fourier transform of the density function, $D(y)$.

Brown and Miller [3] derived Eqs. (1) and (2) as well, using geometric optics. Moreover, they showed that the coherent reflected field was given by

$$R = \left| \frac{\bar{E}}{E_0 \Gamma} \right|_S = \exp(-2(2\pi g)^2) I_0(2(2\pi g)^2) \quad (4)$$

where $I_0(x)$ is the modified Bessel function $J_0(ix)$. They derived this result by using geometric optics, assuming a planar wavefront incident on a Gaussian collection S of sinusoidal surface waves with uniform phase distribution.

Further, $|\bar{E}/E_0 \Gamma|_S$ agreed with Beard's experimental curve (Fig. 1), with a systematic difference in g of 10%. Beard had estimated that wave elevations were within 10% of their correct values.

After reading Ref. 3, Ament suggested to the authors that a short proof of Eq. (4) might be obtained along the lines that he produced Eq. (1) from Eq. (2). In this report we present such a proof.

We assume that the "sea surface," S , has sea wave elevation that is a random variable, y , with

$$y = H \sin \frac{2\pi}{\Lambda} x$$

where Λ is water wavelength, H is distributed as in Eq. (3), and x is distributed uniformly on the interval $(-\Lambda/4, \Lambda/4)$.

It is a straightforward computation to show that the probability density function of y is given by

$$D(y) = \frac{1}{\pi^{3/2} \frac{\sigma_H}{\sqrt{2}}} \int_{|y|}^{\infty} \frac{\exp\left[-\frac{1}{2} \frac{H^2}{\sigma_H^2}\right]}{\sqrt{H^2 - y^2}} dH \quad (5)$$

where

$$\sigma_h = \sigma_y = \sigma_H / \sqrt{2}. \quad (6)$$

(see appendix). We see that

$$\begin{aligned} D(y) &= \frac{1}{\pi^{3/2} \sigma_h} \int_{|y|}^{\infty} \frac{\exp\left[-\frac{1}{4} \frac{H^2}{\sigma_h^2}\right]}{\sqrt{H^2 - y^2}} dH \\ &= \frac{1}{2\pi^{3/2} \sigma_h} \int_1^{\infty} \frac{\exp\left[-\left(\frac{y}{2\sigma_h}\right)^2 x\right]}{\sqrt{x} \sqrt{x-1}} dx, \end{aligned}$$

and using Ref. 4, page 319, 3.383-3,

$$\begin{aligned} D(y) &= \frac{\Gamma(1/2)/\sqrt{\pi}}{2\pi^{3/2} \sigma_h} \exp\left[-\frac{1}{2} \left(\frac{y}{2\sigma_h}\right)^2\right] K_0\left[\frac{1}{2} \left(\frac{y}{2\sigma_h}\right)^2\right] \\ &= \frac{1}{2\pi^{3/2} \sigma_h} \exp\left[\frac{-y^2}{8\sigma_h^2}\right] K_0\left[\frac{y^2}{8\sigma_h^2}\right], \end{aligned} \quad (7)$$

where $K_0(x)$ is a Bessel function of imaginary argument defined by

$$K_0(x) = \int_0^\infty \cos(x \sinh t) dt, \quad x > 0.$$

Substituting Eq. (7) into Eq. (2) we have

$$\begin{aligned} R &= \frac{1}{\pi^{3/2} \sigma_h} \int_0^\infty \cos\left(\frac{4\pi}{\lambda} y \sin \chi\right) \exp\left(-\frac{y^2}{8\sigma_h^2}\right) K_0\left(\frac{y^2}{8\sigma_h^2}\right) dy \\ &= \frac{\sqrt{2}}{\pi^{3/2}} \int_0^\infty x^{-1/2} e^{-x} \cos\left(\frac{8\sqrt{2} \pi \sigma_h \sin \chi}{\lambda} \sqrt{x}\right) K_0(x) dx. \end{aligned}$$

Now using Ref. 4, page 765, 6.755-9 we obtain Eq. (4).

Since Eq. (4) agrees with the experimental data and Eq. (2) is the Fourier transform of $D(y)$, Eq. (7) must be a good approximation for the actual distribution of sea wave elevations (see Fig. 2).

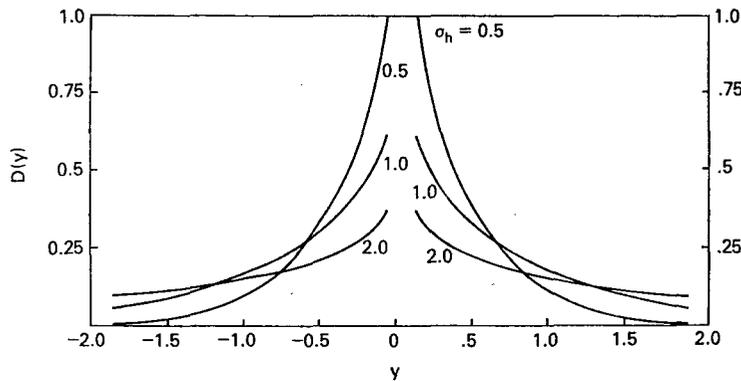


Fig. 2 — Probability density function for sea wave elevations, y (y in feet and the standard deviation of y , σ_h , also in feet)

CONCLUSIONS

In this report we derive the rough surface reflection coefficient for the coherent reflected field with the assumption of a plane wave front incident on a Gaussian collection of sinusoidal surface waves with uniform distribution. The theoretical result agrees with the experimental data, when the systematic difference in the experimental data is taken into account. Furthermore, $D(y)$ as given here is a good approximation for the actual distribution of sea wave elevations.

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Appendix A

Consider the random variable $y = H \sin \theta$, where H is a random variable distributed normally with mean 0 and standard deviation σ_H and θ in a random variable, independent of H , distributed uniformly on the interval $|\theta| < \pi/2$. Let K , F , and D be the densities of H , θ , and y respectively. Then

$$K(H) = \frac{\exp(-H^2/2\sigma_H^2)}{\sigma_H\sqrt{2\pi}}, \quad |H| < \infty$$

and

$$F(\theta) = \begin{cases} \pi^{-1} & |\theta| < \pi/2 \\ 0 & |\theta| \geq \pi/2 \end{cases}.$$

Since θ and H are independent, the joint density, $g(\theta, H)$, of θ and H is given by $K(H)F(\theta)$ or

$$g(\theta, H) = \frac{\exp(-H^2/2\sigma_H^2)}{2^{1/2}\pi^{3/2}\sigma_H}.$$

Let $y = H \sin \theta$, $v = H$ so that $\theta = \text{Sin}^{-1}(y/v)$ and let $f(y, v)$ be the joint density of y and v . Then

$$f(y, v) = |J| g(\text{Sin}^{-1}(y/v), v)$$

where

$$|J| = \begin{vmatrix} \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial v} \\ \frac{\partial H}{\partial y} & \frac{\partial H}{\partial v} \end{vmatrix}.$$

A straightforward computation yields

$$|J| = \frac{1}{\sqrt{v^2 - y^2}}$$

and

$$f(y, H) = \frac{1}{2^{1/2}\pi^{3/2}\sigma_H} \frac{\exp(-H^2/2\sigma_H^2)}{\sqrt{H^2 - y^2}}.$$

Finally, integrate $f(y, H)$ over H to obtain

$$D(y) = \int_{-\infty}^{-|y|} f(y, H) dH + \int_{|y|}^{\infty} f(y, H) dH.$$

Since $f(y, H) = f(y, -H)$ is it clear that $D(y)$ is given by Eq. (5).

Note that the result $\sigma_h = \sigma_y = \sigma_H/\sqrt{2}$ of Eq. (6) follows easily from the definitions of variance, expected value, and the hypothesis that θ and H are independent.