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We consider the initial-value problem for two-dimensional density perturbations of finite amplitude in an incompressible, viscous, stratified fluid. We demonstrate that evaluation of the "drift," or net displacement, field can be reduced to a simple quadrature for a wide class of localized disturbances. Results of possible experimental use are obtained which allow certain characteristics of the initial density perturbation to be deduced from limited knowledge of the drift field and the shape of the region containing the initial density perturbation.

CONTENTS

DRIFT (NET DISPLACEMENT) PRODUCED BY DENSITY PERTURBATIONS IN STRATIFIED FLUIDS

INTRODUCTION

Darwin [1] has shown that interesting and important consequences result from investigation of the movement of material particles in motions of the classical inviscid fluid. (Truesdell and Toupin [2] give additional information on some of the problems considered by Darwin.) A main result he derives is the equality between the mass of fluid occupying the "drift-volume" and the hydrodynamic or virtual mass associated with the body's motion. In this context Darwin uses drift to designate the net displacement of a material particle and drift-volume as the volume of fluid enclosed between the initial and final locations of material surfaces which originally are perpendicular to the body's motion when the body is at infinite distance. The utility of the concept of drift was further demonstrated when Lighthill [3,4] employed the concept in a determination of secondary flows induced by weak shear flows past rigid bodies.

In the next section we reduce the evaluation of the drift field to simple quadratures for a class of two-dimensional initial-value problems involving the "collapse" of a localized density perturbation of finite amplitude in an incompressible, viscous, stratified fluid. (We can apply our analysis to the inviscid case if we assume that the initially localized disturbance propagates to infinity, leaving the finite portions of the fluid in static equilibrium.) Our reduction is made possible by the severe constraints imposed by gravity on the static-equilibrium configurations of a stratified fluid and the condition of incompressibility. We note that Darwin and Lighthill find in some cases the particle displacements as a function of time, whereas our method merely yields the net displacements of particles from their initial positions. Motivated by the connection [1] between drift and hydrodynamic mass for homogeneous fluids, we obtain relationships between the drift field and the initial density perturbation, the "degree of homogeneity" of this perturbation. and the initial potential energy of the disturbance.

The initial-value problem we consider has application to interesting phenomena which arise in nature. Most prominent perhaps is the collapse of the turbulent, mixed wake created by the motion of a self-propelled body through stratified fluid [5,6]. Similar phenomena are the collapse of mixed regions produced in the ocean by instabilities of internal waves [7] and, as Mei [8] pointed out, the eventual flattening of a buoyant plume in the atmosphere. Analytical attempts [8-10] to describe the collapse process use two-dimensional initial-value problems of the type we consider but involve restrictions on the size of the density perturbation. More relevant are the numerical algorithms developed by Wessel [11], Padmanabhan et al. [12], and Young and Hirt [13] for the solution of problems of precisely the type we consider. Our results may offer a means for partially testing numercial schemes of this sort.

We introduce in the next section a quantity called the degree of homogeneity to describe the initial density perturbation. This quantity plays a central role in the dynamics of the collapse process; yet our present understanding of density perturbations arising as the end result of a burst of turbulent mixing in a stratified fluid does not permit its calculation from basic principles. We demonstrate the possible importance of the drift field as an experimental tool by showing in the third section that limited information on the drift field, easily obtained experimentally, provides an estimate of the degree of homogeneity.

Our methods are easily extended to more general situations. For example there is an axisymmetric analog to the problem we consider. Also, our results apply to fluids with a constitutive relation much more general than that of Navier and Stokes. We feel however that to seek utmost generality would detract from the presentation.

THE DIRECT PROBLEM

We consider the motion of an incompressible, viscous, stratified fluid of uniform depth satisfying the equations

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{d\rho}{dt} = 0, \tag{2}$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \mathbf{g} + \nabla \cdot \overleftrightarrow{\mathbf{T}} \tag{3}$$

for all times $t \geqslant 0$. We designate the fluid velocity by $\mathbf{u}$, the density by ρ, the pressure by p, the constant gravitational field by $\mathbf{g}$, and the viscous stress tensor by $\overleftrightarrow{\mathbf{T}}$. The time derivative in (2) and (3) is the material derivative. We suppose the fluid unbounded in the horizontal with a rigid flat bottom boundary. We shall regard the atmosphere above the fluid as also being a stratified viscous fluid; however no direct use of its governing equations will be necessary. Gravity then is the sole external force. We use coordinate frames x, y, z to specify the positions of particles at times $t > 0$ and X, Y, Z to specify their initial positions (at $t = 0$); we take the frames to coincide with y and Y directed vertically upward.

We assume that at $t = 0$ we have in the fluid an initial density of the form (Fig. 1)

$$\rho(X, Y, 0) = \rho_e(Y) + \delta\rho(X, Y) \tag{4}$$

and an initial velocity field $\mathbf{u}$ $(X, Y, 0)$ independent of Z. We further suppose the initial disturbance has the symmetries

$$\delta\rho(X, Y) = \delta\rho(-X, Y), \tag{5a}$$

$$\delta\rho(X, Y) = -\delta\rho(X, -Y), \tag{5b}$$

$$\mathbf{u}(X, Y, 0) = \mathbf{u}(-X, Y, 0) \tag{5c}$$

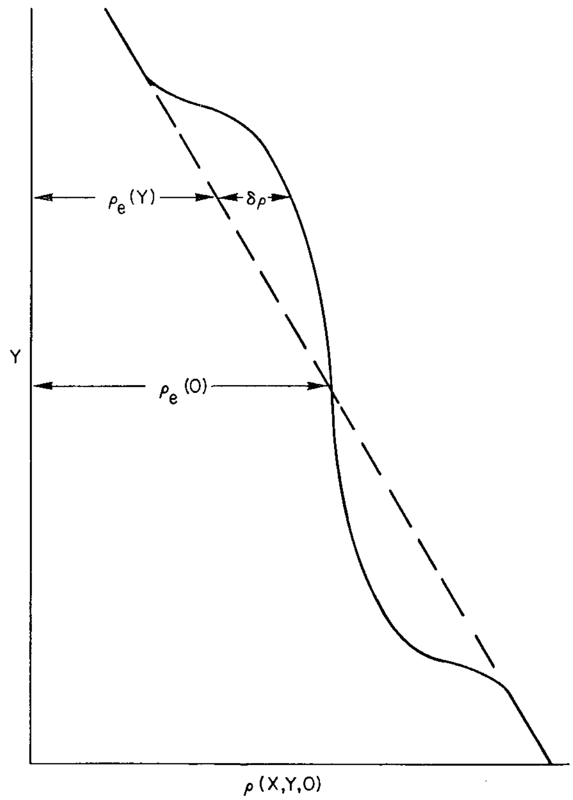


Fig. 1 — The initial density variation along a vertical and associated quantities. The initial density $\rho(X,Y,0)$ is the sum of an equilibrium stratification $\rho_e(Y)$ and a perturbation $\delta\rho$.

and that it vanishes as $|X| \to \infty$. Thus $\rho_e(Y)$ represents a static-equilibrium stratification prevailing at large horizontal distances from the localized disturbance. We can allow a localized disturbance in the atmosphere as well but require that it be symmetric in X.

As time progresses, gravity restores the deformed isopychnals to an appropriate level, and a complex motion, involving the generation and propagation of internal waves, ensues. The form of the initial disturbance indicates that the motion will be two dimensional, independent of Z, and symmetric with respect to the y axis. We therefore suppress any reference to z and examine the behavior in an xy plane. Particles initially at X, Y will move to new locations given by

$$x(X, Y, t) = X + \xi(X, Y, t), \tag{6a}$$

$$y(X, Y, t) = Y + \eta(X, Y, t), \tag{6b}$$

where ξ and η specify the displacement field of particles from their original positions. Ultimately, owing to the action of the viscous stresses, all motion will cease and particle locations will be given by

$$x^*(X,\ Y) = X + \xi^*(X,\ Y), \tag{7a}$$

$$y^*(X,\ Y) = Y + \eta^*(X,\ Y), \tag{7b}$$

where the asterisk denotes the limit as $t \to \infty$. We follow Darwin [1] and refer to the net displacement field ξ^*, η^* as the drift field.

We deal in this section with the direct problem of calculating the drift field from the initial data. Due to the symmetry of the motion, we have $\xi^*(0,\ Y) = 0$. Actually our analysis can be carried out for cases in which none of the symmetries of (5) apply. However we would then be unable to determine $\xi^*(0,\ Y)$ by our methods. It is to avoid such indeterminateness that we impose conditions (5a) and (5c). Condition (5b) is again for convenience. We believe the situation we consider adequately demonstrates the essentials of the method; moreover it seems a reasonable representation of many physical situations.

We shall assume that the initial density $\rho(X,\ Y,\ 0)$ increases with depth in the fluid along any vertical and thereby exclude for convenience the occurence of homogeneous layers. We introduce the degree of homogeneity $\beta(X,\ Y)$ to describe the initial density perturbation. It is defined as

$$\beta(X,\ Y) = \frac{\delta\rho(X,\ Y)}{\rho_e(0) - \rho_e(Y)}\ , \tag{8}$$

in terms of which the initial density may be expressed as

$$\rho(X,\ Y,\ 0) = \beta\ \rho_e(0) + (1 - \beta)\ \rho_e(Y).$$

One consequence of our restriction on $\rho(X,\ Y,\ 0)$ is that $\beta \leqslant 1$. We note that as $\beta \to 1$ in a region, the initial density $\rho(X,\ Y,\ 0)$ approaches a uniform distribution given by $\rho_e(0)$.

The Vertical Component of Drift

We now obtain a determining condition for the vertical component of drift. From (2) we conclude that following a particle we must have

$$\rho(X + \xi,\ Y + \eta,\ t) = \rho_e(Y) + \delta\rho(X,\ Y). \tag{9}$$

All motion subsides as $t \to \infty$, and the fluid assumes a static-equilibrium configuration in which the density ρ^* can depend only on vertical location $y^* = Y + \eta^*$. Taking the limit in (9), we get

$$\rho^*(Y + \eta^*) = \rho_e(Y) + \delta\rho(X,\ Y). \tag{10}$$

We also have that in a viscous fluid a localized disturbance is unable to affect the equilibrium state of the fluid at sufficiently large distances from the disturbance. Consequently

$$\lim_{|X|\to\infty} \eta^* = 0,$$

and a similar limit of (10) yields

$$\rho^*(Y) = \rho_e(Y), \tag{11}$$

which merely asserts that ρ_e is the density stratification ultimately achieved. Using (11) in (10), we obtain

$$\rho_e(Y + \eta^*) = \rho_e(Y) + \delta\rho(X,\ Y) \tag{12}$$

as the condition which yields the vertical drift component in terms of the initial data. We note that in regions where $\delta\rho$ vanishes, (12) indicates that η^* must vanish.

Equation (12) represents an implicit relationship for η^*. We assume for analytical convenience that ρ_e varies linearly over the vertical extent of the initial density perturbation $\delta\rho$. Although more general cases are amenable to analytical treatment, the case of a linear variation provides a good local approximation to many stratifications arising in practice. Equation (12) now yields

$$\eta^*(X,\ Y) = \left(\frac{d\rho_e}{dY}\right)^{-1} \delta\rho, \tag{13}$$

and (8) simplifies to

$$\eta^*(X,\ Y) = -\beta(X,\ Y)Y. \tag{14}$$

These relationships demonstrate the intimate relationship between the vertical drift, the initial density perturbation, and the degree of homogeneity.

An additional physical quantity of interest is the initial potential energy of the disturbance. In the absence of any initial kinetic energy the potential energy represents the sole source of energy for whatever motion occurs during $t \geqslant 0$. Calculation (Appendix A) of the initial potential energy density $\delta P(X,\ Y,\ 0)$ for a particle initially at X, Y yields

$$\delta P(X,\ Y,\ 0) = -\frac{1}{2}\, g \,\frac{d\rho_e}{dY}\, \eta^{*2}\ . \tag{15}$$

Thus knowledge of the vertical drift determines the initial potential energy.

The Horizontal Component of Drift

We now show how the horizontal drift ξ^* can be evaluated from η^*. Conservation of mass in the Lagrangian description and the constancy of the particle density lead [14] to

$$\frac{\partial(x, y)}{\partial(X, Y)} = 1$$

for the Jacobian of the transformation in (6). Passage to the limit $t \to \infty$ (and assuming the interchangeability of this limit with the process of differentiation) yields, when written out,

$$\left(1 + \frac{\partial \eta^*}{\partial Y}\right) \frac{\partial \xi^*}{\partial X} - \frac{\partial \eta^*}{\partial X} \frac{\partial \xi^*}{\partial Y} = - \frac{\partial \eta^*}{\partial Y} . \tag{16}$$

Since η^* is presumed known, (16) represents a linear partial differential equation of the first order for $\xi^*(X, Y)$.

We solve (16) for ξ^* by using the method of characteristics [15] to recast (16) as an initial-value problem involving ordinary differential equations. The equations for the characteristics are

$$\frac{d\xi^*}{d\sigma} = - \frac{\partial \eta^*}{\partial Y} , \tag{17a}$$

$$\frac{dX}{d\sigma} = 1 + \frac{\partial \eta^*}{\partial Y} , \tag{17b}$$

$$\frac{dY}{d\sigma} = - \frac{\partial \eta^*}{\partial X} , \tag{17c}$$

where σ is a parameter identifying points on a characteristic. From the symmetry of our original initial-value problem we have

$$\xi^*(0, Y) = 0 \tag{18}$$

as initial data for the solution of (17). We have therefore reduced the determination of ξ^* to simple quadrature.

A physical interpretation for the characteristics is easily deduced. The characteristics in the XY plane are found by solving

$$\frac{dY}{dX} = \frac{dY}{d\sigma} \bigg/ \left(\frac{dX}{d\sigma}\right) ,$$

which, using (17), we can write as

$$\frac{dY}{dX} = -\frac{\partial}{\partial X}(Y + \eta^*)\bigg/\frac{\partial}{\partial Y}(Y + \eta^*) = \left.\frac{dY}{dX}\right|_{Y+\eta^*}.$$

Thus the characteristics are those curves $Y(X)$ for which $Y + \eta^*$ has a fixed value. We know that $Y + \eta^* = y^*$ gives the final vertical location of a particle initially at X, Y. Since the ultimate state of the fluid is one of static equilibrium with isopychnals horizontal, the initial locus of particles having the same y^* must consist of material particles of the same density. We conclude that the characteristics are those curves specifying the initial forms of the isopychnals.

We have demonstrated in principle then an explicit method for the evaluation of the drift field from the initial data. One possible application of our results may be as a check on various numerical algorithms [11,13], devised for the solution of initial-value problems involving localized disturbances in a stratified fluid. Comparison of the drift fields obtained by the two techniques may indicate whether serious errors accumulate in the numerical algorithms.

Special Examples

To make more explicit the actual form of the drift field, we consider the special initial stratification

$$\rho(X, Y, 0) = \rho_e(Y) + \delta\rho(X, Y), \tag{19a}$$

$$\delta\rho(X, Y) = -\frac{d\rho_e}{dY} Y\,\beta(R), \tag{19b}$$

$$\beta(R) = \beta_0 \exp[-(R/R_0)^\gamma]. \tag{19c}$$

Here β_0, R_0, γ, and $-d\rho_e/dY$ are specified positive constants with $R = (X^2 + Y^2)^{1/2}$. The degree of homogeneity $\beta(R)$ seems, at least intuitively, a reasonable form for describing some density perturbations arising in practice [5]. We shall regard the free surface as being defined by one of the above isopychnals, with $Y \gg R_0$, and take the fluid to have infinite depth.

We find from (19) that the initial form $Y(X, \rho)$ of an isopychnal of density ρ, which is a characteristic curve in the XY plane, is given by

$$Y = \left(\frac{d\rho_e}{dy}\right)^{-1} \frac{\rho - \rho_e(0)}{1 - \beta} \tag{20}$$

and its level at large $|X|$ is given by

$$Y(\infty, \rho) \equiv \lim_{|X| \to \infty} Y(X, \rho) = \left(\frac{d\rho_e}{dY}\right)^{-1} [\rho - \rho_e(0)] . \tag{21}$$

Using (21) in (20), we can write (20) as

$$Y = \frac{Y(\infty, \rho)}{1 - \beta(R)} . \tag{22}$$

This isopychnal intersects the Y axis at $Y(0, \rho)$, which satisfies

$$Y(\infty, \rho) = Y(0, \rho)\left\{1 - \beta[Y(0, \rho)]\right\} . \tag{23}$$

These relations allow us to identify an isopychnal, or characteristic, by specifying either ρ, $Y(\infty, \rho)$, or $Y(0, \rho)$.

We get the vertical component of drift from (14) and (19) to be

$$\eta^* = - \beta(R)Y, \tag{24}$$

which, interestingly, depends linearly on the parameter β_0. The horizontal component of drift results from integration of (17) along the characteristic curves. We found the most convenient method for calculating $\xi^*(X, Y)$ to be as follows. We specify $Y(0, \rho)$ to identify a unique characteristic given by (22) and (23). We parameterize each point (X, Y) on the characteristic by its radial distance R. Then, using (24) in (17) and (18), we obtain

$$\xi^*(X, Y) = \int_0^{X(R)} \frac{\beta(R)\,(1 - \gamma R_0^{-\gamma}\, Y^2\, R^{\gamma-2})}{1 - \beta(R)\,(1 - \gamma R_0^{-\gamma}\, Y^2 R^{\gamma-2})}\, dX(R). \tag{25}$$

We see that the horizontal drift depends nonlinearly on all the parameters β_0, γ, and R_0.

Numerical calculations were performed for the case of a Gaussian degree of homogeneity with $\beta_0 = 0.5$ and $\gamma = 2$. Due to the symmetries involved we can confine ourselves to the first quadrant ($X \geqslant 0$, $Y \geqslant 0$). The results are exhibited in Fig. 2. Figure 2(a) shows the initial form of the isopychnals. It is clear that the density perturbation is confined to a region near the origin whose radial dimension is approximately $3R_0$.

Figure 2(b) portrays the drift field as lines connecting the original locations of particles with their final locations. The right portion of the figure shows the drift field for particles originally on the vertical line $X/R_0 = 3$. Final positions are below or at the same level as original positions. Particles at the top of the figure are displaced downward and to the left, and particles at the bottom of the figure are displaced downward and to the right. Elementary estimates indicate η^* effectively vanishes for $X/R_0 \gtrsim 3$. The resulting smallness of the integrand in (25) indicates that ξ^* effectively becomes independent of X for $X/R_0 \gtrsim 3$. The drift field for points to the right of $X/R_0 = 3$ can be obtained by moving the displayed drift field for $X/R_0 = 3$ to the right an appropriate distance.

A different view of the drift field is afforded by exhibiting the final locus achieved by material particles lying on some specified initial locus. Figure 2(c) shows the final loci of particles whose initial loci were circles. The radii of the circles ranged from $0.1R_0$ to $2R_0$ in steps of $0.1R_0$; these radii are associated with the displayed curves from bottom to top respectively.

An interesting special case of (19) arises when $\gamma \to \infty$. We then have

$$\beta(R) = \beta_0,\ R < R_0, \tag{26a}$$

$$= 0,\ R > R_0, \tag{26b}$$

$$\delta\rho(X,\ Y) = -\frac{d\rho_e}{dY}\ \beta_0\ Y,\ R < R_0, \tag{26c}$$

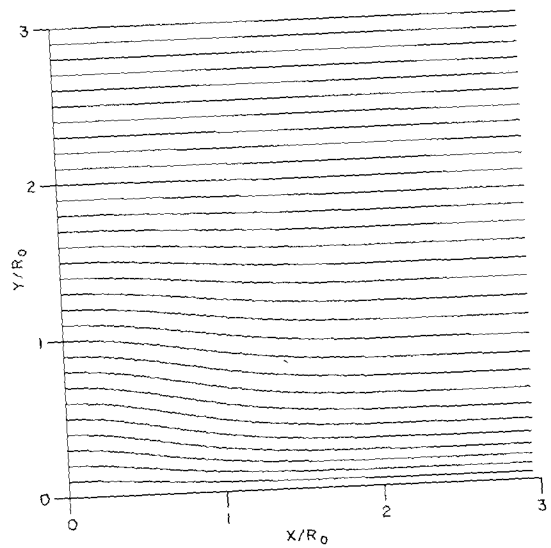
$$= 0,\ R > R_0, \tag{26d}$$

$$\rho(X,\ Y,\ 0) = \rho_e(0) + \frac{d\rho_e}{dY}\ (1 - \beta_0)Y,\ R < R_0, \tag{26e}$$

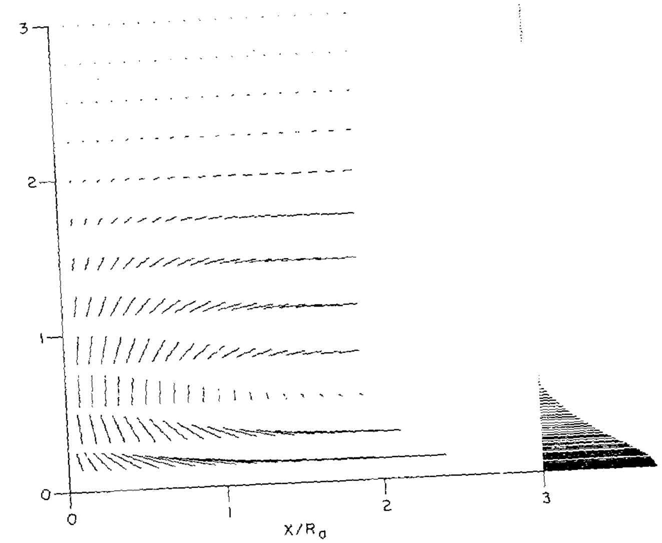
$$= \rho_e(0) + \frac{d\rho_e}{dY}\ Y,\ R > R_0, \tag{26f}$$

which gives a uniform degree of homogeneity β_0 within a circular region. We assume $0 \leqslant \beta_0 < 1$; results for $\beta_0 = 1$ can be obtained as a limiting case. Density perturbations of this type are nearly achievable in practice [16]. Moreover this type of perturbation has been used in linearized inviscid analyses of the collapse [9,10] and in numerical investigations [11-13]. (Reference 12 presents an approximation to the collapse in which motions outside the region containing the density perturbation are neglected. Bell and Dugan [17] question the correctness of the numerical results obtained.) We consider the fluid to have a finite depth, large enough to span the vertical extent of the perturbation.

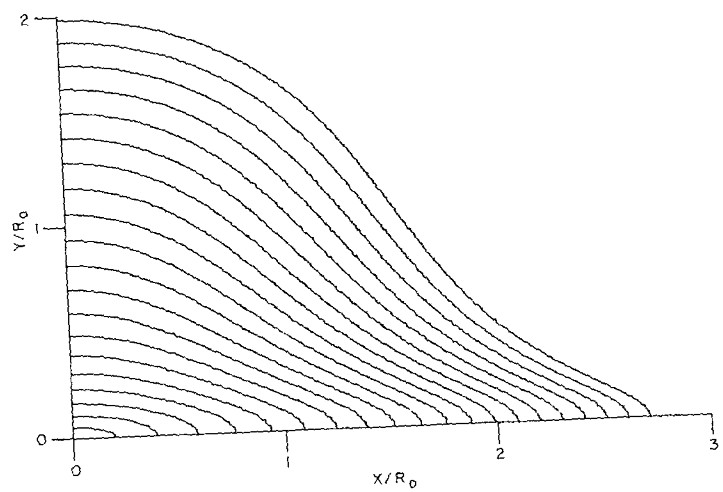
We conclude from (26) that the isopychnals initially are horizontal both inside and outside the circle $R = R_0$. An isopychnal of density ρ, which intersects the Y axis at $Y(0, \rho)$, has by (26) the constant level $Y(0, \rho)$ inside the circular region and has the constant level $Y(\infty, \rho) = (1 - \beta_0)\ Y(0, \rho)$ outside the region. The "limiting," or highest, isopychnal of density ρ_L inside the circle has $Y(0, \rho_L) = R_0$, and the level of this isopychnal outside the circle is $Y_L = (1 - \beta_0)\ R_0$. Those isopychnals outside the circle with $Y_L \leqslant Y(\infty, \rho) \leqslant R_0$ have no continuation into the interior of the circle, and isopychnals with $|Y(\infty, \rho)| > R_0$ are initially undeformed and at the constant level $Y = Y(\infty, \rho)$. The situation is shown in Fig. 3a.



(a) Initial forms of the isopychnals



(b) Drift field indicated by lines connecting initial positions of particles with final positions. The right portion shows drift for particles initially on the vertical line $X/R_0 = 3$.



(c) Final loci of material particles whose initial loci were circles with radii increasing from $0.1R_0$ to $2R_0$ in steps of $0.1R_0$.

Fig. 2 — Results for the drift field when $\beta(R)$ is Gaussian with $\beta_0 = 0.5$ and $\gamma = 2$

Equations (14) and (26) yield

$$\eta^* = -\beta_0 Y, R < R_0, \tag{27a}$$

$$= 0, R > R_0, \tag{27b}$$

and the use of (27) in (17) gives differential equations which are directly integrable for ξ^*. The singular nature of the present problem requires that we apply (17) to four distinct regions (of the first quadrant) which we designate as

$$\text{I:}\quad R < R_0,$$

$$\text{II:}\quad R > R_0,\ 0 \leqslant Y < Y_L,$$

$$\text{III:}\quad R > R_0,\ Y_L \leqslant Y < R_0,$$

$$\text{IV:}\quad Y \geqslant R_0,\ X \geqslant 0.$$

Furthermore we require that the isopychnals be continuous connected curves in the final state achieved by the fluid. We then find from (17), (18), and (27) that

$$\xi^* = \beta_0(1 - \beta_0)^{-1} X \text{ in I,} \tag{28a}$$

$$= (1 - \beta_0)^{-2} \left((1 - \beta_0)^2 R_0^2 - Y^2\right)^{1/2} - \left(R_0^2 - Y^2\right)^{1/2} \text{ in II,} \tag{28b}$$

$$= -(R_0^2 - Y^2)^{1/2} \text{ in III,} \tag{28c}$$

$$= 0 \text{ in IV.} \tag{28d}$$

The drift field for the case of $\beta_0 = 0.5$ is shown in Fig. 3b.

Consider now an initial circular locus of material particles described by

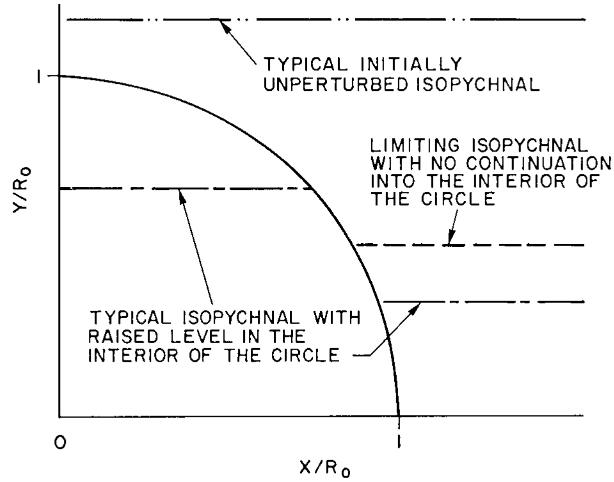
$$X^2 + Y^2 = r^2 \tag{29}$$

with $r < R_0$. The final location of a particle at X, Y on this locus is, from (27) and (28),

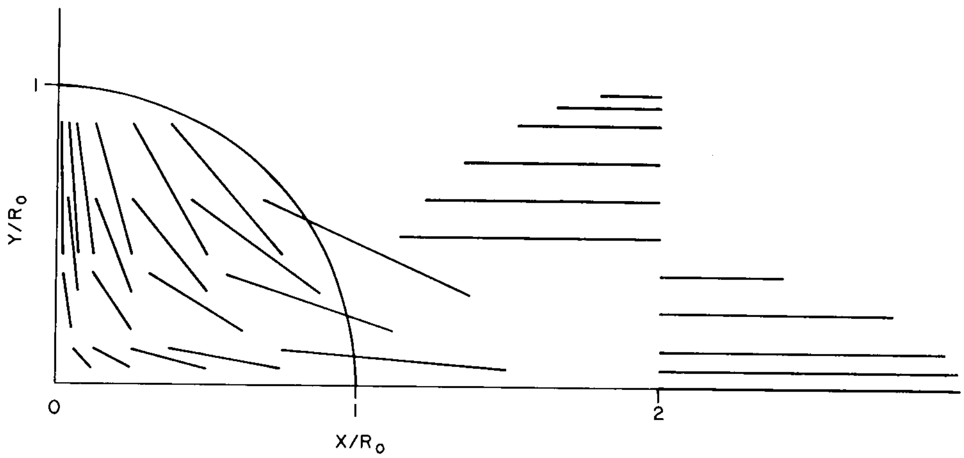
$$x^* = (1 - \beta_0)^{-1} X,$$

$$y^* = (1 - \beta_0) Y.$$

These equations allow us to express X, Y in terms of x^*, y^*; and (29) then can be written as



(a) Initial forms of the isopychnals



(b) Drift field indicated by lines connecting initial positions of particles with final positions

Fig. 3 — Results for the drift field for a constant degree of homogeneity $\beta_0 = 0.5$ inside the circle $R/R_0 = 1$. The circle shows the boundary of the initial density perturbation.

$$\left[\frac{(1-\beta_0)x^*}{r}\right]^2 + \left[\frac{y^*}{(1-\beta_0)r}\right]^2 = 1.$$

Thus a circle of material particles (with $r < R_0$) in the initial state becomes an ellipse in the final state with a vertical half-dimension of $(1-\beta_0)r$ and a half-width of $(1-\beta_0)^{-1}r$. These results extend some results of Hartman and Lewis [9] to finite-amplitude density perturbations and apply to the more general situation of a viscous fluid whose depth may be finite. It is interesting to note that Dugan, Warn-Varnas, and Piacsek [18] obtained the correct expression for the half-width using an approximate argument based on energetics.

THE INVERSE PROBLEM

In the inverse problem we desire to infer the nature of the initial localized density perturbation from such knowledge of the drift field as might be obtained through simple experimental procedure. For example, dye techniques may be used when the density perturbation arises from a brief period of turbulent mixing (as shown by the figures in Ref. 19). We believe our results for this inverse problem may have useful applications in the experimental study of density perturbations caused by the turbulent mixing of stratified fluid [1-13].

We point out that when a specific model, such as (19), for the density perturbation is assumed, the parameters in the model can easily be estimated. If the boundary of the initial density perturbation is known as well as the final configuration of this boundary, then a trial-and-error procedure using different values for the parameters can be employed to calculate the final configuration of the boundary until agreement is obtained with the measured configuration. We shall now give some results which do not require such specific knowledge of the analytical form of the initial density perturbation.

A General Procedure

We suppose the initial density perturbation is confined to a finite region and that for simplicity the density perturbation and the initial velocity field satisfy the symmetries of (5). The only restriction on $\rho_e(Y)$ is that it increase with depth. We also assume that by measurement we know the half-width $\ell(Y)$ of the boundary $\mathcal{C}$ (Fig. 4) outside of which the initial density perturbation vanishes and the final location $B'C'$ of particles initially on the vertical line BC outside $\mathcal{C}$.

Since $\delta\rho$ vanishes outside $\mathcal{C}$, we have by (12) that η^* vanishes outside $\mathcal{C}$ as well. From (16) we conclude that $\xi^* = \xi^*(Y)$ outside $\mathcal{C}$. Thus isopychnals outside $\mathcal{C}$ are horizontal in the initial and final states and at the same level. We consider a segment AB of an initial isopychnal at level Y outside $\mathcal{C}$. The initial form of this isopychnal we describe by $Y_i(X; Y)$. In the final state the segment AB will be located at $A'B'$. The area enclosed by the initial material curve $ABCD$ must, because of incompressibility, equal the area enclosed by $A'B'C'D$. Equivalently, the area enclosed by ABA' must equal the area enclosed by $BB'C'C$.

The area of ABA' is

$$\int_0^{\ell(Y)} [Y_i(X; Y) - Y]\; dX = -\int_0^{\ell(Y)} \eta^*(X, Y_i)\; dX,$$

where η^* denotes the vertical drift for a particle initially located on AB. This area must equal that of $BB'C'C$, which yields

$$-\int_0^{\ell(Y)} \eta^*(X,\ Y_i)\ dX = \int_0^{Y} \xi^*(Y)\ dY,$$

where ξ^* denotes the horizontal drift of particles initially on BC. We can write this result as

$$<\eta^*(X,\ Y_i)> = -\frac{1}{\ell(Y)}\int_0^{Y} \xi^*(Y)\ dY, \tag{30}$$

with $<\eta^*>$ being the average vertical drift for the isopychnal initially at level Y outside $\mathcal{C}$.

The right side of (30) is presumed known by measurement. We see that relatively simple measurements provide the average vertical drift and therefore provide the average initial height $Y - <\eta^*(X,\ Y_i)>$ within $\mathcal{C}$ for each isopychnal. This knowledge provides immediately an "equivalent," initial density perturbation which produces the same drift field outside $\mathcal{C}$ as does the actual density perturbation.

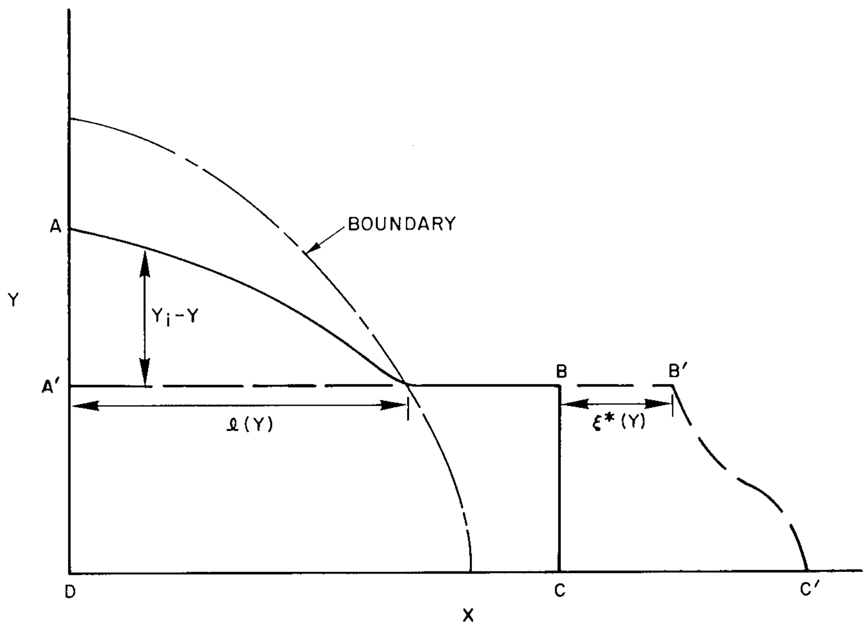


Fig. 4 — Initial and final configurations of a material curve. Boundary $\mathcal{C}$ is the boundary of the region outside of which the initial density perturbation vanishes. $ABCD$ gives the initial location of the material curve, and $A'B'C'D$ gives the final location.

By employing the same methods, but taking account of behavior in all four quadrants of the plane, we can determine $<\eta^*(X, Y_i)>$ for situations in which $\delta\rho(X, Y)$ does not possess any of the symmetries supposed here.

A Special Result

We now consider situations in which it may be supposed that the degree of homogeneity has the form $\beta = \beta(R)$. We further suppose that $\beta(R)$ decreases with R and vanishes as $R \to \infty$. We assume that the density $\rho_e(Y)$ varies linearly over the vertical extent of the initial density perturbation and that the initial velocity field satisfies the symmetry of (5). We see from (8) that $\delta\rho$ with $\beta = \beta(R)$ satisfies the symmetries of (5). Thus the results of the section on the direct problem are available to us.

We now suppose that we know from measurements that some initial circular locus of particles described by

$$X^2 + Y^2 = R_0^2$$

has the elliptical locus

$$\left(\frac{x^*}{a}\right)^2 + \left(\frac{y^*}{b}\right)^2 = 1 \tag{31}$$

in the final state achieved by the fluid. These two loci consist of the same material particles, and each locus must in view of incompressibility enclose the same area; consequently

$$ab = R_0^2 . \tag{32}$$

We know that $y^*(0, R_0) = R_0[1 - \beta(R_0)]$ and that (31) gives $y^*(0, R_0) = b$. Therefore

$$b = R_0[1 - \beta(R_0)], \tag{33a}$$

$$a = R_0/[1 - \beta(R_0)], \tag{33b}$$

where use has been made of (32).

We have $x^*(R_0, 0) = R_0 + \xi^*(R_0, 0)$; but (31) gives $x^*(R_0, 0) = a$, so that, using (33), we get

$$\xi^*(R_0, 0) = \frac{\beta(R_0)}{1 - \beta(R_0)} R_0 . \tag{34}$$

However, the characteristic equations in (17) applied to the characteristic $Y = 0$ yield an alternative expression for $\xi^*(R_0, 0)$, which, when equated with (34) provides

$$\int_0^{R_0} \frac{\beta(R)}{1 - \beta(R)}\, dR = \frac{\beta(R_0)}{1 - \beta(R_0)}\, R_0 \,. \tag{35}$$

The integrand in (35) decreases with R; thus (35) can be satisfied only if $\beta(R)$ has a constant value $\beta_0 = \beta(R_0)$ for $R \leqslant R_0$. Equation (33) allows β_0 to be written as

$$\beta_0 = 1 - (b/a)^{1/2}$$

in terms of parameters describing the final, elliptical locus. The initial density perturbation, by (13) and (14), takes the form

$$\delta\rho = -\beta_0 \frac{d\rho_e}{dY} Y$$

within the circle of radius R_0.

These results may enable experimentalists to easily make estimates of the degree of homogeneity and density perturbation produced by localized turbulent mixing in a stratified fluid.

CONCLUSION

Our results extend to stratified media the importance and utility of the drift field, first indicated by Darwin and further demonstrated by Lighthill in their studies of particle trajectories in a homogeneous, inviscid fluid. We have demonstrated that the drift, or net displacement, field can be calculated exactly for certain localized two-dimensional disturbances in an incompressible viscous stratified fluid. For analytical convenience we restricted the direct problem to a static-equilibrium density field which varies linearly over the vertical extent of the initial density perturbation. We showed the drift field in this case is directly related to the initial density perturbation, to its degree of homogeneity, and to the potential energy of the disturbance. Inverse results of possible use in experimental investigations were found whereby simple measurements of the drift field and shape of the region containing the initial density perturbation provide knowledge about the initial density perturbation.

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Appendix A

CALCULATION OF THE INITIAL POTENTIAL ENERGY OF THE DISTURBANCE

Consider the particle initially located at X, Y. The density of this particle is $\rho(X, Y, 0)$, and at time t the buoyancy force per unit volume (per unit area in the XY plane and per unit length normal to this plane) experienced by this particle is $[\rho(X, Y, 0) - \rho_e(Y + \eta)]\ \mathbf{g}$. The potential energy per unit volume $\delta P(X, Y, t)$ arising from the work done on the particle by this force satisfies

$$\frac{d}{dt}(\delta P) = -\ [\rho(X, Y, 0) - \rho_e(Y + \eta)]\ \mathbf{g} \cdot \mathbf{u} \tag{A1}$$

$$= [\rho(X, Y, 0) - \rho_e(Y + \eta)]\ g\ \frac{d\eta}{dt},$$

where $\frac{d\eta}{dt}$ gives the vertical velocity of the particle.

We can write (A1) as

$$\frac{d}{dt}(\delta P) = \frac{d}{dt}\left\{\int^{\eta} [\rho(X, Y, 0) - \rho_e(Y + \eta')]\ g\ d\eta'\right\} .$$

Integrating this equation from t to infinity, we obtain

$$\delta P(X, Y, t) = -\int_{\eta}^{\eta^*} [\rho(X, Y, 0) - \rho_e(Y + \eta')]\ g\ d\eta', \tag{A2}$$

where we have set $\delta P(X, Y, \infty) = 0$. Thus δP represents the increase in potential energy of a particle over its value in the final equilibrium configuration.

From (A2) and the fact that $\eta(X, Y, 0) = 0$ we get

$$\delta P(X, Y, 0) = -\int_{0}^{\eta^*} [\rho(X, Y, 0) - \rho_e(Y + \eta')]\ g\ d\eta' .$$

But, using (4) and (12), we can write this as

$$\delta P(X, Y, 0) = -\int_{0}^{\eta^*} [\rho_e(Y + \eta^*) - \rho_e(Y + \eta')]\ g\ d\eta' . \tag{A3}$$

Equation (A3) implies that knowledge of the equilibrium stratification and the vertical drift η^* determines the initial potential energy of the disturbance.

For the special case in which ρ_e varies linearly over the vertical extent of the initial disturbance we have

$$\rho_e(Y + \eta^*) - \rho_e(Y + \eta') = \frac{d\rho_e}{dY} \, (\eta^* - \eta') \, ,$$

which, when substituted in (A3), gives

$$\delta P(X,\ Y,\ 0) = -\frac{1}{2}\, g\left(\frac{d\rho_e}{dY}\right) \eta^{*2}. \tag{15}$$