NRL Report 8481

Maximum Likelihood Angle Estimates of Two Signals Using Three Squinted Beams (Symmetric Solution)

B. H. CANTRELL

Radar Analysis Branch
Radar Division

May 29, 1981

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Report 8481	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) MAXIMUM LIKELIHOOD ANGLE ESTIMATES OF TWO SIGNALS USING THREE SQUINTED BEAMS (SYMMETRIC SOLUTION)		5. TYPE OF REPORT & PERIOD COVERED Interim report on a continuing NRL problem 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) B. H. Cantrell		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, DC 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153N; RR-021-05-41; NRL Problem 53-0626-0
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE May 29, 1981 13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Estimation
Angle measurement
Radar

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The maximum likelihood estimates of the elevation angles of two closely spaced targets is considered. For an antenna constructed with three squinted beams a simple closed-form solution for the elevation angle is found. The solution is only valid when the center of the middle beam is directed toward the bisecting line between the sources and the principal axes of the antenna are parallel to and perpendicular to the line connecting the sources. This condition can be obtained with an antenna viewing an emission or reflection over a smooth reflecting sea. The accuracy of solution is studied using simulation procedures. The accuracy is found to be comparable to maximum likelihood solutions using an array antenna.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

CONTENTS

MAXIMUM LIKELIHOOD ANGLE ESTIMATES OF TWO SIGNALS USING THREE SQUINTED BEAMS (SYMMETRIC SOLUTION)

INTRODUCTION

The problem of estimating the angle of arrival of closely spaced targets has received considerable attention [1-19]. Two solution methods, the maximum likelihood estimate (MLE) and the maximum entropy method (MEM) have been used extensively for many problems. For array antennas, White [1] calculated the mean-square error for the two-target case when the signal-to-noise ratio (SNR) was large, a case which provides small errors. This calculation was made for both the case in which the plane of symmetry between the targets was known and the case in which it was not known. Lang [2] compared the MLE with the MEM for the two-target nonsymmetric case. He found that the MLE was only slightly better than the MEM; at most the difference was approximately $\sqrt{2}$. In addition to comparing the accuracy of the MLE and MEM for the two-target case, Trunk et al. [3] compared the MLE accuracy to the Cramer-Rao bound for unbiased estimators. They showed that the MLE can achieve better accuracy than the bound for the lower signal-to-noise ratios because it is a biased estimator.

Besides the accuracy of the estimates, a number of implementations have been studied. White [1] describes a double-null tracking system which approximates the MLE estimate using an array antenna. He also discusses a tempered double-null tracker using multiple beams. Neither of these implementations estimate the angle of arrival on a single pulse; they search for a solution through a tracking loop. Howard [4] uses the MLE method to obtain the angle estimate on a single pulse by a search technique.

Some closed-form solutions for the angle of arrival of two targets under special conditions have been obtained. Peebles and Berkowitz [5] obtained a noise-free solution for two targets and three squinted beams when the antenna patterns could be approximated by polynominals. White [1] obtained a fixed-beam solution under the special conditions of symmetric antenna patterns pointed directly at the plane of symmetry between two targets. Recently a closed-form MLE using three subapertures was obtained for two targets both when the plane of symmetry between them was known and when it was unknown [6]. The accuracy was nearly as good as if all the elements of an array had been used. Since a closed-form MLE using three subapertures had been obtained, we addressed the analogous problem of obtaining a closed-form MLE using three squinted beams. We first reduced the mean square error cost function found in the MLE formulation into as simple a form as possible in terms of the unknown angles of arrival. We then found the angles that minimized the cost and evaluated the accuracy of the estimates. We investigated only the case in which the plane of symmetry separating the sources or targets is known.

Manuscript submitted February 23, 1981

MLE FORMULATION FOR THREE SQUINTED BEAMS AND TWO TARGETS

We begin by defining the geometry and antenna system with the aid of Fig. 1. The antenna lies in the z-y plane with the principal axes of the antenna defined along the z and y axes. This implies that the aperture illumination is factorable in the z and y directions. The antenna patterns can then be written in terms of the product of two individual antenna patterns which are functions of the direction cosines. For large apertures and narrow beams pointed near the bore site axis x, the direction cosine can be approximated by the elevation and azimuth angles θ and ψ, respectively, as shown in Fig. 1. The angles of arrival of the radiation from source 1 and source 2 are defined to be (θ_1, ψ_0) and (θ_2, ψ_0), respectively. We have defined the azimuth ψ_0 of the two sources to be the same and, consequently, there is no cross-leveling of the array with respect to a line connecting the sources.

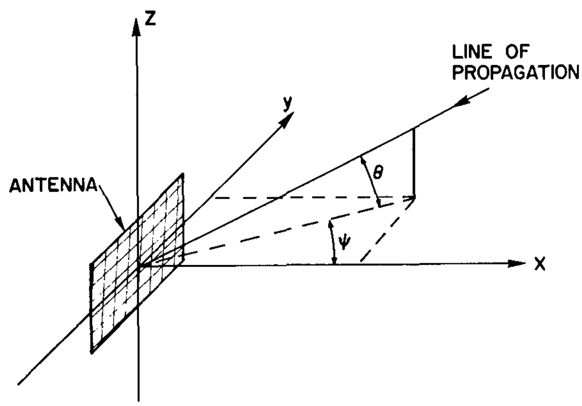

Fig. 1 — Coordinate system

The antenna is constructed so that three squinted beams are formed. The beams are symmetric about the x-z plane and are squinted in the elevation direction θ. Using representative antenna pattern shapes, their antenna pattern factors in the θ direction are shown in Fig. 2.

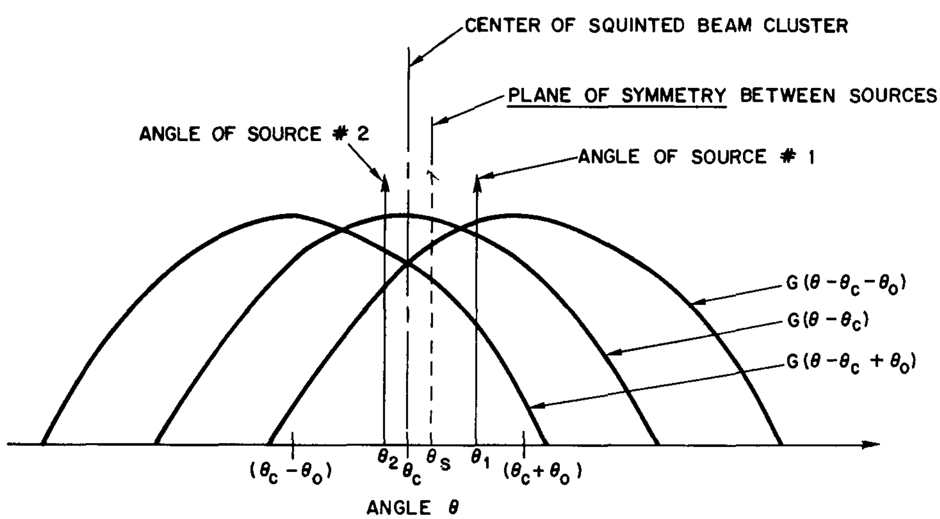

Fig. 2 — Antenna patterns for three squinted beams in θ direction for a fixed angle ψ

The three beams are identical and the separation in angle of adjacent beam centers is denoted by θ_0. The pointing of the three squinted beams is specified by the pointing angle θ_c of the center of the middle beam. The plane of symmetry between the two sources is specified by angle $\theta_s = (\theta_1 - \theta_2)/2$. The antenna pattern factors for the three beams in the θ coordinate are given in terms of the angle of arrival θ of a source by $G(\theta - \theta_c - \theta_0)$, $G(\theta - \theta_c)$, and $G(\theta - \theta_c + \theta_0)$, respectively. The antenna pattern factors in the ψ direction are given by $H(\psi)$ for all three beams. The antenna patterns for the three beams are then given by

$$F(\theta - \theta_c - \theta_0, \psi) = H(\psi)\, G(\theta - \theta_c - \theta_0),$$

$$F(\theta - \theta_c, \psi) = H(\psi)\, G(\theta - \theta_c),$$

and

$$F(\theta - \theta_c + \theta_0, \psi) = H(\psi)\, G(\theta - \theta_c + \theta_0).$$

The antenna is also constructed with the property that when a plane wave sweeps across the aperture, signals will be produced at the output ports of the beamforming devices that are all in phase. These signals after conversion to baseband are denoted by the complex numbers s_1, s_2, and s_3, where the real and the imaginary parts are the time-coincident samples of the in-phase and the quadrature channels of the synchronous detector, respectively. The signals s_1, s_2, and s_3 out of the three beamforming ports can be expressed as the summation of the excitation produced by the plane-wave signals sweeping across the antenna and the additive thermal noise. For two sources in the far field, the received signals are written as

$$s_1 = a_1\, F(\theta_1 - \theta_c - \theta_0, \psi) + a_2\, F(\theta_2 - \theta_c - \theta_0, \psi) + n_1,$$

$$s_2 = a_1 F(\theta_1 - \theta_c, \psi) + a_2\, F(\theta_2 - \theta_c, \psi) + n_2,$$

and

$$s_3 = a_1\, F(\theta_1 - \theta_c + \theta_0, \psi) + a_2\, F(\theta_2 - \theta_c + \theta_0, \psi) + n_3, \tag{2}$$

where a_1 and a_2 are the strengths of the sources 1 and 2, respectively, and n_1, n_2, and n_3 are the thermal noise in each of the channels associated with a beamforming device. The noise is assumed to be uncorrelated and Gaussian distributed with a mean of zero and a standard deviation of σ. Equations (2) can be written in vector form

$$\mathbf{S} = \mathbf{W}\mathbf{A} + \mathbf{N}, \tag{3}$$

where

$$\mathbf{S} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},$$

and

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}.$$

The coefficients of the matrix **W** are the antenna gains of each of the beams in the direction of each of the sources, as described in Eq. (2), and are given by

$$\begin{aligned} w_{11} &= F(\theta_1 - \theta_c - \theta_0, \psi), & w_{12} &= F(\theta_2 - \theta_c - \theta_0, \psi), \\ w_{21} &= F(\theta_1 - \theta_c, \psi), & w_{22} &= F(\theta_2 - \theta_c, \psi), \\ w_{31} &= F(\theta_1 - \theta_c + \theta_0, \psi), & w_{32} &= F(\theta_2 - \theta_c + \theta_0, \psi). \end{aligned} \tag{4}$$

We have now constructed a model of the signals received by an antenna using three squinted beams, excited by two distant sources at different angles. We would now like to estimate the unknown parameters **A** and the angles of arrival θ_1 and θ_2 using the noisy measurements **S**. The estimation technique we will use is the maximum likelihood procedure.

Since **N** is a Gaussian random vector with uncorrelated but equal variance components, maximizing the likelihood function is equivalent to minimizing the square error

$$L = (\mathbf{S} - \mathbf{WA})^* \ (\mathbf{S} - \mathbf{WA}), \tag{5}$$

where the asterisk represents the conjugate transpose. The value of **A** which minimizes L is

$$\mathbf{A} = (\mathbf{W}^*\mathbf{W})^{-1} \mathbf{W}^*\mathbf{S},$$

which expands to

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = E^{-1} \left[\begin{array}{c|c|c} \dfrac{(p_1 - q_1) + p_2(p_1p_2 - q_1q_2)}{w_{21}} & \dfrac{p_2(p_2 - q_2) - q_1(p_1 - q_1)}{w_{21}} & \dfrac{-(p_2 - q_2) - q_1(p_1p_2 - q_1q_2)}{w_{21}} \\ \hdashline \dfrac{-(p_1 - q_1) - q_2(p_1p_2 - q_1q_2)}{w_{22}} & \dfrac{p_1(p_1 - q_1) - q_2(p_2 - q_2)}{w_{22}} & \dfrac{(p_2 - q_2) + p_1(p_1p_2 - q_1q_2)}{w_{22}} \end{array}\right] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{6}$$

where

$$E = (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_1p_2 - q_1q_2)^2,$$

$$p_1 = \frac{w_{11}}{w_{21}}, \qquad q_1 = \frac{w_{12}}{w_{22}},$$

$$q_2 = \frac{w_{31}}{w_{21}}, \qquad p_2 = \frac{w_{32}}{w_{22}}.$$

Equation (5) for the mean-square error L can be written as

$$L = \sum_{i=1}^{3} ||f_i||^2 = \sum_{i=1}^{3} f_i f_i^*, \tag{7}$$

where

$$f_i = s_i - w_{i1}\, a_1 - w_{i2}\, a_2 \tag{8}$$

for $i = 1$, 2, and 3. Substituting the values a_1 and a_2 found in (6) into (8) we obtain

$$f_1 = \frac{(p_2 - q_2)\ [(p_2 - q_2)s_1 - (p_1p_2 - q_1q_2)s_2 + (p_1 - q_1)s_3]}{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_1p_2 - q_1q_2)^2},$$

$$f_2 = \frac{-(p_1p_2 - q_1q_2)\ [(p_2 - q_2)s_1 - (p_1p_2 - q_1q_2)s_2 + (p_1 - q_1)s_3]}{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_1p_2 - q_1q_2)^2},$$

$$f_3 = \frac{(p_1 - q_1)\ [(p_2 - q_2)s_1 - (p_1p_2 - q_1q_2)s_2 + (p_1 - q_1)s_3]}{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_1p_2 - q_1q_2)^2}. \tag{9}$$

Using Eqs. (9) in (5), the mean square error becomes

$$L = \frac{||(p_2 - q_2)s_1 - (p_1p_2 - q_1q_2)s_2 + (p_1 - q_1)s_3||^2}{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_1p_2 - q_1q_2)^2}. \tag{10}$$

The coefficients p_1, q_1, p_2, and q_2 in (9) are complicated nonlinear functions of the angles of arrival and the geometry. Finding the parameters which minimize the error would be difficult in most cases. However, a special case is examined which is of a simpler form than (10).

SOLUTION UNDER SYMMETRY CONDITIONS

We investigate the solution for the angles of arrival of signals when the angular bisecting plane between them is known and both emitters are at the same azimuth ψ. The peak of the center beam in elevation is pointed directly at the bisecting angle between the two sources. In the three squinted beam cluster, the squint all appears in elevation. For these conditions $\theta_c = 0°$, $\theta_1 = -\theta_2$, and the weights become

$$\begin{aligned} w_{11} &= F(\theta_1 - \theta_0, \psi), & w_{12} &= F(-\theta_1 - \theta_0, \psi), \\ w_{21} &= F(\theta_1, \psi), & w_{22} &= F(-\theta_1, \psi), \\ w_{31} &= F(\theta_1 + \theta_0, \psi), & w_{32} &= F(-\theta_1 + \theta_0, \psi). \end{aligned} \tag{11}$$

Using the fact that the antenna patterns are factorable under certain conditions as expressed by Eq. (1), Eq. (11) can be rewritten as

$$\begin{aligned} w_{11} &= H(\psi)\ G(\theta_1 - \theta_0), & w_{12} &= H(\psi)\ G(-\theta_1 - \theta_0), \\ w_{21} &= H(\psi)\ G(\theta_1), & w_{22} &= H(\psi)\ G(-\theta_1), \\ w_{31} &= H(\psi)\ G(\theta_1 + \theta_0), & w_{32} &= H(\psi)\ G(-\theta_1 + \theta_0). \end{aligned} \tag{12}$$

Using the definition of p_1, q_1, p_2, and q_2 we find

$$p_1 = \frac{w_{11}}{w_{21}} = \frac{H(\psi)\ G(\theta_1 - \theta_0)}{H(\psi)\ G(\theta_1)} = \frac{G(\theta_1 - \theta_0)}{G(\theta_1)},$$

$$q_2 = \frac{w_{31}}{w_{21}} = \frac{H(\psi)\ G(\theta_1 + \theta_0)}{H(\psi)\ G(\theta_1)} = \frac{G(\theta_1 + \theta_0)}{G(\theta_1)},$$

$$q_1 = \frac{w_{12}}{w_{22}} = \frac{H(\psi)\ G(-\theta_1 - \theta_0)}{H(\psi)\ G(-\theta_1)} = \frac{G(-\theta_1 - \theta_0)}{G(-\theta_1)},$$

and

$$p_2 = \frac{w_{32}}{w_{22}} = \frac{H(\psi)\ (-\theta_1 + \theta_0)}{H(\psi)\ G(-\theta_1)} = \frac{G(-\theta_1 + \theta_0)}{G(-\theta_1)}.$$

Using the fact that the antenna patterns for the beams have even symmetry, $G(\theta) = G(-\theta)$, we find that $p_1 = p_2$ and $q_1 = q_2$ and define

$$p = p_1 = p_2$$

and

$$q = q_1 = q_2.$$

Using the symmetry conditions in the expression for the mean square error L in Eq. (10), we find

$$L = \frac{||s_1 - Xs_2 + s_3||^2}{2 + X^2}, \tag{13}$$

where

$$X = p + q = \frac{G(\theta_1 - \theta_0) + G[-(\theta_1 + \theta_0)]}{G(\theta_1)}.$$

To minimize the mean square error L with respect to the angle of arrival θ_1 in (13) is equivalent to minimizing L with respect to X and then relating X to θ_1 through a table look up. The minimum value of L is found by finding its stationary points with respect to the real variable X in terms of the complex numbers s_1, s_2, and s_3. The stationary points are found by differentiating the mean square error L with respect to X, setting the result equal to zero, and solving for X. The result is

$$X = \frac{(\eta - 2\gamma)}{\alpha + \beta} \pm \sqrt{\frac{(\eta - 2\gamma)}{\alpha + \beta} + 2}, \tag{14}$$

where

$$\alpha = s_1^* \, s_2 + s_2^* \, s_1,$$

$$\beta = s_2^* \, s_3 + s_3^* \, s_2,$$

$$\gamma = s_2^* \, s_2, \text{ and}$$

$$\eta = s_1 \, s_1^* + s_3 \, s_3^* + s_1 \, s_3^* + s_3 \, s_1^*.$$

Because of the geometry, θ_1 cannot be less than zero degrees, and consequently

$$X \geqslant \frac{G(-\theta_0) + G(-\theta_0)}{G(0)},$$

or

$$X \geqslant \frac{2G(\theta_0)}{G(0)},$$

where $G(\theta_0) = G(-\theta_0)$. If X is less than this zero-angle value due to noise, we limit X to

$$X = \frac{2G(\theta_0)}{G(0)}. \tag{15}$$

The solution is that value of X given by (14) or (15) which makes L in (13) a minimum.

Not only can the angle of arrival θ_1 be estimated, but the estimates of the complex amplitudes can be made using (6). For the special case under consideration, a_1 and a_2 are estimated in terms of p and q by

$$a_1 = \frac{1}{w_{32}} \, \frac{[1 + p(p + q)] \, s_1 + (p - q)s_2 - [1 + q(p + q)] \, s_3}{(p - q) \, [(p + q)^2 + 2]}$$

and

$$a_2 = \frac{1}{w_{32}} \, \frac{-[1 + q(p + q)] \, s_1 + (p - q)s_2 + [1 + p(p + q)] \, s_3}{(p - q) \, [(p + q)^2 + 2]}.$$

To compare the results to previous work, we define a reflection coefficient by

$$\rho = \frac{a_2}{a_1} = \frac{-[1 + q \, (p + q)]s_1 + (p - q)s_2 + [1 + p \, (p + q] \, s_3}{[1 + p \, (p + q)]s_1 + (p - q)s_2 - [1 + q(p + q)]s_3}. \tag{16}$$

If the angle of arrival and stored antenna patterns are known, p and q can be found. Consequently, the reflection coefficient can be estimated. For the latter purposes we define the magnitude $|\rho|$ and phase ϕ of the reflection coefficient ρ.

For a system to be put in operation, a calibration curve of X vs the angle of arrival θ_1 would be found experimentally and stored for reference in a table. For our purposes here, we wish to illustrate the nature of this curve and use it later to indicate the system's expected performance. The normalized antenna patterns chosen are

$$G(\theta \pm \theta_0) = \frac{\sin \pi d(\theta \pm \theta_0)/\lambda}{\pi d(\theta \pm \theta_0)/\lambda} \cdot \frac{\pi^2}{\pi^2 - [\pi d(\theta \pm \theta_0)/\lambda]^2}, \tag{17}$$

where d is the aperture width and λ is the wavelength. The beamwidth B_θ for this antenna pattern is

$$B_\theta = 82.5\lambda/d$$

in degrees. For an antenna aperture width of 2.5 wavelengths and a squint angle of 0.8 beamwidth, the calibration curve of the parameter X vs angle of arrival θ_1 using the postulated antenna pattern (17) is shown in Fig. 3. We find that the curve is very flat for small angles, which indicates that large signal-to-noise ratios will be required to obtain descent-angle estimates for closely spaced targets. We next evaluate quantitatively the performance of the angle-measurement procedure.

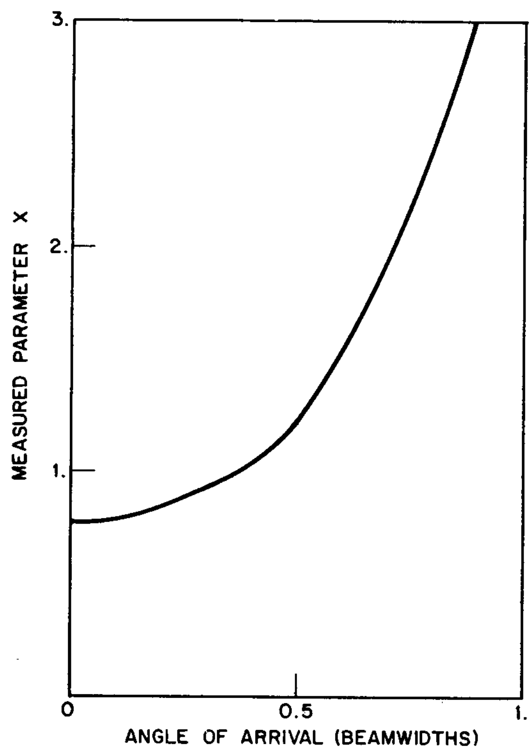

Fig. 3 — Calibration curve

ANGLE ESTIMATE PERFORMANCE

The performance of the MLE angle estimates has been evaluated using a simulation technique. To compare the results to previous work, the signal-to-noise ratio is defined to be

$$\frac{S}{N} = 10 \log \frac{|a_1|^2}{2\sigma^2}.$$

This is the signal-to-noise ratio of one source (the strongest) at the center of the center beam. Using this signal-to-noise ratio, noisy signals are generated using (3), and the angle of arrival θ_1 is estimated. This process is repeated 400 times and the root mean square (RMS) error in the estimates of θ_1, $|\rho|$, and ϕ is computed.

For a 30-dB signal-to-noise ratio, a magnitude of the reflection coefficient of 0.9 and three values of θ_1, the RMS error in estimating θ_1 is plotted vs the relative phase shift ϕ between the sources in Fig. 4. We find that the error is smallest when the effective signal-to-noise ratio is largest (i.e., signals a_1 and a_2 are in phase) and largest when the signals are out of phase. Also, Fig. 4 shows that the closer the signal spacing, the more difficult it is to measure the angle of arrival. The RMS error in estimating the angle of arrival is also shown in Fig. 5 using the same conditions as used in Fig. 4, except that $|\rho| = 0.2$. The effective S/N is more constant with relative phase between the sources, and so the curves are flatter. Again, Fig. 5 shows that it is easier to estimate the angle of widely separated signals.

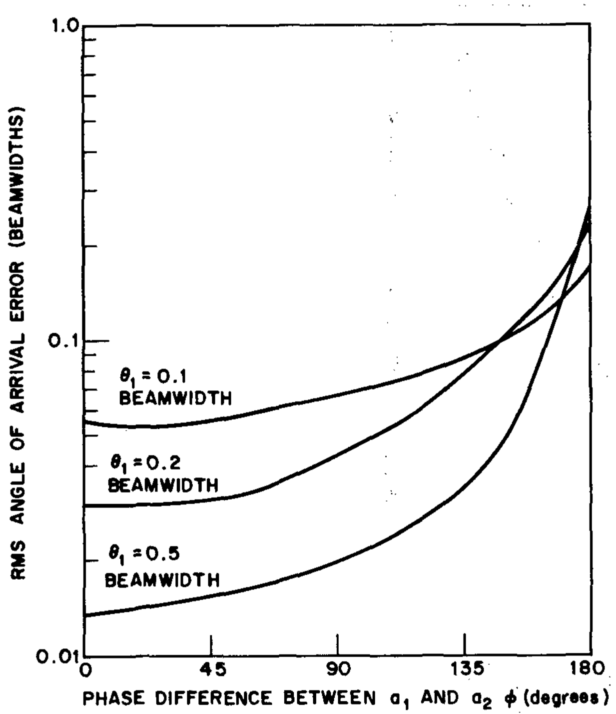

Fig. 4 — RMS error in angle of arrival for $|\rho| = 0.9$ and S/N = 30 dB

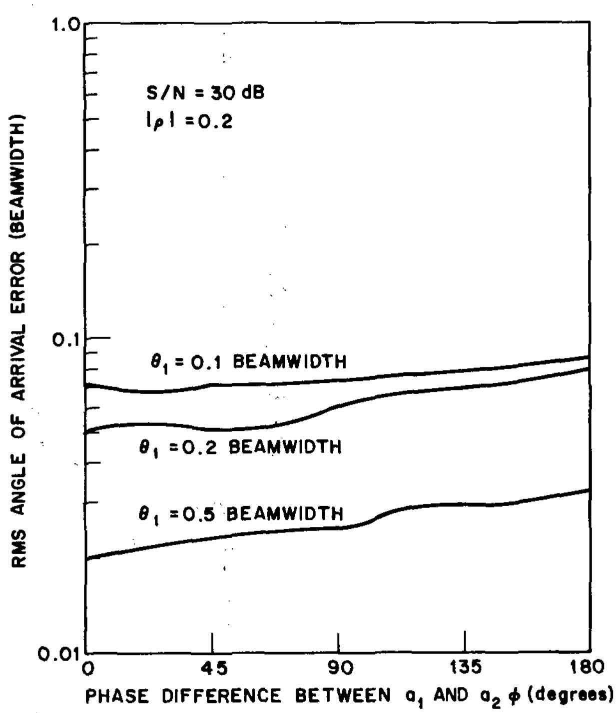

Fig. 5 — RMS error in angle of arrival for $|\rho| = 0.2$ and S/N = 30 dB

Besides estimating the angle of arrivals, we estimated the relative amplitude and phase between the two signals. Using a 30-dB signal-to-noise ratio and a reflection coefficient magnitude of 0.9, the RMS error for $|\rho|$ and ϕ is shown in Figs. 6 and 7, respectively, for three angles of arrival. Again we find that the estimates are better for the more widely separated sources.

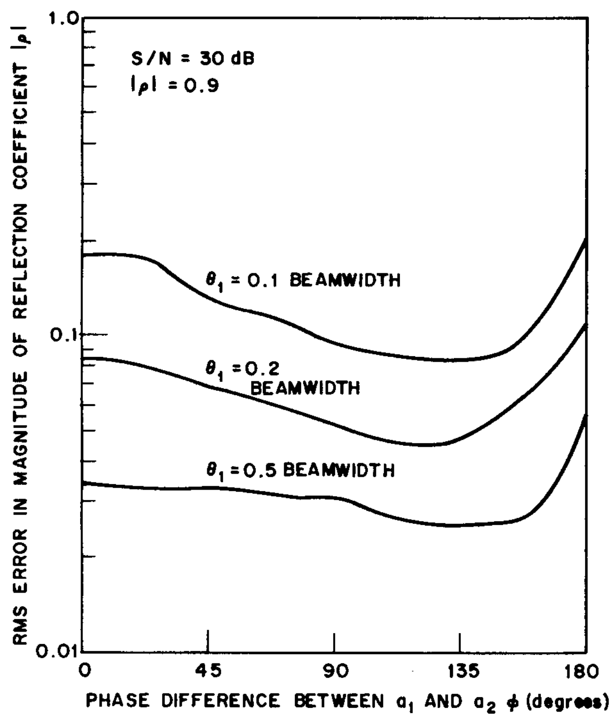

Fig. 6 — RMS error in the magnitude of the reflection coefficient for $|\rho| = 0.9$ and S/N = 30 dB

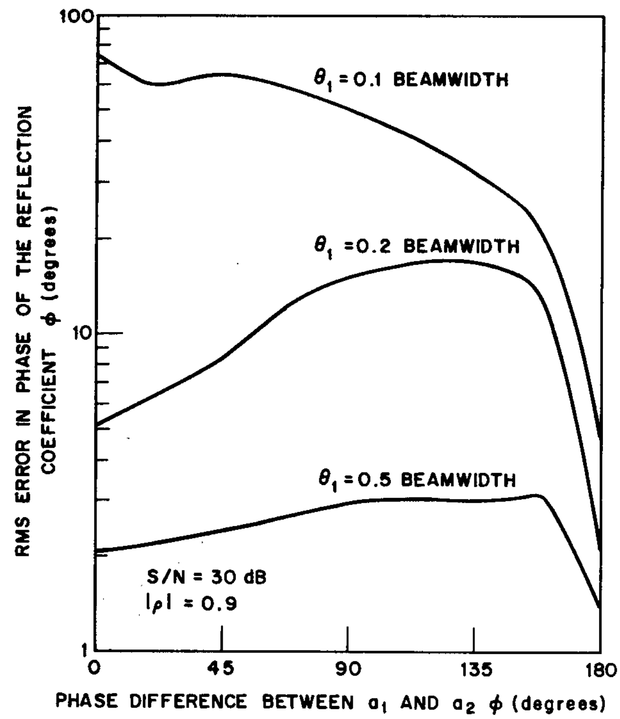

Fig. 7 — RMS error in the phase of the reflection coefficient for $|\rho| = 0.9$ and S/N = 30 dB

We next compare the performance of the MLE solution using three squinted beams to the Cramer-Rao bound results found in Ref. 3 and to the MLE solution using three subapertures [6]. This comparison is shown in Fig. 8, where a 30-dB signal-to-noise ratio, $|\rho| = 0.9$, and two different angles of arrival are used. We find the errors in the MLE solutions using either subaperture or squinted beams are not very different. Sometimes better results than predicted by the Cramer-Rao bound for unbiased estimators can be achieved with biased estimators than with unbiased ones. Although not shown, the estimates for $|\rho|$ and ϕ are about the same for the MLE estimates using subapertures [6] or using squinted beams as described in this paper.

SUMMARY

The estimation of the angle of arrival of two closely spaced sources is studied using an antenna constructed to form three identical squinted beams. A simple closed-form solution for the angle of arrival is found under certain restrictions. The restrictions are that the center of the middle beam is directed toward the bisecting line between the targets and that the principal axes of the antenna are parallel to and perpendicular to the line connecting the targets. This condition can be obtained with an antenna viewing an emission or reflection over a smooth reflecting sea.

The accuracy of the estimates of the angle of arrival and the reflection coefficient was studied using simulation procedures. The best angle estimates, especially for reflection coefficients near 1, were obtained when the signals were in phase and the worst when the signals were out of phase. This can be explained by the high and low effective signal-to-noise ratios under these conditions. The results showed the estimates became better as the target separation increased. Finally, the MLE estimates for three squinted beams were comparable to the MLE estimates for three subapertures; only minor differences were noted.

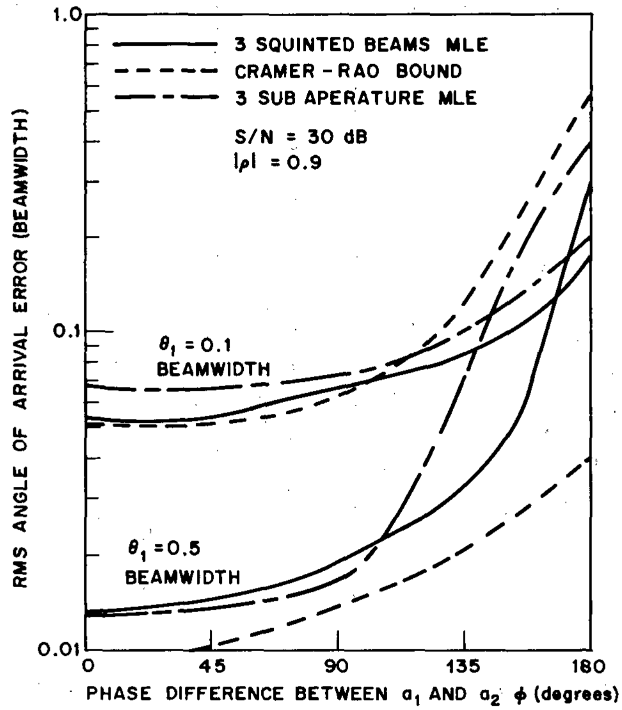

Fig. 8 — Comparison of RMS angle of arrival errors for $|\rho| = 0.9$ and 30 dB S/N

REFERENCES

1. W.D. White, "Low-Angle Radar Tracking in the Presence of Multipath," IEEE Trans. Aerosp. Electron. Syst. **AES-10**, 835-852 (Nov. 1974).

2. P.R. Dax, "Accurate Tracking of Low Elevation Targets over the Sea with a Monopulse Radar," *Radar—Present and Future*, IEE Conference Publication 105, London, 1973, pp. 160-165.

3. G.V. Trunk, B.H. Cantrell, and W.B. Gordon, "Bounds on Elevation Error Estimates of a Target in Multipath," IEEE Trans. Aerosp. Electron. Syst. **AES-15**, 883-887 (Nov. 1979).

4. J.E. Howard, "A Low Angle Tracking System for Fire Control Radars," *IEEE 1975 Internatonal Radar Conference Record,* Arlington, Va., 1975, pp. 412-417.

5. P.Z. Peebles, Jr., and R.S. Berkowitz, "Multiple-Target Monopulse Radar Processing Techniques," IEEE Trans. Aerosp. Electron. Syst. **AES-4**, 845-854 (Nov. 1968).

6. B.H. Cantrell, W.B. Gordon, and G.V. Trunk, "Maximum Likelihood Elevation Angle Estimates of Radar Targets Using Subapertures," IEEE Trans. Aerospace Electron. Syst. **AES-17**, 213-221 (Mar. 1981).

7. A.A. Ksienski and R.B. McGhee, "A Decision Theoretic Approach to the Angular Resolution and Parameter Estimation Problem for Multiple Targets," IEEE Trans. Aerosp. Electron. Syst. **AES-4**, 443-455 (May 1968).

8. G.E. Pollon and G.W. Lank, "Angular Tracking of Two Closely Spaced Radar Targets," IEEE Trans. Aerosp. Electron. Syst. **AES-4**, 541-550 (July 1968).

9. D.K. Barton, "Comment on 'Angular Tracking of Two Closely Spaced Radar Targets,'" IEEE Trans. Aerosp. Electron. Syst. **AES-5**, 350-351 (Mar. 1969).

10. P.Z. Peebles, Jr., "Multipath Angle Error Reduction Using Multiple-Target Methods," IEEE Trans. Aerosp. Electron. Syst. **AES-7**, 1123-1130 (Nov. 1971).

11. D.D. Howard, J. Nessmith, and S.M. Sherman, "Monopulse Tracking Errors Due to Multipath: Cause and Remedies," *IEEE EASCON 71 Record*, 1971, pp. 175-182.

12. D.K. Barton, "Low-Angle Radar Tracking," Proc. IEEE **62**, 687-704 (June 1974).

13. A.J. Berni, "Angle-of-Arrival Estimation Using an Adaptive Antenna Array," IEEE Trans. Aerosp. Electron. Syst. **AES-11**, 278-284 (Mar. 1975).

14. S.S. Reddi, "Multiple Source Location—A Digital Approach," IEEE Trans. Aerosp. Electron. Syst. **AES-15**, 95-105 (Jan. 1979).

15. G.V. Trunk, B.H. Cantrell, and W.B. Gordon, "Probability Density of the Maximum Likelihood Elevation Estimate of Radar Targets," IEEE Trans. Aerosp. Electron. Syst. **AES-15**, 288-290 (Mar. 1979).

16. D.K. Barton, "Mulipath Fluctuation Effects in Track-While-Scan Radar," IEEE Trans. Aerosp. Electron. Syst. **AES-15**, 754-764 (Nov. 1979).

17. W.D. White, "Angular Spectra in Radar Applications," IEEE Trans. Aerosp. Electron. Syst. **AES-15**, 895-899 (Nov. 1979).

18. E.W. Hamlin, P.A. Seay, and W.E. Gordon, "A New Solution to the Problem of Vertical Angle of Arrival of Radio Waves," J. Appl. Phys. **20**, 248-251 (Mar. 1949).

19. S.W. Lang, "Near Optimal Frequency/Angle of Arrival Estimates Based on Maximum Entropy Spectral Techniques," *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, Apr. 1980, pp. 829-832.