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Kineplex typically transmits an ensemble of about 20 pulsed tones, each of which is independently phase-shift keyed in quadrature by two binary bit streams. The Kineplex receiver employs matched filters using differentially coherent detection. Considerations of weight, cost, and size suggest the use of suboptimum but simpler devices employing quantized correlators (QC). Each tone requires a filter, especially to reduce interference from the unwanted tones. In a time-sampled variant of Kineplex detection, the interference from the unwanted tones can be completely eliminated in principle if the tone frequencies and the number of

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samples satisfy certain relations. Elimination of interference results from its coherence, which allows samples of the interfering signals to be summed properly so as to cancel. Although a QC injects quantization noise, reducing coherence, a 1-, 2-, 3-bit QC gave probability of error from interference of 0.33, 0.03, 3×10^{-5}, respectively, for the worst case.

CONTENTS

INTERFERENCE-REJECTION PROPERTIES
OF A QUANTIZED CORRELATOR APPLIED TO A
KINEPLEX TYPE OF DATA TRANSMISSSION SYSTEM

INTRODUCTION

Various data transmission systems, typified by Kineplex,*† transmit simultaneously an ensemble of, say, 20 pulsed tones of about 13.5 ms duration, each of which is independently phase-shift keyed in quadrature by two binary bit streams. The receiver employs matched filters, using differentially coherent detection.† However, considerations of size, weight, cost, and reliability suggest the use of suboptimum but simpler devices employing quantized correlators (QC). Each tone requires a filter not merely to reject Gaussian noise but especially to reduce interference from the unwanted tones. In a Kineplex system the interference from the unwanted tones can be completely eliminated in principle, provided the tone frequencies and the matched-filter integration times satisfy certain relations.

This complete elimination of interference is permitted by the coherence of the interference, which allows the interfering signals to be properly summed so that they cancel. In a time-sampled variant of the Kineplex system, we shall see that again the unwanted tones can, in principle, be completely eliminated, provided certain relations between the tone frequencies and the number of samples are satisfied. However, a QC that sums an r-bit approximation to a superposition of tones injects quantization noise which prevents the interference from totally canceling. The object of this investigation was to determine how the probability of bit error arising from the mutual interference of the pulsed tones varied with the number of quantization bits of the QC.

DEFINITION OF A KINEPLEX SYSTEM

In a Kineplex type of transmission system the receiver obtains a time sequence of baseband waveforms, thus,

$$y = A\sum_{j=1}^{J} \cos\,(\omega_j t+\gamma_j - \mu_j),\ \ 0 \leqslant t < T \tag{1}$$

$$y' = A\sum_{j=1}^{J} \cos[\omega_j(T-t)+ \gamma_j - \mu_j'],\ \ T \leqslant t \leqslant 2T \tag{2}$$

where y is received during the interval $0 \leqslant t \leqslant T$, y' during $T \leqslant t \leqslant 2T$, y'' during

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*E. T. Heald and R. G. Clabaugh, "A Predicted Wave-Signalling Phase Shift Telegraph System," Proc. IRE 45, 316-319 (July 1957).
†S. Stein and J. J. Jones, "Modern Communication Principles," McGraw-Hill, New York, 1967, p. 249.
Note: Manuscript Submitted April 10, 1975.

$2T \leqslant t \leqslant 3T$, etc. Each waveform comprises a superposition of tones of fixed frequencies $f_j = \omega_j/2\pi$. The phase of each tone consists of a random phase γ_j whose shift during an interval 2T is neglibible, plus a phase modulation $\mu_j = \pi/4+m_j\pi/2$, where $m_j = 0, 1, 2, 3$. The information is obtained from the difference in phase between two identical tones in successive frames, $\mu_j' - \mu_j$, which can take on the values 0, $\pi/2$, π, $3\pi/2$ from which the quaternary coded message may be deduced. The problem of how two independent binary data streams are encoded for differential coherent phase-shift-keyed detection so that the data streams may be inferred from the phase difference $\mu_j' - \mu_j$ is discussed at some length in Appendix A. However, the essential problem we address is how to infer $\mu_j' - \mu_j$ for $j = 1, \ldots$ J, despite the other $J-1$ interfering tones.

THE CLASSICAL MATCHED-FILTER KINEPLEX SYSTEM

To obtain the kth signal, the input (Eqs. (1) and (2)) is processed to determine

$$x_c = \frac{1}{T} \int_0^T y \cos \omega_k T \, dt \tag{3}$$

and

$$x_s = \frac{1}{T} \int_0^T y \sin \omega_k t \, dt, \tag{4}$$

with similar expressions for x_c' and x_s'. Let

$$\beta_j = \gamma_j - \mu_j \tag{5}$$

$$\Delta_{jk} = \omega_j - \omega_k$$

$$S_{jk} = \omega_j + \omega_k \; . \tag{6}$$

Substituting Eqs. (1), (2), (5), and (6) into Eqs. (3) and (4) results in

$$x_c = \frac{A}{T} \sum_{j=1}^{J} \left[\frac{\sin (S_{jk}T/2)}{S_{jk}} \cos (S_{jk}T/2+\beta_j) + \frac{\sin (\Delta_{jk}T/2)}{\Delta_{jk}} \cos (\Delta_{jk}T/2+\beta_j) \right] \tag{7}$$

and

$$x_s = \frac{A}{T} \sum_{j=1}^{J} \left[\frac{\sin (S_{jk}T/2)}{S_{jk}} \sin((S_{jk}T/2 + \beta_j) - \frac{\sin (\Delta_{jk}T/2)}{\Delta_{jk}} \sin (\Delta_{jk}T/2+\beta_j) \right] . \tag{8}$$

If the frequencies $f_j = \omega_j/2\pi$ are chosen so that

$$\Delta_{jk}T/2 = \pi(f_j - f_k)T = m_{jk}\pi \tag{9}$$

(m_{jk} an integer), for all j and k, and if $\Delta_{jk} \neq 0$, then every term in the second sum of Eqs. (7) and (8) will vanish except the term for which $\Delta_{jk} = 0$. Also, if the f_j are chosen so that

$$S_{jk} T/2 = \pi (f_j + f_k) T = n_{jk} \pi, \tag{10}$$

then every term in the first sum of Eqs. (7) and (8) will vanish. Accordingly, if Eqs. (9) and (10) are satisfied, then

$$x_c = \frac{A}{2} \cos \beta_k = \frac{A}{2} \cos(\gamma_k - \mu_k) \tag{11}$$

and

$$x_s = -\frac{A}{2} \sin\beta_k = -\frac{A}{2} \sin(\gamma_k - \mu_k). \tag{12}$$

From similarly processing y', one obtains

$$x_c' = \frac{A}{2} \cos(\gamma_k - \mu_k') \tag{13}$$

and

$$x_s' = -\frac{A}{2} \sin(\gamma_k - \mu_k') \tag{14}$$

In general, at the receiver one has Gaussian noise plus a residual amount of interference which does not cancel because of frequency shifts, phase fluctuations, etc. We will assume that the resultant interference behaves like Gaussian noise and that Eqs. (11) - (14) may be written more properly as

$$x_c = \frac{A}{2} \cos(\gamma_k - \mu_k) + n_1 \tag{15}$$

$$x_s = \frac{A}{2} \sin(\gamma_k - \mu_k) + n_2 \tag{16}$$

$$x_c' = \frac{A}{2} \cos(\gamma_k - \mu_k') + n_1' \tag{17}$$

$$x_s' = -\frac{A}{2} \sin(\gamma_k - \mu_k') + n_2' \tag{18}$$

where n_1, n_2, n_1', n_2' are assumed to be statistically independent Gaussian variables, each with zero mean and variance v. In Appendix B we show that if one is given the four observables x_c, x_s, x_c', and x_s', which are defined operationally by Eqs. (3) and (4) and have the structure defined by Eqs. (15) - (18), then the optimum decision procedure from which the value of $\mu_j'-\mu_j$ may be inferred consists in determining the maximum of the four functions F_a, F_b, F_c, and F_d. Thus, if

$$F_a = x_c x_c' + x_s x_s' \text{ is maximum, } \mu_j' - \mu_j = 0; \tag{19a}$$

$$F_b = x_c' x_s - x_c x_s' \text{ is maximum, } \mu_j' - \mu_j = 3\pi/2; \tag{19b}$$

$$F_c = -F_b \text{ is maximum, } \mu_j' - \mu_j = \pi/2 \ ; \tag{19c}$$

and

$$F_d = -F_a \text{ is maximum, } \mu_j' - \mu_j = \pi \ . \tag{19d}$$

TIME-SAMPLED KINEPLEX

Before considering a quantized, time-sampled version of a Kineplex transmission system, it is convenient if we consider first a time-sampled, unquantized version. We assume that synchronization already exists, so that the onset of the different waveform sequences is known. The input waveform of Eq. (1) is sampled and summed with alternating polarity every half period of the desired frequency f_s, determining a sinusoidal component of that frequency. The filtered input is thus

$$y_c = \frac{1}{N_s} \sum_{k=1}^{N_s} (-1)^{k-1} \, y[t+(k-1)P_s/2], \tag{20}$$

where $P_s = 1/f_s$ and N_s = number of samples. Introducing an additional time delay of $P_s/4$ and similarly sampling generates the quadrature component

$$y_s = \frac{1}{N_s} \sum_{k=1}^{N_s} (-1)^{k-1} \, y[t+(k-1)P_s/2+P_s/4] \ . \tag{21}$$

Defining

$$\varphi_j = \omega_j t + \gamma_j - \mu_j \tag{22}$$

and

$$\Delta_j = \pi f_j P_s/2 - \pi/2 = \pi(f_j - f_s)/2f_s \ , \tag{23}$$

substituting Eqs. (1), (22), and (23) into Eq. (20), inverting the order of summation, and summing over k yields

$$y_c = \frac{A}{N_s} \sum_{j=1}^{J} \cos[\varphi_j + (N_s - 1)\Delta_j] \ \sin N_s\Delta_j / \sin \Delta_j \ . \tag{24}$$

Similarly operating on y_s yields

$$y_s = \frac{-A}{N_s} \sum_{j=1}^{J} \sin\,(\varphi_j + N_s \Delta_j] \ \sin\ N_s\Delta_j / \sin\Delta_j \ . \tag{25}$$

To have the interference in Eq. (24) vanish requires that

$$N_s\Delta_j = p\pi \quad (p \text{ an integer}) \tag{26a}$$

and simultaneously that

$$\sin \Delta_j \neq 0. \tag{26b}$$

Since $\Delta_j = 0$ for $j{=}s$, conditions (26) reduce (24) and (25) to

$$y_c = A \cos \varphi_j = A \cos(\omega_j t + \gamma_j - \mu_j) \tag{27}$$

and

$$y_s = -A \sin\varphi_j = -A \sin(\omega_j t + \gamma_j - \mu_j) \ . \tag{28}$$

Similarly, the corresponding filtered in-phase and quadrature components for the succeeding waveform are determined; namely,

$$y_c' \doteq A \cos\varphi_j' \doteq A \cos\,(\omega_j t + \gamma_j - \mu_j') \ . \tag{29}$$

and

$$y_s' \doteq -\mathrm{A} \sin\varphi_s' \doteq -A \sin(\omega_j t + \gamma_j - \mu_j') \ . \tag{30}$$

Since γ_j and hence $\omega_j t + \gamma_j$ are randomly distributed angles, the above expressions for y_c, y_s, y_c' and y_s' are entirely equivalent to the classical matched-filter case of Eqs. (11) - (14), so that the same considerations regarding optimum decision functions apply. Thus the maximum-likelihood estimate consists in attributing that value of $\mu' - \mu$ which is associated with the maximum of the four functions enumerated below.

Case	Function	$(\mu'-\mu)$
a	$F_a = y_c y'_c + y_s y'_s$	0
b	$F_b = y'_c y_s - y_c y'_s$	$3\pi/2$
c	$F_c = -F_b$	$\pi/2$
d	$F_d = -F_a$	π

(31)

QUANTIZED KINEPLEX MODEL

We now consider a quantized time-sampled correlator in which, instead of taking precise samples with alternating polarity at every half period of the desired frequency, the samples are quantized to r bits of the sampled input. A schematic of a 2-bit quantized correlator is shown in Fig. 1. A straightforward analysis similar to that in the preceding section swiftly becomes unreasonably complicated, even for a 1-bit approximation to the sampled input, so that simulation is required.

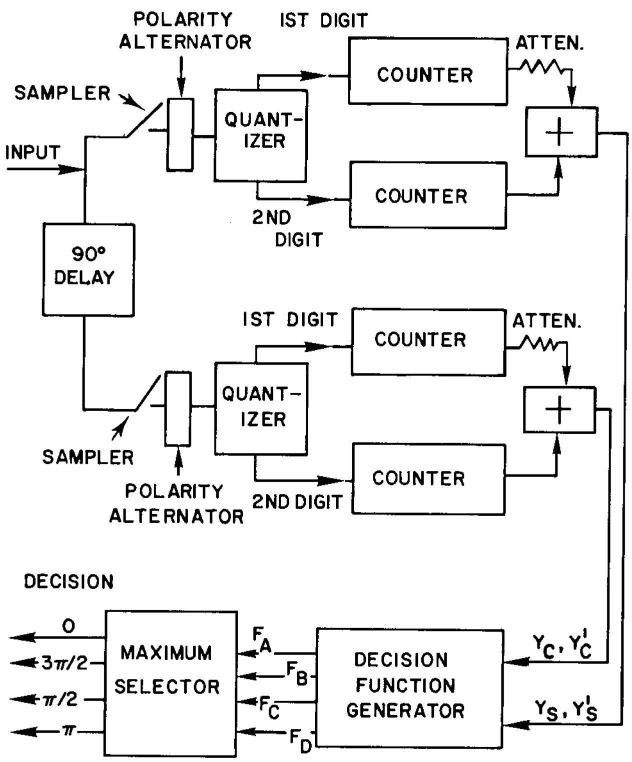

Fig. 1—Kineplex receiver using 2-bit quantized correlator

How to specify the quantizer was solved by Max* who studied the problem of minimizing the mean square error between the input and output of a quantizer for fixed numbers of output levels. In particular, for the case of an input signal with normally distributed amplitude, Max explicitly computed values of the optimum quantizer parameters as well as the mean square error, $\overline{e_r^2} = q_r \overline{y^2}$, to be expected from quantizing a Gaussian variable y to r bits using the optimum quantizer parameters. A Fortran program was written for the CDC 3800 that simulated the quantized, sampled Kineplex type of system of Fig. 1 using the frequencies and the number of samples shown in Table 1 and the optimum quantizer parameters given by Max. The characteristic of such a 2-bit quantizer and values of q_r, $r = 1, \ldots 5$, are shown in Fig. 2.

Table 1
Frequency and Number of Samples

Frequency	Number of Samples N_s	Frequency	Number of Samples N_s
605	11	1815	33
935	17	1925	35
1045	19	2035	37
1155	21	2145	39
1265	23	2255	41
1375	25	2365	43
1485	27	2475	45
1595	29	2915	55
1705	31		

The time-sampled Kineplex model differs from the continuous-time model in that the interference components of the filtered output of the former contain a factor

$$\sin\left[\pi N_s (f_j - f_s)/2f_s\right] / N_s \sin\left[\pi (f_j - f_s)/2f_s\right],$$

whereas the latter contains the factor

$$\sin[\pi(f_j - f_s)T/2] / T(f_j - f_s)/2 = \sin[\pi N_s(f_j - f_s)/2f_s] / N_s \pi (f_j - f_s)/2f_s$$

since $T = N_s/f_s$. It is this difference that requires eliminating frequency 605 (which was used in the original Kineplex system) from the set of frequencies appropriate to the time-sampled model, since, as is easily verified, the third harmonic of 605 coincides with frequency 1815, so that the full energy of the interfering term (1815) equals the signal (605) energy in the time-sampled case. In other words, the allowable frequencies are somewhat more restricted for the time-sampled model than for the continuous model because of the periodic nature of the denominator in the former case.

*J. Max, "Quantizing for Minimum Distortion," IRE Trans. IT-8, 7-12 (Mar. 1960).

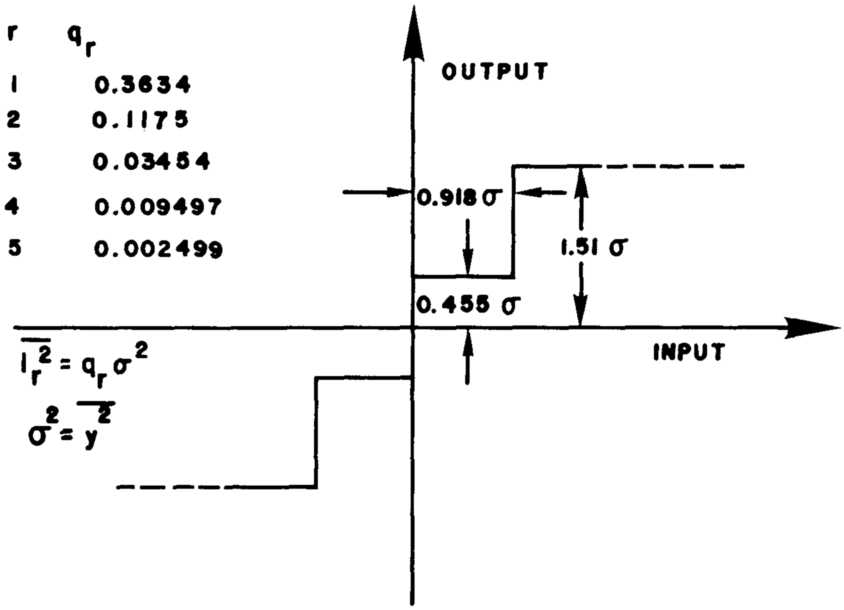

Fig. 2—Characteristic of symmetrical 2-bit optimum quantizer for Gaussian input (after Max)

Returning to Fig. 2, notice that all quantizing parameters are given in terms of $\sigma = (\overline{y^2})^{1/2}$ where y, the input, is given by Eq. (1). Thus, from Eq. (1),

$$\sigma = A\sqrt{J/2} \tag{32}$$

where J equals the number of tones which in our case (Table 1) is 16. For efficient use of the quantizer, it is essential that an auxiliary circuit vary the system gain so that $\overline{y^2}$, averaged over a frame interval T, is essentially constant.

SIMULATION DESCRIPTION AND RESULTS

For each time interval $[0 \leqslant t \leqslant T]$ and $[T \leqslant t \leqslant 2T]$, *the* μ_j were chosen so that $\mu_j = \pi/4 + m_j\pi/2$, with $m_j = 0, 1, 2, 3$, the m_j being randomly and uniformly distributed except for the signal tone, for which $m_s = 0$. The γ_j also were randomly chosen and then fixed for both time intervals. An rth-bit estimate of the input was sampled for $r = 1, \ldots 5$, call it $D_r[y(t+(k-1)P/2]$, and the samples were summed with alternating polarity to determine y_{cr} precisely as in Eq. (20). Similarly to Eq. (21), the y_{sr} were determined, and the values of y_{cr}, y'_{cr}, y_{sr}, y'_{sc} were substituted into Eq. (31). The maximum of the four decision functions was determined, which in turn decided the information state. The experiment was repeated 200 times for each tone tested and was carried out for three different values of σ to determine how critical its value was. The number of successful trials as a function of bit approximation is enumerated in Table 2 for three tones situated in the middle and at either end of the frequency band.; This table shows that the value of σ is not critical. In Table 3 the experiment was repeated with more trials, yielding essentially the same results.

Table 2
Number of Successes in 200 Trials

Tone Frequency, Number of Samples	Bit Approximation					
	σ	1	2	3	4	5
f = 935 N_s = 17	2.89 3.26 4.00	134 134 134	193 194 195	200 200 200	200 200 200	200 200 200
f = 1705 N_s = 31	2.89 3.26 4.00	147 154 154	199 198 200	200 200 200	200 200 200	200 200 200
f = 2475 N_s = 45	2.89 3.26 4.00	184 178 178	199 200 200	200 200 200	200 200 200	200 200 200

Table 3
Number of Successes in 1000 Trials,
with f = 935 and σ = 3.26

Bit Approximation				
1	2	3	4	5
678	961	1000	1000	1000

INDIRECT DETERMINATION OF ERROR PROBABILITY

Since the probability of error is so small for quantizers more accurate than two bits that a nonzero estimate could not be obtained by reasonable sample sizes, we decided to estimate how accurately one could estimate the probability of error from the measured mean and variance of the statistical decision functions, $F_{a,r}$ and $F_{b,r}$ (Eq. (31)). If the subscript r, is dropped the probability of inferring Case a is the probability that (a) $F_a > F_b$, (b) $F_a > -F_b$, and (c) $F_a > -F_a$ (see Eq. (31)). Since condition (c) is fulfilled if (a) and (b) are fulfilled, the probability of inferring Case a simply equals the probability that $F_a > |F_b|$. We assume that F_a and F_b are independent Gaussian variables. Experimentally (and also theoretically) the mean of $F_b = 0$, and the variances of F_a and F_b are essentially equal. If we designate the mean of F_a by a and its variance by s^2, then

$$P_a(x) = \frac{1}{\sqrt{2\pi}\, s} e^{-(x-a)^2/2s^2} \quad , \; -\infty < x \; \infty \tag{33}$$

and

$$P_b(x) = \frac{1}{\sqrt{2\pi}\, s} e^{-x^2/2s^2} \qquad -\infty < x < \infty \tag{34}$$

are the assumed probability densities of F_a and F_b, and

$$P_c(x) = \frac{2}{\sqrt{2\pi}\, s} e^{-x^2/2s^2}, \quad x > 0 \tag{35}$$

is the probability density of $|F_b|$. Then if P designates the probability of success (Case a in the experiments), it equals the probability that $F_a - |F_b| > 0$. Or,

$$P = \int_{z=o}^{\infty} \int_{x=z}^{\infty} p_a(x) p_c(x-z) dx\, dz\,. \tag{36}$$

Carrying out the integration (see Appendix C) yields

$$P = [1+2\, erf(e) + erf^2(e)]/4 \tag{37}$$

where $e = a/2s$. Experimentally determined values of e were used to compute the "theoretical" error probability ($1-P$), given in Table 4, along with the experimental probability. Agreement between the theoretical and experimental error probabilities for r=1 and 2 indicate that the theoretical prediction for r=3 is fairly good. Because of the extremely small error probabilities for r=4 and 5, these are omitted from Table 4.

Table 4
Theoretical and Experimental Probability of Error for 1-, 2-, and 3-bit Sampling Approximations

Number of Samples and Tone Frequency	r (bit approximation)	$e = a/2s$	Theoretical Probability	Experimental Probability
f = 935	1	0.634	0.335	0.33
N_s = 17	2	1.24	0.077	0.03
	3	2.58	3×10^{-5}	0
f = 1705	1	0.767	0.254	0.23
N_s = 31	2	1.71	0.045	0.01
	3	3.53	5×10^{-6}	0
f = 2475	1	1.11	0.113	0.11
N_s = 45	2	2.06	0.0046	0
	3	4.33	$<5 \times 10^{-6}$	0

SUMMARY AND CONCLUSION

A time-sampled Kineplex data communications system employing quantized correlators rather than analog matched filters was analyzed. Simulation studies indicate that the probability of error arising from the mutual interference in such a system can be made negligible.

Appendix A

QUATERNARY DIFFERENTIAL PHASE-SHIFT CODING

We consider two independent binary data streams, a_j, b_j, j=1, . . ., J, which are encoded for the differentially coherent detection of a single differentially encoded quaternary phase-shift-keyed (PSK) carrier.

Let φ_j and ψ_j be the differential encoding of a_j and b_j.

Let

$$y = A\ [\cos(\omega t + \varphi + \gamma) + \sin(\omega t + \psi + \gamma)],\ \ 0 < t < T$$

designate a quaternary PSK carrier pulse, where ψ and φ can take on the values 0 or π, and γ is an arbitrary constant.

Since $\sin\varphi = \sin\psi = 0$,

$$y = A\ [\cos(\omega t + \gamma)\cos\varphi + \sin(\omega t + \gamma)\cos\psi]. \tag{A1}$$

If $\cos\alpha = \dfrac{\cos\varphi}{\sqrt{2}}$ and $\sin\alpha = \dfrac{\cos\psi}{\sqrt{2}}$, Eq. (1) becomes

$$y = \sqrt{2}\ A\cos(\omega t + \gamma - \alpha). \tag{A1a}$$

Figure A1 correlates the pair (φ, ψ) and underneath ($\cos\varphi$, $\cos\psi$) with the corresponding angle α. Let one also be given the succeeding waveform

$$y' = \sqrt{2}A\cos(\omega t + \gamma - \alpha'), \tag{A2}$$

where α' is the value of α occurring in the emission immediately following y, say in the interval ($T \leqslant t \leqslant 2T$) where the new values of φ, ψ are φ', ψ'. Here we refer the functional form of Eq. (A2) to the preceding interval, $0 < t < T$.

We multiply the input y (Eq. (1a)) by $\cos\omega t$ and integrate to obtain

$$x_1 = \frac{1}{T}\int_0^T y\cos\omega t\,dt = \frac{A}{\sqrt{2}}\cos(\gamma - \alpha), \tag{A3}$$

where we have neglected the double frequency. Similarly,

$$x_2 = \frac{1}{T}\int_0 y\sin\omega t\,dt = -\frac{A}{\sqrt{2}}\sin(\gamma - \alpha). \tag{A4}$$

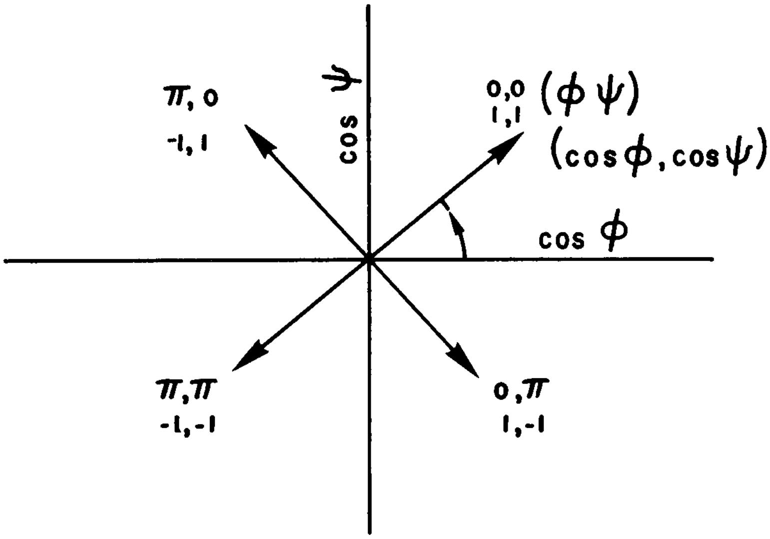

Figure A1—Graphical representation of ϕ, ψ or cos ϕ, cos ψ as a function of $\alpha = \tan^{-1}(\cos\psi/\cos\phi)$

Also, from Eq. (2),

$$x_1' = \frac{1}{T}\int_0^T y' \cos\omega t \, dt = \frac{A}{\sqrt{2}} \cos(\gamma - \alpha') \tag{A5}$$

and

$$x_2' = \frac{1}{T}\int_0^T y' \sin\omega t \, dt = -\frac{A}{\sqrt{2}} \sin(\gamma - \alpha'). \tag{A6}$$

To eliminate the random variable γ from the observables x_1, x_2, x_1', and x_2', we form

$$z_1 = x_1 x_1' + x_2 x_2' = \frac{A^2}{2} \cos(\alpha' - \alpha) \tag{A7}$$

and

$$z_2 = x_1 x_2' - x_2 x_1' = \frac{A^2}{2} \sin(\alpha' - \alpha). \tag{A8}$$

To relate φ, ψ and φ', ψ', the corresponding values of φ, ψ in the succeeding waveform, to $(\alpha' - \alpha)$, one observes values of

$$z_1 = A^2[\cos(\alpha' - \alpha)] = \frac{A^2}{2}(\cos\varphi \cos\varphi' + \cos\psi \cos\psi') \tag{A9}$$

and

$$z_2 = A^2 \sin(\alpha'-\alpha) = \frac{A^2}{2} (\cos\psi' \cos\phi - \cos\varphi' \cos\psi) \quad \text{(A10)}$$

for all possible (φ, ψ) and (φ', ψ') values. This is shown in Table A1, where we have omitted the factor $A^2/2$ which is irrelevant.

Table A1
Parameters z_1, z_2 as a Function of (φ, ψ) and (φ^1, ψ^1)

	Values of φ^1, ψ^1			
$\varphi\ \psi$	0 0	0 π	π 0	$\pi\ \pi$
0 0	1,0	0,-1	0,+1	-1,0
0 π	0,+1	1,0	-1,0	0,-1
π 0	0,-1	-1,0	1,0	0,+1
$\pi\ \pi$	-1,0	0,+1	0,-1	1,0

In Table A2 we have rearranged Table A1 to indicate more clearly which 2-sequences of φ and ψ are equivalent with respect to the indicator pair $(z_1, z_2) = \cos(\alpha'-\alpha), \sin(\alpha'-\alpha)$

Table A2
Values of (φ, ψ) and (φ^1, ψ^1) Corresponding to a Unique Value of (z_1, z_2)

Case	z_1, z_2	Quaternary Coding	Info. Code	$\alpha' - \alpha$
a	1, 0	0,0 0,0 π,π π,π 0,0 π,π 0,0 π,π	0=a 0=b	0
b	0,-1	0,0 0,π π,0 π,π 0,π π,π 0,0 π,0	0=a 1=b	$-\pi/2$
c	0,+1	0,π 0,0 π,π π,0 0,0 π,0 0,π π,π	1=a 0=b	$+\pi/2$
d	-1, 0	0,π 0,π π,0 π,0 0,π π,0 0,π π,0	1=a 1=b	$\pm\pi$

Thus, the four possible encodings indicated on the first line are equivalent to the observation $z_1 = 1, z_2 = 0$. These situations are those for which there is no change both in φ and in ψ. Similarly, $z_1 = 1, z_2 = 0$ is the indication corresponding to the case in which there is a change in both φ and ψ. On the other hand, it is not possible to differentiate from the case in which there is a change in φ and no change in ψ from the case in which there is no change in φ and a change in ψ, since the observable (z_1, z_2) does not define these cases uniquely. It is instructive to graph the set of equivalent 2-sequence pairs, as in Fig. A2.

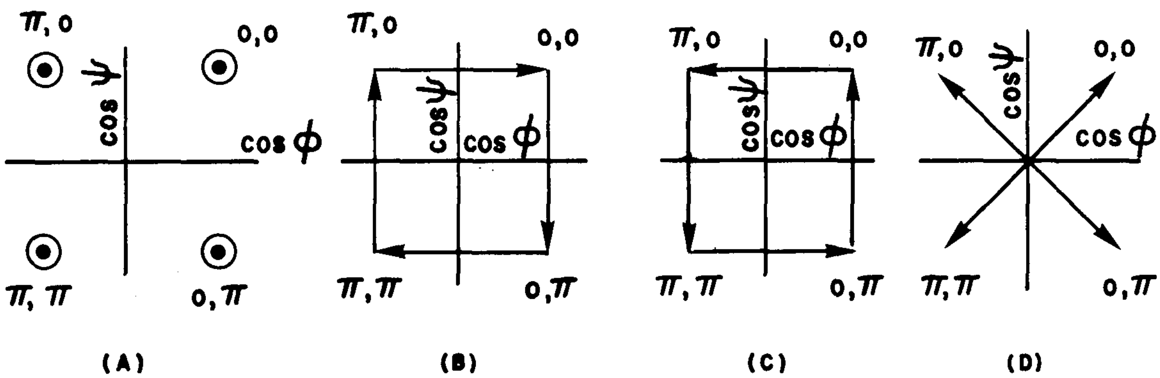

Fig. A2—Graphical representation of Table 2, in which φ, ψ and ϕ', ψ' are represented by the initial and terminal points of an arrow. In Fig. A2a, the arrow has zero length.

Thus Fig. 2d depicts the possible state transitions corresponding to $z_1, z_2 = -1{,}0$. Here, the vectors designating the transformations indicate a transition, say, from the initial state $\varphi = 0$, $\psi = 0$, to the succeeding state π, π, or conversely. Also from 0, π to π, 0 and conversely. Thus, Case d in Fig. 2d corresponds to a rotation from the initial to the final state vector of $\pm\pi$ radians. Case a corresponds to a rotation of 0°, Case b corresponds to the set of clockwise rotations of 90°, and Case c corresponds to the set of counterclockwise rotations of 90°. Since the sets of initial to succeeding state transitions do not overlap and are exhaustive, we can map each of the four possible transition sets corresponding to a given value of z_1, z_2 into a quaternary information symbol. One such mapping is shown in the fourth column of Table A2, headed "Information Code." With this mapping or code, if $\{a_j\}$ and $\{b_j\}$ are two binary messages where

$$\{a_j\} = 1\ 0\ 0\ 1\ 1\ 0\ 1$$

$$\{b_j\} = 1\ 1\ 0\ 1\ 0\ 1\ 1\ ,$$

and if 0, 0 are used as initial reference digits, the encoding of $\{a\}$, $\{b\}$ into φ, ψ sequences would be

$$\varphi :\ 0\ \pi\ \pi\ \pi\ 0\ 0\ 0\ \pi$$
$$\psi :\ 0\ \pi\ 0\ 0\ \pi\ 0\ \pi\ 0\ .$$

Thus, of the four possible pairs of 2-sequences corresponding to $a_1=1$, $b_1=1$, the only pairs initially 0 are $\varphi = 0,\pi$ and $\psi = 0,\pi$. Similarly for $a_2 = 0$, $b_2 = 1$, the pairs $\varphi = \pi,\pi$ and $\psi = \pi, 0$ are the only admissible ones beginning with π. And so on.

To determine the information code from the observation z_1, z_2 (see Table A2), one first determines whether $|z_1|$ or $|z_2|$ is the larger. If the former, then one determines whether $z_1 > -z_1$; otherwise, whether $z_2 > -z_2$. Or, one determines the maximum of the quantities z_1, $-z_2$, z_2, $-z_1$ and infers cases a, b, c, or d accordingly.

Appendix B

DERIVING THE OPTIMUM ESTIMATOR

We are given sample values of the four variables $x^0 = x_c^0, x_s^0, x_c'^0, x_s'^0$ whose functional form is given by Eqs. (15)-(18):

$$x_c = \frac{A}{2}\cos(\gamma_k - \mu_k) + n_1 \quad \text{(B1)}$$

$$x_s = \frac{A}{2}\sin(\gamma_k - \mu_k) + n_2 \quad \text{(B2)}$$

$$x_c' = \frac{A}{2}\cos(\gamma_k - \mu_k') + n_1' \quad \text{(B3)}$$

$$x_s' = -\frac{A}{2}\sin(\gamma_k - \mu_k') + n_2' \quad \text{(B4)}$$

The probability of getting the particular sample values x^0 varies, of course, with the particular values of μ, μ'. According to the principle of maximum likelihood, the optimum decision on the value of μ, μ' to be inferred from the observed values x^0 is that value of μ, μ' which maximizes the probability density of x at x^0.

From Eqs. (B1)-(B4) the probability density that x will assume the value X is

$$P = \frac{1}{4\pi^2 v} \exp\left(\frac{-1}{2v}\left\{[X_c - \frac{A}{2}\cos(\gamma-\mu)]^2 + [X_s + \frac{A}{2}\sin(\gamma-\mu)]^2 + [X_c' - \frac{A}{2}\cos(\gamma-\mu')]^2 + [X_s' + \frac{A}{2}\sin(\gamma-\mu')^2]\right\}\right) . \quad \text{(B5)}$$

After simplifying, the argument of the exponent becomes

$$\left(-\frac{1}{2v}\left\{X_c{}^2 + X_s{}^2 + X_c'^2 + X_s'^2 + \frac{A^2}{2} - A\left[\cos\gamma(X_c \cos\mu + X_s \sin\mu + X_c' \cos\mu' + X_s' \sin\mu') + \sin\gamma(X_c \sin\mu - X_s\cos\mu + X_c' \sin\mu' - X_s' \cos\mu')\right]\right\}\right). \quad \text{(B6)}$$

Letting

$$B = \exp\left\{-\frac{1}{2v}\left[X_c^2 + X_s^2 + X_c'^2 + A^2/2\right]\right\}\Big/ 4\pi^2 v,$$

we can rewrite (B5) as

$$P = B\ \exp\left\{\frac{A}{2v}\left[X_c^2 + X_s^2 + X_c'^2 + X_s'^2 + 2\ (X_cX_c' + X_sX_s')\cos(\mu'-\mu)\right.\right.$$
$$\left.\left. + 2(X_cX_s' - X_sX_c')\sin(\mu'-\mu)\right]^{1/2}\cos(\gamma-\eta)\right\}$$

where

$$\tan\eta = C/D,$$

and

$$C = X_c\sin\mu - X_s\cos\mu + X_c'\sin\mu' - X_s'\cos\mu'$$

$$D = X_c\sin\mu - X_s\cos\mu + X_c'\sin\mu' - X_s'\cos\mu'\ .$$

Since γ is randomly and uniformly distributed over the interval $[0,2\pi]$, the average probability of X is

$$P = \frac{1}{2\pi}\int_0^{2\pi} P(\gamma)d\gamma.$$

Or,

$$\overline{P} = BI_o\left\{\frac{A}{2v}\left[X_c^2 + X_s^2 + X_c'^2 + X_s'^2 + 2(X_cX_c' + X_sX_s')\cos(\mu' - \mu)\right.\right.$$
$$\left.\left. + 2(X_cX_s' - X_sX_c')\sin(\mu'-\mu)\right]^{1/2}\right\},$$

since

$$I_o(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{r\cos\varphi}d\varphi,$$

where $I_o(r)$ is the zeroth-order modified Bessel function. Since $I_o(r)$ is a monotonically increasing even function of r, the value of $\mu'-\mu$ which maximizes any monotonically increasing function of the argument of I_o, such as the square, also maximizes I_o. Accordingly, for given X_c, X_s, X_c', X_s' the value of $\mu'-\mu$ which maximizes

$$F = (X_cX_c' + X_sX_s')\cos(\mu'-\mu) + (X_cX_s' - X_sX_c')\sin(\mu'-\mu)$$

maximizes $\overline{P}$. Since $(\mu'-\mu)$ can take on only the four values 0, $\pi/2$, π, and $3\pi/2$, F can only take on the values

$$F_a = X_c X_c' + X_s X_s' \qquad (\mu' - \mu = 0)$$

$$F_b = X_c' X_s - X_c X_s' \qquad (\mu' - \mu = 3\pi/2)$$

$$F_c = -F_b \qquad (\mu' - \mu = \pi/2)$$

$$F_d = -F_a \qquad (\mu' - \mu = \pi).$$

Accordingly, to determine which value of $(\mu' - \mu)$ is the best inference, one determines the maximum of F_a, F_b, F_c, F_d and infers the associated value of $(\mu' - \mu)$.

Appendix C

INTEGRATION OF EQUATION (36)

Substituting Eqs. (35) and (33) into Eq. (36) results in

$$P = \int_{z=0}^{\infty} \int_{x=z}^{\infty} \frac{1}{\pi s^2} e^{-(x-a)^2/2s^2} e^{-(x-z)^2/2s^2} dx\, dz.$$

If $x/\sqrt{2}\, s = x'$, $a/\sqrt{2}\, s = a'$, and $z/\sqrt{2}\, s = z'$, then

$$P = \int_{z'=0}^{\infty} \int_{x'=z'}^{\infty} \frac{2}{\pi} e^{-(x'-a')^2} e^{-(x'-z')^2} dx'dz'-$$

Let $x'-a' = x$ and $z'-a' = z$. Then

$$P = \frac{2}{\pi} \int_{-a'}^{\infty} \int_{x=z}^{\infty} e^{-x^2} e^{-(x-z)^2} dx\, dz.$$

If we let $\sqrt{2x} = x'$, and $\sqrt{2z} = z'$, then

$$P = \frac{1}{\pi} \int_{-\sqrt{2a'}}^{\infty} \int_{z'}^{\infty} e^{-\left[x'^2 - x'z' + \frac{z'^2}{4} + \frac{z'^2}{4}\right]} dx'\, dz'.$$

Or,

$$P = \frac{1}{\pi} \int_{-\sqrt{2a'}}^{\infty} e^{-z'^2/4} \int_{z'}^{\infty} e^{-(x'-z'/2)^2} dx'\, dz'.$$

Let $x' - z'/2 = x$ and $z'/2 = z$; then

$$P = \frac{2}{\pi} \int_{-a'/\sqrt{2}}^{\infty} e^{-z^2} \int_{z}^{\infty} e^{-x^2} \, dx \, dz.$$

Or, from Pierce,*

$$P = \frac{1}{\sqrt{\pi}} \int_{-a'/\sqrt{2}}^{\infty} e^{-z^2} (1 - \operatorname{erf} z) \, dz.$$

Or, since $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^2} \, dx$ and since $a'/\sqrt{2} = a/2$,

$$P = \frac{1}{2} \left[\operatorname{erf}(a/2) + 1 - \int_{-a|2}^{\infty} \operatorname{erf} z \, d(\operatorname{erf} z) \right]$$

or

$$P = \frac{1}{4} \left[1 + 2 \operatorname{erf}(a/2) + \operatorname{erf}^2(a/2 \right] .$$

*B. O. Pierce, *A Short Table of Integrals,* 3rd rev. ed., Ginn & Company, Boston, 1929, p 116.