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Computing the Grazing Angle of Specular Reflection

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Three methods for computing the grazing angle of specular reflection are given. A FORTRAN program that employs one of the methods and computes the grazing angle to an arbitrary degree of accuracy is also provided.

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COMPUTING THE GRAZING ANGLE OF SPECULAR REFLECTION

INTRODUCTION

In this report we give three methods for computing the grazing angle of specular reflection. The first was developed by Fishback [1,2] in 1943 and refined somewhat by Blake [3] in 1980, and it gives an approximation to the grazing angle. The second is an iterative method, whose derivation is easy and provides for the computation of the grazing angle to any degree of accuracy specified. The third method shows that there is a closed form explicit expression for the grazing angle. This in turn also allows for a computation to an arbitrary degree of accuracy. A FORTRAN program of the second method is included in Appendix B.

GENERAL REMARKS

The geometry of the spherical-earth problem and definitions of various angles and distances are given in Fig. 1.

The problem of computing the grazing angle of specular reflection is that of computing ψ in Fig. 1, i.e., finding ψ an angle between the tangent line, l, to the circle and $\overline{AR}$ where ψ is also the angle between l and $\overline{TR}$.

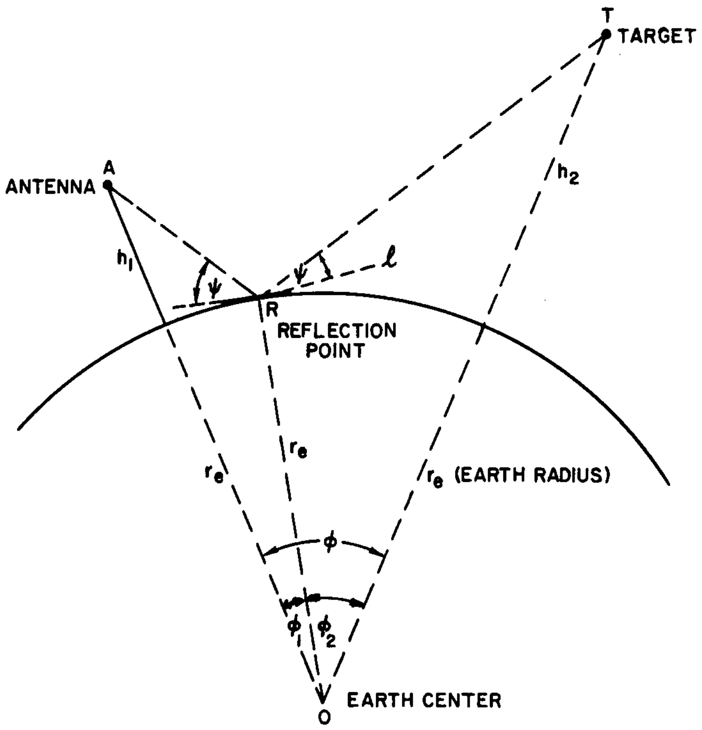

Fig. 1 — Geometry of spherical-earth specular reflection

Manuscript submitted May 20, 1982.

We assume that $\phi = (\phi_1 + \phi_2)$; h_1, h_2, and r_e are given. Define $k_i = \dfrac{r_e}{r_e + h_i}$, $i = 1, 2$.

METHOD ONE

In method one. let

$$s = r_e \phi, \tag{1}$$

$$p^2 = \frac{4}{3}\left[r_e(h_1 + h_2) + \frac{1}{4}s^2\right], \tag{2}$$

and

$$q = \sin^{-1}\left[2p^{-3} r_e s (h_2 - h_1)\right]. \tag{3}$$

Then when h_1 and h_2 are very much less than r_e, Fishback approximates ϕ_1 by

$$\phi_1 \doteq \frac{1}{2}\phi - \frac{p}{r_e}\sin\left(\frac{q}{3}\right). \tag{4}$$

Finally, the grazing angle ψ is given by

$$\psi = \tan^{-1}(cot\ \phi_1 - k_1\ csc\ \phi_1). \tag{5}$$

The error in the approximation for ϕ_1 (and consequently for ψ) is not generally known. However, if $h_1 = h_2$, then $\phi_1 = 1/2\phi$ and Eq. (5) provides the exact result for the grazing angle.

METHOD TWO

Method two uses an iterative procedure for computing the grazing angle to any degree of accuracy.

First, applying the law of sines to triangles OAR and ORT, we obtain

$$\phi_1 + \psi = \cos^{-1}(k_1 \cos\psi) \tag{6}$$

and

$$\phi_2 + \psi = \cos^{-1}(k_2 \cos\psi). \tag{7}$$

Adding the equations (and recalling that $\phi = \phi_1 + \phi_2$) we have,

$$\psi = g(\psi) \tag{8}$$

where,

$$g(\psi) = \frac{1}{2}\left[\cos^{-1}(k_1 \cos\psi) + \cos^{-1}(k_2 \cos\psi) - \phi\right]. \tag{9}$$

If we choose ψ_0 arbitrarily and define

$$\psi_{i+1} = g(\psi_i), \quad i = 0, 1, 2, \ldots \tag{10}$$

then by the results of Appendix A,

$$\lim_{i \to \infty} \psi_i \tag{11}$$

exists and is the unique solution of Eq. (8), i.e., (11) is the grazing angle.

Approximations to the grazing angle are given by the terms of the sequence ψ_0, ψ_1, ..., with successive terms providing more accurate approximations. In practice, when the relative difference of successive terms of this sequence differ in absolute value by no more than a predetermined constant, we obtain the grazing angle to our required degree of accuracy. A FORTRAN program implementing method two is given in Appendix B.

To obtain a rapid convergence for method two, we might use method one to obtain the initial value, ψ_0. In addition, method two may be useful in real-time computation of the grazing angle, since an angle once computed may be used as the initial value for an update computation.

METHOD THREE

The third method produces, in principle, an explicit expression for the grazing angle.

Let $$U = \exp(i\phi) \text{ and } Z = \exp(2i\psi).$$

Then replacing U and Z in Eq. (8), we obtain the following quartic (see Appendix C for derivation):

$$\alpha Z^4 + \beta Z^3 + CZ^2 + \bar{\beta} Z + \bar{\alpha} = 0, \tag{12}$$

where,

$$A = U - k_1 k_2,$$

$$\alpha = UA,$$

$$\beta = k_1^2 + k_2^2 - 2k_1 k_2 U,$$

and

$$C = 2\mathrm{Re}\left[\beta - U\bar{A}\right].$$

Since Eq. (12) is a quartic, the roots can be exhibited explicitly using the classical method of Ferrari and Cardan [4]. At least one of these roots lies on the unit circle. Let Z_* designate any of those roots on the unit circle. Then

$$\psi_* \equiv \frac{1}{2} \cos^{-1} \{\mathrm{Re}\,(Z_*)\}, \tag{13}$$

and that unique value of ψ_* that satisfies Eq. (8) is the grazing angle.

Rather than use the method of Ferrari and Cardan to find the four roots of Eq. (12), it is easier and more efficient to solve the quartic numerically on a computer using a polynomial root finder routine.

CONCLUSION

Three methods for computing the grazing angle of specular reflection are given. The first provides an approximation where the error is not known. Methods two and three will provide computations good to any degree of accuracy. Method two, an iterative procedure, may be especially useful in real time computation.

REFERENCES

1. W.T. Fishback, "Simplified Methods of Field Intensity Calculations in the Interface Region," Report 461, Radiation Laboratory, M.I.T., December 1943.

2. D.E. Kerr, *Propagation of Short Radio Waves*, McGraw-Hill Book Co., N.Y., 1951.

3. L.V. Blake, *Radar Range-Performance Analyses*, Lexington Books, Lexington, Mass., 1980.

4. G.A. Korn and T.M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Book Co., 1968.

Appendix A

PROOF OF CONVERGENCE TO THE GRAZING ANGLE

Let X be an arbitrary set, d a metric on X, and f a contraction mapping of X to itself. (f is called a contraction mapping if there is a positive constant $k < 1$ such that

$$d(f(x),\ f(y)) \leqslant k\, d(x,y) \quad \text{for all } x,\ y \text{ in } X).$$

The following result may be found in [A1].

THEOREM: Every contraction mapping, f, on a complete metric space has a unique fixed point, i.e., there is a unique x in X such that $f(x) = x$. Furthermore, for an arbitrary x_0 in X, the sequence given by $x_{n+1} = f(x_n)$, $n = 0, 1, 2 \ldots$ converges to x.

As a special case, let X be the reals and for x,y in X let $d(x,y) = |x - y|$. Then (X,d) is a complete metric space.

Now let f be a differentiable function defined on X and suppose furthermore that there is a constant k, $0 < k < 1$ such that $|f'(x)| \leqslant k$ for all x in X. Then f is a contraction mapping. This may be seen as follows. Let x and y be real numbers, $x < y$. By the mean value theorem there is a number ξ, $x \leqslant \xi < y$ such that

$$f(x) - f(y) = (x - y) f'(\xi).$$

Hence,

$$|f(x) - f(y)| = |x - y|\,|f'(\xi)| \leqslant k\ |x - y|.$$

These results may be summarized in the following.

COROLLARY: Let R be the reals, k a constant $0 < k < 1$, and f a differentiable function on R such that $|f'(x)| \leqslant k$ for all x in R. Then there is a unique real number a such that

$$a = f(a).$$

Moreover, if x_0 is an arbitrary real number and $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$, then

$$\lim_{n \to \infty} x_n = a.$$

With respect to Method 2, we have given the differentiable function

$$g(x) = \frac{1}{2}\left[\cos^{-1}(k_1 \cos x) + \cos^{-1}(k_2 \cos x) - \phi\right].$$

Differentiating, we have

$$\begin{aligned} 2g'(x) &= \frac{k_1 \sin x}{\sqrt{1 - k_1^2 \cos^2 x}} + \frac{k_2 \sin x}{\sqrt{1 - k_2^2 \cos^2 x}}, \\ &= \pm\, k_1 \sqrt{\frac{1 - \cos^2 x}{1 - k_1^2 \cos^2 x}} \pm k_2 \sqrt{\frac{1 - \cos^2 x}{1 - k_2^2 \cos^2 x}}. \end{aligned}$$

Recalling that

$$k_i = \frac{r_e}{r_e + h_i} < 1, \quad i = 1, 2$$

we have

$$|2g'(x)| \leqslant k_1 \sqrt{\frac{1 - \cos^2 x}{1 - k_1^2 \cos^2 x}} + k_2 \sqrt{\frac{1 - \cos^2 x}{1 - k_1^2 \cos^2 x}}$$

$$\leqslant k_1 + k_2 < 2,$$

or

$$|g'(x)| \leqslant \frac{k_1 + k_2}{2} < 1.$$

Hence, by the Corollary, there is a unique number a such that

$$a = g(a).$$

Moreover, if x_0 is arbitrary and $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$, then

$$\lim_{n \to \infty} x_n = a.$$

REFERENCE

A1. G. Bachman and L. Narici, *Functional Analysis*, Academic Press, N.Y., 1966.

Appendix B

A FORTRAN PROGRAM OF METHOD TWO

```
SOURCE LISTING              ASC FAST FORTRAN COMPILER        RELEASE FTFX0529.P294/80

  STATEMENT                 CP OPTIONS = (M,X)              DATE = 05/06/82(82.126)

      SUBROUTINE GRAZE(H1,H2,THETA,RE,PHI)
C
C     THIS ROUTINE COMPUTES THE GRAZING ANGLE PHI FOR SPHERICAL
C     EARTH SPECULAR REFLECTION.
C
C     H1= ANTENNA HEIGHT.
C     H2= TARGET HEIGHT.
C     RE= EARTH EFFECTIVE RADIUS OR EARTH RADIUS.
C     THETA IS CENTRAL ANGLE IN RADIANS.
C
C     PHI MUST BE INITIALIZED IN CALLING ROUTINE: PHI .NE. 0.
C     PHI= GRAZING ANGLE OUTPUT IN RADIANS.
C     RELATIVE ERROR IN PHI IS 10**-8, BUT CAN BE DECREASED BY CHANGING
C     VALUE OF TOL IN DATA STATEMENT.
C
C     INPUTS AND OUTPUTS ARE IN REAL*8:REAL*4 CAN BE USED BY REMOVING
C     IMPLICIT STATEMENT AND ADJUSTING TOL.
C
C     H1, H2, RE, THETA ARE INPUTS. UNITS FOR H1, H2, RE MUST BE
C     CONSISTENT.
C
C
      IMPLICIT REAL*8(A-H,O-Z)
C
      DATA TOL/1.D-8/
C
      RK1=RE/(H1+RE)
      RK2=RE/(H2+RE)
C
      A=DARCOS(RK1)+DARCOS(RK2)-THETA
      IF( A .GE. 0.D0 .AND. A .LE. 1.D-15 ) GOTO 11
C
   20 G=0.5D0*(DARCOS(RK1*DCOS(PHI) )+DARCOS(RK2*DCOS(PHI))-THETA)
C
      RTST=(G-PHI)/PHI
      RTST=DABS(RTST)
C
      IF(RTST.LE. TOL ) GOTO 10
C
      PHI=G
C
      GOTO 20
C
   10 PHI=G
C
      RETURN
C
   11 PHI=0.D0
      RETURN
      END
```

Appendix C

DERIVATION OF THE QUARTIC

Beginning with Eq. (8), we have

$$\cos(2\psi + \phi) = k_1 k_2 \cos^2\psi - \sqrt{1 - k_1^2 \cos^2\psi}\ \sqrt{1 - k_2^2 \cos^2\psi}\ .$$

Squaring, we obtain

$$(1 - k_1^2 \cos^2\psi)(1 - k_2^2 \cos^2\psi) = [k_1 k_2 \cos^2\psi - \cos(2\psi + \phi)]^2,$$

or

$$1 - (k_1^2 + k_2^2)\cos^2\psi = \cos^2(2\psi + \phi) - 2k_1 k_2 \cos^2\psi \cos(2\psi + \phi). \qquad \text{(C1)}$$

Let $\qquad U = \exp(i\phi)$ and $Z = \exp(i2\psi)$,

so that

$$\cos^2\psi = \frac{1}{4}(Z + \bar{Z} + 2), \qquad \text{(C2)}$$

and

$$\cos(2\psi + \phi) = \frac{1}{2}[UZ + \overline{UZ}]. \qquad \text{(C3)}$$

Multiplying both sides of Eq. (C1) by $4U^2Z^2$ and substituting Eq. (C2) and Eq. (C3) into Eq. (C1) we have

$$4U^2Z^2 - U^2(k_1^2 + k_2^2)(Z^3 + 2Z^2 + Z)$$
$$= (U^4Z^4 + 2U^2Z^2 + 1) - Uk_1k_2(Z^2 + 2Z + 1)(U^2Z^2 + 1).$$

Then grouping the terms in powers of Z gives

$$U^3AZ^4 + U^2\beta Z^3 + U^2(2\mathrm{Re}\,(\beta - U\bar{A}))\,Z^2 + U^2\bar{\beta}Z + U\bar{A} = 0.$$

Now dividing by U^2 gives Eq. (12).