

Estimation of Radar Range Biases at Multiple Sites

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ESTIMATION OF RADAR RANGE BIASES AT MULTIPLE SITES

INTRODUCTION

An interesting and important problem the Navy has been concerned with for years is called "Gridlock." Primarily, it is concerned with using radars to form a relative navigation system. This can be achieved because radars located on different ships should see common targets at the same points in space. In general, there is a large number of identifiable parameters which must be estimated such as latitudes, longitudes, speeds, heights, orientations, range biases, azimuth biases, etc. Since these parameters are often coupled, often the general estimation problem can become very complex. It would be desirable to formulate the problem so that some of the parameters could be estimated independently from the rest. If this could be done, several small estimation problems could be solved, and probably a more stable solution could be obtained.

A means of estimating the range bias error which is independent of a ship or platform's position and orientation is studied. The method to be described is based on the requirement that two non-colocated radars must measure the distance between two targets as the same. The algorithm is first formulated. The performance of the estimation is studied using synthetically generated data and real recorded data. The recorded data were obtained simultaneously using SPS-39 radars located at the Johns Hopkins University/Applied Physics Laboratory (JHU/APL) and the Naval Research Laboratory (NRL), Chesapeake Bay Detachment (CBD). Finally, the results are summarized.

MATHEMATICAL FORMULATION

Figure 1 illustrates the geometry of the problem. Coordinate systems are established at each radar site with the x-axis oriented due east, the y-axis oriented due north and the z-axis oriented outward along a ray joining the site and the Earth's center. Azimuth is measured counterclockwise from the x-axis, elevation is measured from the x-y plane and range is the length of the position vector whose initial point is the origin of a radar site's coordinate system to a point in space.

The difference vectors from sites 1 and 2, \vec{D}_1 and \vec{D}_2 , can be described as functions of the following elements:

$$\vec{D}_1 = F(r_{11}, r_{12}, az_{11}, az_{12}, el_{11}, el_{12}), \quad (1)$$

and

$$\vec{D}_2 = F(r_{21}, r_{22}, az_{21}, az_{22}, el_{21}, el_{22}), \quad (2)$$

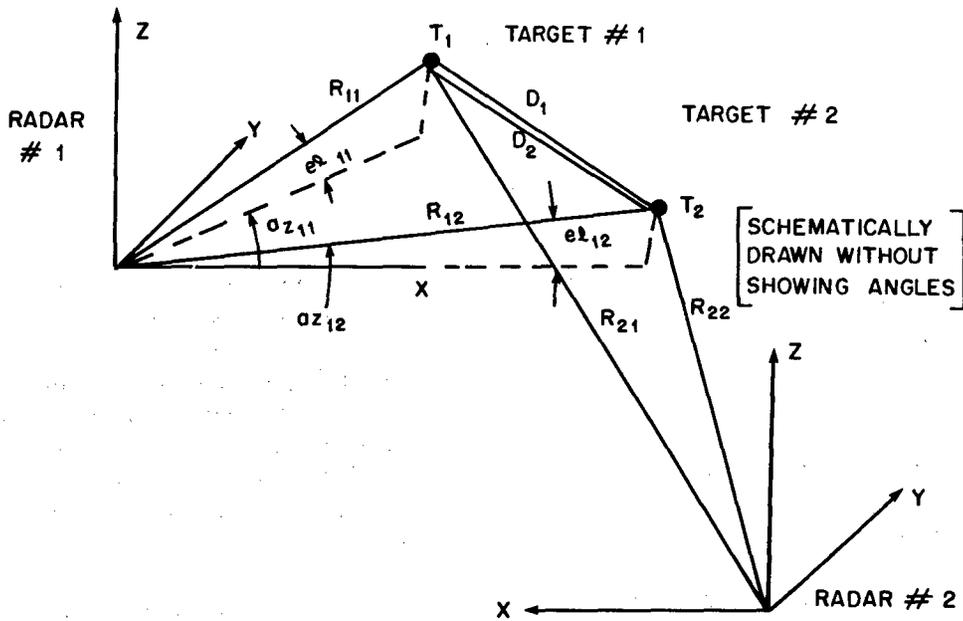


Fig. 1 — Geometric representation of problem

where from Fig. 1

$$\begin{aligned}
 r_{11} &= r_{T_{11}} + b_{r_1} + n_{r_{11}}, \\
 r_{12} &= r_{T_{12}} + b_{r_1} + n_{r_{12}}, \\
 az_{11} &= az_{T_{11}} + b_{az_1} + n_{az_{11}}, \\
 az_{12} &= az_{T_{12}} + b_{az_1} + n_{az_{12}}, \\
 el_{11} &= el_{T_{11}} + b_{el_1} + n_{el_{11}}, \\
 el_{12} &= el_{T_{12}} + b_{el_1} + n_{el_{12}}, \\
 r_{21} &= r_{T_{21}} + b_{r_2} + n_{r_{21}}, \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 el_{22} &,
 \end{aligned}
 \tag{3}$$

and

$r_{T_{ij}}$ = the true range measurement for target j ; from radar i ,

b_{r_i} = the range bias in radar i ,

$n_{r_{ij}}$ = the noise present in radar i 's range measurement to target j ,

$az_{T_{ij}}$ = the true azimuth measurement for target j ; from radar i ,

b_{az_i} = the azimuth bias in radar i ,

$n_{az_{ij}}$ = the noise present in radar i 's azimuth measurement to target j ,

$e\ell_{T_{ij}}$ = the true elevation measurement for target j ; from radar i ,

$b_{e\ell_i}$ = the elevation bias in radar i ,

and

$n_{e\ell_{ij}}$ = the noise present in radar i 's elevation measurement to target j .

Referring again to Fig. 1, the location of target j from radar i , \vec{T}_{ij} , is given in vector form using the unit vectors \vec{i} , \vec{j} , and \vec{k} by

$$\vec{T}_{11} = r_{11} \cos e\ell_{11} \cos az_{11} \vec{i} + r_{11} \cos e\ell_{11} \sin az_{11} \vec{j} + r_{11} \sin e\ell_{11} \vec{k}, \quad (4)$$

and

$$\vec{T}_{12} = r_{12} \cos e\ell_{12} \cos az_{12} \vec{i} + r_{12} \cos e\ell_{12} \sin az_{12} \vec{j} + r_{12} \sin e\ell_{12} \vec{k} \quad (5)$$

as seen from radar 1 and

$$\vec{T}_{21} = r_{21} \cos e\ell_{21} \cos az_{21} \vec{i} + r_{21} \cos e\ell_{21} \sin az_{21} \vec{j} + r_{21} \sin e\ell_{21} \vec{k}, \quad (6)$$

and

$$\vec{T}_{22} = r_{22} \cos e\ell_{22} \cos az_{22} \vec{i} + r_{22} \cos e\ell_{22} \sin az_{22} \vec{j} + r_{22} \sin e\ell_{22} \vec{k} \quad (7)$$

as seen from radar 2. The distance between targets 1 and 2 as measured by radar 1 is

$$D_1^2 = |\vec{T}_{12} - \vec{T}_{11}|^2, \quad (8)$$

and as measured by radar 2 is

$$D_2^2 = |T_{22} - T_{21}|^2, \quad (9)$$

where $| \cdot |$ represents the magnitude of the vector.

Using Eqs. (4) through (7), the distances D_1 and D_2 are given by

$$D_1^2 = r_{11}^2 + r_{12}^2 - 2r_{11}r_{12} \left[\sin e\ell_{21} \sin e\ell_{22} + \cos e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22}) \right], \quad (10)$$

and

$$D_2^2 = r_{21}^2 + r_{22}^2 - 2r_{21}r_{22} \left[\sin e\ell_{21} \sin e\ell_{22} + \cos e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22}) \right]. \quad (11)$$

Approximations for D_1 and D_2 are obtained by expanding the functions in a truncated Taylor's series in terms of the noise and bias errors. If the function D_1 and D_2 are nearly linear functions in terms of range, azimuth, and elevation over the range of values of the noise and bias errors, only the first two terms of the Taylor series need be kept and the approximation becomes linear. In the case we will consider, the elevation and azimuth bias errors are assumed to be zero or previously removed by another process. Actually it can be shown that D_1 and D_2 are not dependent on a constant azimuth bias. Assuming the elevation bias is zero, the linear approximation for D_1 is

$$\tilde{D}_1 = \bar{D}_1 + [C_{11}] b_{r_1} + [C_{12}] \begin{bmatrix} n_{r_{11}} \\ n_{r_{12}} \\ n_{az_{11}} \\ n_{az_{12}} \\ n_{e\ell_{11}} \\ n_{e\ell_{12}} \end{bmatrix}, \quad (12)$$

where

$$[C_{11}] = \left[\frac{\partial D_1}{\partial b_{r_1}} \right], \quad [C_{12}] = \left[\frac{\partial D_1}{\partial n_{r_{12}}} \quad \frac{\partial D_1}{\partial n_{r_{12}}} \quad \frac{\partial D_1}{\partial az_{11}} \quad \dots \quad \frac{\partial D_1}{\partial n_{e\ell_{12}}} \right],$$

\tilde{D}_1 is the approximate value, \bar{D}_1 is the true or mean value at the same time, C_{11} is a 1-by-1 matrix derived in the appendix, and C_{12} is a matrix of partial derivatives which is also derived in the appendix. Similarly, D_2 can be approximated as

$$\tilde{D}_2 = \bar{D}_2 + [C_{21}][b_{r_2}] + [C_{22}] \begin{bmatrix} n_{r_{21}} \\ n_{r_{22}} \\ n_{az_{21}} \\ n_{az_{22}} \\ n_{e\ell_{21}} \\ n_{e\ell_{22}} \end{bmatrix} \quad (13)$$

where

$$[C_{21}] = \left[\frac{\partial D_2}{\partial b_{r_2}} \right] \text{ and } [C_{22}] = \left[\begin{array}{cccccc} \frac{\partial D_2}{\partial n_{r_{21}}} & \frac{\partial D_2}{\partial n_{r_{22}}} & \frac{\partial D_2}{\partial n_{az_{21}}} & \dots & \dots & \frac{\partial D_2}{\partial n_{e\ell_{22}}} \end{array} \right].$$

If the measurements, Eqs. (12) and (13), are made in time coincidence, the residue error $\Delta D = \bar{D}_1 - \bar{D}_2$ between the measurements is

$$\Delta D = [C_{11}][b_{r_1}] - [C_{21}][b_{r_2}] + [C_{12}] \begin{bmatrix} n_{r_{11}} \\ n_{r_{12}} \\ n_{az_{11}} \\ n_{az_{12}} \\ n_{e\ell_{11}} \\ n_{e\ell_{12}} \end{bmatrix} - [C_{22}] \begin{bmatrix} n_{r_{21}} \\ n_{r_{22}} \\ n_{az_{21}} \\ n_{az_{22}} \\ n_{e\ell_{21}} \\ n_{e\ell_{22}} \end{bmatrix} \quad (14)$$

Equation (14) can be rewritten as

$$\Delta D = HX + N, \quad (15)$$

where

$$H = \left[[C_{12}] - [C_{22}] \right], \quad X = \begin{bmatrix} b_{r_1} \\ b_{r_2} \end{bmatrix},$$

and

$$N = \left[[C_{12}] - [C_{22}] \right] \begin{bmatrix} n_{r_{11}} \\ n_{r_{12}} \\ \cdot \\ \cdot \\ n_{e\ell_{12}} \\ n_{r_{21}} \\ n_{r_{22}} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ n_{e\ell_{22}} \end{bmatrix},$$

Equation (15), without the noise, is the form of one linear equation with two unknowns. If another equation using at least one other target is formed like (15), we would have two linear equations and two unknowns which would have a unique solution. This system of equations is represented by

$$R = GX + S, \tag{16}$$

where

$$R = \begin{bmatrix} \Delta D_1 \\ \Delta D_2 \end{bmatrix}, \quad G = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad X = \begin{bmatrix} b_{r_1} \\ b_{r_2} \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

The subscripts on D , H , and N denote the values in Eq. (15) corresponding to the first equation and second equation formed by examining distances between different target pairs. Equation (16) is

now in the form of the observation equation used in the least-square estimator. The vector of range bias errors, X , is the state vector with S being a matrix of measurement noises.

The smoothed range bias errors are estimated recursively by a least-square filter from Ref. 1. The three steps involved in the recursive algorithm are as follows:

Step 1. Calculate the prediction observation $\Delta(k)$;

$$\Delta(k+1) = \Delta(k) + G'C^{-1}R(k) ; \quad (17)$$

Step 2. Calculate a new covariance matrix $P(k+1)$,

$$P^{-1}(k+1) = P^{-1}(k) + G'C^{-1}G ; \quad (18)$$

Step 3. Calculate a new smoothed estimate $\tilde{x}(k)$;

$$\tilde{X}(k) = P(k+1)\Delta(k+1) \quad (19)$$

where C is the covariance matrix of the noise S .

The following sections describe the performance of this algorithm with simulated and real data.

SIMULATION RESULTS

The simulation used to test the algorithm consisted of two stationary platforms; platform one had (x, y, z) coordinates, (10, 8, 0) measured in nautical miles, while platform two was located at the origin. On each simulated iteration, two stationary targets were randomly generated in any one of the four quadrants. The radar at the two platforms then made their own range, azimuth, and elevation measurements on each target as illustrated in Fig. 1. All of these measurements were corrupted with zero-mean noise in range, azimuth, and elevation with standard deviations of 360 m (1200 ft), 0.5° , and 1° respectively.

The filter's ability to accurately approximate range bias was tested using the bias conditions:

	Platform 1	Platform 2
Range bias, nmi	1	-2
Azimuth bias	0	0
Elevation bias	0	0

The results are illustrated in Fig. 2. Approximately ten iterations were required for the filter to accurately obtain the induced biases.

These results verify, under simulation, that a simple algorithm based on requiring two non-collocated radars to measure the same distance between targets in space can accurately estimate range bias errors. More realistic testing is required to verify operation of the algorithm in ship-board environments. In the next section the results of testing with real data will be discussed.

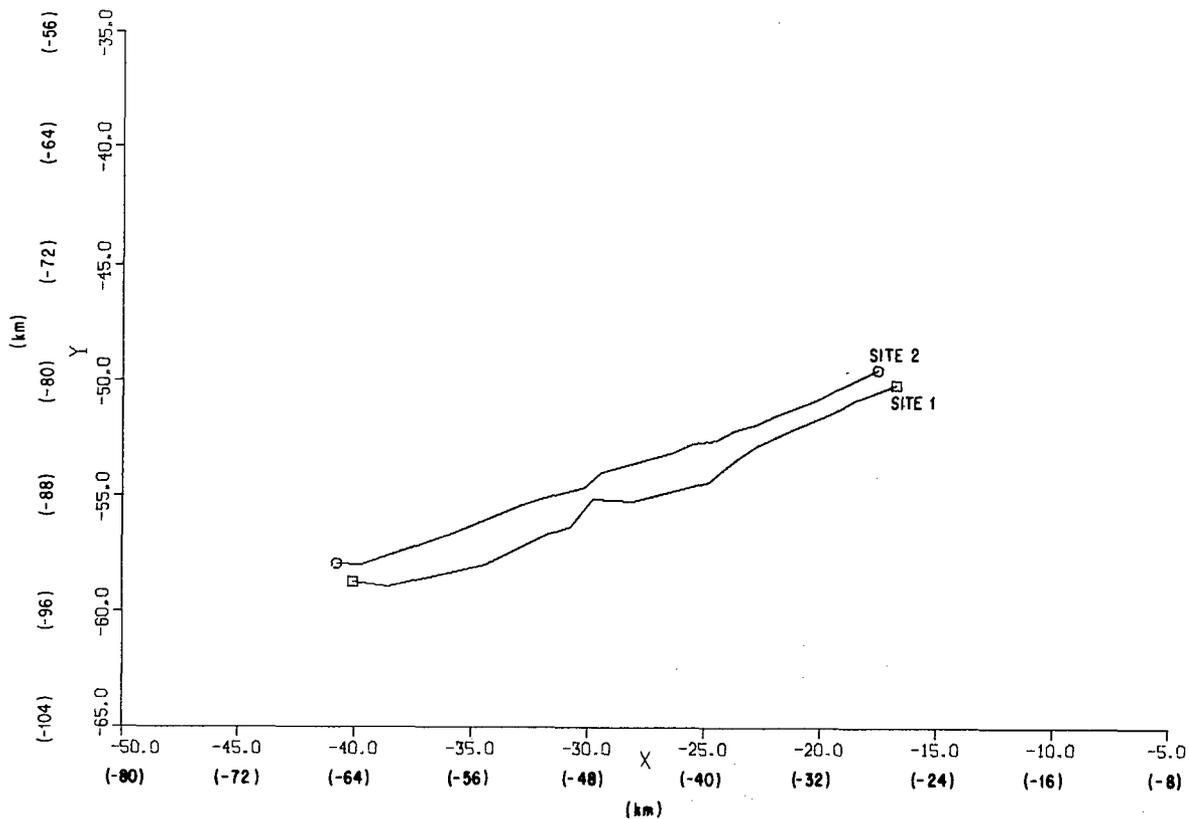


Fig. 3 — Target A as tracked by two radar sites without bias removed

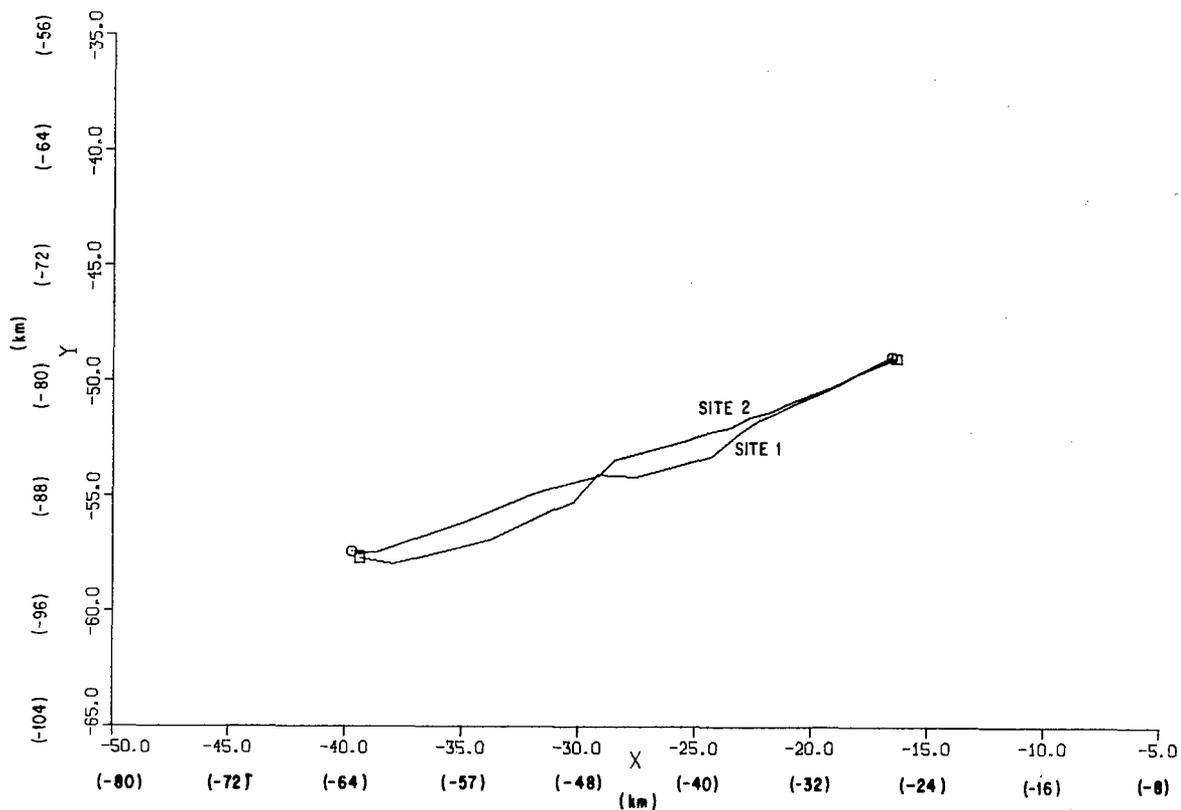


Fig. 4 — Target A with bias removed

These range biases were obtained by taking the average on biases obtained on several runs. In Figs. 3 and 4, as well as the others, APL is located at the origin with NRL located approximately 56.5 km (35 mi) southeast. Figures 5 and 6, again, represent a track as seen by both sites without and with biases respectively. Figures 7 and 8 are similar to the above but produced results which were not as accurate.

SUMMARY

An algorithm was developed to measure the range bias errors that can occur in radars. The method was based on the condition that two non-colocated radars must measure the same distance between targets. Because only distances between points were being measured and compared, it was not necessary to know the radars' location or orientation. This fact is advantageous, because the entire "Gridlock" problem can then be solved in independent stages.

In this study the elevation bias error was assumed to be zero or previously removed. However, in a more expanded study this requirement is not necessary and could be included in the same estimation process presented. The linearization of the equations seemed to work quite well and did not present a problem with the types of data studied.

Excellent results were obtained in removing bias errors under controlled simulation conditions. However, when the recorded data from the APL and NRL sites were used, the results did not look quite as good. Time did not permit a full assessment of the discrepancies found in the real data.

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2. A. Grindlay, "Radar Bias Error Removal Algorithm for a Multiple-Site System," NRL Report 8467, April 1981.

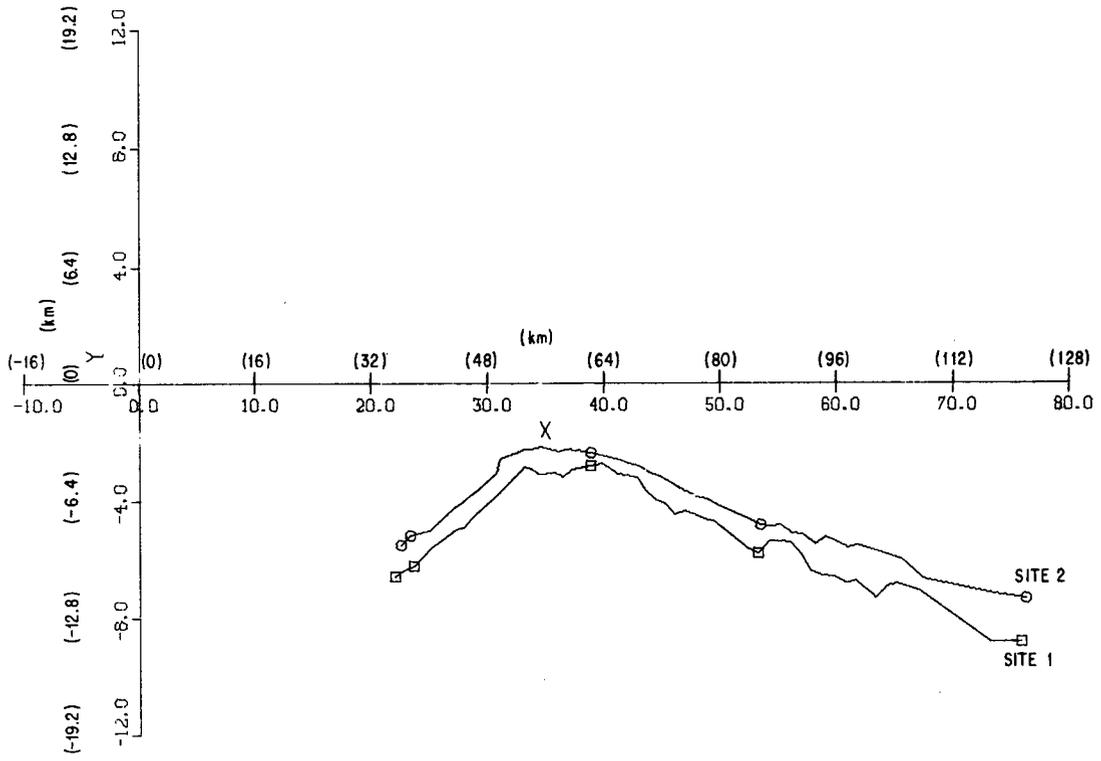


Fig. 5 — Target B as tracked by two radar sites without bias removed

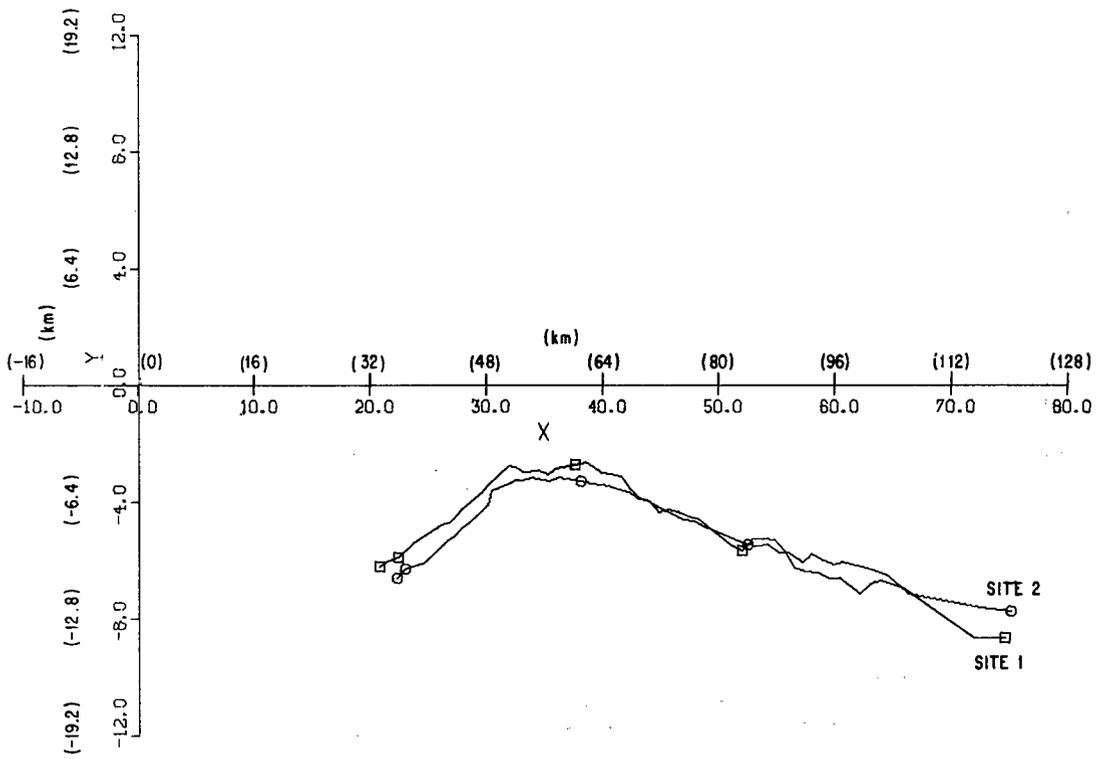


Fig. 6 — Target B with bias removed

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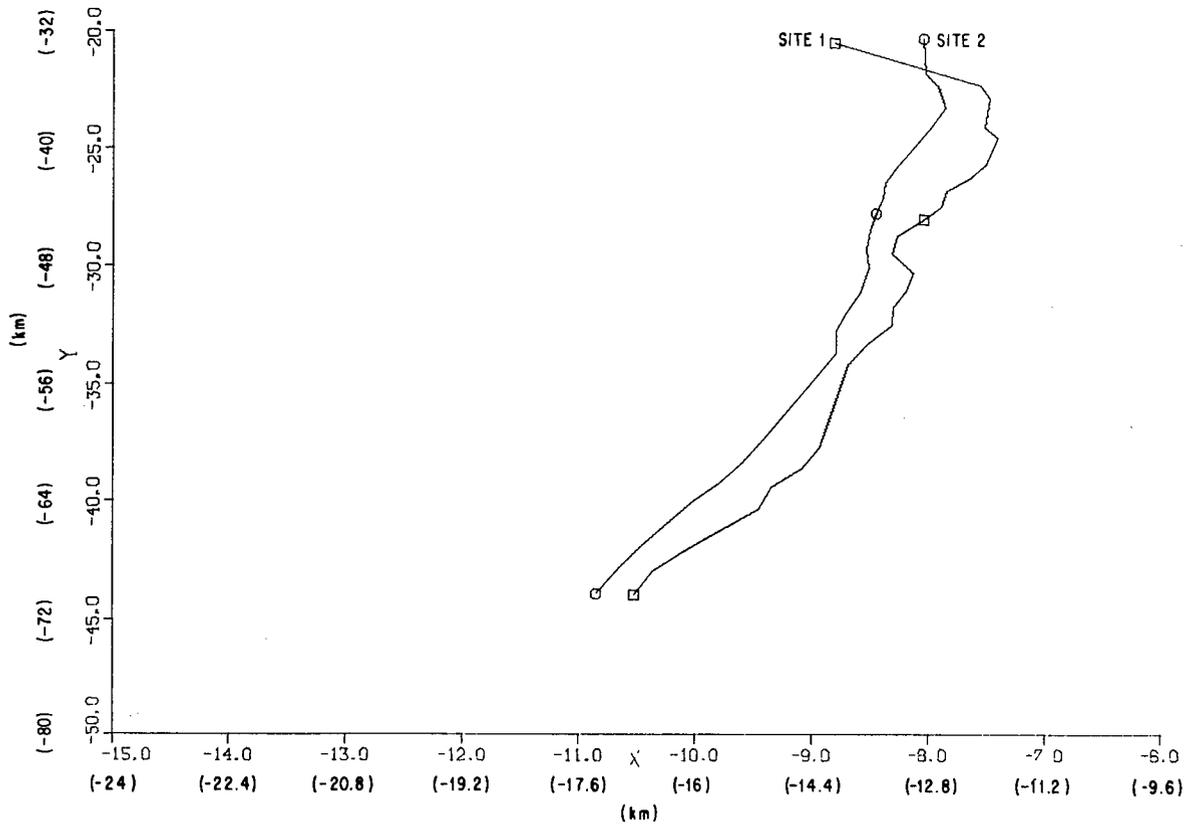


Fig. 7 — Target C as tracked by two radar sites without bias removed

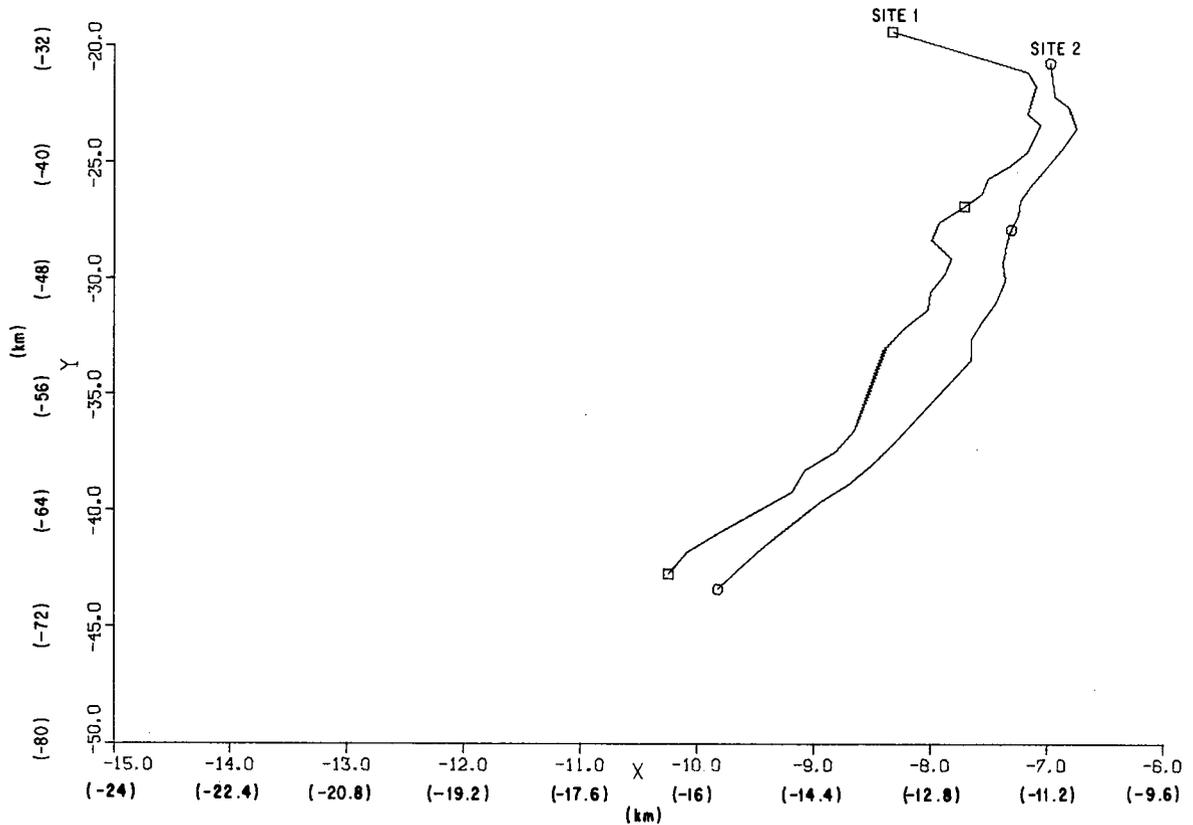


Fig. 8 — Target C with bias removed

Appendix

MATRIX COEFFICIENTS

The matrix

$$[C_{11}] = \left[\frac{\partial D_1}{\partial b_{r_1}} \right]$$

is given by

$$\frac{\partial D_1}{\partial b_{r_1}} = \frac{\partial \sqrt{F}}{\partial b_{r_1}} = \frac{1}{2\sqrt{F}} \frac{\partial F}{\partial b_{r_1}}, \quad (\text{A1})$$

where

$$F = r_{11}^2 + r_{12}^2 - 2r_{11}r_{12} \left[\sin e\ell_{11} \sin e\ell_{12} + \cos e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12}) \right]. \quad (\text{A2})$$

Recalling from Eq. (3) the models of r_{11} , r_{12} , $e\ell_{11}$, etc., and then differentiating Eq. (A2) with respect to b_{r_1} yields,

$$\begin{aligned} \frac{\partial D_1}{\partial b_{r_1}} = \frac{1}{2\sqrt{F}} & \left[2r_{11} + 2r_{12} - 2 \sin e\ell_{11} \sin e\ell_{12} (r_{11} + r_{12}) \right. \\ & \left. - 2 \cos e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12}) (r_{11} + r_{12}) \right]. \end{aligned} \quad (\text{A3})$$

Collecting terms yields

$$\frac{\partial D_1}{\partial b_{r_1}} = \frac{1}{\sqrt{F}} \left[(r_{11} + r_{12}) (1 - \sin e\ell_{11} \sin e\ell_{12} - \cos e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12})) \right]. \quad (\text{A4})$$

By replacing the subscripts in Eq. (A3) we arrive at $[C_{21}]$

$$\frac{\partial D_2}{\partial b_{r_2}} = \frac{1}{\sqrt{F_2}} \left[(r_{21} + r_{22}) (1 - \sin e\ell_{21} \sin e\ell_{22} - \cos e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22})) \right] \quad (\text{A5})$$

where

$$F_2 = r_{21}^2 + r_{22}^2 - 2r_{21}r_{22} \left[\sin e\ell_{21} \sin e\ell_{22} + \cos e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22}) \right].$$

The matrix

$$[C_{12}] = \begin{bmatrix} \frac{\partial D_1}{\partial n_{r_{11}}} & \frac{\partial D_1}{\partial n_{r_{12}}} & \frac{\partial D_1}{\partial n_{az_{11}}} & \frac{\partial D_1}{\partial n_{az_{12}}} & \frac{\partial D_1}{\partial n_{e\ell_{11}}} & \frac{\partial D_1}{\partial n_{e\ell_{12}}} \end{bmatrix}$$

is a matrix made up of position elements differentiated with respect to the measurement noises. Recalling again from Eq. (3) the models of r_{11} , r_{12} , $e\ell_{11}$, etc., and expanding the first element yields:

$$\frac{\partial D_1}{\partial n_{r_{11}}} \frac{\partial \sqrt{F}}{\partial n_{r_{11}}} = \frac{1}{2\sqrt{F}} \frac{\partial F}{\partial n_{r_{11}}}$$

with F from Eq. (A2).

Differentiating Eq. (A2) with respect to $n_{r_{11}}$ yields

$$\begin{aligned} \frac{\partial D_1}{\partial n_{r_{11}}} = \frac{1}{2\sqrt{F}} & \left[2r_{11} - 2r_{12} \sin e\ell_{11} \sin e\ell_{12} \right. \\ & \left. - 2r_{12} \cos e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12}) \right]. \end{aligned} \quad (A6)$$

Collecting terms yields

$$\frac{\partial D_1}{\partial n_{r_{11}}} = \frac{1}{\sqrt{F}} \left[r_{11} - r_{12} (\sin e\ell_{11} + \sin e\ell_{12} + \cos e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12})) \right]. \quad (A7)$$

Similarly differentiating the remaining elements of C_{12} yields

$$\frac{\partial D_1}{\partial n_{r_{12}}} = \frac{1}{\sqrt{F}} \left[r_{12} - r_{11} (\sin e\ell_{11} \sin e\ell_{12} + \cos e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12})) \right], \quad (A8)$$

$$\frac{\partial D_1}{\partial n_{az_{11}}} = \frac{1}{\sqrt{F}} \left[r_{11} r_{12} \cos e\ell_{11} \cos e\ell_{12} \sin (az_{11} - az_{12}) \right], \quad (A9)$$

$$\frac{\partial D_1}{\partial n_{az_{12}}} = -\frac{1}{\sqrt{F}} \left[r_{11} r_{12} \cos e\ell_{11} \cos e\ell_{12} \sin (az_{11} - az_{12}) \right], \quad (A10)$$

$$\frac{\partial D_1}{\partial n_{e\ell_{11}}} = \frac{1}{\sqrt{F}} \left[r_{11} r_{12} \left[\sin e\ell_{11} \cos e\ell_{12} \cos (az_{11} - az_{12}) - \cos e\ell_{11} \sin e\ell_{12} \right] \right], \quad (A11)$$

and

$$\frac{\partial n_1}{\partial n_{e\ell_{12}}} = \frac{1}{\sqrt{F}} \left[r_{11} r_{12} \left[\cos e\ell_{11} \sin e\ell_{12} \cos (az_{11} - az_{12}) - \sin e\ell_{11} \cos e\ell_{12} \right] \right]. \quad (\text{A12})$$

By replacing the subscripts in Eqs. (A6) through (A11), we arrive at the elements of C_{22} , with F_2 defined by Eq. (A5):

$$\frac{\partial D_2}{\partial n_{r_{21}}} = \frac{1}{\sqrt{F_2}} \left[r_{21} - r_{22} \left[\sin e\ell_{21} \sin e\ell_{22} + \cos e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22}) \right] \right], \quad (\text{A13})$$

$$\frac{\partial D_2}{\partial n_{r_{22}}} = \frac{1}{\sqrt{F_2}} \left[r_{22} - r_{21} \left[\sin e\ell_{21} \sin e\ell_{22} + \cos e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22}) \right] \right], \quad (\text{A14})$$

$$\frac{\partial D_2}{\partial n_{az_{21}}} = \frac{1}{\sqrt{F_2}} \left[r_{21} r_{22} \cos e\ell_{21} \cos e\ell_{22} \sin az_{21} \right], \quad (\text{A15})$$

$$\frac{\partial D_2}{\partial n_{az_{22}}} = -\frac{1}{\sqrt{F_2}} \left[r_{21} r_{22} \cos e\ell_{21} \cos e\ell_{22} \sin az_{22} \right], \quad (\text{A16})$$

$$\frac{\partial D_2}{\partial n_{e\ell_{21}}} = \frac{1}{\sqrt{F_2}} \left[r_{21} r_{22} \left[\sin e\ell_{21} \cos e\ell_{22} \cos (az_{21} - az_{22}) - \cos e\ell_{21} \sin e\ell_{22} \right] \right], \quad (\text{A17})$$

and

$$\frac{\partial D_2}{\partial n_{e\ell_{22}}} = \frac{1}{\sqrt{F_2}} \left[r_{21} r_{22} \left[\cos e\ell_{21} \sin e\ell_{22} \cos (az_{21} - az_{22}) - \sin e\ell_{21} \cos e\ell_{22} \right] \right]. \quad (\text{A18})$$