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Further studies have been made on the effect of viscosity and on the spectral-shape dependence on the energy flux of a gravity-capillary spectrum. Viscous effects have been included as a perturbation of the inviscid results and also in a more exact manner, producing an effective decoupling of gravity and capillary waves, suggesting that the growth of gravity-capillary waves in the front face of the spectrum must be mostly dependent on the energy flux from the wind. However, at an early stage of development, the nonlinear resonant interactions may still play a significant contribution to the growth of the gravity-capillary wave spectrum.

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20. Abstract (cont.)

The investigation is concluded with a brief description on the effect of nonlinear resonant interactions and surface tension to the second-order contributions to the doppler spectrum of radio waves backscattered from a water-wave system.

CONTENTS

FURTHER RESULTS ON NONLINEAR ENERGY TRANSFER IN GRAVITY-CAPILLARY WAVE SPECTRA, WITH VISCOUS CONSIDERATIONS

INTRODUCTION

The evolution of the two-dimensional spectrum $F(\mathbf{x}, \mathbf{k}, \mathbf{t})$ of a water-wave system under the influence of a number of expansible processes (which are weak in the mean), can conveniently be described in phase-space $(\mathbf{x}, \mathbf{k})$ representation by the radiative-transfer equation [1, 2]

$$\frac{\partial F}{\partial t} + \dot{x}_i \frac{\partial F}{\partial x_i} + \dot{k}_i \frac{\partial F}{\partial k_i} = S_{in} + S_{n\ell} + S_{ds} , \qquad (1)$$

where $\mathbf{x}$ is the horizontal coordinate vector, $\mathbf{k}$ is the vector wavenumber, $\mathbf{t}$ is time, $\dot{x}_i = \partial\sigma(\mathbf{x}, \mathbf{k})/\partial k_i$, $\dot{k}_i = - \partial\sigma(\mathbf{x}, \mathbf{k})/\partial x_i$, and $\sigma(\mathbf{x}, \mathbf{k})$ is the angular frequency. S_{in}, $S_{n\ell}$ and S_{ds} are respectively the source functions representing the input from the atmosphere, the nonlinear energy transfer produced by wave-wave resonant interactions, and the losses from dissipative processes (for example, wave breaking and viscous damping).

Of great significance has been the finding by Hasselmann et al. [2] during the JONSWAP experiments of 1969 that for given wind conditions the development of the gravity spectrum and the attainment of an equilibrium state by the spectrum are crucially controlled by the self-stabilization features of the conservative nonlinear energy-transfer mechanism. For example, it was found there that for short fetches, about 80 percent of the maximal wave growth in the forward face of the spectrum could be attributed to nonlinear energy transfer.

On the other hand, gravity-capillary waves are known to be a great deal more sensitive to the wind than gravity waves are. Therefore, it should be important to learn to what degree the conservative nonlinear resonant interactions participate in the development of the gravity-capillary wave spectrum.

In this report we will not answer this question per se, since this would require an integration of the radiative-transfer equation with all the source functions. This time we will satisfy ourselves to investigate in more detail the dependence of the nonlinear energy transfer of a gravity-capillary wave spectrum on the shape and state of development of the spectrum.

Previously we have found that nonlinear energy transfer in a gravity-capillary wave spectrum provides a new mechanism for the transfer of energy of short gravity waves towards the capillary region, where ultimately it is dissipated by viscosity and wave breaking [3].

Note: Manuscript submitted May 30, 1975.

The present results suggest that wave growth in the forward face of the gravity-capillary wave spectrum may be directly related to the input from the wind, since the positive transfer of energy toward the small wavenumbers is negligible in most cases. However, in the early stages of development of the spectrum, just after it has been generated by a turbulent resonant mechanism [4] for example, it is possible that the nonlinear energy-transfer mechanism may still play a significant role in the development of the spectrum.

The effect of viscosity has been considered anew, and it is shown, with a treatment of forced vibrations in the presence of small dissipative forces, that in practice a decoupling of gravity waves and capillary waves will occur. This may be further proof that no positive transfer of energy can take place from capillary to gravity waves [5].

In any case, if viscosity is not introduced at all in the analysis for the energy transfer of gravity-capillary waves, quite unrealistic results do occur under certain conditions. For those conditions it is found that a line spectrum in the gravity-capillary region interacting with a background spectrum will actually tend to grow.

Combining the results on resonant interaction theory for gravity-capillary waves with perturbation electromagnetic scattering theory [6, 7], the effect of surface tension and resonant interactions on the second-order contributions to the doppler spectrum of radiowaves backscattered from a water-wave system is obtained.

NONLINEAR ENERGY TRANSFER DUE TO RESONANT WAVE-WAVE INTERACTIONS

Nonlinear energy transfer in a wave spectrum has been obtained for infinitesimal waves by Hasselmann [8, 9] and Valenzuela and Laing [3] for an ideal incompressible deep fluid and irrotational motion under constant atmospheric pressure. In these investigations the surface displacement ζ, the velocity potential ϕ, and the mean energy of the wave system E are expanded in perturbation series (the expansion parameter being the wave slope)

$$\left.\begin{aligned} \zeta &= {}_1\zeta + {}_2\zeta + {}_3\zeta + \ldots && \text{(1a)} \\ \phi &= {}_1\phi + {}_2\phi + {}_3\phi + \ldots && \text{(1b)} \\ E &= {}_2E + {}_4E + {}_6E + \ldots = {}_2E + \Delta E && \text{(1c)} \end{aligned}\right\}$$

for given initial conditions.

In the linear approximation (that is, ${}_1\zeta$, ${}_1\phi$, and ${}_2E$) the water surface is composed of a superposition of noninteracting sinusoidal wave components, and the statistics of the linear approximation to the surface displacement are homogeneous, stationary, and Gaussian. Thus the surface is completely determined by a two-dimensional wavenumber spectrum.

For gravity waves for which resonant interactions are of third order, nonstationary contributions are present in the term ${}_6E$ of the energy expansion. These interactions were investigated by Hasselmann [8, 9] by means of a fifth-order perturbation analysis, finding that for a gravity-wave spectrum the nonlinear energy transfer is a cubic function of the spectrum:

$$S^g_{n\ell} = \iiiint_{-\infty}^{\infty} T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)(\omega_4 F_1 F_2 F_3 + \omega_3 F_1 F_2 F_4$$

$$- \omega_2 F_1 F_3 F_4 - \omega_1 F_2 F_3 F_4)\delta(\omega_4 + \omega_3 - \omega_2 - \omega_1) d^2\mathbf{k}_1 d^2\mathbf{k}_2, \qquad (2)$$

where $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ and $\omega_1 + \omega_2 = \omega_3 + \omega_4$.

For gravity-capillary waves the resonant interactions occur at second order, and the nonstationary energy contributions now appear in the term ${}_4E$ [3]. The nonlinear source function for a gravity-capillary wave spectrum is given by

$$S^{g-c}_{n\ell} = \iint_{-\infty}^{\infty} T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)(\omega_3 F_1 F_2 - \omega_2 F_1 F_3 - \omega_1 F_2 F_3)\delta(\omega_3 - \omega_2 - \omega_1) d^2\mathbf{k}_1$$

$$+ 2 \iint_{-\infty}^{\infty} T(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2)(\omega_3 F_1 F_2 - \omega_2 F_1 F_3 + \omega_1 F_2 F_3)\delta(\omega_3 - \omega_2 + \omega_1) d^2\mathbf{k}_1, \qquad (3)$$

where $\mathbf{k}_3 = \mathbf{k}_2 \pm \mathbf{k}_1$ and $\omega_3 = \omega_2 \pm \omega_1$.

In Eqs. (2) and (3) we have used the notation $F_i = F(\mathbf{k}_i)$, $\omega_i = \omega(\mathbf{k}_i)$ and $\delta(\ )$ is the Dirac delta function; $T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ and $T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ are the respective transfer coefficients, which are given in detail in the original papers (except for some printing errors). For example the coupling coefficient $D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2}$ given in Ref. 3 is incorrect. The correct expression used in all the numerical calculations presented in the original expression and in this report is

$$D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = \frac{i}{2}\left\{(\omega_1 + \omega_2)(k_1 k_2 - \mathbf{k}_1 \cdot \mathbf{k}_2) + \omega_1\omega_2(\omega_1 + \omega_2)\left(\frac{k_1}{g + Tk_2^2} + \frac{k_2}{g + Tk_1^2}\right)\right.$$

$$\left. - (g + Tk^2)\left[\frac{\omega_2(k_1^2 + \mathbf{k}_1 \cdot \mathbf{k}_2)}{g + Tk_2^2} + \frac{\omega_1(k_2^2 + \mathbf{k}_1 \cdot \mathbf{k}_2)}{g + Tk_1^2}\right]\right\} . \qquad (4)$$

The main contribution to $S^g_{n\ell}$ comes from two pairs of nearly parallel gravity waves of almost identical wavenumbers. On the other hand, the main contribution to $S^{g-c}_{n\ell}$ comes from a triad of nearly parallel waves, but now the wave components may have quite different wavenumbers [10].

As usual the perturbation results for nonlinear energy transfer are valid only if ΔE, the energy contributed by higher order terms, is small compared with ${}_2E$, the mean energy of the linear approximation to the surface. An additional constraint is imposed on the interaction time of a line spectrum with a background wave spectrum: it must be larger than the corresponding wave period.

DEPENDENCE OF THE NONLINEAR ENERGY TRANSFER OF GRAVITY-CAPILLARY WAVES ON THE SPECTRAL SHAPE

For gravity waves the magnitude and position of the positive low-frequency lobe of the energy transfer is determined by the falloff rate of the spectrum toward small frequencies [2]. To investigate this effect on the nonlinear energy transfer for gravity-capillary waves, we have used various exponential factors on the radial part of the spectrum (for the two-dimensional spectrum we have used the separable form $F(\mathbf{k}) = S(k, \alpha) = S(k)S(\alpha)$). Accordingly the radial part of the spectrum has been expressed as

$$S(k) = 10^{-2}k^{-4} \exp\left\{-\left(\frac{C}{K}\right)^n\right\}, \tag{5}$$

where C is a nondimensional constant, $K = k/k_m$ is a nondimensional wavenumber, $k_m \approx 3.65$ cm^{-1} is the wavenumber of 1.7-cm waves, and n is an integer taken as 1, 2, or 4.

In Fig. 1 the positive small-wavenumber lobe of the energy transfer is shown as a function of the falloff rate of the spectrum toward small wavenumbers for a $\cos^2 \alpha$ angular spreading factor (normalized over a half plane) for a spectral peak at 0.375 k_m.

A noticeable shift of the positive small-wavenumber lobe of the energy transfer is observed as the falloff rate of the spectrum increases. However, the shift of the lobe toward the spectral peak with increasing falloff rate is in the opposite direction than for gravity waves [2]. The increase in amplitude of the lobe is mostly a reflection of the increase in amplitude of the spectral peak with falloff rate.

The dependence of the energy transfer on the power law of the spectrum in the equilibrium range is not too strong for small changes in the power law of the equilibrium range (Fig. 2).

However, if a more realistic falloff rate is adopted for the spectrum in the viscous range, say for wavenumbers greater than 10 cm^{-1}, a drastic reduction and disappearance of the positive small-wavenumber lobe occurs. This can be corroborated by adding a factor

$$\left[1 + \left(\frac{k}{k_\nu}\right)^4\right]^{-4/3} \tag{6}$$

to the radial part of the spectrum, Eq. (5). Surprisingly the factor (6) is that suggested by Heisenberg for the viscous subrange of a turbulent spectrum [11], which also seems to apply to the viscous part of the wave spectrum for light winds and short fetches $k_\nu = 15$ cm^{-1}, in accordance with unpublished spectral measurements with radar by

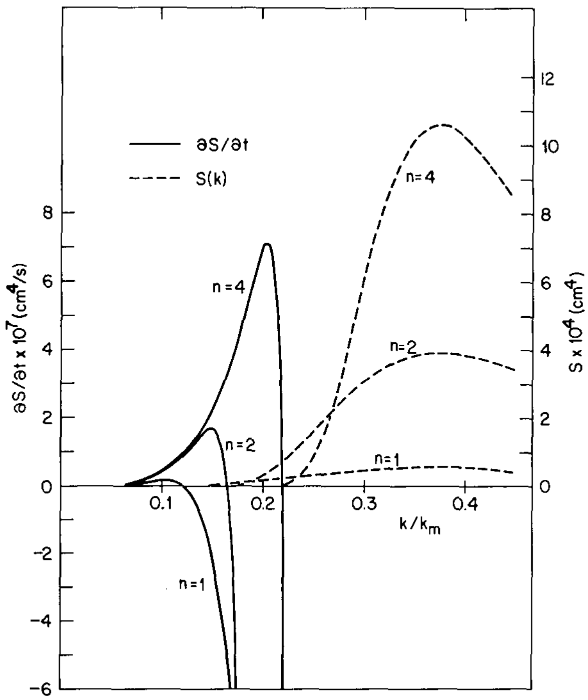

Fig. 1—Small-wavenumber positive lobe of the energy transfer for a gravity-capillary wave spectrum as a function of dropoff rate of the spectrum towards small wavenumbers. $S(k) = 0.01k^{-4} \exp\{-(C/K)^n\}$, $S(\alpha) = (2/\pi)\cos^2\alpha$ and $\alpha_3 = 0^0$. The spectral peak is at $0.375\ k_m$.

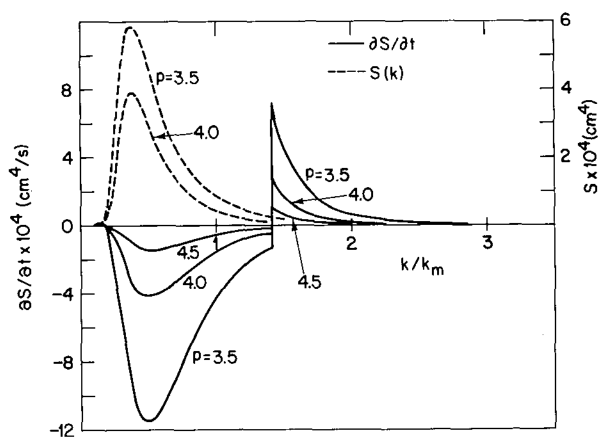

Fig. 2—Energy transfer for a gravity-capillary wave spectrum as a function of power law for the equilibrium range of the spectrum. $S(k) = 0.01k^{-p} \exp\{-(C/K)^2\}$, $S(\alpha) = 2/\pi \cos^2\alpha$ and $\alpha_3 = 0^0$. The spectral peak is at $0.375\ k_m$.

J. W. Wright. The apparent decoupling of capillary waves from gravity waves may be a support of Hasselmann's arguments [5] against a positive transfer of energy and momentum for gravity-capillary waves toward the gravity region. Later on in this report we will find that a treatment of forced vibrations in the presence of small dissipative forces will also confirm the decoupling between gravity and capillary waves.

From the previous results we can immediately conclude that the growth of waves in the forward face of a gravity-capillary wave spectrum, at least at this intermediate stage of development of the spectrum, cannot be due to the nonlinear source function and must be wind dependent.

However, when the spectrum is at an earlier stage of development, the nonlinear energy transfer may still play a significant role in the development of the gravity-capillary wave spectrum. For example, when the spectral peak is closer to k_m, the wavenumber of 1.7-cm waves, some dramatic changes take place in the shape of the energy transfer: now a positive lobe in the forward face of the spectrum is most evident, but in this case it cannot be reduced with viscous effects.

In Fig. 3, the energy flux is shown for a spectral peak at 0.92 k_m; now the energy is quite different from that previously obtained. The large positive lobe of the energy flux on the forward face of the spectrum suggests that nonlinearities may play a more direct role in the growth of the spectrum at this stage of development. However, in the final analysis the growth of the spectrum will depend on the net balance of the various participating mechanisms: S_{in}, the input from the wind, which in the linear approximation is given by Miles' instability mechanism [12]; $S_{n\ell}$, the nonlinear source function; and S_{ds}, the source function for the dissipative processes. In Fig. 3 a noticeable shift of the small-wavenumber lobe of the energy transfer occurs with the change in the falloff rate of the spectrum toward small wavenumbers.

In some cases we have found that the energy transfer becomes quite sensitive to the directional properties of the spectrum; this is illustrated in Fig. 4 for a spectral peak at 0.776 k_m. Very surprisingly, under these conditions the energy flux is totally positive along the wind direction for the $\cos^2 \alpha$ and $\cos^4 \alpha$ spreading factors.

In most calculations on the energy transfer we have found a significant amount of wave energy being scattered into angles greater than 90° (the crosswind direction), but for backscattering ($\alpha = 180°$) it vanishes. All this energy against the wind direction could explain the fact that in many instances gravity-capillary waves have been observed travelling upwind in wave tanks (private communication by W. C. Keller).

VISCOUS CONSIDERATIONS

It is quite evident that an exact viscid theory for the nonlinear energy transfer in a gravity-capillary wave spectrum is not possible. However, in the hierarchy of the various viscous approximations, to first order, viscosity may be introduced as a correction to the inviscid energy flux. We include a viscous source term to Eq. (1), so that

$$\frac{DF}{Dt} = S_{n\ell} - 4\nu k^2 F \tag{7}$$

where ν is the kinematic viscosity of the water.

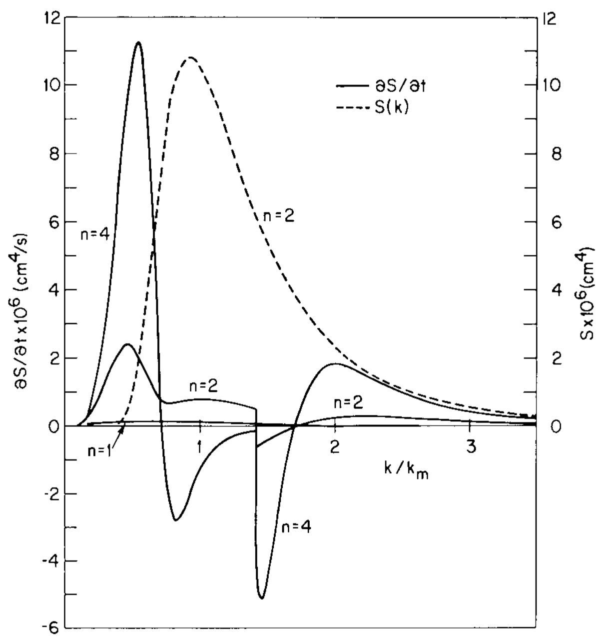

Fig. 3—Energy transfer for a gravity-capillary wave spectrum as a function of dropoff rate toward small wavenumbers of the spectrum. $S(k) = 0.01k^{-4} \exp\{-(C/K)^n\}$, $S(\alpha) = 2/\pi \cos^2 \alpha$ and $\alpha_3 = 0^0$. The spectral peak is at $0.92\ k_m$.

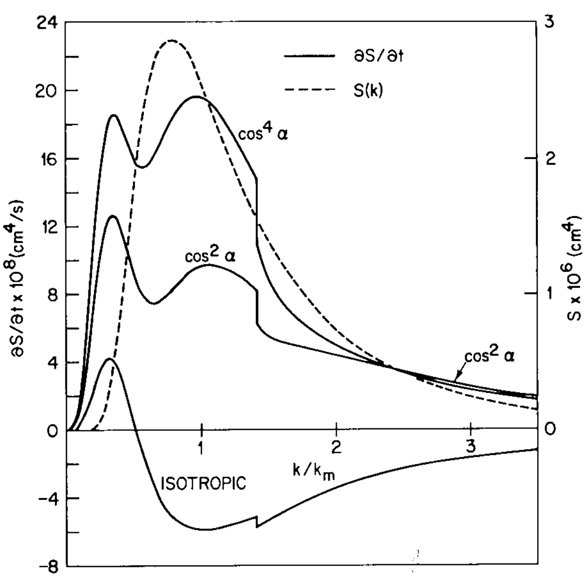

Fig. 4—Energy transfer for a gravity-capillary wave spectrum as a function of angular-spreading factor. $S(k) = 0.01k^{-4} \exp\{-3.104/K)\}$ and $\alpha_3 = 0^0$. The spectral peak is at $0.776\ k_m$.

This correction does not affect the actual dynamics of the resonant interactions; it merely gives a guideline for the validity of the inviscid results. That is, when the time constant of viscous dissipation is large compared to the wave period, the inviscid analysis applies.

In Fig. 5 we have included viscosity as a correction to the nonlinear energy transfer for the case when the spectral peak is at 0.375 k_m.

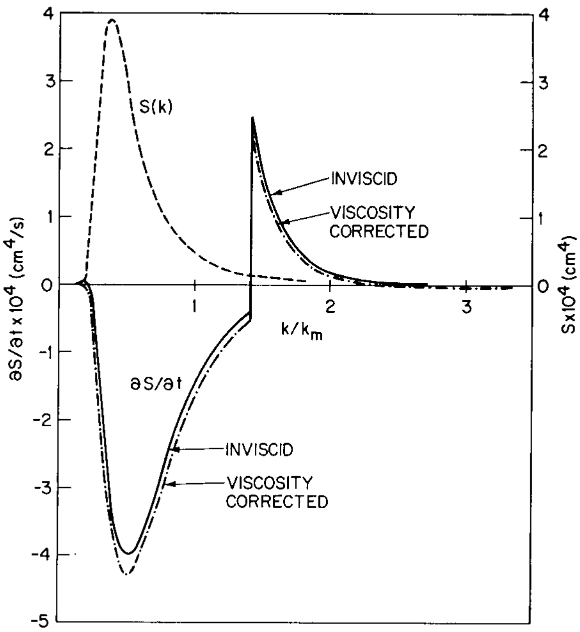

Fig. 5—Energy transfer for a gravity-capillary wave spectrum corrected for viscosity. $S(k) = 0.01k^{-4} \exp\{-(0.53/K)^2\}$, $S(\alpha) = 2/\pi \cos^2 \alpha$ and $\alpha_3 = 0^0$. The spectral peak is at 0.375 k_m.

In the previous section it was found that the shape of the energy transfer changed radically as the spectral peak approached k_m. Surprisingly enough, the interaction time between a line spectrum located at $\mathbf{k}_\ell$ and a background wave spectrum peaked at wavenumbers greater than $2^{-1/2}$ k_m is negative for certain ranges of $\mathbf{k}_\ell$ (indicating growth of the line spectrum); thus instabilities occur. This is demonstrated in Fig. 6. However, when viscosity is introduced, even as a perturbation, the instabilities cease. Thus this indicates that viscosity may be very important in the early stages of development of a gravity-capillary wave spectrum.

In Fig. 7, the decay (interaction) time is shown for a spectral peak at 0.92 k_m as a function of falloff rate of the spectrum toward small wavenumbers. It is illustrated that even for the most peaked spectra (n = 4), the instabilities cease when viscosity is included.

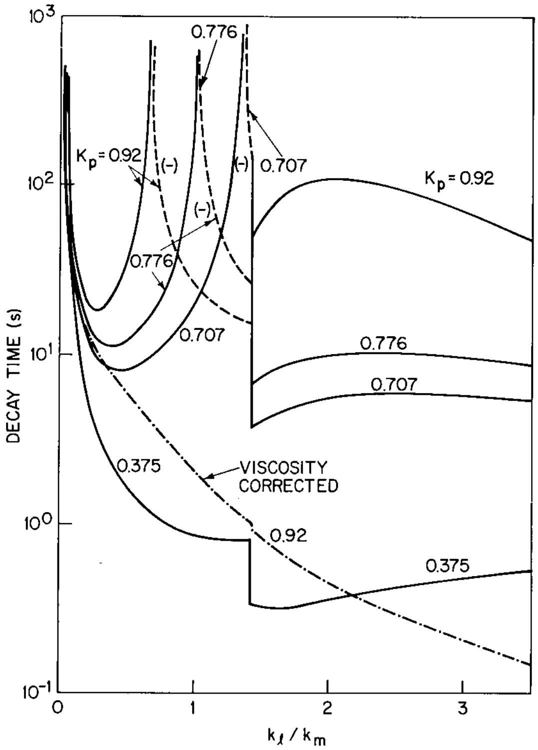

Fig. 6—Decay (interaction) time for a line spectrum at $\mathbf{k}_\ell$ with a background wave spectrum as a function of the position of the spectral peak of the spectrum. $S(k) = 0.01k^{-4} \exp\{-(C/K)\}$, $S(\alpha) = 2/\pi \cos^2 \alpha$ and $\alpha_\ell = 0^0$. The position of the spectral peak in nondimensional wavenumber is at $K_p = C/4$.

The effect of viscosity on the dynamics of the energy transfer can be obtained by a more formal treatment on forced vibrations in the presence of small dissipative forces, where we are not including boundary-layer effects. For example, solutions of the differential equation

$$\left(\frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} + b\right)\psi = h(t)e^{-ct}, \tag{8}$$

for given initial conditions are investigated (a, b, and c being constants). Using the weak-decay assumption (Appendix A), it is possible to show that the covariance of ψ is given by

$$\overline{\psi\,\psi^*} = \frac{\pi}{2(\omega')^2}\,\frac{(e^{-at} - e^{-2ct})}{(2c - a)}\,[H(\omega') + H(-\omega')] + \text{other terms} \tag{9}$$

with $\omega' = [b - (a^2/4)]^{1/2}$, $b = \omega^2$, and $H(\omega')$ is the spectrum of $h(t)$.

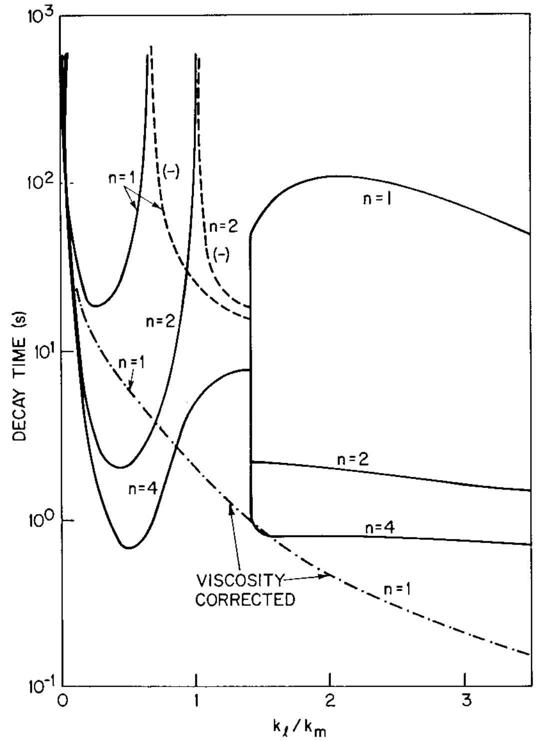

Fig. 7—Decay (interaction) time for a line spectrum at $\mathbf{k}_\ell$ with a background spectrum as a function of dropoff rate toward small wavenumbers of the spectrum. $S(k) = 0.01k^{-4} \exp\{-(C/K)^n\}$, $S(\alpha) = 2/\pi \cos^2 \alpha$ and $\alpha_\ell = 0^0$. The spectral peak is at $0.92\ k_m$.

For $at < 1$ and $2ct < 1$, Eq. (9) reduces to the lossless case. A second asymptotic formula needed in the perturbation analysis for the energy transfer calculation can be modified in a similar fashion.

An application of these results to the nonlinear energy transfer of a gravity-capillary wave spectrum (Appendix B) for infinitesimal times yields the inviscid results, Eq. (3).

For intermediate times, say $\nu k_3^2 t < 1$ and $\nu k_1^2 t > 1$ together with $k_1^2 \approx k_2^2$, a reduction and a change of sign of the energy transfer occurs. (This case arises for difference resonant interactions with $\mathbf{k}_3 = \mathbf{k}_2 - \mathbf{k}_1$ and $\omega_3 = \omega_2 - \omega_1$, $\mathbf{k}_3$ being in the gravity region and $\mathbf{k}_1$, $\mathbf{k}_2$ being in the capillary region.) The same conditions prevail in the formation of the positive low-frequency lobe which was discussed in regard to Fig. 1.

For large times the energy transfer vanishes, as it should if no other positive source function is included.

Hence, this analysis also verifies the results obtained earlier in regard to the decoupling of gravity waves and capillary waves when the wavenumber spectrum was allowed to have a faster falloff rate in the viscous range.

APPLICATION OF THE WAVE-WAVE RESONANT INTERACTION THEORY TO THE SECOND-ORDER CONTRIBUTIONS TO THE DOPPLER SPECTRUM OF RADIO WAVES BACKSCATTERED FROM A WIND-WAVE SYSTEM

The results on wave-wave resonant interactions for a gravity-capillary wave spectrum [3] can be combined with perturbation electromagnetic scattering theory [6, 7] for the development of a generalized theory for the second-order contributions to the doppler spectrum of radio waves backscattered from a wind-wave system [13]. The previous results by Hasselmann [14] for a constant transfer coefficient and by Barrick [15] in more exact numerical calculations for HF radio waves (frequencies in the 3-to-30-MHz range) are contained as a special case of ours.

In the generalized theory we include the effect of capillarity (surface tension), the radar polarization, the angle of incidence, and the lossy dielectric properties of the water surface. For HF radio frequencies our results are in basic agreement with those obtained by Barrick. However, for higher radio frequencies (microwaves), as the surface-tension effects become important in the properties of the water waves contributing to the electromagnetic backscattering, a second-order contribution appears at the Bragg frequency ω_B because of the wave-wave resonant interactions. (ω_B is the radian frequency of Bragg resonant water waves, which are of wavelength half the wavelength of the radio wave at grazing incidence.) Resonant interactions will also occur for gravity waves which are responsible for the backscattering of lower radio frequencies, but these will be present at higher order.

Surface tension also produces a shift of the secondary lines, present at $2^{1/2}\omega_B$ and $2^{3/4}\omega_B$ for HF radio waves, toward ω_B and $2^{1/2}\omega_B$ respectively. In addition, for crosswind conditions the doppler spectrum now does not vanish for frequencies above ω_B, as it is the case for HF radio waves.

The second-order contributions of the doppler spectrum at the Bragg frequency represent the nonstationary and non-Gaussian statistical properties of the wind-wave system. A detail analysis on these results has been published elsewhere [16].

CONCLUSIONS

The results obtained for the nonlinear energy transfer for a gravity-capillary wave spectrum are in agreement with experimental observations in regard to the sensitivity of gravity-capillary waves to the wind. However, when the gravity-capillary wave spectrum is at an earlier stage of development with the spectral peak near k_m (the wavenumber of 1.7-cm waves), the nonlinear resonant interactions may still play a significant role in the development of the spectrum.

The investigation has also shown that viscosity may play an important role in the overall energy balance of the gravity-capillary wave spectrum and in the dynamics of the nonlinear resonant interactions themselves, producing a net decoupling between gravity and capillary waves.

However, an application of nonlinear energy transfer for gravity-capillary waves to the overall energy balance of a gravity wave spectrum should await a consistent theory for the energy transfer of a spectrum also including the third-order gravity wave-wave resonant interactions [8] and the modification of short waves by long gravity waves treatable by the WKBJ approximation [5].

All the results for the energy transfer in a gravity-capillary wave spectrum have been obtained using the classical dispersion relation for gravity-capillary waves. However, the dispersion relation for wind waves is a great deal more complex, because of the wind-produced surface current which is highly sheared with depth [17].

In the investigation it was also described briefly that surface tension and resonant interactions should produce significant contributions to the doppler spectrum of radio and microwaves backscattered from a water wave system.

REFERENCES

1. K. Hasselmann, "Weak-Interaction Theory of Ocean Waves," in *Basic Developments in Fluid Dynamics* (M. Holt, editor), Vol. 2, Academic Press, New York, 1968, p. 117.
2. K. Hasselmann et al., "Measurements of Wind-Wave Growth and Swell Decay During the Joint North Sea Wave Project (JONSWAP)," D. Hydrogr. Inst., Suppl. A(8^0), No. 12 (1973).
3. G.R. Valenzuela and M.B. Laing, *J. Fluid Mech.*, **54**, 507 (1972).
4. O.M. Phillips, *J. Fluid Mech*, **2**, 417 (1957).
5. K. Hasselmann, *J. Fluid Mech.*, **50**, 189 (1971).
6. G.R. Valenzuela, *Trans. IEEE* **AP-15**, 552 (1967).
7. G.R. Valenzuela, *Radio Science* **3**, 1057 (1968).
8. K. Hasselmann, *J. Fluid Mech.* **12**, 481 (1962).
9. K. Hasselmann, *J. Fluid Mech.* **15**, 273 and 385 (1963).
10. L.F. McGoldrick, *J. Fluid Mech.* **21**, 305 (1965).
11. G.K. Batchelor, *The Theory of Homogeneous Turbulence*, Cambridge Univ. Press, Cambridge, England, 1953.
12. J.W. Miles, *J. Fluid Mech.* **13**, 433 (1962).
13. G.R. Valenzuela, "Second-Order Contributions to the Doppler Spectrum of Radar Sea-Echo," URSI meeting, Aug. 21 - 24, 1973, Boulder, Colorado.
14. K. Hasselmann, *Nature Physical Science* **229**, 16 (1971).

15. D.E. Barrick, *Remote Sensing of the Troposphere*, (V.E. Derr, editor) Chapter 12, U.S. Government Printing Office, Washington, D.C., 1972.

16. G.R. Valenzuela, *J. Geophys. Res.* 79, 5031 (1974).

17. J.W. Wright, and W.C. Keller, *Phys. Fluids* 14, 466 (1971).

APPENDIX A
GENERALIZED ASYMPTOTIC EXPANSIONS

Consider the solution of the linear second-order differential equation with small losses or weak damping ($a^2 \ll b$)

$$\left(\frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} + b\right)\psi = f(t)\ , \qquad \text{(A1)}$$

for $\psi = \partial\psi/\partial t = 0$ at $t = 0$ and $f(t)$ a given random driving function with a continuous spectral-density function.

The solution of Eq. (A1) for the prescribed initial conditions is

$$\psi = \frac{e^{\lambda_1 t}}{2i\omega'}\int_0^t f(\tau)e^{\lambda_2\tau}e^{a\tau}\,d\tau - \frac{e^{\lambda_2 t}}{2i\omega'}\int_0^t f(\tau)e^{\lambda_1\tau}e^{a\tau}\,d\tau\ , \qquad \text{(A2)}$$

where $\lambda_{1,2} = -\ a/2 \pm i[b - (a^2/4)]^{1/2}$ and $\omega' = [b - a^2/4]^{1/2}$.

With Eq. (A2) the covariance function of ψ can be derived:

$$\overline{\psi\psi^*} = \overline{\frac{e^{-at}}{4(\omega')^2}\int_0^t\int_0^t f(\tau)f^*(\tau')\,d\tau d\tau'}$$

$$\overline{\times\ e^{a(\tau+\tau')/2}\left(e^{i\omega'(t-\tau)} - e^{-i\omega'(t-\tau)}\right)\left(e^{-i\omega'(t-\tau')} - e^{i\omega'(t-\tau')}\right)}, \qquad \text{(A3)}$$

where ψ^* and f^* are the complex conjugates of ψ and f respectively and the bar over a quantity means the ensemble average.

Before proceeding to obtain the results, let $f(t) = h(t)e^{-ct}$, where c is a small quantity and $h(t)$ is a stationary random function with a continuous spectral density $H(\omega)$.

After substituting the explicit expression for $f(t)$ in Eq. (A3), transforming the double integrals into a single integral by the change of variables $\tau_1 = \tau - \tau'$ and $\tau_2 = \tau + \tau'$ and integrating over τ_2, exchanging the order of averaging and integration, using the weak-decay assumption, and performing some algebra, it is possible to show that the covariance function of ψ is given by

$$\overline{\psi\psi^*} = \frac{\pi}{4(\omega')^2}\frac{(e^{-at} - e^{-2ct})}{c - \frac{a}{2}}[H(\omega') + H(-\omega')] + \frac{(2c - a)e^{-2ct}\sigma_h^2(t)}{8(\omega')^2\left[(\omega')^2 + \left(c - \frac{a}{2}\right)^2\right]}$$

$$+ \frac{\sigma_h^2(t)e^{-at}}{8(\omega')^2}\left[\frac{e^{-2i\omega' t}}{i\omega' - \left(c - \frac{a}{2}\right)} - \frac{e^{2i\omega' t}}{i\omega' + \left(c - \frac{a}{2}\right)}\right], \tag{A4}$$

where $H(\omega) = \frac{1}{2\pi}\int_0^t \overline{h(\tau_1 + \tau')h^*(\tau')e^{-i\omega\tau_1}}\, d\tau_1$ and $\sigma_h^2(t) = \int_0^t \overline{h(\tau_1 + \tau')h^*(\tau')}\, d\tau_1$.

Thus for $a \to 0$, $c \to 0$, and a time averaging, Eq. (A4) reduces to the lossless case

$$\overline{\psi\psi^*} \sim \frac{\pi t}{2\omega^2}[H(\omega) + H(-\omega)], \tag{A5}$$

which is Eq. (3.9) of Ref. 8.

A second formula needed in the analysis, Eq. (3.14) of Ref. 8, has been modified in a similar manner using the weak-decay assumption. For example,

$$\mathcal{J}_2(\omega, -\omega', \omega'', \omega + \omega'; t) \approx \frac{e^{i\omega t}[e^{-(a'+a''')t} - e^{-at}]}{2i\omega(a - a' - a''')[(\omega'')^2 - (\omega + \omega')^2]} + \text{other terms} \tag{A6}$$

and

$$e^{-i\omega t}\mathcal{J}_2(\omega, -\omega', \omega'', \omega + \omega'; t) = \frac{-\pi[e^{-(a'+a''')t} - e^{-at}]}{4\omega(\omega + \omega')(a - a' - a''')}[\delta(\omega'' + \omega' + \omega)]$$

$$+ \delta(\omega'' - \omega' - \omega)] + \text{other terms}, \tag{A7}$$

where a, a', and a''' are the damping rates of the waves ω, ω', and ω''' respectively.

Of course, Eqs. (A4) and (A7) are approximations which apply for a weakly decaying process. A more exact analysis would actually result in the broadening of the delta functions, and resonances would disappear because of the viscous terms in the angular frequencies.

APPENDIX B
COVARIANCE PRODUCTS

To apply the analysis on forced vibrations with small dissipative forces (Appendix A) to the nonlinear energy transfer in gravity-capillary wave spectra, we use the following approximate representations for the first lowest order terms in the velocity potential (neglecting quadratic contributions of ν):

$$_1\phi_{\mathbf{k}} = {_1\Phi^s_{\mathbf{k}}} e^{-2\nu k^2 t} e^{-is\omega_{\mathbf{k}} t} e^{i\mathbf{k}\cdot\mathbf{x}}; \tag{B1}$$

$$_2\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ s_1,s_2}} \frac{D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} e^{-2\nu(k_1^2+k_2^2)t} e^{-i(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2})t} {_1\Phi^{s_1}_{\mathbf{k}_1}} {_1\Phi^{s_2}_{\mathbf{k}_2}}}{[\omega^2_{\mathbf{k}} - (s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2})^2 + 4i\nu(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2})(k_1^2+k_2^2-k^2) + O(\nu^2k^4)]}; \tag{B2}$$

$$_3\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}\\ s_1,s_2,s_3}} \frac{D^{s_1,s_2,s_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} e^{-2\nu(k_1^2+k_2^2+k_3^2)t} e^{-i(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2}+s_3\omega_{\mathbf{k}_3})t} {_1\Phi^{s_1}_{\mathbf{k}_1}} {_1\Phi^{s_2}_{\mathbf{k}_2}} {_1\Phi^{s_3}_{\mathbf{k}_3}}}{[\omega^2_{\mathbf{k}} - (s_1\omega_{\mathbf{k}}+s_2\omega_{\mathbf{k}_2}+s_3\omega_{\mathbf{k}_3})^2 + 4i\nu(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2}+s_3\omega_{\mathbf{k}_3})(k_1^2+k_2^2+k_3^2-k^2) + O(\nu^2k^4)]} \tag{B3}$$

where the notation follows Ref. 3.

Using the formulas derived in Appendix A, it can be shown that

$$\overline{|_2\phi^+_{\mathbf{k}}|^2} \approx \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ s_1,s_2}} \frac{\pi|D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2}|^2}{\omega^2_{\mathbf{k}}} \overline{|_1\Phi^{s_1}_{\mathbf{k}_1}|^2}\,\overline{|_1\Phi^{s_2}_{\mathbf{k}_2}|^2}$$

$$\times \frac{[e^{-4\nu k^2 t} - e^{-4\nu(k_1^2+k_2^2)t}]}{4\nu(k_1^2+k_2^2-k^2)} \delta(\omega_{\mathbf{k}}+s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2}) \tag{B4}$$

and

$$2\,\Re e\,\overline{({_1\phi^-_{-\mathbf{k}}}\,{_3\phi^+_{\mathbf{k}}})} = -\,2\pi \sum_{\mathbf{k}_1,s_1} \overline{|_1\Phi^+_{\mathbf{k}_1}|^2}\,\overline{|_1\Phi^{s_1}_{\mathbf{k}_1}|^2} \left\{ \frac{D^{-s_1,+}_{-\mathbf{k}_1,\mathbf{k}+\mathbf{k}_1} D^{+,s_1}_{\mathbf{k},\mathbf{k}_1}}{\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+s_1\omega_{\mathbf{k}_1})} \right.$$

$$\times\ \delta(\omega_{\mathbf{k}+\mathbf{k}_1} - \omega_{\mathbf{k}} - s_1\omega_{\mathbf{k}_1})$$

$$\left. + \frac{D^{-s_1,-}_{-\mathbf{k}_1,\mathbf{k}+\mathbf{k}_1} D^{+,s_1}_{\mathbf{k},\mathbf{k}_1}}{\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+s_1\omega_{\mathbf{k}_1})} \delta(\omega_{\mathbf{k}+\mathbf{k}_1} + \omega_{\mathbf{k}} + s_1\omega_{\mathbf{k}_1}) \right\}.$$

$$\frac{e^{-2\nu k^2 t}\,[e^{-2\nu(k_1^2+|\mathbf{k}_1+\mathbf{k}|)^2 t} - e^{-2\nu k^2 t}]}{2\nu(k^2 - k_1^2 - |\mathbf{k}+\mathbf{k}_1|^2)} . \tag{B5}$$

Obviously, now the energy transfer will be time dependent, and we must investigate its magnitude as a function of time.

For small times, $4\nu k^2 t < 1$, and $4\nu(k_1^2 + k_2^2)t < 1$, the inviscid results should follow. For intermediate time, such that $4\nu k^2 t < 1$ but $4\nu k_1^2 t > 1$ and $k_1^2 \approx k_2^2$, we find

$$\frac{\partial}{\partial t}(\overline{|_2\phi_{\mathbf{k}}^+|^2}) \propto -\frac{1}{2}\left(\frac{k}{k_1}\right)^2 \tag{B6a}$$

and

$$\frac{\partial}{\partial t}[2\,\mathcal{R}_e\,\overline{(_1\phi_{-\mathbf{k}}^-\,{}_3\phi_{\mathbf{k}}^+)}] \propto -\left(\frac{k}{k_1}\right)^2 . \tag{B6b}$$

Thus for intermediate times for relatively small $|\mathbf{k}|$ and large $|\mathbf{k}_1| \approx |\mathbf{k}_2|$ the energy transfer for gravity-capillary wave spectra should be reduced in magnitude and reversed in sign from the inviscid result. These conditions apply for difference interactions with $\mathbf{k}$ in the gravity region and $\mathbf{k}_1$, $\mathbf{k}_2$ in the capillary region, which indicates that in practice a decoupling of gravity and capillary waves will occur because of viscous effects. For very large times the energy transfer will vanish, as it should, unless a positive source function is introduced.